Aggregate dynamics under payoff heterogeneity: status-quo bias and non-aggregability

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Preliminary[1]

Abstract

We consider a binary Bayesian game with large population where agents have additively separative payoff heterogeneity, and we investigate dynamic relationship between the aggregate strategy (the action distribution aggregated over all agents) and the strategy composition (the joint distribution of action and payoff type). When each agent's decision follows the best response dynamic with constant revision rate, Ely and Sandholm (2005) prove that the dynamic of aggregate strategy is independent from the strategy composition. We introduce stochastic status-quo biases into BRD; then, the revision rate is positively correlated with the incentive of revision. We verify that stationarity of the aggregate strategy requires balance between inflows and outflows in the strategy composition. The aggregate strategy exhibits instability even when the strategy composition is close to an equilibrium composition, due to the pressure of sorting the composition.

Keywords: best response dynamic, Bayesian games, aggregate dynamics, learning speed, payoff heterogeneity

JEL classification: C73, C62, C61

1 Introduction

There has been a number of research that study how exogeneous heterogeneity in learning or decision protocols affect aggregate dynamics. While agents in these models are simply assumed to have exogeneously different rules, it is natural for economists to rationalize difference in learning protocols by difference in incentives of learning. In this paper, an agent is supposed to overcome a status-quo bias and revise his action more frequently if he can expect larger improvement of

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[1] For the most recent version, visit `http://sites.temple.edu/zusai/research/nonaggbrd/`.

payoff. Then, payoff heterogeneity induces heterogeneity in revision frequency. We focus on a symmetric two-action Bayesian games and find the dynamic of the aggregate strategy significantly depends on the strategy composition and it generally exhibits instability even in a neighborhood of an equilibrium composition.

In the standard best response dynamic (BRD), proposed by Gilboa and Matsui (1991) and Hofbauer (1995), the revision opportunity of each agent arrives according to a Poisson process with a constant rate. Ely and Sandholm (2005) formulate a Bayesian version of best response dynamic by embedding payoff heterogeneity. The Bayesian best response dynamic (B-BRD) is *aggregable* in the sense that the evolution of aggregate behavior depends on the strategy composition only through the current aggregate behavior.

Aggregability simplifies dynamic analysis, as the aggregate behavior is just a distribution on the action set while the strategy composition is a joint distribution on the action set and the type space. On the other hand, this means that aggregable dynamic cannot capture possible effects of the strategy composition on the aggregate behavior.

In any evolutionary dynamics, social change is driven by the group of revising agents; transition of the aggregate behavior is determined by the aggregation of the change in their strategy composition. The constant revision rate of the standard BRD implies that the receipt of a revision opportunity is independent of the current social state and the agent's current action. As the revision opportunity is not biased by the agent's current action, the strategy composition of the revising agents before their revision coincides with that of the society; so is the aggregate behavior. After the revision, they switch to their optimal actions, which is determined in a Bayesian population game by the current aggregate behavior. Since both old and new aggregate strategies of the revising agents are described by the current aggregate behavior of the society, change in the social aggregate behavior is simply determined by its current state.

In reality, we expect the frequency of switching actions to depend on the incentive to switch.[2] Zusai (2014) defines a version of BRD, the *tempered best response dynamic* (tBRD), where the revision rate increases with the payoff difference between the agent's current action and the optimal action. Here we extend the tBRD to a two-strategy Bayesian population game. In the tBRD, the receipt of a revision opportunity by an agent of a given type is positively correlated with that type's incentive to revise. Hence the Bayesian tBRD (B-tBRD) is not aggregable, since the aggregation of revising agents' previous actions may differ from the aggregate behavior of the society. This non-aggregability without exogenous heterogeneity in revision protocols would be useful to select robust implications from agent based dynamics, especially to check the robustness to non-aggregable distributional effects on the aggregate dynamic.

First, we verify that the stationarity of an aggregate strategy requires balance between inflows and outflows in the strategy composition. So, even if the aggregate strategy is in aggregate equilibrium and stationary under the B-BRD, it may not be stationary under the B-tBRD when the

[2] Experimental and empirical research report significance of status-quo bias in real economic choices: see Samuelson and Zeckhauser (1988); Hartman, Doane, and Woo (1991); Madrian and Shea (2001). Also the theory of industrial organization notes the significance of consumers' switching costs in market competition (Klemperer, 1995).

underlying strategy composition is not an equilibrium composition. Second, even if the strategy composition is close to an equilibrium composition, the aggregate strategy can deviate away from the aggregate equilibrium. This suggests that the pressure to sort the composition is stronger than the aggregate dynamic. These results are obtained without imposing a specific form on the functional relationship between the payoff difference and the revision rate. So they robustly hold as long as a revision opportunity of strategy is positively correlated with the incentive of revision.

In other contexts, a number of other authors have used status-quo bias or switching costs to model insensitivity to negligible payoff improvements. On the implementation problems, Lou, Yin, and Lawphongpanich (2010) and Szeto and Lo (2006) consider (deterministic) status-quo biases in a model of traffic congestion. The former investigates congestion pricing in a static equilibrium model, and the latter studies a dynamic traffic assignment model without pricing. Zusai (2014) argues that the tempered BRD can be interpreted as the BRD with stochastic status-quo biases: even if the revision opportunity arrives at a constant rate, a revising agent does not take the optimal action when the realized status-quo bias exceeds the payoff deficit of his current action. As the status-quo bias varies stochastically, larger payoff deficit implies higher probability that a revising agent actually switches to the optimal action. So this paper is the first study that investigates the implication of such stochastic switching costs on aggregate dynamic in a Bayesian population game.

Evolutionary game theorists and macroeconomists have considered the implications of heterogeneity in evolutionary dynamics and learning process on aggregate dynamics and controls, by introducing exogeneous heterogeneity in revision or learning protocols into evolutionary or adaptive dynamics: see Schuster, Sigmund, Hofbauer, Gottlieb, and Merz (1981), Golman (2009, Chapter 5), and Sawa and Zusai (2014) in evolution; Evans, Honkapohja, and Marimon (2001) and Giannitsarou (2003); Honkapohja and Mitra (2006) in macroeconomics. In contrast to these models, the tBRD generates heterogeneous learning speeds endogenously, combining payoff heterogeneity with stochastic status-quo biases. So all heterogeneity in our model results from payoff heterogeneity. This enables us to consider distributional effects of heterogeneity on aggregate social dynamics.

This paper proceeds as follows. The next section defines a two-strategy Bayesian population game. In the following two sections, we define Bayesian BRD and Bayesian tBRD for these games, and compare their implications for aggregate dynamics. In the last section, we discuss implications on dynamic implementation of the social optimum. Lengthy proofs are given in the appendix.

2 The binary-choice Bayesian population game

Here we model a two-strategy Bayesian population game. An agent in the population chooses his action from the two options, IN and OUT. The payoff of IN is a function of the mass of the agents who choose IN in the entire society and the function is common to all agents, while the payoff of OUT is constant and heterogeneous.

The society Ω consists of continuously many agents. Let $\mathcal{B}_\Omega$ be a σ-algebra and $\mathbb{P}_\Omega : \mathcal{B}_\Omega \to [0, 1]$

be the probability measure over Ω. Each agent $\omega \in \Omega$ chooses action $a(\omega)$ from IN (action I) and OUT (action O). An agent who is IN is called a participant. We restrict the action profile $a : \Omega \to \{I, O\}$ to be a $\mathcal{B}_\Omega$-measurable function. The **aggregate participation rate** over all the population is $\bar{x} = \mathbb{P}_\Omega(a(\omega) = I) \in [0, 1]$. We also call it the aggregate strategy.

