# Entrants' reputation and industry dynamics*

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**Abstract**

This paper studies the entry-exit dynamics of an experience good industry. Consumers observe noisy signals of past firm behavior and hold common beliefs regarding their types, or reputations. There is a small chance that firms may independently and unobservably be exogenously replaced. The market is perfectly competitive: entry is free, and all participants are price-takers. Entrants have an endogenous reputation $\mu_E$. In the steady-state equilibrium, $\mu_E$ is the lowest reputation among active firms: Firms that have done poorly leave the market, and some re-enter under a new name. This endogenous replacement of names drives the industry dynamics. The main predictions include: Exit probabilities are higher for younger firms, inept firms, and firms with worse reputations, and competent firms have stochastically larger reputations than inept firms both in the population as a whole and within each cohort, and thus are able to live longer and charge higher prices.

**JEL Classification**: C7, D8, L1
**Keywords**: reputation, industry dynamics, free entry, exit and entry rates

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# 1 Introduction

This paper studies the exit-entry dynamics of a perfectly competitive industry, where the driving force is consumer beliefs regarding product quality. A distinctive element of the analysis is the explicit consideration of an option to exit and re-enter under a new name. In the non-revealing, pure strategy equilibrium that is our focus, firms choose to exercise this option whenever their reputation falls below some threshold. This threshold is precisely the entrants' reputation, which is endogenously determined. Hence, the entrants' reputation not only determines the price level (through a free-entry condition), but affects all industry dynamics.

The predominant explanation for firm dynamics in the literature is based on technological shocks. Firms exit either because others drove them out through product and process innovation, as in the creative destruction hypothesis, or because of adverse shocks that increased their production costs. While such shocks certainly explain an important part of the observed dynamics, there is more to the story. On the one hand, the empirical literature finds that a significant number of firms of different sizes change their names rather than leaving the market. Changing names is a strategy aimed at affecting consumer beliefs, not at controlling costs. On the other hand, the literature also finds reputation-driven exit: In some industries, there is evidence that exit is more likely subsequent to poor consumer reviews or complaints (McDevitt, 2011; Cabral and Hortacsu, 2010).

The incidence of name changing is not only disparate within an industry, but also among industries. McDevitt (2011) finds that about 8% of residential plumbing services firms in Illinois changed names within one year, while Wu (2010) finds that this frequency is much smaller among CRSP-listed companies: about 0,5% per year on average in the 1925-2000 period. Our model sheds some light on this issue: The frequency of name changes is shown to depend on individual characteristics like type and age, and on industry characteristics as well.

To address these issues, we develop an adverse selection model with imperfect public monitoring along the lines of Mailath and Samuelson (2001). There are two types of firm: competent and inept. Reputations are the common belief regarding a firm's type. There is perfect competition in the sense of Gretsky, Ostroy, and Zame (1999): There is a continuum of price-taking firms and consumers and there is free entry. The equilibrium price is increasing in the seller reputation. The incumbents' reputation is the Bayesian update of a common prior given an observed history of (imperfect, public) signals; the entrants' reputation $\mu_E$ is also a consistent belief. In equilibrium, firms exit when their reputations fall below $\mu_E$. As competent firms obtain stochastically larger signals than inept firms, their expected present value is higher at each reputation level. As a result, competent firms always want to participate–be it with their old names or new ones. In the steady state, there may be exit and entry flows while the industry as a whole is stagnant. The exit probability is found to depend on firms' characteristics, like type and age. In particular, the reputation of competent firms stochastically dominates that of inept firms and the reputation of older firms dominates that of younger firms as well. Yet there is always heterogeneity both within and between cohorts; in fact, the reputation distributions for all cohorts have full support.

The endogeneization of the entry-level reputation is important for two reasons. On the one hand, the analysis shows that even though the firms that exit are those with reputations lower than $\mu_E$, the equilibrium exit rate (or turnover) does not necessarily depend monotonically on the entry-level reputation. Rather, a change in a parameter that increases $\mu_E$ may also shift the reputation distributions in such a way that firms fall below $\mu_E$ less often. Thus, treating $\mu_E$ as a parameter may be misleading. On the other hand, $\mu_E$ is an important determinant of the equilibrium price level, as potential entrants that would enjoy a reputation $\mu_E$ if they enter are the ones that must be indifferent between entering or staying out.

Modelling the entrants' reputation presents some challenges, as $\mu_E$ and the steady-state reputation distributions are jointly determined. At all times some firms change their names and start anew at this level of reputation affecting the distributions. Reciprocally, the fraction of competents among entrants depends on how many competent firms choose to change their names, which is determined by the reputation distribution. Using a fixed-point argument, we show that there is a unique pair of mutually-consistent entry-level reputation and steady-state reputation distribution.

Our analysis shows that a key determinant of the level of industry turnover is the mass of competents that are born outside the market. The argument is purely informational: The (consistent) entry-level reputation may be higher than the average exiting firm's reputation only if there are other competent firms among entrants. Hence, a necessary condition (which will prove also sufficient) for the existence of positive turnover is that there are competents being born outside the market, so that firms that have renamed themselves pool with them when re-entering. That said, a small mass of newborn competents can "sustain" large flows of exit and entry.

## Related literature

By now there is a large body of empirical literature on the dynamics of firms within an industry. Among the most salient patterns that have consistently been found are[1]: (1) The presence of sizeable entry and exit rates even in industries that are scarcely growing, with significant heterogeneity across industries (Dunne, Roberts, and Samuelson, 1988); (2) younger firms are –ceteris paribus– more likely to exit and also (3) more likely to charge lower prices (Foster, Haltiwanger, and Syverson, 2008).

A recent strand of the literature adds a number of regularities related to firms' reputations. McDevitt (2011) focuses on an industry where firms with widely different track records compete with each other, and where exit, entry, and name changes occur frequently. Similarly, Cabral and Hortacsu (2010) study the reputational mechanism of eBay, an online auctioneer. These studies confirm previous findings and add that: (4) the probability that a given seller will exit the market increases as its reputation worsens (Cabral and Hortacsu, 2010); and (5) the firms that are more likely to change names or exit are those with worse or shorter track records (McDevitt, 2011). Moreover, while these studies find a strong positive relation between survival and age, quality may be the common causal link between those variables, explaining their apparent correlation (Thompson, 2005).

The theoretical literature has investigated a number of possible explanations for the five patterns above. One strand asks whether such dynamics can be the result of individual productivity shocks in a perfectly competitive market for a homogeneous good (the seminal paper of Hopenhayn, 1992, stands out). A related strand looks at the combination of

[1] The empirical literature has also given a great deal of attention to firm growth and firm size. We will abstract from this issue by assuming that all fims have a capacity constraint of one unit.

productivity shocks and financial frictions (Cooley and Quadrini, 2001, Albuquerque and Hopenhayn, 2004, Clementi and Hopenhayn, 2006) or labor market frictions (Hopenhayn and Rogerson, 1993). While (1) and (2) are consistent with this view, the law of one price is at odds with (3). Also, the empirical concepts of reputation and track records do not have a theoretical counterpart in this setting. The same is true in Fishman and Rob (2003), in which industry dynamics are driven by consumer inertia in a context of search costs and older firms sell more because they have a larger customer base.

On the other hand, there is a large body of theoretical literature that looks at the creation and maintenance of firms' reputations in markets for experience goods (e.g., Klein and Leffler, 1981, Fudenberg and Levine, 1989, and Mailath and Samuelson, 2001, to name just a few; Mailath and Samuelson, 2006, and Bar-Isaac and Tadelis, 2008, present comprehensive expositions of the literature.) This literature discusses primarily the monopoly case. In spite of this, some papers still manage to look at entry and exit decisions. For instance, Bar-Isaac (2003) assumes that the firm has the option to leave the market. When the firm knows its own type, in equilibrium the high-quality firm never leaves, while the low-quality firm plays a strictly mixed strategy at low levels of reputation–i.e., below some threshold. The mixed strategy is such that the post-exit reputation of any firm that has crossed the threshold becomes the threshold. Having a strictly positive probability of exiting, the low-quality type eventually leaves; this implies that there is complete separation in the long run. Board and Meyer-ter Vehn (2010) extend this analysis by incorporating moral hazard and the possibility of entry, and focus their analysis on the investment and exit decisions over the life cycle of the firm. In this equilibrium, the entry-level reputation coincides with the threshold as well.

Within the strand of the literature that looks at reputation dynamics in competitive markets, some papers focus on markets in which the information flow to potential customers is quite limited and fundamentally different from that to existing customers–namely, private monitoring; Hörner (2002) and Rob and Fishman (2005) stand out. Instead, we want to examine markets where information–albeit imperfect–flows constantly to potential customers as well; for instance, the eBay feedback system (Cabral and Hortacsu, 2010), or the complaint record of plumbing firms (McDevitt, 2011). Indeed, Internet-related technological progress moves an increasing number of markets into this category by providing means of communication among customers; one example of this is the role of TripAdvisor, Expedia, etc. in the travel industry.

Tadelis (1999) is one of the first papers to formally analyze competition under imperfect public monitoring. It presents an adverse-selection model with a continuum of firms. However, the author focuses on an equilibrium where firms leave the market after one bad outcome; this means that active firms either don't have any history (they are new), or they must have impeccable records. Tadelis (2002) develops a similar model, under moral hazard. While this kind of model can explain certain stylized facts of industry dynamics, like the differences in pricing and probability of exit between cohorts, it cannot explain the observed heterogeneity in these variables after controlling by age: all firms of the same age must have the same records and reputation. In particular, it cannot account for observations (4) and (5) beyond age.

Our paper contributes to recent literature on reputation under competition that features heterogeneous reputations. Some papers do not consider entry (e.g., Vial, 2010); others take the entry-level reputation as an exogenous parameter (e.g., Ordoñez, 2013), while others obtain it independently from the reputation distributions because of their focus on mixed strategies (e.g., Atkeson, Hellwig, and Ordonez, 2012). In our paper the entry-level reputation is endogenously determined, and depends only on informational variables; the price function fulfills the market-clearing role. In contrast, in Bar-Isaac (2003), Board and Meyer-ter Vehn (2010) and Atkeson, Hellwig, and Ordonez (2012), the price function is determined by consumer valuations and $\mu_E$ adjusts to clear the market, through a zero-profit condition.

The rest of the paper is organized as follows: Section 2 presents the model. Section 3 introduces the equilibrium concept. Section 4, the core of the paper, analyzes the dynamics of the industry in the steady state. Section 5 discusses the existence and uniqueness of the consistent reputation distribution and entry-level reputation, and examines the relationship between the replacement rate and the turnover ratio. All proofs are contained in the Appendix.

# 2 The model

## 2.1 Preliminaries

We consider an infinitely repeated game in which, at every date $t = 0, 1, 2, ...$, a market for a given service opens. Firms are long-run players, while consumers are not. Instead, at every stage there is a different generation of short-lived consumers.

The service is an experience good as per Nelson (1970): its quality is ex ante unobservable to buyers. There is no communication among consumers. Since consumers only live for one period, the information each one obtains as a result of consuming the service is not transferred to the next generation, but lost altogether. Hence, quality is also unobservable ex post. Nevertheless, after consumption takes place, an imperfect signal $r$ of the quality each active firm provided is publicly observed.

Each generation of consumers is of mass 1. In contrast, there is an unlimited supply of potential firms. Each individual may consume or produce at most one unit per period. Hence, while all consumers may purchase, not all firms will be able to sell. A firm is "active at $t$" if it produces at time $t$, and "inactive" otherwise. Consumers are homogeneous; their willingness to pay is high enough so that they all buy. As a consequence, the mass of active firms (in equilibrium) is 1.

There are two types of firm: competent ($H$) and inept ($L$). Competent firms are those that can only produce a high-quality variety of the service, while inept firms can only produce a low-quality one. The total mass of active competent firms is denoted by $m^H$; in the steady-state equilibrium we focus on, the mass of competent firms is constant over time and less than 1. Types are privately observed. Thus, this is a pure adverse selection model.

Each active firm may die. A dead firm is replaced immediately by a newly born firm that inherits its name. While consumers are aware of this replacement process, they do not observe it. The process is assumed to be i.i.d. across time and firms. $\lambda$ denotes the probability of dying, and $\theta$ the probability that a dead firm is replaced by a competent one.

This replacement process ensures that throughout any history there is never almost certainty about any firm's type. In effect, Cripps, Mailath, and Samuelson (2004) show that the adverse selection model with imperfect monitoring needs a mechanism for replenishing uncertainty about types in order for doubts about players' types to persist in the long run. One such mechanism is given by information frictions, such as limited memory (Liu and Skrzypacz, 2009), coarse observability (e.g., in Ekmekci, 2011, in which consumers observe discrete ratings rather than full histories), or costly observation of records (Liu, 2011). A second, related approach is that of Tadelis (1999) and Tadelis (2002), where consumers forget certain aspects of a history (what the author calls "reputation reduction"), with the same effect. Another mechanism is provided by "trembles," as in Levine and Martinelli (1998). The approach we follow is the one advanced in Mailath and Samuelson (2001): By adding an unobservable replacement process, consumers are never certain of who they are dealing with. The replacement of exiting firms may be plainly exogenous (our choice) or endogenous as it is in the literature on the possibility of trading names.[2]

Firm names play a key role in our model. Consumers only observe each firm's name and the history of public signals since the last spell of uninterrupted use of that name. They don't know if that name has always belonged to that firm, or if it was first used by one or more of its predecessors. In that sense, a firm's reputation is really the reputation of the name they are currently using.