Each participant gets the payoff $F(\bar{x})$ from being IN, given the aggregate participation rate $\bar{x}$. We assume that $F : [0, 1] \to \mathbb{R}$ is continuously differentiable and bounded; thus it is Lipschitz continuous, say with Lipshitz constant L_F. There is heterogeneity in the payoff from OUT: an agent of type $\theta \in \Theta := [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}$ gains payoff θ from being OUT. The function $\theta : \Omega \to \Theta$ is assumed to be $\mathcal{B}_\Omega$-measurable. Let $\mathcal{B}_\Theta$ be a σ-algebra on Θ and $\mathbb{P}_\Theta$ be the probability measure of types over Θ, generated from $(\Omega, \mathcal{B}_\Omega, \mathbb{P}_\Omega)$ by θ. To avoid trivial indeterminacy of equilibrium, assume that $\mathbb{P}_\Theta$ has no atom. Let $P_\Theta : \Theta \to [0, 1]$ be the cumulative distribution function of θ, i.e., $P_\Theta(\theta) := \mathbb{P}_\Theta((-\infty, \theta])$ and $p_\Theta : \Theta \to \mathbb{R}_+$ be the density function. Assume that p_Θ is bounded above and always positive: there exists $\bar{p}_\Theta$ such that

$$0 < p_\Theta(\theta) \le \bar{p}_\Theta \qquad \text{for all } \theta \in \Theta.$$

The joint distribution of type and action, i.e, the **participants' composition** is expressed by a probability measure $\mathbb{X}$ on Θ such that $\mathbb{X}(B_\Theta) := \mathbb{P}_\Omega(a(\omega) = 1 \text{ and } \theta(\omega) \in B_\Theta)$ for each measurable set $B_\Theta \in \mathcal{B}_\Theta$. $\mathbb{X}$ is dominated by $\mathbb{P}_\Theta$ in the sense that

$$\mathbb{P}_\Theta(B_\Theta) = 0 \quad \Longrightarrow \quad \mathbb{X}(B_\Theta) = 0 \qquad \text{for any } B_\Theta \in \mathcal{B}_\Theta.$$

It follows from Radon-Nikodym theorem that there exists a $\mathcal{B}_\Theta$-measurable function $x : \Theta \to \mathbb{R}$ such that

$$\mathbb{X}(B_\Theta) = \int_{B_\Theta} x(\theta) d\mathbb{P}_\Theta(\theta) \qquad \text{for any } B_\Theta \in \mathcal{B}_\Theta.$$

x is the density function of $\mathbb{X}$; $x(\theta)$ is the *proportion* of participants among the type-θ agents. Like Ely and Sandholm (2005), we sometimes call x a **Bayesian strategy**, imagining a Bayesian game where a player chooses a strategy (a contingent action plan) before he knows his own type.[3] The aggregate participation rate is expressed in terms of the participants' composition via[4]

$$\bar{x} = \mathbb{X}(\Theta) = \mathbb{E}_\Theta x.$$

The space of strategy compositions is $\mathcal{X}_\Theta$, the set of probability measures over Θ that is dominated by $\mathbb{P}_\Theta$. We adopt the variational norm on this space:[5] for any two probability measures

[3]In a Bayesian game, we distinguish a 'player' and an 'agent.' A player comes to the game before knowing its type, and decides on a plan of the action contingent on the realized type: a Bayesian strategy is this contingent plan of one player. In a Bayesian *population* game, an agent comes to the game after knowing his type and decides on an action; the Bayesian strategy is essentially an empirical joint distribution of type and actions.

[4]Here $\mathbb{E}_\Omega$ is the expectation operator on the probability space $(\Omega, \mathcal{B}_\Omega, \mathbb{P}_\Omega)$, while $\mathbb{E}_\Theta$ is that on $(\Theta, \mathcal{B}_\Theta, \mathbb{P}_\Theta)$: i.e., $\mathbb{E}_\Omega f := \int_\Omega f(\omega) d\mathbb{P}_\Omega(\omega)$ for a $\mathcal{B}_\Omega$-measurable function $f : \Omega \to \mathbb{R}$ and $\mathbb{E}_\Theta \tilde{f} := \int_\Theta \tilde{f}(\theta) d\mathbb{P}_\Theta(\theta)$ for a $\mathcal{B}_\Theta$-measurable function $\tilde{f} : \Theta \to \mathbb{R}$.

[5]Here we import formulation and properties of topology on our strategy space from the models on evolutionary

$\mathbb{X}, \mathbb{X}' \in \mathcal{X}_\Theta$ with densities x, x', the distance between $\mathbb{X}$ and $\mathbb{X}'$ is defined as

$$\|\mathbb{X} - \mathbb{X}'\| := \mathbb{E}_\Theta |x - x'| = \int_\Theta |x(\theta) - x'(\theta)| d\mathbb{P}_\Theta(\theta).$$

Let $M(\pi, \theta) \in [0,1]$ be the set of optimal probabilities of IN that maximize the expected payoff given the payoff vector $(\pi, \theta) \in \mathbb{R}^2$:

$$M(\pi,\theta) = \arg\max_{y\in[0,1]} y\pi + (1-y)\theta = \begin{cases} \{1\} & \text{if } \theta < \pi, \\ [0,1] & \text{if } \theta = \pi, \\ \{0\} & \text{if } \theta > \pi. \end{cases}$$

Given $\bar{x}$, IN and OUT are indifferent for an agent with type $\theta = F(\bar{x})$; we call such a type the **indifferent type** given $\bar{x}$. The type $\theta = P_\Theta^{-1}(\bar{x})$ is called the **marginal type**. If the composition is *sorted*, i.e., if agents take IN from the lowest type until the aggregate participation rate reaches $\bar{x}$, then the marginal type is the threshold between the participants and the nonparticipants.

In a Nash equilibrium of this binary-choice game, every agent correctly predicts the participation rate and plays the best response to it. Precisely speaking, the Bayesian strategy $x : \Theta \to [0,1]$ is a **Bayesian equilibrium** if for almost all $\theta \in \Theta$, we have $x(\theta) \in M(F(\mathbb{E}_\Theta x), \theta)$,

$$\text{i.e., } x(\theta) \begin{cases} = 1 & \text{if } \theta < F(\mathbb{E}_\Theta x), \\ \in [0,1] & \text{if } \theta = F(\mathbb{E}_\Theta x), \\ = 0 & \text{if } \theta < F(\mathbb{E}_\Theta x). \end{cases} \tag{1}$$

This means that almost every player's action is optimal given the aggregate state.

In a Bayesian equilibrium, the composition is sorted and the marginal type matches the indifferent type: that is, the aggregate participation rate $\bar{x} = \mathbb{E}_\Theta x$ satisfies

$$\bar{x} = P_\Theta(F(\bar{x})). \tag{2}$$

We say that the aggregate participation rate $\bar{x}$ is an **aggregate equilibrium** if it satisfies (2), and that the Bayesian strategy x is a **Bayesian pseudo-equilibrium** if its aggregation $\bar{x} = \mathbb{E}_\Theta x$ satisfies (2). A Bayesian equilibrium is a Bayesian pseudo-equilibrium but the converse is not true when the participants' composition is not sorted. We should also note that for any aggregate equilibrium $\bar{x}^*$, there is a Bayesian equilibrium $x^* \in \Sigma$ with this aggregate state $\mathbb{E}_\Theta x^* = \bar{x}^*$: namely $x^*(\theta) = 1$ for almost all types $\theta < F(\bar{x}^*)$ and $x^*(\theta) = 0$ for almost all types $\theta > F(\bar{x}^*)$.

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dynamics in a continuous action space such as Oechssler and Riedel (2001): see Cheung (2013) for a well-organized summary. The density-based formula of the variational norm comes from Theorem 5 in Oechssler and Riedel (2001).

3 Aggregability of Bayesian BRD

The best response dynamic, defined by Gilboa and Matsui (1991) and Hofbauer (1995), is an evolutionary dynamic based on myopic optimization in a population game. Each agent switches to the best action based on the current payoff (myopic optimization) only when he gets an opportunity to revise his action. The revision opportunity arrives according to a Poisson process with arrival rate 1. Ely and Sandholm (2005) extend BRD to a dynamic of Bayesian strategy, **Bayesian best response dynamic (B-BRD)**.

In our binary-choice Bayesian population game, B-BRD is defined as

$$\dot{\mathbb{X}}(B_\Theta) = \mathbb{P}_\Omega[B_\Theta \cap (-\infty, F(\mathbb{X}(\Theta))] - \mathbb{X}(B_\Theta) \qquad \text{for any } B_\Theta \in \mathcal{B}_\Theta,$$

or equivalently $\dot{x}_t(\theta) \in M(F(\mathbb{E}_\Theta x_t), \theta) - x_t(\theta)$,

$$\text{i.e.,} \qquad \dot{x}_t(\theta) \begin{cases} = 1 - x_t(\theta) & \text{if } \theta < F(\mathbb{E}_\Theta x_t), \\ \in [-x_t(\theta), 1 - x_t(\theta)] & \text{if } \theta = F(\mathbb{E}_\Theta x_t), \\ = -x_t(\theta) & \text{if } \theta > F(\mathbb{E}_\Theta x_t) \end{cases} \tag{3}$$

for each $\theta \in \Theta$. Notice that $\dot{x}_t(\theta) = 0$ is equivalent to (1). That is, a stationary point of B-BRD coincides with a Bayesian equilibrium. If a Bayesian strategy x is not in equilibrium, there are (a positive measure of) types of agents who want to switch the action and their revision changes the Bayesian strategy.

In general, the dynamic of the aggregate participation rate $\bar{x}$ depends on the composition x_t. Ely and Sandholm (2005) show that B-BRD is **aggregable** in the sense that $\dot{\bar{x}}_t$ depends on x_t only through its aggregation $\bar{x}_t = \mathbb{E}_\Theta x_t$. Indeed, the aggregate dynamic takes the simple form

$$\dot{\bar{x}}_t = \mathbb{E}_\Theta \dot{x}_t = P_\Theta(F(\bar{x}_t)) - \bar{x}_t. \tag{4}$$

That is, the future aggregate state $\{\bar{x}_t\}_{t \geq 0}$ is perfectly predicted from the current aggregate state $\bar{x}_0$, without knowing the participants' composition x_0.

In particular, once the aggregate state falls in an aggregate equilibrium, it stays there forever even if the participants' composition is not a Bayesian equilibrium. To provide a contrast with later results, let us rephrase this as a theorem.

Theorem 1 (Aggregability of B-BRD and stationarity of aggregate equilibrium, Ely and Sandholm 2005)**.** *Under B-BRD with the constant revision rate 1, the aggregate state $\bar{x}_t = \mathbb{E}_\Theta x_t$ follows the aggregate dynamic* (4)*. In particular, an aggregate equilibrium* (2) *is a rest point of this aggregate dynamic and vice versa.*

Rough proof. Under B-BRD (3), the dynamic of the aggregate state $\bar{x}_t = \mathbb{E}_\Theta x_t$ is

$$\begin{aligned}\dot{\bar{x}}_t = \mathbb{E}_\Theta \dot{x}_t(\theta) &= \int_{\underline{\theta}}^{F(\bar{x}_t)} (1 - x_t(\theta)) dP_\Theta(\theta) + \int_{F(\bar{x}_t)}^{\bar{\theta}} (-\sigma_t(\theta)) dP_\Theta(\theta) \\ &= \left\{ \int_{\underline{\theta}}^{F(\bar{x}_t)} dP_\Theta(\theta) - \int_{\underline{\theta}}^{\bar{\theta}} x_t(\theta) dP_\Theta(\theta) \right\} = \{P_\Theta(F(\bar{x}_t)) - \bar{x}_t\}.\end{aligned}$$

So we obtain (4). It follows that (2) is equivalent to $\dot{\bar{x}} = 0$. □

The computation in this proof gives us the intuition behind the aggregation of B-BRD. Under B-BRD, each agent revises his strategy with a common and constant rate 1. In particular, receipts of revision opportunities are independent of an agent's type and action. Thus the current aggregate participation rate of the revising agents is just $\bar{x}_t$ before their revision; it becomes $P_\Theta(F(\bar{x}_t))$ after the revision because the types below the indifferent type $\theta = F(\bar{x}_t)$ switch to IN and the proportion of such types among the revising agents is the same as that in the whole society, namely $P_\Theta(F(\bar{x}_t))$. As the mass of revising agents is scaled down to 1 per unit time, these transitions of revising agents yield the aggregate dynamic (4).

4 Non-aggregability of Bayesian tBRD

4.1 Bayesian tempered BRD

The assumption of a constant revision rate is crucial to obtain the aggregability of B-BRD. The constant revision rate ensures that every agent revises his action at the same rate regardless of the incentive for revision. This assumption, while greatly simplifying the analysis, may not always be appropriate.

Zusai (2014) proposes a variant of BRD, the **tempered best response dynamic (tBRD)**, where an agent becomes less likely to revise action as the payoff improvement from revision becomes smaller. He considers a population game with finitely many separate populations. The tBRD can be interpreted as the standard BRD with stochastic status-quo biases or switching costs.

Here we extends tBRD to a single-population Bayesian game and define Bayesian tBRD. In the B-tBRD, the revision rate is 'tempered' by an increasing function $Q : \mathbb{R}_+ \to [0, 1]$ of the difference between the agent's current payoff and the maximal payoff. The rate is thus $Q(|F(\bar{x}) - \theta|)$ for a type-θ agent; in particular, it is $Q(0)$ if he is taking the optimal one. As in Zusai (2014), we assume that $Q : \mathbb{R}_+ \to [0, 1]$ is continuously differentiable and satisfies $Q(0) = 0$, $Q(q) > 0$ for all $q > 0$. Besides, we assume the existence of $\bar{q} \in (0, \infty]$ such that Q is strictly increasing in $[0, \bar{q})$ and

$Q(\bar{q}) = 1$. Then the dynamic of the Bayesian strategy is

$$\dot{x}_t(\theta) = Q(|F(\mathbb{E}_\Theta x_t) - \theta|)\{M(F(\mathbb{E}_\Theta x_t), \theta) - x_t(\theta)\}$$
$$= \begin{cases} Q(F(\mathbb{E}_\Theta x_t) - \theta)\{1 - x_t(\theta)\} & \text{if } \theta < F(\mathbb{E}_\Theta x_t), \\ 0 & \text{if } \theta = F(\mathbb{E}_\Theta x_t), \\ -Q(\theta - F(\mathbb{E}_\Theta x_t))x_t(\theta) & \text{if } \theta > F(\mathbb{E}_\Theta x_t). \end{cases} \tag{5}$$

This is the **Bayesian tempered best response dynamic** (B-tBRD).

We can establish unique existence of the path $\{x_t\}$ from arbitrary $x_0 \in \Sigma$ and its Lipschitz continuity in x_0 by Theorem A.3 in Ely and Sandholm (2005), which they use to prove those properties for the standard Bayesian BRD. Similar to the B-BRD, a stationary point of the B-tBRD coincides with a Bayesian equilibrium.