When a name is not used in one period, it is forgotten by consumers together with its associated signal history. The next time the firm becomes active, it will have to do so under a new name. Hence, maintaining a name requires remaining active without interruption. In addition, the firm may also choose to change its name at the beginning of each stage; we assume that this is done by exiting the market and reentering immediately.

Thus, a firm may have different names at different times according to the (endogenous) name-changing process. Moreover, the same name may pass from one firm to another under the (exogenous) replacement (or birth-death) process, in which the latter inherits the former's history.

On the other hand, a mass $\eta$ of competent firms is also born each period among inactive firms. $\eta$ is assumed to be smaller than 1, for otherwise the adverse selection problem would vanish. Furthermore, $\eta$ needs to be much smaller than 1 (specifically, $\eta < \lambda(1-\lambda)(1-\theta)$) in order to avoid the trivial equilibrium in which all firms want to change their names at all times.

The timeline for the stage game is shown in Figure 1. Consumers' beliefs refer to the probability that a given name belongs to a competent firm, conditional on all available information. Consumers have common priors and observe the same events, so they have common beliefs. We refer to consumers' belief as the name's reputation. At the beginning of each stage, each incumbent is endowed with a **prior** reputation $\overline{\mu}_t$. Then, should the incumbent decide to produce (be active) at $t$, it must choose whether to change its name to a new one–which will carry the reputation associated with a name with no history $\mu_E$– or keep its old name. The updated reputation is the firm's **interim** reputation, denoted by $\mu_t$. This is the reputation the firm will have when the market opens. After trading ends, the signal $r_t$ will be publicly observed, and the replacement process occurs (yet is unobserved by consumers); the Bayesian update of the interim reputation given $r_t$, which takes into account the possibility of having been replaced, will be the firm's **posterior** reputation. This posterior will be the next period's prior, and so is denoted by $\overline{\mu}_{t+1}$.

[2]See, for example, Tadelis (1999) and Mailath and Samuelson (2001). A different strand of this literature looks at the case in which trading names is observable, as in Wang (2011) and Hakenes and Peitz (2007).

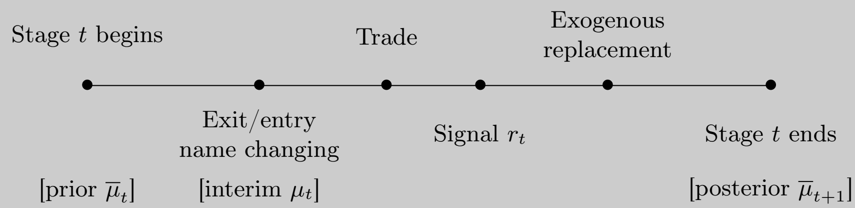


Figure 1: Timeline for the date-$t$ stage game

## 2.2 Signals and utility outcomes

The signal $r$ refers to any piece of information that is publicly available to consumers. For instance, if the firms were schools, $r$ could be the score percentile on a standardized test; if the firms were academic journals, $r$ could be their impact factor; if the firms were health care providers, $r$ could be their medical malpractice track records; if the firms were car makers, $r$ could be consumer reports, and so on.

The signal $r$ lies in the open unit interval $(0, 1)$. When a firm provides high quality, its signal is distributed according to the cdf $F_H$; when it provides low quality, it is distributed according to the cdf $F_L$. The pdf's are denoted by $f_H$ and $f_L$, respectively. We assume monotone likelihood ratio:

**Assumption 1** (Monotone likelihood ratio)**.** *The likelihood ratio* $R(r) \equiv \frac{f_H(r)}{f_L(r)}$ *is a monotonically increasing bijection from* $(0, 1)$ *to* $(0, \infty)$.

This assumption implies that $F_H(r) \leq F_L(r)$ for all $r$, this is to say, the signal conditional on $H$ first-order stochastically dominates the signal conditional on $L$.

The utility of the service is $u_H$ if the quality is high, and $u_L$ if it is low, with $u_H > u_L \geq 0$. Then, the expected utility when purchasing from a reputation-$\mu$ provider is:

$$E[u|\mu] = u_L + \mu(u_H - u_L) - p$$

Some authors normalize $u_H = 1$ and $u_L = 0$. Others (e.g., Mailath and Samuelson, 2001) assume that the ex post utility is the signal itself, so that the utility when purchasing from a type-$\tau$ provider is the expected value of the signal conditional on the quality that a type-$\tau$ provider gives, i.e., $u_H = \int_0^1 r dF_H$ and $u_L = \int_0^1 r dF_L$ . In the latter case, $u_H > u_L$ follows from the stochastic dominance assumption. Regardless of the interpretation and/or normalization, what matters is that consumers' willingness-to-pay is (linearly) increasing in $\mu$.

## 2.3 Firms

At every stage $t$, each firm chooses whether to produce or not (i.e., remain active in the case of active firms, or enter in the case of inactive firms). In addition, those active firms that choose to produce must decide whether to keep their previous name or change it (by exiting and reentering immediately at no cost). Those firms that were inactive do not have this choice.

The date-$t$ profits $\pi_t$ are given by:

$$\pi(\mu_t) = \begin{cases} p(\mu_t) - c & \text{if active} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $c$ is the production cost, and $p(\mu)$ the competitive price for a service that is of high quality with probability $\mu$. $\mu$ reflects consumers' beliefs and is endogenously determined in equilibrium, taking into consideration the equilibrium naming and production policies.

The production and naming decisions jointly maximize the expected, discounted profits:

$$(1-\delta)\sum_{t=0}^{\infty}(1-\lambda)^t \delta^t E[\pi_t|\tau], \tag{2}$$

where $\delta \in (0,1)$ is a discount factor. The expectation depends on the firm's type, as this affects the signal distribution.

Firms have heterogeneous and ever-changing reputations. Let $\overline{G}_t$ denote the cdf of prior reputations at the beginning of stage $t$ of those firms that were active at $t-1$. Stage $t$ begins with firms' exit and entry decision. $G_t$ denotes the cdf of interim reputations of those firms that chose to be active at $t$. Thus, $G_t$ and $\overline{G}_t$ differ because some active firms chose to exit, some to re-enter, and some inactive firms decided to enter (the last two with a reputation $\mu_E$).

# 3 Equilibrium

By equilibrium we mean stationary, Markov perfect, steady-state equilibrium. Each firm's state variable is the (commonly known) prior reputation $\overline{\mu}_t$, and its (privately known) type $\tau \in \{H, L\}$. Consumers do not distiguish among entering firms, regardless of whether they were active or not in the previous period. Thus, all entrants will have the same interim reputation level, denoted by $\mu_E$.

An equilibrium is a set of strategies for consumers and firms, consumers' beliefs, and a price function, such that beliefs are consistent, the strategies are optimal, and the market clears. Moreover, in our steady-state equilibrium, reputation distributions are constant over time.

As is common in this kind of model, many equilibria are supported by different beliefs. For instance, there is a trivial equilibrium with no reputation-building: All consumers believe only inept firms are active at all times, so that every active firm has a null reputation and is indifferent as to whether to produce or not. Instead, we look at a reputational equilibrium–an equilbrium where the reputation of each firm is affected by its signals–where competent firms are always active and more reputable firms are paid higher prices. In this equilibrium, a constant mass $m^H = \frac{\lambda\theta+\eta}{\lambda} \in (0,1)$ of competent firms trades at every date.

### 3.1 Definition of reputational equilibrium

The reputational equilibrium is defined by:

**Strategies:** Consumers always buy and are indifferent among providers; competent firms are always active, as are inept firms with prior reputation above $\mu_E$ while those with prior reputation below $\mu_E$ are indifferent; all firms that remain active change their names whenever their reputation falls below $\mu_E$.

**Beliefs:** Beliefs are consistent with Bayes' rule and the equilibrium strategies. Upon observation of the name choice, a prior probability of being competent $\overline{\mu}$ is updated to an interim reputation $\mu$ according to:

$$\mu = \begin{cases} \overline{\mu} & \text{if old name} \\ \mu_E & \text{if new name} \end{cases} \tag{3}$$

Then, upon observation of a signal $r$, the interim is updated to a posterior $\overline{\mu}'$ :

$$\overline{\mu}' = \lambda\theta + (1-\lambda)\frac{f_H(r)\,\mu}{f_H(r)\,\mu + f_L(r)\,(1-\mu)}, \tag{4}$$

Let $\varphi(\mu, r)$ denote the right-hand-side of (4). Let $\tilde{r}(x,\mu)$ denote the signal that a firm of current reputation $\mu$ requires to get a reputation $\overline{\mu}' = x$ in the next period, and $\tilde{\mu}(x,r)$ denote the interim reputation that a firm requires to get a posterior $x$ after a signal $r$. Both functions are defined implicitly by:[3]

$$\begin{aligned} x =& \varphi(\mu, \tilde{r}(x,\mu)) \\ x =& \varphi(\tilde{\mu}(x,r), r). \end{aligned}$$

The steady-state conditional distributions of prior and interim reputations satisfy:

$$\begin{pmatrix} \overline{G}(x|H) \\ \overline{G}(x|L) \end{pmatrix} \equiv \begin{pmatrix} \frac{m^H(1-\lambda+\lambda\theta)}{m^H-\eta} & \frac{\left(1-m^H\right)\lambda\theta}{m^H-\eta} \\ \frac{m^H\lambda(1-\theta)}{1-m^H+\eta} & \frac{\left(1-m^H\right)(1-\lambda\theta)}{1-m^H+\eta} \end{pmatrix} \begin{pmatrix} \int_0^{\tilde{r}(x,\mu_E)} G(\tilde{\mu}(x,r)\,|H)\,dF_H \\ \int_0^{\tilde{r}(x,\mu_E)} G(\tilde{\mu}(x,r)\,|L)\,dF_L \end{pmatrix} \tag{5}$$

where the conditional distributions of interim reputations are given by:

$$G(x|H) = \begin{cases} 0 & \text{if } x < \mu_E \\ \frac{1}{m^H}\left(\eta + \left(m^H - \eta\right)\overline{G}(x|H)\right) & \text{if } x \geq \mu_E \end{cases}$$

$$G(x|I) = \begin{cases} 0 & \text{if } x < \mu_E \\ \frac{1}{1-m^H}\left(-\eta + \left(1 - m^H + \eta\right)\overline{G}(x|I)\right) & \text{if } x \geq \mu_E \end{cases} \tag{6}$$

and the entry-level reputation satisfies:

$$\mu_E = \frac{\left(m^H - \eta\right)\overline{G}(\mu_E|H) + \eta}{\overline{G}(\mu_E)} \tag{7}$$

whenever $\overline{G}(\mu_E) > 0$.

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[3] Specifically,

$$\begin{aligned} \widetilde{r}(x,\mu) &= R^{-1}\left(\frac{x-\lambda\theta}{1-\lambda+\lambda\theta-x}\,\frac{1-\mu}{\mu}\right) \\ \tilde{\mu}(x,r) &= \frac{x-\lambda\theta}{(1-\lambda+\lambda\theta-x)\,R(r) + x - \lambda\theta} \end{aligned}$$

for $x, \mu \in (\lambda\theta, 1-\lambda+\lambda\theta)$, where $R(r) \equiv {}^{f_H(r)}/_{f_L(r)}$. Assumption (1) ensures that $R^{-1}$ exists.

**Prices:** The equilibrium price function $p(\mu)$ makes consumers indifferent among providers:

$$\frac{\partial E[u|\mu]}{\partial \mu} = 0, \tag{8}$$

willing to buy:

$$E[u|\mu] \geq 0, \tag{9}$$

and inept, inactive firms indifferent between entering or staying out:

$$v(\mu_E, L) = 0, \tag{10}$$

where $v(\mu, \tau)$ is the value function of a firm of type $\tau$ with interim reputation $\mu$. The existence of one such function in the end crucially depends on the cost level $c$ being small enough.

Equation 4 is Bayes' rule upon consideration of the possibility of replacement. In effect, the probability of a firm being competent is the probability of replacement ($\lambda$) times the probability of being competent conditional on replacement ($\theta$), plus the probability of no replacement $(1-\lambda)$ times the probability of competent given the signal $r$ and not having been replaced. Since $f_H$ and $f_L$ have full support, Equation 4 is well-defined for all $\mu$ and $r$.

On the equilibrium path, Equation 3 derives from Bayes' rule: When competent and inept firms are expected to do the same–namely, keep their names if the prior is larger than the threshold $\mu_E$–the action is uninformative and the interim is equal to the prior. Off the equilibrium path, Equation 3 embeds the assumption that off-equilibrium moves are uninformative as well.[4]

We look at a steady state, where the mass of competent firms is constant over time. As each generation of consumers is of mass 1, market clearing is obtained when a mass 1 of firms are active. If the mass of competent firms that trade at each date is denoted as $m^H$, then the mass of active inept firms must be $1 - m^H$. Then, the mass $m^H$ of active competent firms at each date must be equal to $(1-\lambda) m^H + \lambda\theta + \eta$, the sum of:

- active competents that survived: $(1-\lambda) m^H$
- newly born competents that replace active competents that died: $\lambda\theta m^H$
- newly born competents that replace active inepts that died: $\lambda\theta \left(1 - m^H\right)$
- newly born competents that enter with a name with no history: $\eta$

The mass of active competent firms at each date is thus given by $m^H = \frac{\lambda\theta+\eta}{\lambda} > \eta$. This implies that after the replacement process, the mass of competents is reduced to $m^H - \eta$.

Equation 5 shows the posterior distributions of competent and inept firms, respectively, as a funcion of the previous period's interim reputation distributions. In effect, the competent firms at the beginning of any particular date are those that were competent

[4] Notice that these off-equilibrium beliefs can be obtained by taking the limit of $\mu$ from any sequence of completely mixed pooling naming strategies that converge to 1. Notice also that this equilibrium satisfies Cho and Kreps' Intuitive Criterion.

in the previous period and did not change to inept (the fraction $m^H(1-\lambda+\lambda\theta)/m^H-\eta$), plus those that were inept and changed to competent (the fraction $(1-m^H)\lambda\theta/m^H-\eta$); similarly, the active inept firms at the beginning of a particular date are those that were inept and did not change to competent (the fraction $(1-m^H)(1-\lambda\theta)/1-m^H+\eta$), plus those that were competent and changed to inept (the fraction $m^H\lambda(1-\theta)/1-m^H+\eta$). From each particular group of firms, the firms with reputation smaller than or equal to $x$ are those that in the previous period had an interim reputation no greater than $\tilde{\mu}(x,r)$ whose signal realization was $r$ –of which there is a mass $\int_0^{\tilde{r}(x,\mu_E)} G(\tilde{\mu}(x,r)|H)\,dF_H$ that originally were competent and $\int_0^{\tilde{r}(x,\mu_E)} G(\tilde{\mu}(x,r)|L)\,dF_L$ that originally were inept; the integrals go from 0 to $\tilde{r}(x,\mu_E)$ because a signal higher than $\tilde{r}(x,\mu_E)$ is required to reach a reputation $x$ today only if the interim reputation was smaller than $\mu_E$, of which there are none.

Equation 6 shows the interim reputation distributions, which differ from the prior distributions in two respects: (1) the entry of a mass $\eta$ of newborn competent firms, which replaces an equal mass of inept firms that exit; and (2) the changing of names by the firms with priors lower than $\mu_E$ that remain active.

In equilibrium, the entry-level reputation $\mu_E$ must coincide with the fraction of competent firms among the group of entrants (if nonempty):

$$\mu_E = \frac{\text{Competent entrants}}{\text{Entrants}}.$$

In the steady state, the mass of entrants (i.e., new names) is equal to the mass of firms that exit (i.e., the mass of lost names $\overline{G}(\mu_E)$); the competent "entrants," in turn, are the newborn competents (with mass $\eta$) plus the competents that exited in order to change their names (with mass $(m^H-\eta)\,\overline{G}(\mu_E|H)$). Plugging in the ratio we get Equation 7, which is thus a consistency condition.

Section 5 below shows that there is a (unique) pair of steady-state distributions and threshold $\mu_E$ that jointly satisfy equations 5, 6, and 7.

## 3.2 Characterization of reputational equilibrium

**Prices.** Consumers are indifferent among providers (Equation 8) if $p$ is of the form:

$$p(\mu) = \alpha + (u_H - u_L)\,\mu,$$

where $\alpha$ is some constant, to be determined by the free-entry condition (Equation 10). All consumers will buy if $\alpha$ is smaller than $u_L$, which is ultimately a condition over the cost $c$.

**Value function.** The value function for a firm of type $\tau$ and interim reputation $\mu$ is given by:

$$v(\mu,\tau) = \alpha - c + (u_H - u_L)\,\mu + \delta(1-\lambda)\int_0^1 v(\max\{\varphi(\mu,r),\mu_E\},\tau)\,dF_\tau$$

where

$$\mu = \max\{\overline{\mu},\mu_E\}.$$

The value function is increasing in $\mu$ for both types. On the other hand, since the signals for a competent firm are stochastically larger than those of an inept one, the value for the competent type is larger than the value for an inept type at any reputation level:

$$v(\mu, H) > v(\mu, L)$$

It follows that the free-entry condition applies only to inept firms; hence, Equation (10) must hold. As the flow payoff is linear in $\alpha$, for each cost level $c$ there will be one $\alpha$ such that Equation (10) holds.

**Distributions.** The a priori distributions $\overline{G}(x|\tau)$ have support $[\lambda\theta, 1-\lambda+\lambda\theta]$ because the likelihood ratio is onto; consequently, the interim distributions $G(x|\tau)$ have support $[\mu_E, 1-\lambda+\lambda\theta]$. All reputation distributions are absolutely continuous because the signal distributions are. The corresponding probability density functions of prior and interim reputations conditional on type $\tau$ will be denoted by $\overline{g}(.|\tau)$ and $g(.|\tau)$, respectively.

# 4 Industry dynamics

This section describes the dynamics of entry, exit, and reputations within the steady-state reputational equilibrium. Exit occurs when a name was used at some date and not at the next. The mass of exiting names–or turnover rate–in equilibrium is $\overline{G}(\mu_E)$. Some of the firms that exit may re-enter immediately under a new name. Those that don't become inactive or "retire." The decisions to exit, retire, and re-enter are endogenous. There is also the exogenous replacement process which we call "death" and affects a mass $\lambda$ of firms.

## 4.1 Exit rates and turnover

The empirical literature on industry dynamics finds that there is considerable heterogeneity among industries in terms of entry and exit rates. However, these rates are sizeable even in industries that are neither growing nor shrinking. Typically, within an industry the gross entry and exit rates are similar to each other, but at the same time they are orders of magnitude larger than the net rates (Dunne, Roberts, and Samuelson, 1988).

Our focus is on a steady state, where the net entry rate is zero. Still, there is a constant renewal (exit and entry), given by $\overline{G}(\mu_E)$–the fraction of incumbents that leave the market, or turnover rate. If there is entry, there must be exiting. The newly born competents will definitely enter, as well as the competent firms that change their names. In addition, some inept firms must also enter so that the reputation of entrants is not perfect; these inepts may come from the group of inactive firms or from the group of exiting firms indistinctly. When competent firms are born among inactive firms (i.e., $\eta > 0$), the entry-level reputation $\mu_E$ must be strictly larger than $\lambda\theta$, so that a positive mass of firms chooses to retire, thereby making room for the entrants.

When no competent firms are born among inactive firms (i.e., $\eta = 0$), the entry-level reputation $\mu_E$ cannot be larger than $\lambda\theta$ and, as a consequence, there are no exit or entry flows. The reason behind this is the strong adverse selection the entering firms (i.e., firms operating under a new name) would suffer from. As consumers are aware that no new competent firms are born, they understand that the only competents in that group must be the firms that decided to change their names because their reputation fell below $\mu_E$. Consistency requires that the fraction of competents among entrants is the average prior reputation in the group, which is strictly smaller than the exit threshold $\mu_E$; yet the fraction of competents among entrants must also be exactly equal to $\mu_E$. As these conditions are contradictory, the group of entrants must be empty. This is summarized in the following proposition:

**Proposition 1** (Name turnover)**.** *A necessary and sufficient condition for the turnover rate of names $\overline{G}(\mu_E)$ to be strictly positive in equilibrium is that the inflow of newborn competents is strictly positive:*

$$\overline{G}\left(\mu_E\right) > 0 \iff \mu_E > \lambda\theta \iff \eta > 0$$

Next we look at the rate of name exiting within particular subpopulations: by type, reputation and signal. Our model has a number of predictions, some of which have been investigated empirically. Cabral and Hortacsu (2010), in a study of eBay auctions, find that the probability that a name exits the market increases as its reputation declines (as defined by eBay's reputation mechanism.) In turn, McDevitt (2011) studies the plumbing services market in Illinois and finds that, all else being equal, the firms that are more likely to change names or retire are those with worse track records–a variable that resembles the history of public signals in our model. Proposition 2 establishes that this is exactly what we should expect.

**Proposition 2** (Name dynamics)**.** *The exit probability in the next period is higher for inept firms than for competent ones, conditional on reputation; the lower the reputation, the higher the probability (both conditional and unconditional on type), and the lower the current signal, the higher the probability (both conditional and unconditional on type).*

The exit probability is the probability that the firm's reputation falls below the threshold $\mu_E$. The first part of the proposition follows from the fact that competent firms' signals are stochastically larger than inept firms'; the second part follows from the fact that, according to Bayes' rule, the posterior probability of an event is increasing in the prior; and the last part follows from the monotone likelihood ratio assumption.

## 4.2 Age

The empirical literature finds systematic differences between firms of different ages. Foster, Haltiwanger, and Syverson (2008) find that younger firms (i.e., younger names) are more likely to exit, and charge lower prices, than older firms. In the same vein, McDevitt (2011) finds that the exit probability is monotonically decreasing with age.

In the steady state, the group of age $a$ looks exactly the same as the group of age 0, $a$ periods into the future. In this sense, studying the cross-sectional variation (across cohorts) is equivalent to studying the evolution over time of a given cohort.

If no new competents were born among inactive firms ($\eta = 0$), then there would be no entry nor exiting, this is to say, no dynamics. In this case, it is not possible to distinguish among cohorts, as all names are introduced at the same time. Age is irrelevant, as it is completely detached from reputation. This contrasts with Hörner's model (Hörner, 2002), where age and reputation are biunivocally related, as only firms with perfect records survive.

On the other hand, as soon as there is a positive mass of competents born outside the market ($\eta > 0$), an entry/exit flow emerges. Let $a$ denote the age of a name that was introduced $a$ periods ago in the market and has been kept throughout (this, regardless of whether the firm that carries it has died and been replaced in that lapse or not). All such names conform the cohort $a$; the prior mass of cohort $a$ is denoted by $\overline{m}_a$, and the interim mass by $m_a$. Let $\overline{G}(.|\tau, a)$ denote the prior reputation distribution of the set of firms of type $\tau$ and cohort $a$, and $\overline{m}_a^\tau$ its mass; similarly, $G(.|\tau, a)$ and $m_a^\tau$ denote the interim reputation distributions and the cohort's mass (i.e., after exit). The corresponding probability density functions of prior and interim reputations conditional on type $\tau$ and age $a$ will be denoted by $\overline{g}(.|\tau, a)$ and $g(.|\tau, a)$, respectively.

At any date, a new cohort of mass $\overline{G}(\mu_E)$ enters. Out of them, a fraction $\mu_E$, is competent: $m_0^H = \mu_E \overline{G}(\mu_E)$. As all new names carry the same reputation $\mu_E$, we have for $\tau \in \{C, I\}$:

$$G(x|\tau, 0) = \begin{cases} 0 & \text{if } x < \mu_E \\ 1 & \text{if } x \geq \mu_E \end{cases}$$

As time goes by, in each period two changes occur: (i) The birth-death process shifts the masses of competent and inept firms within the cohort according to:

$$\begin{pmatrix} \overline{m}_{a+1}^H \\ \overline{m}_{a+1}^L \end{pmatrix} = \begin{pmatrix} 1-\lambda+\lambda\theta & \lambda\theta \\ \lambda(1-\theta) & 1-\lambda\theta \end{pmatrix} \begin{pmatrix} m_a^H \\ m_a^L \end{pmatrix} \tag{11}$$

and (ii) The mass of surviving names in each subpopulation $\tau$ shrinks by a factor of $\left(1-\overline{G}(\mu_E|\tau, a+1)\right)$, as those firms that exit are not replaced by other firms from the same cohort. Hence:

$$m_{a+1}^\tau = \overline{m}_{a+1}^\tau \left(1-\overline{G}(\mu_E|\tau, a+1)\right) \tag{12}$$

In turn, the evolution of the prior reputation distributions in a given cohort at different ages is given by:

$$\begin{pmatrix} \overline{G}(x|H, a+1) \\ \overline{G}(x|L, a+1) \end{pmatrix} = \begin{pmatrix} \frac{(1-\lambda+\lambda\theta)m_a^H}{\overline{m}_{a+1}^H} & \frac{\lambda\theta m_a^L}{\overline{m}_{a+1}^H} \\ \frac{\lambda(1-\theta)m_a^H}{\overline{m}_{a+1}^L} & \frac{(1-\lambda\theta)m_a^L}{\overline{m}_{a+1}^L} \end{pmatrix} \begin{pmatrix} \int_0^{\tilde{r}(x,\mu_E)} G(\tilde{\mu}(x,r)|H, a)\, dF_H \\ \int_0^{\tilde{r}(x,\mu_E)} G(\tilde{\mu}(x,r)|L, a)\, dF_L \end{pmatrix} \tag{13}$$

For $a > 0$, the distributions of interim reputations relate to the priors' as follows:

$$G(x|\tau, a) = \begin{cases} 0 & \text{if } x < \mu_E \\ \frac{\overline{G}(x|\tau,a)-\overline{G}(\mu_E|\tau,a)}{1-\overline{G}(\mu_E|\tau,a)} & \text{if } x \geq \mu_E \end{cases} \tag{14}$$

Equation 13 is analogous to Equation 5; the only difference is in the weights. In the population of active firms as a whole the total mass and the ratio of competent to inept are constant over time. In contrast, not only is each cohort losing mass over time, but also each type does so at different rates. Similarly, Equation 14 resembles Equation 6; they differ in that within each cohort there is only exiting and no entry.

It is clear that the interim reputation of older cohorts first-order stochastically dominates the cohort of age 0. As the equilibrium price is a linear function of the firms' interim reputation, the price distributions inherit the properties of the interim reputation distributions. In particular, the price that older cohorts charge first-order stochastically dominates that of entrants. This is consistent with the findings of Foster, Haltiwanger, and Syverson (2008).

On the other hand, the exit (or name-changing) decision is made in response to the prior reputation. The next proposition shows that the prior reputation of older cohorts also first-order stochastically dominates that of age 1, which implies that the exit probability of older firms is smaller. This is also consistent with the findings of Foster, Haltiwanger, and Syverson (2008) and McDevitt (2011).