Like the (non-Bayesian) tBRD, the B-tBRD (5) can be interpreted as the standard B-BRD with stochastic status-quo biases. First we assume that the revision opportunity of each individual agent arrives according to a Poisson process with constant arrival rate > 0, like the standard BRD. Then, we introduce stochastic status-quo biases. Once an agent gets a revision opportunity at time t, he draws a value $q \in [0, \bar{q}]$ of status-quo bias from a continuous distribution Q. Then he decides the next action a' given his current action a, so as to maximize the genuine payoff plus the status-quo bias:[6]

$$\arg \max_{a' \in A = \{O, I\}} F_{a'}(\bar{x}; \theta) + q\mathbb{1}(a' = a).$$

The optimal revision is as follows.

An action-O player switches to I	if $F(\bar{x}) > \theta + q$, i.e. $q < F(\bar{x}) - \theta$;
An action-I player switches to O	if $F(\bar{x}) + q < \theta$, i.e. $q < \theta - F(\bar{x})$.

That is, an agent receiving a revision opportunity switches his action if his current action is not optimal and also he draws a status-quo bias satisfying $q < |F(\bar{x}) - \theta|$; otherwise, a revising agent sticks with his current action. In sum, a suboptimal action player switches to the optimal action at the rate $Q(|F(\bar{x}) - \theta|)$, and an optimal action player never switches. We thus have the same Bayesian dynamic as (5).

4.2 Stationarity condition and instability of aggregate equilibrium

While the incentive-dependent revision rate changes each type's revision rate, it also changes the direction of the transition of the aggregate participation rate. The dynamic of the aggregate participation rate $\bar{x}_t$ is obtained from aggregation of the transition of the participate rates over all

[6] The indicator function $\mathbb{1}(P(\cdot)) : \mathcal{Z} \to \mathbb{R}$ of a propositional function (predicate) $P : \mathcal{Z} \to \{\text{true}, \text{false}\}$ means $\mathbb{1}(P(z)) = 1$ if the proposition $P(z)$ is true and $\mathbb{1}(P(z)) = 0$ if it is false.

the types:

$$\dot{\bar{x}}_t = \mathbb{E}_\Theta \dot{x}_t(\theta) = \bar{V}(x_t) := \int_{\underline{\theta}}^{F(\bar{x}_t)} Q(F(\bar{x}_t)-\theta)(1-x_t(\theta))dP_\Theta(\theta) + \int_{F(\bar{x}_t)}^{\bar{\theta}} \{-Q(\theta-F(\bar{x}_t))x_t(\theta)\}dP_\Theta(\theta).$$

In an infinitesimal period of time $[t, t+dt]$, the net increase of aggregate participation rate $d\bar{x}_t$ under B-tBRD (5) is approximated as $d\bar{x}_t \approx V(x_t)dt$. We decompose $V(x_t)$ as $V(x_t) = \tilde{Q}_t(\tilde{y}_t - \tilde{x}_t)$ with

$$\begin{aligned}
\tilde{Q}_t &:= \int_{\underline{\theta}}^{F(\bar{x}_t)} Q(F(\bar{x}_t)-\theta)(1-x_t(\theta))dP_\Theta(\theta) + \int_{F(\bar{x}_t)}^{\bar{\theta}} \{Q(\theta-F(\bar{x}_t))x_t(\theta)\}dP_\Theta(\theta), \\
\tilde{y}_t &:= \int_{\underline{\theta}}^{F(\bar{x}_t)} Q(F(\bar{x}_t)-\theta)(1-x_t(\theta))dP_\Theta(\theta)/\tilde{Q}, \\
\tilde{x}_t &= \int_{F(\bar{x}_t)}^{\bar{\theta}} Q(\theta-F(\bar{x}_t))x_t(\theta)dP_\Theta(\theta)/\tilde{Q}.
\end{aligned}$$

In this infinitesimal period, a type-θ player of a suboptimal action receives a revision opportunity with probability $Q(|F(\bar{x}_t)-\theta|)dt$. In the definition of $\tilde{Q}_t$, the first integral multiplied with dt is the mass of revising agents who have types smaller than $F(\bar{x}_t)$ and took OUT before the revision, and the second integral multiplied with dt is the mass of those who have type larger than $F(\bar{x}_t)$ and took IN. The sum $\tilde{Q}_t dt$ is thus the mass of all revising agents in this period. $\tilde{x}_t$ is the proportion of the latter group to all the revising agents, namely the participation rate among revising agents before their revision. After revision, this rate changes to $\tilde{y}$, since the former group enters and the latter exits.

If the revision opportunity were independent from action and type, then $\tilde{Q}_t, \tilde{y}_t$ and $\tilde{x}_t$ are simply $1, P_\Theta(F(\bar{x}_t))$ and $\bar{x}_t$, respectively; the the last equation $\tilde{dt}(\tilde{y}_t - \tilde{x}_t)$ would reduce to $P_\Theta(F(\bar{x}_t)) - \bar{x}_t$, the aggregate dynamic under B-BRD (4). But it does not under B-tBRD and thus $\dot{\bar{x}}_t$ can be different from (4). In particular, the sign of $\tilde{y}_t - \tilde{x}_t$ may be different from the sign of $P_\Theta(F(\bar{x}_t)) - \bar{x}_t$. So the dynamic of the aggregate participation rate $\bar{x}_t$ under B-tBRD is different from the aggregate of B-BRD, not only in the speed (mass of revising agents $\tilde{Q}$) but also in the direction.

Under B-tBRD the current state of the aggregate participation rate $\bar{x}_t$ is not sufficient to predict its transition $\dot{\bar{x}}_t$. In particular, an aggregate equilibrium is not necessarily stationary, unless the underlying Bayesian strategy is exactly a Bayesian equilibrium. The following theorem states a necessary and sufficient condition to keep the aggregate participation rate at an aggregate equilibrium. Recall that $\bar{q} \in (0, \infty]$ is the upper bound of status-quo bias.

Theorem 2 (Stationarity condition of an aggregate equilibrium under B-tBRD)**.** *Assume a continuous type distribution P_Θ with density p_Θ. Suppose the aggregate state $\bar{x}_t$ is an aggregate equilibrium at time 0: $\bar{x}_0 = P_\Theta(F(\bar{x}_0))$. Let $\bar{x}^* = \bar{x}_0$ and $\theta^* = F(\bar{x}^*)$. Then, $\bar{x}_t = \bar{x}^*$ for all $t \in [0, \Delta]$ with some $\Delta > 0$ if and only if*

$$(1 - x(\theta^* - q))p_\Theta(\theta^* - q) = x(\theta^* + q)p_\Theta(\theta^* + q) \quad \text{for all } q \in [0, \bar{q}]. \tag{6}$$

Indeed $\bar{x}_t = \bar{x}^*$ for all $t \in [0, \infty)$ if this condition holds.

The left hand side is the density of those who have type $\theta = \theta^* - q$ and play Out at time 0, and the right hand side is the density of those who have type $\theta = \theta^* + q$ and play In at time 0. Noticing that the Bayesian strategy moves from x_0 to the Bayesian equilibrium (1) with $\mathbb{E}_\Theta x^* = \bar{x}^*$, provided that $\bar{x}_t \equiv \bar{x}^*$; the former group gradually enters and the latter exits. The stationarity of aggregate equilibrium under B-tBRD thus requires balancing these entries and exits at each level of type difference $q = |\theta - \theta^*| \in (0, \bar{q})$, not only balancing the entries and exits aggregated over all types.

The upper bound $\bar{q}$ matters to keep the aggregate equilibrium as a stationary point of the aggregate dynamic. The shape of the distribution function $Q(\cdot)$ does not appear in this stationarity condition (6), but it determines the size of divergence from the aggregate equilibrium when this condition is violated.

In general, the stationarity condition (6) imposes a non-trivial restriction on the type distribution P_Θ and the composition x_0 at time 0. Assume that density p_Θ is continuous. If x_0 is continuous in θ at θ^*, (6) implies $p_\Theta(\theta^*) = 1/2$. In particular, if $x_0(\theta) = \bar{x}^*$ for all θ, the aggregate equilibrium must be $\bar{x}^* = 1/2$. In contrary, suppose that x_0 is discontinuous at θ^* such that $x_0(\theta) = x_L$ for all $\theta < \theta^*$, $x_0(\theta) = x_H$ for all $\theta > \theta^*$ and $x_L P_\Theta(\theta^*) + s_H(1 - P_\Theta(\theta^*)) = \bar{x}^*$. Then, (6) implies $x_L + x_H = 1$ and thus $\bar{x}^* = 1/2$. In these two cases, any aggregate equilibria other than $\bar{x}^* = 1/2$ cannot be stationary.

In aggregate BRD, stability of an aggregate equilibrium is easily determined from F'. In particular, if F' is positive, the aggregate equilibrium is unstable. This holds in the aggregation of B-tBRD as well. Somewhat surprisingly, an aggregate equilibrium cannot be stable even around a Bayesian equilibrium, whatever F' is; in any neighborhood of a Bayesian equilibrium x^*, we can find a Bayesian strategy at which the aggregate strategy move away from $\mathbb{E}_\Theta x^*$.

Theorem 3. *If $\mathbb{X}^*$ satisfies either of the following two conditions, then, for any $\bar{\varepsilon} > 0$, there is a Bayesian strategy $\mathbb{X}^\dagger$ such that $\|\mathbb{X}^\dagger - \mathbb{X}^*\| < \bar{\varepsilon}$, $\bar{x}^\dagger := \mathbb{E}_\Theta x^\dagger > \bar{x}^*$ and $\dot{\bar{x}} = \bar{V}(x^\dagger) > 0$.*

1. *$\mathbb{X}^*$ is not a Bayesian equilibrium but its aggregate $\bar{x}^*$ is in the aggregate equilibrium, its density x^* satisfies the balancing condition (6) and thus $\bar{V}(\mathbb{X}^*) = 0$. Further, $F'(\bar{x}^*) > 0$.*

2. *$\mathbb{X}^*$ is a Bayesian equilibrium with density x^* and aggregate $\bar{x}^*$. Assume that $F(\bar{x}^*)$ belongs to the interior of Θ.*

5 Discussion: dynamic implementation of social optimum

In the tempered best response dynamic proposed by Zusai (2014), frequency of revision increases with incentive of revision, namely the payoff deficit of the current action. This paper applies the tBRD to a binary-choice game with heterogeneous payoff types. The tBRD makes the dynamic of aggregate participation rate depend on the participants' composition not only through the current aggregate participation rate, unlike the aggregate dynamic under the standard BRD with a constant revision rate. We thus propose tBRD as a tool to generate non-aggregable perturbation of BRD in population games with payoff heterogeneity.

One application of Bayesian evolutionary dynamics is dynamic pricing schemes used by a social planner to implement a socially optimal aggregate behavior.[7] For example, consider a congestion pricing efficient utilization of toll and free lanes,[8] with commuters who have different values of time. The central planner evaluates the social state primarily in terms of the allocation of commuters (aggregate behavior), i.e. the numbers of commuters (utilization level) on each lane.[9]

The optimal control under Bayesian BRD is a bang-bang control. It indeed achieves the most efficient aggregate behavior in the shortest time. As long as the utilization of the toll lane is below the efficient level, the central planner should keep the toll at zero (or the lowest feasible toll). Once the utilization level reaches the efficient level, the toll should be raised so as to keep this utilization level as a Nash equilibrium. Still the strategy composition may change: namely individual commuters may still switch their lanes. But inflow and outflow are offset thanks to the constant revision rate of B-BRD.

In contrast, under Bayesian tBRD, this equilibrium toll may not maintain the efficient aggregate behavior, depending on the strategy composition. When the strategy composition is not in equilibrium, there are players of suboptimal actions who could gain some payoff improvement from switching to the optimal action. Under the B-tBRD, the degree of payoff improvement can break the balance between inflow and outflow. Compare two groups of commuters, namely those who are currently taking the free lane but would switch to toll lane (inflow) and those who are taking the toll lane but would switch to free lane (outflow). Provided that the current utilization level is in aggregate equilibrium given the equilibrium toll, the masses of these two groups are the same and thus the inflow and outflow are offset under the BRD. But, suppose that the payoff improvements of the former group is larger than that of the latter. Then, under the tBRD, the former group switches to the free lane in greater numbers than the latter group switches to the toll lane; the inflow gets larger than the outflow and thus the utilization level overshoots the efficient level even with the equilibrium toll. So the bang-bang control does not work, as it can leave such disparity in the degree of payoff improvement. Instead, the tBRD requires one to smooth the control depending on the strategy composition, so as to reduce the payoff disparity.

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[7]See Sandholm (2002, 2005, 2007) for dynamic pricing schemes that robustly works under a variety of evolutionary dynamics.

[8]Dynamic congestion pricing on toll lanes is actually implemented in Minnesota (MnPass), Texas, etc.

[9]In the long run, the strategy composition would converge to a Bayesian equilibrium given the utilization level.

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A Proofs

A.1 Theorem 2

Proof of the "if" part in Theorem 2. The time derivative $\dot{\bar{x}}$ at time 0 is

$$\begin{aligned}\dot{\bar{x}}_0 &= \mathbb{E}\dot{x}_0(\theta)\\ &= \int_{\underline{\theta}}^{\theta^*} Q(\theta^*-\theta)(1-x_t(\theta))p_\Theta(\theta)d\theta + \int_{\theta^*}^{\bar{\theta}} \{-Q(\theta-\theta^*)x_t(\theta)\}p_\Theta(\theta)d\theta\\ &= \int_0^{\theta^*-\underline{\theta}} Q(q)(1-x_t(\theta^*-q))p_\Theta(\theta^*-q)dq + \int_0^{\bar{\theta}-\theta^*} \{-Q(q)x_t(\theta^*+q)\}p_\Theta(\theta^*+q)dq\\ &= \int_0^{\bar{q}} Q(q)\left\{(1-x_t(\theta^*-q))p_\Theta(\theta^*-q)-x_t(\theta^*+q)p_\Theta(\theta^*+q)\right\}dq\\ &\quad+\int_{\bar{q}}^{\infty} Q(q)\left\{(1-x_t(\theta^*-q))p_\Theta(\theta^*-q)-x_t(\theta^*+q)p_\Theta(\theta^*+q)\right\}dq. \end{aligned} \tag{7}$$

On the other hand, the aggregate equilibrium $\bar{x}_0 = \bar{x}^*$ implies

$$\begin{aligned} P_\Theta(\theta^*) - \bar{x}^* = &\int_0^{\bar{q}} \left\{(1-x_t(\theta^*-q))p_\Theta(\theta^*-q)-x_t(\theta^*+q)p_\Theta(\theta^*+q)\right\}dq\\ &+\int_{\bar{q}}^{\infty} \left\{(1-x_t(\theta^*-q))p_\Theta(\theta^*-q)-x_t(\theta^*+q)p_\Theta(\theta^*+q)\right\}dq. \end{aligned} \tag{8}$$

The stationarity condition (6) implies that the first integrals in these two equations (7) and (8) are both zero. Further, by $\bar{x}_0 = \bar{x}^* = P_\Theta(\theta^*)$, the second integral in (8) is zero, which implies the second one in (7) is also zero since $Q(q) = 1$ for all $q \geq \bar{q}$. Therefore, the stationarity condition implies $\dot{\bar{x}}_0 = 0$.