**Proposition 3** (Reputation across cohorts)**.** *The prior reputation of firms of age 1 is first-order stochastically dominated by the prior reputation of firms of any older cohort, both conditional and unconditional on types. Formally,* $(\forall a \in \mathbb{N})\,(\forall x \in [\lambda\theta, 1-\lambda+\lambda\theta])$ ,

$$\overline{G}\left(x|\tau, a\right) \leq \overline{G}\left(x|\tau, 1\right) \text{ and } \overline{G}\left(x|a\right) \leq \overline{G}\left(x|1\right)$$

with strict inequality in the interior for $a > 1$.

Figure 2 shows the family of prior reputation distributions by cohorts in a numerical example where older firms (i.e., names) have stochastically better reputations than younger ones. This means that exit rates are monotonically decreasing in age, and the price distributions are also ordered by first-order stochastic dominance. Notice also that not even the limit distributions are degenerate.

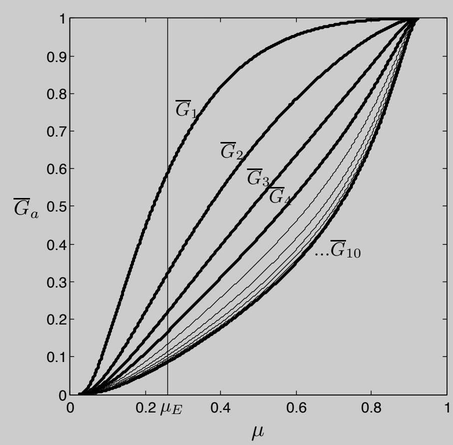



Figure 2: Reputation distributions for different ages, $\overline{G}_a\left(\mu\right)$

Notes: $r|_H \sim Be\left(r \mid 3, 2\right)$ and $r|_L \sim Be\left(r \mid 2, 3\right)$. The parameters are $\lambda = 0.1$, $m^H = 0.5$ and $\eta = 0.025$. $\mu_E$ is 0.26. $\overline{G}_a\left(\mu\right) \equiv \overline{G}\left(\mu|a\right)$.

### 4.3 Differences across types

In this subsection, we compare interim and prior reputations between competent and inept firms, both conditional and not conditional on age. This comparison relates straightforwardly to the prices and exit rates of competent and inept firms. The main result is that competent firms charge higher prices than inept ones (in a stochastic sense), and have lower exit probabilities.

We wish to compare the distributions over (prior, interim) reputations for competent and inept firms. Yet, reputations are themselves probabilities. This fact imposes a particular structure on the random variable $\overline{\mu}$ (resp., $\mu$) and its density $\overline{g}$ (resp., $g$):

**Lemma 1** (Consistency)**.** *For any $x \in [\lambda\theta, 1-\lambda+\lambda\theta]$, the probability of being competent conditional on the firm's prior reputation being $x$ is exactly $x$, both conditional and not conditional on age:*

$$\frac{\overline{m}_a^H \overline{g}(x|H,a)}{\overline{m}_a \overline{g}(x|a)} = x \qquad (15) \qquad\qquad \frac{\left(m^H - \eta\right)\overline{g}(x|H)}{\overline{g}(x)} = x \qquad (16)$$

*Moreover, for any $x \in [\mu_E, 1-\lambda+\lambda\theta]$ the probability of being competent conditional on the firm's interim reputation being $x$ is also $x$, both conditional and not conditional on age:*

$$\frac{m_a^H g(x|H,a)}{m_a g(x|a)} = x \qquad (17) \qquad\qquad \frac{m^H g(x|H)}{g(x)} = x \qquad (18)$$

Lemma 1 establishes that the probability of being competent for a given reputation level must be equal to that reputation level. This implies that the mean of the reputation distribution must coincide with the fraction of competent firms in the relevant reference group. Thus, we must have:

$$E[\overline{\mu}|a] = \frac{\overline{m}_a^H}{\overline{m}_a} \qquad\qquad E[\overline{\mu}] = m^H$$

$$E[\mu|a] = \frac{m_a^H}{m_a} \qquad\qquad E[\mu] = m^H - \eta$$

An immediate consequence of Lemma 1 is that the likelihood ratios $\frac{\overline{g}(x|H,a)}{\overline{g}(x|L,a)}$ and $\frac{g(x|H,a)}{g(x|L,a)}$ are increasing in $x$, and so are $\frac{\overline{g}(x|H)}{\overline{g}(x|L)}$ and $\frac{g(x|H)}{g(x|L)}$. In other words, consistency of beliefs implies that the monotone likelihood ratio property–which is assumed for the distributions of signals conditional on types–extends to the prior and interim distributions of reputations conditional on types within any cohort and in the population as a whole. Hence, we get:

**Corollary 1** (Reputation across types by cohort)**.** *Within each cohort $a \geq 1$, and in the population as a whole, the (prior, interim) reputation of competent firms first-order stochastically dominates the (prior, interim) reputation of inept firms:* $(\forall a \in \mathbb{N})$,

$$\overline{G}(x|H,a) \leq \overline{G}(x|L,a) \qquad\qquad \overline{G}(x|H) \leq \overline{G}(x|L)$$

$$G(x|H,a) \leq G(x|L,a) \qquad\qquad G(x|H) \leq G(x|L)$$

$\forall x \in [\lambda\theta, 1-\lambda+\lambda\theta]$.

In view of the linear connection between prices and interim reputations, and the connection between exit rates and prior distributions, Corollary 1 implies that:

**Proposition 4** (Prices and exit rates across types)**.** *Competent firms charge higher prices (in a stochastic sense) and exit less often than inept firms, both within each cohort and throughout the population.*

# 5 Reputation of entrants and incumbents

## 5.1 Existence and uniqueness

This section discusses the existence and uniqueness of a consistent entry-level reputation and reputation distributions for competent and inept firms.

The fraction of competent firms among entrants is given by the right-hand side of Equation 7, which we define as the function $\psi$:

$$\psi(x) \equiv \frac{\left(m^H - \eta\right)\overline{G}(x|H) + \eta}{\overline{G}(x)} \tag{19}$$

where the denominator doesn't vanish as long as $x > \lambda\theta$. Thus, Equation 7 says that any consistent entry-level reputation $\mu_E$ is a fixed point of the function $\psi$.

Assume $\eta > 0$. It turns out that $\psi$ has a unique fixed point. In effect, the sets $\{x : x > \psi(x)\}$ and $\{x : x < \psi(x)\}$ are nonempty and $\psi$ is continuous, decreasing when $\psi > x$ and increasing when $\psi < x$ so that $\psi' = 0$ if and only if $x = \psi(x)$.

The function $\psi(x)$ is constructed under the assumption that firms change their names if and only if their reputation falls below the cutoff $x$ and all competents enter. Consider the value $x \in (\lambda\theta, 1 - \lambda + \lambda\theta)$ that satisfies $\left(m^L + \eta\right)\overline{G}(x|L) = \eta$, namely, that all inepts that exit are replaced by newborn competents that enter. In that case, all entrants are competent and their reputation is the highest possible: $\psi(x) = 1$, and therefore $\{x : x < \psi(x)\}$ is nonempty. On the other hand, at $x = 1 - \lambda + \lambda\theta$ all firms exit and $\psi(x) = m^H < 1 - \lambda + \lambda\theta$, so that $\{x : x > \psi(x)\}$ is nonempty.

Fix any $x$ in the set $\{x : x < \psi(x)\}$. If the cutoff point $x$ increases, the additional firms that exit have on average a reputation of about $x$, i.e., $x$ is the marginal reputation of the group of entrants. As the marginal reputation $x$ must be equal to the fraction of competents in that group (Equation 16), and since $x$ is smaller than $\psi(x)$, $\psi$ decreases. Similarly, $\psi$ is increasing in the set $\{x : x > \psi(x)\}$. It follows that there is a unique value of $\mu_E$ in which the cutoff point coincides with the expected fraction of competents in the group of entrants, i.e., a unique fixed point of $\psi$. This is illustrated in Figure 3.

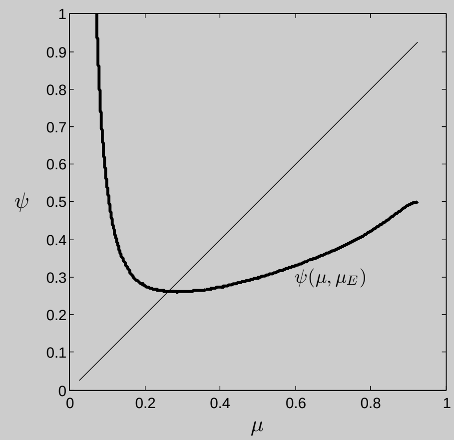



Figure 3: Consistent entry-level reputation as a fixed point

Notes: $r|_H \sim Be\,(r \mid 3,2)$ and $r|_L \sim Be\,(r \mid 2,3)$. The parameters are $\lambda = 0.1$, $m^H = 0.5$ and $\eta = 0.025$. The resulting $\mu_E$ is 0.26.

Indeed, if only competents were to enter, all active firms would want to pool themselves with them by replacing their names. A consistent belief is such that the group of competent firms pools with precisely enough low reputation firms so that $\mu_E$ is the fraction of competents among entrants, and no active firm outside that group would gain by joining it. The preceding argument shows that there can be only one such consistent $\mu_E$ for a given pair of reputation distributions.

If, on the other hand, $\eta = 0$, the set $\{x : x < \psi\,(x)\}$ is empty, meaning that no entry-level reputation greater than $\lambda\theta$ can be consistent. Yet, Lemma 1 implies that $\lim_{x\to\lambda\theta^+} \psi\,(x) = \lambda\theta$, meaning that $\mu_E = \lambda\theta$ satisfies the consistency condition of Equation 7 in a limiting sense.

Now think of $\mu_E$ as a parameter. Notice that the steady-state interim distributions themselves do depend on it. Clearly, the parameter $\mu_E$ defines their support: No firm would ever keep its name should its reputation fall below that threshold. As there would be a point mass at $\mu_E$ every period (namely, the mass of firms that enter), $\mu_E$ not only determines the support but also the shape of the distributions.

Replacing Equation 5 in Equation 6, we see that the pair of interim distributions $(G\,(\cdot|H),\, G\,(\cdot|L))$ is also the fixed point of some operator. Proposition 5 below establishes the existence and uniqueness of this fixed point which depends parametrically on $\mu_E$. Furthermore, there exists a unique $\mu_E$ and a pair of distributions that jointly satisfy equations 5, 6, and 7.

**Proposition 5.** *If $\eta > 0$, there exists a unique tuple of steady-state distributions for competent and inept firms, and an entry-level reputation $\mu_E \in \left(\lambda\theta, m^H\right)$, that jointly satisfy equations 5, 6, and 7.*
*If $\eta = 0$, there exists a unique pair of steady-state distributions for competent and inept firms with $\mu_E = \lambda\theta$ that jointly satisfies equations 5 and 6, and such that $\lim_{x\to\lambda\theta^+} \psi\left(x\right) = \lambda\theta$.*

Summing up, in this section we have proven that there is a unique steady-state reputation distribution for each type of firm $G\left(\cdot|H\right)$ and $G\left(\cdot|L\right)$, and a unique, consistent entry-level reputation $\mu_E$. When no competent firms are born among inactive firms ($\eta = 0$), this $\mu_E$ is the lowest possible, and as a consequence all firms want to keep their names at all times and there are no exit-entry flows. Any reputation is better than $\mu_E$. Nevertheless, the threat of entry is not without consequences; rather, it serves the purpose of keeping prices down. On the other hand, when new competent firms are born among inactive firms ($\eta > 0$), $\mu_E$ is such that there are exit-entry flows, and at all times a positive mass of firms chooses to change their names.

## 5.2 Exogenous and endogenous replacement rates

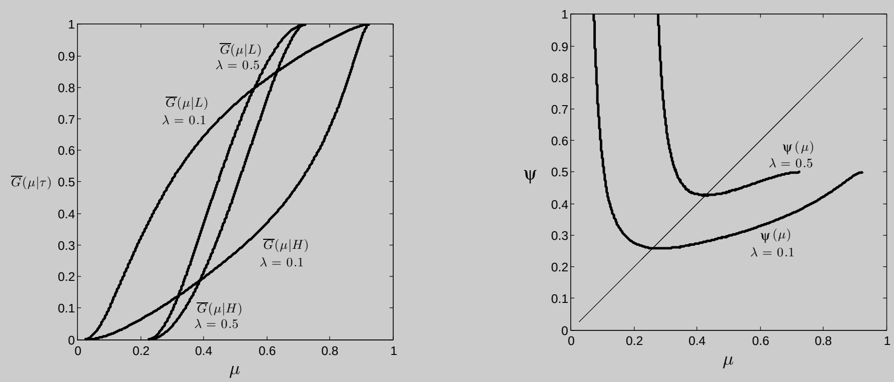
In our model there are two replacement processes: that of firms (namely, the exogenous replacement after death, determined by the replacement probability $\lambda$ and the probability of being replaced by a competent firm $\theta$), and that of names (driven by the endogenous naming strategy). The former, although empirically unobservable, affects the latter.

The industry dynamics of two markets with distinct exogenous replacement processes are different even if they are alike in terms of mass of active competent firms $m^H$ and mass of newborn competent firms $\eta$. This is so because the replacement process affects the "depreciation rate" of information: The larger the probability of death, the less important old signals become.

Recall that $\theta$, $\lambda$ , $\eta$ and $m^H$ jointly satisfy $m^H = \theta + \frac{\eta}{\lambda}$. We want to look at a change in $\lambda$ compensated by $\theta$ so that only the replacement process is affected, while the mass of competent firms $m^H$ remains the same (holding $\eta$ fixed). Proposition 6 discusses how the exogenous replacement process affects the entry-level reputation.

**Proposition 6** (entry-level reputation and replacement)**.** *A joint increase in $\lambda$ and $\theta$ such that $m^H$ remains constant increases the entry-level reputation $\mu_E$.*

Intuitively, a higher level of $\lambda$ implies a lower informativeness of histories, as the past becomes less useful in predicting a firm's current type. As a consequence, the reputation distributions of competent and inept firms move closer to each other while their support shrinks. By this mechanism, the fraction of competent firms among those below the threshold $\mu_E$ increases, so that $\mu_E$ increases. This is depicted in Figure 4.

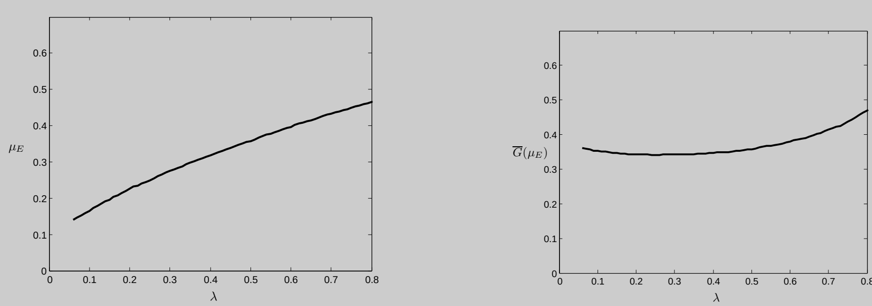



(a) $\overline{G}(\mu|H)$ and $\overline{G}(\mu|L)$ for different values of $(\lambda, \theta)$

(b) $\psi(\mu)$ for different values of $(\lambda, \theta)$

Figure 4: Distributions and consistent entry-level reputation for different values of $(\lambda, \theta)$

Notes: $r|_H \sim Be\,(r \mid 3, 2)$ and $r|_L \sim Be\,(r \mid 2, 3)$. The parameters are $m^H = 0.5$ and $\eta = 0.025$. $\theta$ is set to $\theta = m^H - \frac{\eta}{\lambda}$




(a) $\mu_E$ for different values of $\lambda$

(b) $\overline{G}(\mu_E)$ for different values of $\lambda$

Figure 5: Entry-level reputation and turnover for different values of $(\lambda, \theta)$

Notes: $r|_H \sim Be\,(r \mid 4, 2)$ and $r|_L \sim Be\,(r \mid 2, 4)$. The parameters are $m^H = 0.5$ and $\eta = 0.025$. $\theta$ is set to $\theta = m^H - \frac{\eta}{\lambda}$.

As for the turnover rate, it not only depends on $\mu_E$ but also on the population-wide reputation distribution–which shifts when $\lambda$ and $\theta$ increase. Figure 5 shows an example in which the turnover rate $\overline{G}\,(\mu_E)$ does not vary monotonically with $\lambda$, even though $\mu_E$ does. The larger $\mu_E$ means that each firm will replace its name (thereby erasing its history) in a larger number of states. However, the increase in $\mu_E$ also shifts the reputation distribution. The result may be a decrease or an increase in the frequency of name changes, as Figure 5(b) illustrates.

## 6 Concluding remarks

We presented a model of firm dynamics in which the stochastic movement of individual reputations, coupled with an option to change names, is the driving force of the market. The model features pure adverse selection because of our focus on industry dynamics. In fact, the same dynamics would be obtained in a high-quality equilibrium of the model with moral hazard, in which the competent firms have a quality choice as well. In a companion paper (Vial and Zurita, 2013) we focus on the incentives for high-quality production.

One distinctive feature of our model is that in equilibrium there is heterogeneity both within and across cohorts with regard to reputation and pricing. This stems from the fact that consumers are never sure about a firm's type. The uncertainty about types is not resolved even in the long run due to the unobservable *replacement* process. This process could be changed to an unobservable *type change* process, with minor modifications but without affecting the dynamics of the industry–which are the focus of this paper.

The industry dynamics studied here hinge on the ability of disgraced firms to improve their reputation by changing their names. In a rational expectations equilibrium this is only possible if they can pool with a group of entrants of better "quality" than their fellow exiting firms. In turn, this possibility depends on competent firms being willing to enter when inept firms are just indifferent. In our model, this follows from the assumptions of monotone likelihood ratio–which implies that competent firms have an easier road to higher reputations–and competent and inept firms having the same production cost. Naturally, the latter assumption can be relaxed somehow, as long as the value function for competent firms doesn't become smaller than the value function for inept firms. Otherwise, if all inept firms would prefer to enter, the pool of entrants would be no better than the pool of exiting firms, and the driving force of the industry dynamics studied here would be lost.

The model showed the importance of the entry-level reputation as an equilibrating variable of the market. This message is likely to extend to other environments. For instance, consider the case in which firms could invest, or pay a cost, to become competent prior to entry. The fact that competent firms have an advantage over inept firms would generate a willingness-to-pay for acquiring competency. The mass of newborn competents $\eta$ should adjust until the marginal firm is indifferent between investing or not. A different $\eta$ leads to a different entry-level reputation, which in turn affects the payoff advantage of competent firms. Board and ter Vehn (2013) discuss the incentives to invest in product quality for a monopolist.

We restricted our analysis to the case in which the only way to enter the industry is with an unknown name. Some authors have explored the possibility that a firm purchases another firm's name. Yet another interesting situation is the possibility of entering with a name whose history was acquired in a different market–namely, umbrella branding. There are plenty of avenues for future research.

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# Appendices

# A Proof of Proposition 2

Notice first that:

$$\frac{\partial \tilde{r}\left(x,\mu\right)}{\partial x} > 0 \quad \frac{\partial \tilde{r}\left(x,\mu\right)}{\partial \mu} < 0$$

and

$$\frac{\partial \tilde{\mu}\left(x,r\right)}{\partial x} > 0 \quad \frac{\partial \tilde{\mu}\left(x,r\right)}{\partial r} < 0.$$

In effect, for a given prior $\mu$, the signal that is required to achieve a higher posterior $x$ is higher; and the signal required to achieve a given posterior $x$ is inversely related to the prior $\mu$. Similarly, for a given signal $r$, achieving a higher posterior $x$ requires a higher prior $\mu$; and if the signal is higher, then a lower prior is required for the same posterior $x$.

Observe that:

$$\Pr\left(\text{exit at } t+1|\mu_t,\tau_t\right) = F_{\tau_t}\left(\tilde{r}\left(\mu_E,\mu_t\right)\right). \tag{20}$$

The first-order stochastic dominance assumption implies that:

$$\Pr\left(\text{exit at } t{+}1|\mu_t,H\right) = F_H\left(\tilde{r}\left(\mu_E,\mu_t\right)\right) \leq F_L\left(\tilde{r}\left(\mu_E,\mu_t\right)\right) = \Pr\left(\text{exit at } t+1|\mu_t,L\right),$$

i.e., inept firms have a higher exit probability than competent firms of the same reputation.

On the other hand, direct computation of the derivative of Equation 20 yields:

$$\frac{\partial \Pr\left(\text{exit at } t+1|\mu_t,\tau_t\right)}{\partial \mu_t} = f_{\tau_t}\left(\tilde{r}\left(\mu_E,\mu_t\right)\right)\frac{\partial \tilde{r}\left(\mu_E,\mu_t\right)}{\partial \mu_t} < 0$$

since $\frac{\partial \tilde{r}(\mu_E,\mu_t)}{\partial \mu_t} < 0$. This is to say, the exit probability is decreasing in the prior reputation for each type. This also holds for the unconditional probability. In effect,

$$\Pr\left(\text{exit at } t+1|\mu_t\right) = \mu_t F_H\left(\tilde{r}\left(\mu_E,\mu_t\right)\right) + \left(1-\mu_t\right)F_L\left(\tilde{r}\left(\mu_E,\mu_t\right)\right);$$

taking the derivative yields:

$$\frac{\partial \Pr\left(\text{exit at } t+1|\mu_t\right)}{\partial \mu_t} = \left(F_H - F_L\right) + \left(\mu_t f_H + \left(1-\mu_t\right)f_L\right)\frac{\partial \tilde{r}\left(\mu_E,\mu_t\right)}{\partial \mu_t} < 0.$$

Regarding signals, the probability of exit is that of having a reputation smaller than $\tilde{\mu}\left(\mu_E,r_t\right)$:

$$\Pr\left(\text{exit at } t+1|r_t,\tau_t\right) = G\left(\tilde{\mu}\left(\mu_E,r_t\right)|\tau_t\right).$$

Hence,

$$\frac{\partial \Pr\left(\text{exit at } t+1|r_t,\tau_t\right)}{\partial r_t} = g\left(\tilde{\mu}\left(\mu_E,r_t\right)|\tau_t\right)\frac{\partial \tilde{\mu}\left(\mu_E,r_t\right)}{\partial r_t} < 0$$

since $\frac{\partial \tilde{\mu}(x,r_t)}{\partial r_t} < 0$. Also, the unconditional (on type) probability of exit is given by: $\Pr\left(\text{exit at } t{+}1|r_t\right) = \left(\left(m^H-\eta\right)G\left(\tilde{\mu}\left(\mu_E,r_t\right)|H\right) + \left(1-m^H+\eta\right)G\left(\tilde{\mu}\left(\mu_E,r_t\right)|L\right)\right)$, so that:

$$\frac{\partial \Pr\left(\text{exit at } t{+}1|r_t\right)}{\partial r_t} = \left(\left(m^H-\eta\right)g\left(\tilde{\mu}\left(\mu_E,r_t\right)|H\right) + \left(1-m^H+\eta\right)g\left(\tilde{\mu}\left(\mu_E,r_t\right)|L\right)\right)\frac{\partial \tilde{\mu}\left(x,r_t\right)}{\partial r_t} < 0.$$

□

# B Proof of Lemma 1

Consider the claim "the probability of being competent conditional on the firm's reputation being $x$ must be $x$." We first prove that if this claim holds for the prior reputations within cohort $a$, then it also holds for the interim reputations within cohort $a$.

Assume that $\frac{\overline{m}_a^H \overline{g}(x|H,a)}{\overline{m}_a \overline{g}(x|a)} = x$ for all $x \in [\lambda\theta, 1-\lambda+\lambda\theta]$. Then, $\frac{\overline{m}_a^H \overline{g}(x|H,a)}{\overline{m}_a^L \overline{g}(x|L,a)} = \frac{x}{1-x}$, and as $\frac{\overline{m}_a^H \overline{g}(x|H,a)}{\overline{m}_a^L \overline{g}(x|L,a)} = \frac{m_a^H g(x|H,a)}{m_a^L g(x|L,a)}$ for all $x \in [\mu_E, 1-\lambda+\lambda\theta]$, we conclude that $\frac{m_a^H g(x|H,a)}{m_a g(x|,a)} = x$ for all $x \in [\mu_E, 1-\lambda+\lambda\theta]$.

Next, we proceed by induction. We prove first that if the claim holds for the prior reputations within cohort $a$, it must also hold for the prior reputations within cohort $a+1$.

Taking the derivative of $\overline{G}(x|H,a+1)$ with respect to $x$, and noticing that $G(\tilde{\mu}(x,\tilde{r}(x,\mu_E))|\tau,a) = G(\mu_E|\tau,a) = 0$, we obtain:

$$\overline{g}(x|H,a+1) = \frac{\int_0^{\tilde{r}(x,\mu_E)} \left[(1-\lambda+\lambda\theta) m_a^H g(\tilde{\mu}(x,r)|H,a) f_H(r) + \lambda\theta m_a^L g(\tilde{\mu}(x,r)|L,a) f_L(r)\right] \frac{\partial\tilde{\mu}(x,r)}{\partial x} dr}{\overline{m}_{a+1}^H}$$

Rewriting $m_a^L g(\tilde{\mu}(x,r)|L,a)$ as $\frac{1-\tilde{\mu}(x,r)}{\tilde{\mu}(x,r)} m_a^H g(\tilde{\mu}(x,r)|H,a)$ we get:

$$\overline{g}(x|H,a+1) = \frac{\int_0^{\tilde{r}(x,\mu_E)} \frac{m_a^H g(\tilde{\mu}(x,r)|H,a)}{\tilde{\mu}(x,r)} \left[(1-\lambda+\lambda\theta)\tilde{\mu}(x,r) f_H(r) + \lambda\theta(1-\tilde{\mu}(x,r)) f_L(r)\right] \frac{\partial\tilde{\mu}(x,r)}{\partial x} dr}{\overline{m}_{a+1}^H}$$

But replacing $\mu$ by $\tilde{\mu}(x,r)$ in Bayes' rule (Equation 4), we obtain:

$$x = \lambda\theta + (1-\lambda)\frac{\tilde{\mu}(x,r) f_H(r)}{\tilde{\mu}(x,r) f_H(r) + (1-\tilde{\mu}(x,r)) f_L(r)}$$

and hence:

$$\overline{m}_{a+1}^H \overline{g}(x|H,a+1) = x\int_0^{\tilde{r}(x,\mu_E)} \frac{m_a^H g(\tilde{\mu}(x,r)|H,a)}{\tilde{\mu}(x,r)} \left[\tilde{\mu}(x,r) f_H(r) + (1-\tilde{\mu}(x,r)) f_L(r)\right] \frac{\partial\tilde{\mu}(x,r)}{\partial x} dr$$