Besides, for any $q \in [0, \bar{q}]$, the two types $\theta^* - q$ and $\theta^* + q$ satisfy

$$\frac{d}{dt}\log\{1 - x_t(\theta^* - q)\} = -Q(q) = \frac{d}{dt}\log\sigma_t(\theta^* + q),$$

as long as $\bar{x}_t = \bar{x}^*$ and hence $F(\bar{x}_t) = \theta^*$. So $\bar{x}_t = \bar{x}^*$ keeps the stationarity condition (6) satisfied at time t. Therefore, once (6) holds at time 0, we have $\bar{x}_t = \bar{x}^*$ for all $t \geq 0$. □

In the proof of the "only if" part, we use the following lemma.

Lemma 1. *Let a random variable κ^i $(i = 1, 2)$ have a continuous c.d.f. G^i with density g^i. Besides, assume that a function $\beta : \mathbb{R} \to [0, \bar{b}] \subset \mathbb{R}$ be nondecreasing and especially strictly increasing in an interval $\bar{K} \subset \mathbb{R}$ with $\inf\{\beta(k)|k \in \bar{K}\} = 0$ and $\sup\{\beta(k)|k \in \bar{K}\} = \bar{b}$. Suppose that there exists $\epsilon > 0$ such that*

$$\int_{-\infty}^{+\infty} \exp(-\beta(k)\tau)g^1(k)dk = \int_{-\infty}^{+\infty} \exp(-\beta(k)\tau)g^2(k)dk \quad \textit{for all } \tau \in (-\epsilon, \epsilon). \tag{9}$$

Then, we have

$$g^1(k) = g^2(k) \quad \textit{for all } k \in \bar{K}.$$

Proof of Lemma 1. Define function $B^i : \mathbb{R} \to \mathbb{R}$ as $B^i(b) := \mathbb{P}[\beta(\kappa^i) \le b] = G^i(\sup \beta^{-1}(b))$ ($i = 1, 2$). Then, it is the cumulative distribution function of the random variable $\beta(\kappa^i)$ with a bounded support, $\operatorname{supp}(B^i) = [0, \bar{b}]$. Each side of the assumption (9) is

$$\int_{-\infty}^{\infty} \exp(-\beta(k)\tau) g^i(k) dk = \int_0^{\bar{b}} e^{-b\tau} dB^i(b),$$

namely the moment generating function of each distribution function B^i.

So the assumption (9) means that these two moment generating functions coincide each other for $\tau \in (-\epsilon, \epsilon)$. As these two have bounded support $[0, \bar{b}]$, this implies their identity $B^1(b) = B^2(b)$ for all $b \in \mathbb{R}$ (Billingsley, 1979, p.253). $B^1 \equiv B^2$ means

$$G^1(\beta^{-1}(b)) = G^2(\beta^{-1}(b)) \quad \text{for all } b \in [0, \bar{b}];$$

because β is non-decreasing and strictly increasing in $\bar{K}$, it is equivalent to

$$G^1(k) = G^2(k), \quad \text{i.e., } g^1(k) = g^2(k) \quad \text{for all } k \in \bar{K}.$$

□

Proof of the "only if" part in Theorem 2. As long as the aggregate state stays at $\bar{x}_0$, the indifferent type remains the same, i.e., $F(\bar{x}_t) = F(\bar{x}_0) = \theta^*$. This implies that the revision rate of type θ is constant, i.e. $Q(|F(\bar{x}_t) - \theta|) = Q(|\theta^* - \theta|)$ for $t \in [0, \Delta]$.

We explicitly obtain the path $\{x_t(\theta) | t \in [0, \Delta]\}$ from $x_0(\theta)$. Fix a moment of time $T \in [0, \Delta)$ and express the path as the transition from time $T \in [0, \Delta]$. For all $\tau \in [-T, \Delta - T]$,

$$x_{T+\tau}(\theta) = \begin{cases} 1 - (1 - x_T(\theta)) \exp(-Q(\theta^* - \theta)\tau) & \text{if } \theta < \theta^*, \\ x_T(\theta^*) & \text{if } \theta = \theta^*, \\ x_T(\theta) \exp\left[-Q\left((\theta - \theta^*)\tau\right)\right] & \text{if } \theta > \theta^*. \end{cases}$$

The aggregate state is thus expressed as

$$
\begin{aligned}
\bar{x}_{T+\tau} &= \int_{\underline{\theta}}^{\theta^*} \{1-(1-x_T(\theta))\exp(-Q(\theta^*-\theta)\tau)\} p_\Theta(\theta) d\theta \\
&\quad + \int_{\theta^*}^{\bar{\theta}} \{x_T(\theta)\exp\left[-Q\left((\theta-\theta^*)\tau\right)\right]\} p_\Theta(\theta) d\theta \\
&= P_\Theta(\theta^*) - \int_0^{\theta^*-\underline{\theta}} \exp(-Q(q)\tau)(1-x_T(\theta^*-q)) p_\Theta(\theta^*-q) dq \\
&\quad + \int_0^{\bar{\theta}-\theta^*} \exp(-Q(q)\tau) x_T(\theta^*+q) p_\Theta(\theta^*+q) dq \\
&= P_\Theta(\theta^*) - M_T^1 \int_0^{\theta^*-\underline{\theta}} \exp(-Q(q)\tau) y_T^1(q) dq + M_T^2 \int_0^{\bar{\theta}-\theta^*} \exp(-Q(q)\tau) y_T^2(q) dq,
\end{aligned}
$$

where $Y_T^i \in \mathbb{R}$ and $y_T^i : \mathbb{R} \to \mathbb{R}_+$ $(i = 1, 2)$ are given by

$$
\begin{aligned}
Y_T^1 &:= \int_0^{\theta^*-\underline{\theta}} (1-x_T(\theta^*-q)) p_\Theta(\theta^*-q) dq, \\
y_T^1(q) &:= \begin{cases} (1-x_T(\theta^*-q)) p_\Theta(\theta^*-q)/Y_T^1 & \text{if } q \in [0, \theta^*-\underline{\theta}] \\ 0 & \text{otherwise,} \end{cases} \\
Y_T^2 &:= \int_0^{\bar{\theta}-\theta^*} x_T(\theta^*+q) p_\Theta(\theta^*+q) dq, \\
y_T^1(q) &:= \begin{cases} x_T(\theta^*+q) p_\Theta(\theta^*+q)/Y_T^2 & \text{if } q \in [0, \bar{\theta}-\theta^*] \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Y_T^1 is the mass of those who have types below θ^* and play OUT at time T and Y_T^2 is the mass of those who have above θ^* and play IN at time T. Notice that $Y_T^1 - Y_T^2 = P_\Theta(\theta^*) - \mathbb{E}_\Theta x_T$. The function y_T^i is the density function of this type difference in the mass of Y_T^i.

The aggregate state staying at the aggregate equilibrium $\bar{x}_T = \bar{x}^* = P_\Theta(\theta^*)$ implies $Y_T^1 = Y_T^2$. Hence the aggregate state $\bar{x}_{T+\tau}$ remains at $\bar{x}^* = P_\Theta(\theta^*)$ if and only if for all $\tau \in [-T, \Delta - T]$,

$$
\int_{-\infty}^{+\infty} \exp(-Q(q)\tau) y_T^1(q) dq = \int_{-\infty}^{+\infty} \exp(-Q(q)\tau) y_T^2(q) dq.
$$

In particular, $\bar{x}_{T+\tau} = \bar{x}^*$ for all $t \in [0, \Delta]$ if and only if the above equation holds for each $T = \Delta/i$ and all $\tau \in (-\Delta/i, \Delta/i)$ with any $i = 2, 3, \ldots$. Each side of the equation is the moment generating function of $-Q(q)$ under each of the two conditional density functions y_T^1, y_T^2 of the type difference q. Notice that function $Q(q)$ is strictly increasing in the range $(0, \bar{q})$ with $Q(0) = 0$ and $Q(\bar{q}) = 1$.