Replacing $\frac{1-\tilde{\mu}(x,r)}{\tilde{\mu}(x,r)} m_a^H g(\tilde{\mu}(x,r)|H,a)$ by $m_a^L g(\tilde{\mu}(x,r)|L,a)$ and noticing that:

$$\int_0^{\tilde{r}(x,\mu_E)} \left[m_a^H g(\tilde{\mu}(x,r)|H,a) f_H(r) + m_a^L g(\tilde{\mu}(x,r)|L,a) f_L(r)\right] \frac{\partial\tilde{\mu}(x,r)}{\partial x} dr = \overline{m}_{a+1}\overline{g}(x|a+1),$$

we conclude that:

$$\frac{\overline{m}_a^H \overline{g}(x|H,a)}{\overline{m}_a \overline{g}(x|a)} = x \Rightarrow \frac{\overline{m}_{a+1}^H \overline{g}(x|H,a+1)}{\overline{m}_{a+1}\overline{g}(x|a+1)} = x$$

Finally, we prove that the claim holds for prior reputations within cohort $a = 1$. As the interim reputation distribution for new firms is degenerate at $\mu_E$ (both for competent and inept firms), then $\int_0^{\tilde{r}(x,\mu_E)} G(\tilde{\mu}(x,r)|\tau,0) dF_\tau = F_\tau(\tilde{r}(x,\mu_E))$. Replacing this expression in Equation 13 and taking the derivative with respect to $x$, we obtain:

$$\begin{pmatrix} \overline{g}(x|H,1) \\ \overline{g}(x|L,1) \end{pmatrix} = \frac{\partial\tilde{r}(x,\mu_E)}{\partial x} \begin{pmatrix} \frac{(1-\lambda+\lambda\theta)m_0^H}{\overline{m}_1^H} & \frac{\lambda\theta m_0^L}{\overline{m}_1^H} \\ \frac{\lambda(1-\theta)m_0^H}{\overline{m}_1^L} & \frac{(1-\lambda\theta)m_0^L}{\overline{m}_1^L} \end{pmatrix} \begin{pmatrix} f_H(\tilde{r}(x,\mu_E)) \\ f_L(\tilde{r}(x,\mu_E)) \end{pmatrix}$$

On the other hand, if we plug $\tilde{r}(x,\mu_E)$ instead of $r$ into Bayes' rule (Equation 4), we obtain:

$$x = \frac{(1-\lambda+\lambda\theta)\mu_E f_H(\tilde{r}(x,\mu_E)) + \lambda\theta(1-\mu_E) f_L(\tilde{r}(x,\mu_E))}{\mu_E f_H(\tilde{r}(x,\mu_E)) + (1-\mu_E) f_L(\tilde{r}(x,\mu_E))} \tag{21}$$

Using these two expressions and rearranging, we get:

$$\frac{\overline{m}_1^H \overline{g}(x|H,1)}{\overline{m}_1 \overline{g}(x|1)} = x \tag{22}$$

where $\overline{m}_1\overline{g}(x|1) = \overline{m}_1^H \overline{g}(x|H,1) + \overline{m}_1^L \overline{g}(x|L,1)$. In other words, consistency requires that the probability of being competent conditional on age $a = 1$ and on the firm's reputation being $x$ should be exactly $x$.

The probability density function unconditional on age satisfies:

$$g(x|\tau) = \frac{1}{m^\tau}\sum_{a=1}^{\infty} m_a^\tau g(x|\tau,a) \text{ and } g(x) = \textstyle\sum_{a=1}^{\infty} m_a g(x|a)$$

Replacing $m_a^H g(x|H,a)$ with $x m_a g(x|a)$ and rearranging, we obtain:

$$\frac{m^H g(x|H)}{g(x)} = x$$

Hence, $\frac{m^H g(x|H)}{m^L g(x|L)} = \frac{x}{1-x}$. Noticing that for $x \in [\mu_E, 1-\lambda+\lambda\theta]$, $g(x|H) = \frac{(m^H-\eta)}{m^H}\overline{g}(x|H)$ and $g(x|L) = \frac{(1-m^H+\eta)}{1-m^H}\overline{g}(x|L)$ we arrive at the desired result for prior reputations. □

# C Proof of Proposition 3

Let $\begin{pmatrix} \gamma_a^H & 1-\gamma_a^H \\ \gamma_a^L & 1-\gamma_a^L \end{pmatrix} \equiv \begin{pmatrix} \frac{(1-\lambda+\lambda\theta)m_a^H}{\overline{m}_{a+1}^H} & \frac{\lambda\theta m_a^L}{\overline{m}_{a+1}^H} \\ \frac{\lambda(1-\theta)m_a^H}{\overline{m}_{a+1}^L} & \frac{(1-\lambda\theta)m_a^L}{\overline{m}_{a+1}^L} \end{pmatrix}$. Then Equation 13 can be written as

$$\overline{G}\left(x|\tau,a+1\right)=\gamma_a^\tau\int_0^{\tilde{r}(x,\mu_E)}G\left(\tilde{\mu}\left(x,r\right)|H,a\right)dF_H+\left(1-\gamma_a^\tau\right)\int_0^{\tilde{r}(x,\mu_E)}G\left(\tilde{\mu}\left(x,r\right)|L,a\right)dF_L.$$

As $G\left(\tilde{\mu}\left(x,r\right)|\tau,a\right)\leq G\left(\tilde{\mu}\left(x,r\right)|\tau,0\right)$,

$$\begin{aligned}\overline{G}\left(x|\tau,a+1\right)&\leq\gamma_a^\tau\int_0^{\tilde{r}(x,\mu_E)}G\left(\tilde{\mu}\left(x,r\right)|H,0\right)dF_H+\left(1-\gamma_a^\tau\right)\int_0^{\tilde{r}(x,\mu_E)}G\left(\tilde{\mu}\left(x,r\right)|L,0\right)dF_L\\&=\overline{G}\left(x|\tau,1\right)+\left(\gamma_a^\tau-\gamma_0^\tau\right)\left(\int_0^{\tilde{r}(x,\mu_E)}G\left(\tilde{\mu}\left(x,r\right)|H,0\right)d\left(F_H-F_L\right)\right).\end{aligned}$$

Evaluating the integral,

$$\overline{G}\left(x|\tau,a+1\right)\leq\overline{G}\left(x|\tau,1\right)+\left(\gamma_a^\tau-\gamma_0^\tau\right)\left(F_H\left(\tilde{r}\left(x,\mu_E\right)\right)-F_L\left(\tilde{r}\left(x,\mu_E\right)\right)\right).$$

However, $\left(\gamma_a^\tau-\gamma_0^\tau\right)>0\Leftrightarrow\frac{m_a^H}{m_a}>\mu_E$; moreover, $\frac{m_a^H}{m_a}=E\left[\mu|a\right]>\mu_E$. On the other hand, $F_H\left(\tilde{r}\left(x,\mu_E\right)\right)-F_L\left(\tilde{r}\left(x,\mu_E\right)\right)<0$ because of the stochastic dominance assumption (Assumption 1). □

# D Proof of Proposition 5

We proceed in two steps. First, the entry-level reputation is assumed to be an exogenous parameter $y\in(0,1)$. Under this assumption, Lemma 5 shows that there is a unique steady-state distribution pair for competent and inept firms. That this pair exists and is unique is important because we want to focus on steady-state equilibria.

Second, the entry-level reputation $y$ is endogenized by requiring it to be consistent: $y=\mu_E$. Indeed, consistency implies that the fraction of competent firms among those active firms whose histories grant them a given reputation $\mu$ is precisely $\mu$, and similarly, that the fraction of competent firms among entrants (if any) is precisely $\mu_E$. These two properties turn out to be closely related. In the steady state new firms will enter (and some old ones will exit) if and only if new competent firms are born among inactive firms.

Consider the system of integral equations defined by:

$$\begin{pmatrix}\overline{G}_{t+1}\left(x|H\right)\\\overline{G}_{t+1}\left(x|L\right)\end{pmatrix}\equiv\begin{pmatrix}\frac{m^H(1-\lambda+\lambda\theta)}{m^H-\eta}&\frac{\left(1-m^H\right)\lambda\theta}{m^H-\eta}\\\frac{m^H\lambda(1-\theta)}{1-m^H+\eta}&\frac{\left(1-m^H\right)(1-\lambda\theta)}{1-m^H+\eta}\end{pmatrix}\begin{pmatrix}\int_0^{\tilde{r}(x,y)}G_t\left(\tilde{\mu}\left(x,r\right)|H\right)dF_H\\\int_0^{\tilde{r}(x,y)}G_t\left(\tilde{\mu}\left(x,r\right)|L\right)dF_L\end{pmatrix}\tag{23}$$

and

$$\begin{aligned}G_t\left(x|H\right)&=\begin{cases}0&\text{if }x<y\\\frac{1}{m^H}\left(\eta+\left(m^H-\eta\right)\overline{G}_t\left(x|H\right)\right)&\text{if }x\geq y\end{cases}\\G_t\left(x|I\right)&=\begin{cases}0&\text{if }x<y\\\frac{1}{1-m^H}\left(-\eta+\left(1-m^H+\eta\right)\overline{G}_t\left(x|I\right)\right)&\text{if }x\geq y\end{cases}\end{aligned}\tag{24}$$

Replacing Equation 24 in Equation 23 and rearranging, we get:

$$\begin{aligned}\begin{pmatrix}\overline{G}_{t+1}\left(x|H\right)\\\overline{G}_{t+1}\left(x|L\right)\end{pmatrix}&=\eta\begin{pmatrix}\frac{(1-\lambda+\lambda\theta)}{m^H-\eta}&-\frac{\lambda\theta}{m^H-\eta}\\\frac{\lambda(1-\phi)}{1-m^H+\eta}&-\frac{(1-\lambda\theta)}{1-m^H+\eta}\end{pmatrix}\begin{pmatrix}F_H\left(\tilde{r}\left(x,y\right)\right)\\F_L\left(\tilde{r}\left(x,y\right)\right)\end{pmatrix}\\&+\begin{pmatrix}(1-\lambda+\lambda\theta)&\frac{\lambda\theta\left(1-m^H+\eta\right)}{m^H-\eta}\\\frac{\lambda(1-\theta)\left(m^H-\eta\right)}{1-m^H+\eta}&(1-\lambda\theta)\end{pmatrix}\begin{pmatrix}\int_0^{\tilde{r}(x,y)}\overline{G}_t\left(\tilde{\mu}\left(x,r\right)|H\right)dF_H\\\int_0^{\tilde{r}(x,y)}\overline{G}_t\left(\tilde{\mu}\left(x,r\right)|L\right)dF_L\end{pmatrix}\end{aligned}\tag{25}$$

Define the right-hand side of Equation 25 as the operator $T$ in the set of pairs of continuous, normalized[5] functions $\left(\overline{G}(\cdot|H), \overline{G}(\cdot|L)\right)$ endowed with the following metric:

$$\rho\left(\left(\overline{G}(\cdot|H), \overline{G}(\cdot|L)\right), \left(\overline{G}'(\cdot|H), \overline{G}'(\cdot|L)\right)\right) = \max\left\{\rho_\infty\left(\overline{G}(\cdot|H), \overline{G}'(\cdot|H)\right), \rho_\infty\left(\overline{G}(\cdot|L), \overline{G}'(\cdot|L)\right)\right\},$$

where:

$$\rho_\infty\left(\overline{G}(\cdot|\tau), \overline{G}'(\cdot|\tau)\right) = \sup_{x\in[\lambda\theta, 1-\lambda+\lambda\theta]} \left|\overline{G}^\tau(x|\tau) - \overline{G}'(x|\tau)\right|$$

for $\tau \in \{H, L\}$. The supremum is taken over $x \in [\lambda\theta, 1-\lambda+\lambda\theta]$ since the domains of $\overline{G}$ and $\overline{G}'$ are always contained in this interval.

Notice that equations 23 and 24 coincide with 5 and 6, respectively, in the steady state. Hence, the steady-state reputation distributions $\overline{G}(\cdot|H)$ and $\overline{G}(\cdot|L)$ described in equations 5 and 6 are a fixed point of $T$. Since $T$ depends parametrically on $y$, so do $\overline{G}(\cdot|H)$ and $\overline{G}(\cdot|L)$.

We start by establishing that:

**Lemma 2.** *The operator $T$ has a unique fixed point.*

*Proof.* Notice that there are no firms with reputation either below $y$ or above $1-\lambda+\lambda\theta$ after entry-exit decisions are made, and that:

1. $\tilde{\mu}(x, \tilde{r}(x, y)) = y$; this is to say, the previous reputation of a firm who obtained a signal $\tilde{r}(x, y)$ that changed its reputation from $y$ to $x$ was $y$;
2. $\tilde{\mu}(x, r) < y \Leftrightarrow r > \tilde{r}(x, y)$: Those firms that have a reputation $x$ today and had a reputation lower than $y$ in the previous period are those that obtained signals of at least $\tilde{r}(x, y)$; and
3. $\tilde{\mu}(x, r) > 1-\lambda+\lambda\theta \Leftrightarrow r < \tilde{r}(x, 1-\lambda+\lambda\theta)$ : Those firms that had a higher reputation than $1-\lambda+\lambda\theta$ in the previous period and have a reputation $x$ today are those whith signals lower than $\tilde{r}(x, 1-\lambda+\lambda\theta)$.

Using these facts, the distance between $\overline{G}_{t+1}(\cdot|\tau)$ and $\overline{G}'_{t+1}(\cdot|\tau)$ can be bounded as follows:

$$\rho_\infty\left(\overline{G}_{t+1}(\cdot|\tau), \overline{G}'_{t+1}(\cdot|\tau)\right) \leq \beta \cdot \rho\left(\left(\overline{G}_t(\cdot|H), \overline{G}_t(\cdot|L)\right), \left(\overline{G}'_t(\cdot|H), \overline{G}'_t(\cdot|L)\right)\right)$$

where $\beta \in (0, 1)$ is defined by:

$$\beta = \max\left\{\sup_{x\in[\lambda\theta, 1-\lambda+\lambda\theta]} \left(F_H(\tilde{r}(x, y)) - F_H(\tilde{r}(x, 1-\lambda+\lambda\theta))\right), \sup_{x\in[\lambda\theta, 1-\lambda+\lambda\theta]} \left(F_L(\tilde{r}(x, y)) - F_L(\tilde{r}(x, 1-\lambda+\lambda\theta))\right)\right\}.$$

It follows that:

$$\rho\left(\left(\overline{G}_{t+1}(\cdot|H), \overline{G}_{t+1}(\cdot|L)\right), \left(\overline{G}'_{t+1}(\cdot|H), \overline{G}'_{t+1}(\cdot|L)\right)\right) \leq \beta\rho\left(\left(\overline{G}_t(\cdot|H), \overline{G}_t(\cdot|L)\right), \left(\overline{G}'_t(\cdot|H), \overline{G}'_t(\cdot|L)\right)\right)$$

i.e., $T$ is a contraction mapping with modulus $\beta$.

On the other hand, the set of continuous, bounded real functions endowed with the sup norm is complete. Moreover, the subset of normalized functions is closed,[6] thereby complete. Hence, by Banach's fixed point theorem, $T$ has a unique fixed point, which is a pair of continuous and normalized functions.

Notice that if $y$ is consistent, $\overline{G}(x|H)$ and $\overline{G}(x|L)$ are increasing functions because $G(x|H)$ and $G(x|L)$ are non-negative in the whole domain, while $\tilde{r}(x, y)$ is increasing in $x$. Thus, $\overline{G}(x|H)$ and $\overline{G}(x|L)$ are not only normalized and continuous, but also increasing–i.e., they are distribution functions. □

[5] The function $\overline{G}^\tau$ is normalized if $\overline{G}^\tau(\lambda\theta) = 0$ and $\overline{G}^\tau(1-\lambda+\lambda\theta) = 1$.

[6] See Lemma 1 in Vial (2010) for a proof.

The absolute continuity of $F_H$ and $F_L$ implies that $\overline{G}(x|H)$ and $\overline{G}(x|L)$ are absolutely continuous, with common support $[\lambda\theta, 1-\lambda+\lambda\theta]$. The parameter $y$ affects the operator $T$, and through it the modulus of the contraction and the limiting distributions. Moreover, the limiting distributions are continuous in $y$, as they are the fixed point of a contraction.[7] We write $T^y$ and $\overline{G}^y(x|\tau)$ to emphasize this dependence when necessary. Similarly, the parameters $\lambda$, $\theta$ and $\eta$ also affect the operator $T$ and the limiting distributions.

We now endogenize the entry-level reputation $y$. When taking into consideration the dependence of the distributions on $y$, Equation 19 should be written as:

$$\psi(x,y) \equiv \frac{\left(m^H-\eta\right)\overline{G}^y(x|H)+\eta}{\left(m^H-\eta\right)\overline{G}^y(x|H)+\left(m^L+\eta\right)\overline{G}^y(x|L)}. \tag{26}$$

Define the function

$$\sigma(\mu) \equiv \psi(\mu,\mu), \tag{27}$$

for $\mu > \lambda\theta$. A consistent entry-level reputation satisfies $\mu_E = \psi(\mu_E, y)$ for given distributions; now we need to verify that those distributions were generated by the same entry-level reputation: $y = \mu_E$. In other words, we need to prove that $\sigma(\mu)$ has a unique fixed point. We begin by observing that:

**Lemma 3.** *Assume that $\eta > 0$. Then $\sigma$ has at least one fixed point.*

*Proof.* The function $f(\mu) \equiv \overline{G}^\mu(\mu|L)$ is continuous, with $f(\lambda\theta) = 0$ and $f(1-\lambda+\lambda\theta) = 1$. Hence, by the intermediate value theorem there is at least one $\xi \in (\lambda\theta, 1-\lambda+\lambda\theta)$ such that $f(\xi) = \frac{\eta}{1-m^H+\eta}$, and so $\sigma(\xi) = 1$. We also know that $\sigma$ is continuous in its domain, and that $\sigma(\xi) - \xi = 1 - \xi > 0$ and $\sigma(1-\lambda+\lambda\theta) - (1-\lambda+\lambda\theta) = m^H - (1-\lambda+\lambda\theta) < 0$ (as $\eta < \lambda(1-\lambda)(1-\theta)$). Also by the intermediate value theorem, there is at least one $\mu \in (\xi, 1-\lambda+\lambda\theta)$ such that $\sigma(\mu) - \mu = 0$. Hence, there is at least one $\mu_E \in (\lambda\theta, 1-\lambda+\lambda\theta)$ such that $\psi(\mu_E, \mu_E) = \mu_E$. □

The next step is to establish uniqueness.

**Lemma 4.** *Assume that $\eta > 0$. Then $\sigma'(\mu_E) = 0$ if $\mu_E$ is a fixed point. Hence, the fixed point is unique.*

*Proof.* We prove that $\frac{\partial\psi}{\partial x} = 0$ and $\frac{\partial\psi}{\partial y} = 0$ at $x = y = \mu_E$, from which we deduce that $\sigma'(\mu_E) = 0$ since:

$$\sigma'(\mu_E)\,d\mu_E = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy.$$

Taking the derivative of Equation 26 we obtain:

$$\frac{\partial\psi}{\partial x}(x,y) = \frac{\overline{g}^y(x)}{\overline{G}^y(x)}\left(\frac{\left(m^H-\eta\right)\overline{g}^y(x|H)}{\overline{g}^y(x)} - \psi(x,y)\right)$$

By Lemma 1, $\frac{(m^H-\eta)\overline{g}^y(x|H)}{\overline{g}^y(x)} = x$. Moreover, at a fixed point $\psi(x,y) = x$. Hence, $\frac{\partial\psi}{\partial x} = 0$ at $x = y = \mu_E$. In words, the entrants' reputation $\psi(x,y)$ increases when the exit reputation level increases if and only if the firms that leave and reenter after this change have a higher reputation than those that are already replacing their names. At the fixed point, however, those firms have exactly the same average reputation, so moving the cutoff point will have no effect on the entrants' reputation.

As for $y$, it affects $\psi$ through the distributions $\overline{G}^y(\cdot|H)$ and $\overline{G}^y(\cdot|L)$. Since the steady-state distributions are the fixed point of a contraction mapping in a complete metric space, they can be obtained as the limit of the sequence of distributions $\left\{\overline{G}_t^y(\cdot|H), \overline{G}_t^y(\cdot|L)\right\}$ defined by iterating $T^y$ starting from any pair $\overline{G}_0(\cdot|H)$ and $\overline{G}_0(\cdot|L)$, where $\overline{G}_t^y(\cdot|H)$ and $\overline{G}_t^y(\cdot|L)$ denote the $t$-th iteration of $T^y$. Define $\psi_t(x,y)$ as $\psi_t(x,y) \equiv \frac{(m^H-\eta)\overline{G}_t^y(x|H)+\eta}{\overline{G}_t^y(x)}$. We will show that $\{\psi_t(\mu_E, y)\}$ is a constant sequence when the starting point is $\overline{G}_0^y(\cdot|\tau) \equiv \overline{G}^{\mu_E}(\cdot|\tau)$ (i.e., the steady-state distributions under $T^{\mu_E}$) and $y = \mu_E + dy$ is infinitesimally different from $\mu_E$; hence $\frac{\partial\psi}{\partial y} = 0$ when evaluated at $x = y = \mu_E$.

After one iteration we obtain:

$$\overline{G}_1^y(x|\tau) = \overline{G}_0^y(x|\tau) + \frac{\partial T_\tau^y}{\partial y}(x)\,dy$$

[7] See De la Fuente (2000), Chapter 2, Theorem 7.18.

where $T^y_H$ and $T^y_L$ denote the first and second entries of the operator $T^y$, respectively.

From direct computation of the derivative of the right-hand-side of Equation 25 (with fixed distributions):

$$\begin{pmatrix} \frac{\partial T^y_H}{\partial y} \\ \frac{\partial T^y_L}{\partial y} \end{pmatrix}(x) = \frac{\partial \widetilde{r}(x,y)}{\partial y} \begin{pmatrix} \frac{m^H(1-\lambda+\lambda\theta)}{m^H-\eta} & \frac{\lambda\theta(1-m^H)}{m^H-\eta} \\ \frac{m^H\lambda(1-\theta)}{1-m^H+\eta} & \frac{(1-\lambda\theta)(1-m^H)}{1-m^H+\eta} \end{pmatrix} \begin{pmatrix} G^y_t(y|H) f_H(\tilde{r}(x,y)) \\ G^y_t(y|L) f_L(\tilde{r}(x,y)) \end{pmatrix} \tag{28}$$

When evaluating Equation 28 at $t=0$, taking into account that $\frac{(m^H-\eta)\overline{G}_0(x|H)+\eta}{\overline{G}_0(x)} = \mu_E$, and rearranging, we obtain:

$$\begin{pmatrix} \frac{\partial T^y_H}{\partial y} \\ \frac{\partial T^y_L}{\partial y} \end{pmatrix}(x) = \overline{G}^y_0(\mu_E) \frac{\partial \widetilde{r}(x,y)}{\partial y} \begin{pmatrix} \frac{(1-\lambda+\lambda\theta)\mu_E f_H(\tilde{r}(x,\mu_E))+\lambda\theta(1-\mu_E)f_L(\tilde{r}(x,\mu_E))}{m^H-\eta} \\ \frac{\lambda(1-\theta)\mu_E f_H(\tilde{r}(x,\mu_E))+(1-\lambda\theta)(1-\mu_E)f_L(\tilde{r}(x,\mu_E))}{1-m^H+\eta} \end{pmatrix} \tag{29}$$

Moreover, using Equation 21 this expression reduces to:

$$\begin{pmatrix} \frac{\partial T^y_H}{\partial y} \\ \frac{\partial T^y_L}{\partial y} \end{pmatrix}(x) = \overline{G}^y_0(\mu_E) \frac{\partial \widetilde{r}(x,y)}{\partial y} \left(\mu_E f_H(\tilde{r}(x,\mu_E)) + (1-\mu_E) f_L(\tilde{r}(x,\mu_E))\right) \begin{pmatrix} \frac{x}{m^H-\eta} \\ \frac{1-x}{1-m^H+\eta} \end{pmatrix}$$

Hence the pair $\left(\overline{G}^y_1(\cdot|H), \overline{G}^y_1(\cdot|L)\right)$ can be written as:

$$\begin{pmatrix} \overline{G}^y_1(\cdot|H) \\ \overline{G}^y_1(\cdot|L) \end{pmatrix}(x) = \begin{pmatrix} \overline{G}^y_0(\cdot|H) \\ \overline{G}^y_0(\cdot|L) \end{pmatrix}(x) + \overline{G}^y_0(\mu_E)\, \upsilon_0(x,y) \begin{pmatrix} \frac{x}{m^H-\eta} \\ \frac{1-x}{1-m^H+\eta} \end{pmatrix} \tag{30}$$

with $\upsilon_0(x,y) \equiv \frac{\partial \tilde{r}(x,y)}{\partial y}\left(\mu_E f_H(\tilde{r}(x,\mu_E)) + (1-\mu_E) f_L(\tilde{r}(x,\mu_E))\right) dy$, while $\psi_1(\mu_E, y)$ can be written as:

$$\psi_1(\mu_E, y) = \frac{(m^H-\eta)\,\overline{G}^y_0(\mu_E|H) + \mu_E \overline{G}^y_0(\mu_E)\, \upsilon_0(\mu_E, y) + \eta}{\overline{G}^y_0(\mu_E)\left(1 + \upsilon_0(\mu_E, y)\right)}$$

As $\frac{(m^H-\eta)\overline{G}^y_0(\mu_E|H)+\eta}{\overline{G}^y_0(\mu_E)} = \mu_E$, we conclude that

$$\psi_1(\mu_E, y) = \psi_0(\mu_E, y) = \mu_E.$$

We now look at higher iterations of $T^y$. The change of variables $\mu = \tilde{\mu}(x,r)$ and $r = \tilde{r}(x,\mu)$ inside the integral in Equation 25 allows it to be written as:

$$T\begin{pmatrix} \overline{G}^y_t(\cdot|H) \\ \overline{G}^y_t(\cdot|L) \end{pmatrix}(x) = \eta \begin{pmatrix} \frac{(1-\lambda+\lambda\theta)}{m^H-\eta} & -\frac{\lambda\theta}{m^H-\eta} \\ \frac{\lambda(1-\theta)}{1-m^H+\eta} & -\frac{(1-\lambda\theta)}{1-m^H+\eta} \end{pmatrix} \begin{pmatrix} F_H(\tilde{r}(x,y)) \\ F_L(\tilde{r}(x,y)) \end{pmatrix}$$
$$- \begin{pmatrix} (1-\lambda+\lambda\theta) & \frac{\lambda\theta(1-m^H+\eta)}{m^H-\eta} \\ \frac{\lambda(1-\theta)(m^H-\eta)}{1-m^H+\eta} & (1-\lambda\theta) \end{pmatrix} \begin{pmatrix} \int_y^{\tilde{\mu}(x,0)} \overline{G}^y_t(\mu|H) f_H(\tilde{r}(x,\mu)) \frac{\partial \tilde{r}(x,\mu)}{\partial \mu} d\mu \\ \int_y^{\tilde{\mu}(x,0)} \overline{G}^y_t(\mu|L) f_L(\tilde{r}(x,\mu)) \frac{\partial \tilde{r}(x,\mu)}{\partial \mu} d\mu \end{pmatrix}$$