According to Lemma 1, the above equation means that these two conditional distributions should be the same in this range. Therefore, $\bar{x}_{T+\tau} = \bar{x}^*$ for all $t \in [0, \Delta]$ only if

$$
y_{\Delta/i}^1(q) = y_{\Delta/i}^2(q) \quad \text{for all } q \in [0, \bar{q}]
$$

for each $i = 2, 3, \ldots$. Continuity of x_t in time t implies continuity of y_t^i in time t. As this equation holds for $i = 2, 3, \ldots$, we thus obtain $y_0^1(q) = y_0^2(q)$, i.e. (6). □

A.2 Theorem 3

First, we decompose the aggregate dynamic $\bar{V}$ into $\bar{V} = \bar{V}_0 + \Delta\bar{V}_1 + \Delta\bar{V}_2 + \Delta\bar{V}_3$ with

$$
\begin{aligned}
\bar{V}_0(x) &:= \int_{-\infty}^{\theta^*} Q(\theta^* - \theta)(1 - x(\theta))dP_\Theta(\theta) + \int_{\theta^*}^{+\infty} \{-Q(\theta - \theta^*)x(\theta)\}dP_\Theta(\theta), \\
\Delta\bar{V}_1(x) &= \int_{-\infty}^{\theta^*} \{Q(F(\bar{x}) - \theta) - Q(\theta^* - \theta)\}(1 - x(\theta))dP_\Theta(\theta), \\
\Delta\bar{V}_2(x) &= \int_{\theta^*}^{F(\bar{x})} \{Q(F(\bar{x}) - \theta)(1 - x(\theta)) + Q(\theta - \theta^*)x(\theta)\}dP_\Theta(\theta), \\
\Delta\bar{V}_3(x) &= -\int_{F(\bar{x})}^{+\infty} \{Q(\theta - F(\bar{x})) - Q(\theta - \theta^*)\}x(\theta)dP_\Theta(\theta).
\end{aligned}
$$

With $\bar{x}^* =: \mathbb{E}_\Theta x^*$ and $\theta^* := F(\bar{x}^*)$, we have the linear approximation of Q as

$$
Q(F(\bar{x}) - \theta) = Q(\theta^* - \theta) + Q'(\theta^* - \theta)F'(\bar{x}^*)(\bar{x} - \bar{x}^*) + o(\bar{x} - \bar{x}^*),
$$

where o is Landau's little-o, i.e., $|o(\delta)/\delta| \to 0$ as $\delta \to 0$. It follows that

$$
\Delta\bar{V}_1(x) = F'(\bar{x}^*)(\bar{x} - \bar{x}^*) \left\{ \int_{-\infty}^{\theta^*} Q'(\theta^* - \theta)(1 - x(\theta))dP_\Theta(\theta) + o(\bar{x} - \bar{x}^*) \right\}.
$$

Similarly, we obtain

$$
\Delta\bar{V}_3(x) = F'(\bar{x}^*)(\bar{x} - \bar{x}^*) \left\{ \int_{F(\bar{x})}^{+\infty} Q'(\theta - \theta^*)x(\theta)dP_\Theta(\theta) + o(\bar{x} - \bar{x}^*) \right\}.
$$

Presume that $F(\bar{x}) \geq \theta^*$. Then, $\Delta\bar{V}_2(x)$ is $o(\bar{x}^* - \bar{x})$, since $\Delta V_2(x) \geq 0$ and

$$
\begin{aligned}
\Delta\bar{V}_2(x) &\leq \int_{\theta^*}^{F(\bar{x})} \{Q(F(\bar{x}) - \theta^*)(1 - x(\theta)) + Q(F(\bar{x}) - \theta^*)x(\theta)\}dP_\Theta(\theta) = Q(F(\bar{x}) - \theta^*)\}\mathbb{P}_\Theta((\theta^*, F(\bar{x}))) \\
&\leq Q(F(\bar{x}) - \theta^*)\bar{p}_\Theta(F(\bar{x}) - \theta^*) \\
&= \{Q'(0) + F'(\bar{x}^*)(\bar{x}^* - \bar{x}) + o(\bar{x}^* - \bar{x})\}\bar{p}_\Theta\{(F'(\bar{x}^*)(\bar{x}^* - \bar{x}) + o(\bar{x}^* - \bar{x})\} \\
&= Q'(0)F'(\bar{x}^*)^2\bar{p}_\Theta(\bar{x}^* - \bar{x})^2 + o((\bar{x}^* - \bar{x})^2).
\end{aligned}
$$

In the first inequality, we use the assumption that Q is increasing and $F(\bar{x}) \geq \theta^*$. For the second equality, notice that, in both two parts, $\bar{x}^*$ is an aggregate equilibrium and thus $\theta^* = F(\bar{x}^*)$.

Part 1. First, since x^* is not a Bayesian equilibrium but $\bar{x}^*$ is an aggregate equilibrium, the participant composition $\mathbb{X}^*$ should satisfy

$$P_\Theta(\theta^*) - \mathbb{X}^*((-\infty, \theta^*]) = \mathbb{X}^*((\theta^*, +\infty)) > 0.$$

Choose $\varepsilon \in (0, \bar{\varepsilon})$ arbitrarily. Define Bayesian strategy $x^\dagger$ as

$$x^\dagger(\theta) := \begin{cases} (1-\varepsilon)x^*(\theta) + \varepsilon & \text{if } \theta < \theta^*, \\ x^*(\theta) & \text{otherwise.} \end{cases}$$

This Bayesian strategy induces the aggregate participation rate $\bar{x}^\dagger$ such that

$$\begin{aligned} \bar{x}^\dagger := \mathbb{E}_\Theta x^\dagger &= \mathbb{E}_\Theta x^* + \int_{-\infty}^{\theta^*} \varepsilon(1 - x^*(\theta)) dP_\Theta(\theta) \\ &= \bar{x}^* + \varepsilon \underbrace{\{P_\Theta(\theta^*) - \mathbb{X}^*((-\infty, \theta^*])\}}_{A>0} > \bar{x}^*. \end{aligned}$$

Observe that

$$\begin{aligned} \bar{V}_0(x^\dagger) = &\int_{-\infty}^{\theta^*} Q(\theta^* - \theta)(1 - x^\dagger(\theta)) dP_\Theta(\theta) \int_{\theta^*}^{+\infty} \{-Q(\theta - \theta^*) x^\dagger(\theta)\} dP_\Theta(\theta) \\ =&V(x^*) + \int_{-\infty}^{\theta^*} Q(\theta^* - \theta)(x^*(\theta) - x^\dagger(\theta)) dP_\Theta(\theta) + \int_{\theta^*}^{+\infty} Q(\theta - \theta^*)\{x^*(\theta) - x^\dagger(\theta)\} dP_\Theta(\theta) \\ =&\int_{-\infty}^{\theta^*} Q(\theta^* - \theta)\varepsilon(1 - x^*(\theta)) dP_\Theta(\theta) \\ =&\varepsilon \underbrace{\int_{-\infty}^{\theta^*} Q(\theta^* - \theta)(1 - x^*(\theta)) dP_\Theta(\theta)}_{B>0} > 0. \end{aligned}$$

In the second equality, we use the assumption that x^* satisfies the balancing condition and thus $\bar{V}(x^*) = 0$. The last inequality comes from the fact that $P_\Theta(\theta^*) - \mathbb{X}^*((-\infty, \theta^*]) > 0$ and thus $1 - x^*(\theta) > 0$ in a positive measure subset of $(-\infty, \theta^*]$.

We have

$$\int_{-\infty}^{\theta^*} Q'(\theta^* - \theta)(1 - x^\dagger(\theta)) dP_\Theta(\theta) = (1 - \varepsilon) \underbrace{\int_{-\infty}^{\theta^*} Q'(\theta^* - \theta)(1 - x^*(\theta)) dP_\Theta(\theta)}_{C\geq 0} \geq 0,$$

since $Q' \geq 0$ and $1 \geq x^*(\theta)$. Thus,

$$\begin{aligned} \Delta V_1(x) &= F'(\bar{x}^*) A\varepsilon\{(1-\varepsilon)C + o(A\varepsilon)\} \\ &= F'(\bar{x}^*) AC\varepsilon + o(\varepsilon). \end{aligned}$$

We have

$$\int_{F(\bar{x}^\dagger)}^{+\infty} Q'(\theta-\theta^*)x^\dagger(\theta)dP_\Theta(\theta) = \int_{F(\bar{x}^\dagger)}^{+\infty} Q'(\theta-\theta^*)x^*(\theta)dP_\Theta(\theta)$$
$$= \underbrace{\int_{\theta^*}^{+\infty} Q'(\theta-\theta^*)x^*(\theta)dP_\Theta(\theta)}_{D\geq 0} - \int_{\theta^*}^{F(\bar{x}^\dagger)} Q'(\theta-\theta^*)x^*(\theta)dP_\Theta(\theta).$$

The latter integral is non-negative and bounded by a linear function of ε, because

$$\int_{\theta^*}^{F(\bar{x}^\dagger)} Q'(\theta-\theta^*)x^*(\theta)dP_\Theta(\theta) \leq \bar{Q}'\bar{p}_\Theta(F(\bar{x}^\dagger)-\theta^*) = \bar{Q}'\bar{p}_\Theta(A\varepsilon+o(\varepsilon)).$$

Here $\bar{Q}'$ is the maximum of $Q'(q)$ in $[0, F(\bar{x}^\dagger)-\theta^*]$; recall that Q is continuously differentiable. Hence,

$$\Delta V_3(x) = F'(\bar{x}^*)A\varepsilon\left\{D - \int_{\theta^*}^{F(\bar{x}^\dagger)} Q'(\theta-\theta^*)x^*(\theta)dP_\Theta(\theta) + o(A\varepsilon)\right\}$$
$$= F'(\bar{x}^*)AD\varepsilon + o(\varepsilon).$$

Therefore, we have

$$\bar{V}(x^\dagger) = \{B + F'(\bar{x}^*)A(C+D)\}\varepsilon + o(\varepsilon)$$

When ε is sufficiently small, $\dot{\bar{x}} = \bar{V}(x^\dagger)$ is positive. □

Part 2. Let $e \in \mathbb{R}_+$ be small enough to meet $e < \bar{q}/4$ and $F(\bar{x}^*)-4e, F(\bar{x}^*)+e \in \Theta$. Choose $w \in \mathbb{R}_+$ arbitrarily such that

$$\int_e^{2e} Q(q)dq \Big/ \int_{3e}^{4e} Q(q)dq < w < 1.$$

As Q is increasing in $[0,\bar{q}]$ and $4e < \bar{q}$, the fraction on the LHS is smaller than 1. Then, choose $\varepsilon \in (0,\bar{\varepsilon})$ so small that $\mathbb{X}^\dagger$ defined from density $x^\dagger$ given below satisfies $F(\mathbb{E}_\Theta x^\dagger) < \theta^* + e$, $x^\dagger(\theta) \in (0,1)$ for all θ, and $\|\mathbb{X}^\dagger - \mathbb{X}^*\| < \varepsilon$:

$$x^\dagger(\theta) := \begin{cases} x^*(\theta) + \varepsilon/p_\Theta(\theta) = \varepsilon/p_\Theta(\theta) & \text{if } \theta \in [\theta^*+e, \theta^*+2e), \\ x^*(\theta) - w\varepsilon/p_\Theta(\theta) = 1 - w\varepsilon/p_\Theta(\theta) & \text{if } \theta \in [\theta^*-4e, \theta^*-3e), \\ x^*(\theta) & \text{otherwise.} \end{cases}$$

This Bayesian strategy induces the aggregate participation rate $\bar{x}^\dagger$ such that

$$\begin{aligned}\bar{x}^\dagger := \mathbb{E}_\Theta x^\dagger = \mathbb{E}_\Theta x^* + \int_{\theta^*+e}^{\theta^*+2e} \frac{\varepsilon}{p_\Theta(\theta)} dP_\Theta(\theta) - \int_{\theta^*-4e}^{\theta^*-3e} \frac{w\varepsilon}{p_\Theta(\theta)} dP_\Theta(\theta) \\ = \bar{x}^* + \underbrace{(1-w)e}_{A'>0}\varepsilon > \bar{x}^*.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\bar{V}_0(x^\dagger) &= \int_{\underline{\theta}}^{\theta^*} Q(\theta^*-\theta)(1-x^\dagger(\theta))dP_\Theta(\theta) + \int_{\theta^*}^{\bar{\theta}} \{-Q(\theta-\theta^*)x^\dagger(\theta)\} dP_\Theta(\theta) \\ &= \int_{\theta^*-4e}^{\theta^*-3e} Q(\theta^*-\theta)\frac{w\varepsilon}{p_\Theta(\theta)} dP_\Theta(\theta) - \int_{\theta^*+e}^{\theta^*+2e} Q(\theta-\theta^*)\frac{\varepsilon}{p_\Theta(\theta)} dP_\Theta(\theta) \\ &= \varepsilon' \underbrace{\left(w\int_{3e}^{4e} Q(q)dq - \int_{e}^{2e} Q(q)dq\right)}_{B'>0} > 0\end{aligned}$$

The definition of $x^\dagger$ implies

$$\begin{aligned}&\int_{-\infty}^{\theta^*} Q'(\theta^*-\theta)(1-x^\dagger(\theta))dP_\Theta(\theta) \\ &= \int_{\theta^*-4e}^{\theta^*-3e} Q'(\theta^*-\theta)\frac{w\varepsilon}{p_\Theta(\theta)} dP_\Theta(\theta) = w\varepsilon \int_{\theta^*-4e}^{\theta^*-3e} Q'(\theta^*-\theta)d\theta \\ &= \varepsilon \underbrace{w(Q(4e)-Q(3e))}_{C'>0} > 0.\end{aligned}$$

It follows that

$$\begin{aligned}\Delta V_1(x) &= F'(\bar{x}^*)A'\varepsilon\{\varepsilon C' + o(A'\varepsilon)\} \\ &= F'(\bar{x}^*)A'C'\varepsilon^2 + o(\varepsilon^2) = o(\varepsilon).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\int_{F(\bar{x}^\dagger)}^{+\infty} Q'(\theta-\theta^*)x^\dagger(\theta)dP_\Theta(\theta) \\ &= \int_{\theta^*+e}^{\theta^*+2e} Q'(\theta-\theta^*)\frac{\varepsilon}{p_\Theta(\theta)} dP_\Theta(\theta) = \varepsilon \int_{\theta^*+e}^{\theta^*+2e} Q'(\theta-\theta^*)d\theta \\ &= \varepsilon \underbrace{(Q(2e)-Q(e))}_{D'>0} > 0\end{aligned}$$

Note that $\theta^* + e > F(\bar{x}^\dagger)$. Hence,

$$\begin{aligned}\Delta V_3(x^\dagger) &= F'(\bar{x}^*)A'\varepsilon(\varepsilon D' + o(A'\varepsilon)) \\ &= F'(\bar{x}^*)A'D'\varepsilon^2 + o(\varepsilon^2) = o(\varepsilon).\end{aligned}$$

Therefore, we have

$$\bar{V}(x^\dagger) = B'\varepsilon + o(\varepsilon).$$

Hence, when $\varepsilon^\dagger$ is sufficiently small, $\dot{\bar{x}} = \bar{V}(x^\dagger)$ is positive. □