Applying this operator to $\begin{pmatrix} \overline{G}^y_t(\cdot|H) \\ \overline{G}^y_t(\cdot|L) \end{pmatrix}$ we obtain:

$$\begin{pmatrix} \overline{G}^y_{t+1}(\cdot|H) \\ \overline{G}^y_{t+1}(\cdot|L) \end{pmatrix}(x) = \begin{pmatrix} \overline{G}^y_t(\cdot|H) \\ \overline{G}^y_t(\cdot|L) \end{pmatrix}(x) - \overline{G}^y_0(\mu_E) \begin{pmatrix} (1-\lambda+\lambda\theta) & \frac{\lambda\theta(1-m^H+\eta)}{m^H-\eta} \\ \frac{\lambda(1-\theta)(m^H-\eta)}{1-m^H+\eta} & (1-\lambda\theta) \end{pmatrix}$$
$$\times \begin{pmatrix} \frac{\int_y^{\tilde{\mu}(x,0)} \upsilon_t(\mu,y)\mu f_H(\tilde{r}(x,\mu)) \frac{\partial \tilde{r}(x,\mu)}{\partial \mu} d\mu}{m^H-\eta} \\ \frac{\int_y^{\tilde{\mu}(x,0)} \upsilon_t(\mu,y)(1-\mu) f_L(\tilde{r}(x,\mu)) \frac{\partial \tilde{r}(x,\mu)}{\partial \mu} d\mu}{1-m^H+\eta} \end{pmatrix} \tag{31}$$

where $\upsilon_t(x,y) \equiv -\int_y^{\tilde{\mu}(x,0)} \upsilon_{t-1}(\mu,y)\left(\mu f_H(\tilde{r}(x,\mu)) + (1-\mu) f_L(\tilde{r}(x,\mu))\right) \frac{\partial \tilde{r}(x,\mu)}{\partial \mu} d\mu$. Rearranging and using Equation 21, we get:

$$\begin{pmatrix} \overline{G}^y_{t+1}(\cdot|H) \\ \overline{G}^y_{t+1}(\cdot|L) \end{pmatrix}(x) = \begin{pmatrix} \overline{G}^y_t(\cdot|H) \\ \overline{G}^y_t(\cdot|L) \end{pmatrix}(x) + \overline{G}^y_0(\mu_E)\, \upsilon_t(x,y) \begin{pmatrix} \frac{x}{m^H-\eta} \\ \frac{1-x}{1-m^H+\eta} \end{pmatrix}$$

Accordingly,

$$\psi_{t+1}(\mu_E, y) = \frac{(m^H-\eta)\,\overline{G}^y_t(\mu_E|H) + \mu_E \overline{G}^y_0(\mu_E)\, \upsilon_t(\mu_E, y) + \eta}{\overline{G}^y_t(\mu_E) + \overline{G}^y_0(\mu_E)\, \upsilon_t(\mu_E, y)}.$$

We now prove by induction on $t$ that $\psi_t(\mu_E, y) = \mu_E$ for all $t$. By assuming that $\psi_t(\mu_E, y) = \mu_E$, we deduce that $\psi_{t+1}(\mu_E, y) = \mu_E$. Indeed:

$$\psi_t(\mu_E, y) = \mu_E \Leftrightarrow \mu_E \overline{G}_t^y(\mu_E) = \left(m^H - \eta\right) \overline{G}_t^y(\mu_E|H) + \eta.$$

Replacing, we get:

$$\psi_{t+1}(\mu_E, y) = \frac{\mu_E \overline{G}_t^y(\mu_E) + \mu_E \overline{G}_0^y(\mu_E)\, \upsilon_t(\mu_E, y)}{\overline{G}_t^y(\mu_E) + \overline{G}_0^y(\mu_E)\, \upsilon_t(\mu_E, y)} = \mu_E,$$

as asserted. Moreover, $\psi_1(\mu_E, y) = \mu_E$ had already been established.

Thus, $\psi_t(\mu_E, y) = \psi_0(\mu_E, y) = \mu_E$ for all $t > 0$, and therefore $\frac{\partial \psi}{\partial y}(x, y) = 0$ when $x = y = \mu_E$. □

Finally, we conclude that $\mu_E < m^H$:

**Lemma 5.** *Assume that $\eta > 0$. Then the consistent entry-level reputation is lower than the fraction of competents among active firms after the exit-entry process takes place: $\mu_E < m^H$.*

*Proof.* Notice that $\psi(\mu_E, \mu_E) = \mu_E$ and $\psi(1 - \lambda + \lambda\theta, \mu_E) = m^H$. Moreover,

$$\frac{\partial \psi}{\partial x}(x, \mu_E) = \left( \frac{\left(m^H - \eta\right) \overline{g}^{\mu_E}(x|H) + \left(1 - m^H + \eta\right) \overline{g}^{\mu_E}(x|I)}{\overline{G}^{\mu_E}(x)} \right) (x - \psi(x, \mu_E)).$$

This is strictly positive in the interval $(\mu_E, 1 - \lambda + \lambda\theta)$. Hence, $\mu_E < m^H$. □

# E Proof of Proposition 6

We want to look at the effect of $\lambda$ on the fixed point of the function $\sigma(\mu)$ defined in Equation 27. The change in $\lambda$ is compensated by a change in $\theta$ so that only the replacement process is affected, i.e., in this proof $\theta$ is set to $\theta = m^H - \frac{\eta}{\lambda}$, while the mass of active competent firms $m^H$ and the flow of competent entrants $\eta$ are fixed. As $\sigma'(\mu_E) = 0$, the (compensated) change in $\lambda$ affects $\mu_E$ only through the distributions $\overline{G}(\cdot|H)$ and $\overline{G}(\cdot|L)$.

Consider an initial value $\lambda_0 > 0$, and the corresponding pair of distributions $\overline{G}^{\mu_E;\lambda_0}(\cdot|H)$ and $\overline{G}^{\mu_E;\lambda_0}(\cdot|L)$, where $\mu_E$ is the entry-level reputation when the replacement probability is $\lambda_0$ and $\theta = m^H - \frac{\eta}{\lambda_0}$. As in the proof of Lemma 4, we analyze the sequence of distributions $\left\{\overline{G}_t^{\mu_E;\lambda}(\cdot|H), \overline{G}_t^{\mu_E;\lambda}(\cdot|L)\right\}$ defined by iterating the operator $T^{\mu_E;\lambda}$ starting from $\overline{G}_0^{\mu_E;\lambda}(\cdot|H) \equiv \overline{G}^{\mu_E;\lambda_0}(\cdot|H)$ and $\overline{G}_0^{\mu_E;\lambda}(\cdot|L) \equiv \overline{G}^{\mu_E;\lambda_0}(\cdot|L)$ and with $\lambda = \lambda_0 + d\lambda$ infinitesimally larger than $\lambda_0$, and the associated sequence $\{\psi_t(\mu_E, \mu_E; \lambda)\}$ defined by $\psi_t(x, \mu_E; \lambda) \equiv \frac{(m^H - \eta)\overline{G}_t^{\mu_E;\lambda}(x|H) + \eta}{\overline{G}_t^{\mu_E;\lambda}(x)}$. We will show that $\lim_{t\to\infty} \psi_t(\mu_E, \mu_E; \lambda) > \mu_E$, and hence the entry-level reputation obtained whith $\lambda > \lambda_0$ is larger than $\mu_E$.

After one iteration of the operator $T^{\mu_E;\lambda}$ we obtain:

$$\overline{G}_1^{\mu_E;\lambda}(x|\tau) = \overline{G}_0^{\mu_E;\lambda}(x|\tau) + \frac{\partial T_\tau^{\mu_E;\lambda}}{\partial \lambda}(x)\, dy$$

where $T_H^{\mu_E;\lambda}$ and $T_L^{\mu_E;\lambda}$ denote the first and second entries of the operator $T^{\mu_E;\lambda}$, respectively.

Taking the derivative of the right-hand-side of Equation 25 (with fixed distributions and with $\theta = m^H - \frac{\eta}{\lambda}$) and rearranging, we get:

$$\begin{pmatrix} \frac{\partial T_H^{\mu_E;\lambda}}{\partial \lambda} \\ \frac{\partial T_L^{\mu_E;\lambda}}{\partial \lambda} \end{pmatrix}(x) = -\frac{m^H - \eta - x}{1 - \lambda} \begin{pmatrix} \overline{g}_t^{\mu_E;\lambda}(x|H) \\ \overline{g}_t^{\mu_E;\lambda}(x|L) \end{pmatrix}$$
$$- \left( \int_0^{\tilde{r}(x,y)} G_t^{\mu_E;\lambda}(\tilde{\mu}(x, r)|L)\, dF_L - \int_0^{\tilde{r}(x,y)} G_t^{\mu_E;\lambda}(\tilde{\mu}(x, r)|H)\, dF_H \right) \begin{pmatrix} -\frac{m^H(1-m^H)}{m^H-\eta} \\ \frac{m^H(1-m^H)}{1-m^H+\eta} \end{pmatrix} \quad (32)$$

As the distributions $\overline{G}_0^{\mu_E;\lambda}(\cdot|\tau)$ for $\tau \in \{H, L\}$ are steady-state distributions, then they satisfy:

$$\overline{G}_0^{\mu_E;\lambda}(x|L) - \overline{G}_0^{\mu_E;\lambda}(x|H) = \frac{m^H\left(1-m^H\right)(1-\lambda)}{\left(m^H-\eta\right)\left(1-m^H+\eta\right)}\left(\int_0^{\tilde{r}(x,y)} G_0^{\mu_E;\lambda}\left(\tilde{\mu}(x,r)|L\right)dF_L - \int_0^{\tilde{r}(x,y)} G_0^{\mu_E;\lambda}\left(\tilde{\mu}(x,r)|H\right)dF_H\right)$$

and also

$$\frac{\left(m^H-\eta\right)\overline{g}_0^{\mu_E;\lambda}(x|H)}{\overline{g}_0^{\mu_E;\lambda}(x)} = x.$$

Hence, evaluating Equation 32 at $t=0$ and rearranging, we obtain:

$$\begin{pmatrix} \frac{\partial T_H^{\mu_E;\lambda}}{\partial\lambda} \\ \frac{\partial T_L^{\mu_E;\lambda}}{\partial\lambda} \end{pmatrix}(x) = -\frac{\left(m^H-\eta\right)\left(1-m^H+\eta\right)}{1-\lambda}\left(\overline{G}_0^{\mu_E;\lambda}(x|L) - \overline{G}_0^{\mu_E;\lambda}(x|H)\right)\begin{pmatrix} -\frac{1}{m^H-\eta} \\ \frac{1}{1-m^H+\eta} \end{pmatrix} - \frac{m^H-\eta-x}{1-\lambda}\overline{g}_0^{\mu_E;\lambda}(x)\begin{pmatrix} \frac{x}{m^H-\eta} \\ \frac{1-x}{1-m^H+\eta} \end{pmatrix} \quad (33)$$

It follows that the pair $\left(\overline{G}_1^{\mu_E;\lambda}(\cdot|H), \overline{G}_1^{\mu_E;\lambda}(\cdot|L)\right)$ can be written as:

$$\begin{pmatrix} \overline{G}_1^{\mu_E;\lambda}(\cdot|H) \\ \overline{G}_1^{\mu_E;\lambda}(\cdot|L) \end{pmatrix}(x) = \begin{pmatrix} \overline{G}_0^{\mu_E;\lambda}(\cdot|H) \\ \overline{G}_0^{\mu_E;\lambda}(\cdot|L) \end{pmatrix}(x) + \alpha_0(x,\mu_E)\begin{pmatrix} \frac{1}{m^H-\eta} \\ -\frac{1}{1-m^H+\eta} \end{pmatrix} + \beta_0(x,\mu_E)\begin{pmatrix} \frac{x}{m^H-\eta} \\ \frac{1-x}{1-m^H+\eta} \end{pmatrix}$$

where

$$\alpha_0(x,\mu_E) \equiv \frac{\left(m^H-\eta\right)\left(1-m^H+\eta\right)}{1-\lambda}\left(\overline{G}_0^{\mu_E;\lambda}(x|L) - \overline{G}_0^{\mu_E;\lambda}(x|H)\right)d\lambda > 0$$

and

$$\beta_0(x,\mu_E) \equiv -\frac{m^H-\eta-x}{1-\lambda}\overline{g}_0^{\mu_E;\lambda}(x)\,d\lambda < 0,$$

while $\psi_1(\mu_E,\mu_E;\lambda)$ can be written as:

$$\psi_1(\mu_E,\mu_E;\lambda) = \frac{\left(m^H-\eta\right)\overline{G}_0^{\mu_E;\lambda}(\mu_E|H) + \alpha_0(\mu_E,\mu_E) + \mu_E\beta_0(\mu_E,\mu_E) + \eta}{\overline{G}_0^{\mu_E;\lambda}(\mu_E) + \beta_0(\mu_E,\mu_E)}$$

where $\frac{\left(m^H-\eta\right)\overline{G}_0^{\mu_E;\lambda}(y|H)+\eta}{\overline{G}_0^{\mu_E;\lambda}(y)} = \psi_0(\mu_E,\mu_E;\lambda) = \mu_E$. Rearranging, we obtain:

$$\psi_1(\mu_E,\mu_E;\lambda) = \psi_0(\mu_E,\mu_E;\lambda) + \frac{\alpha_0(\mu_E,\mu_E)}{\overline{G}_1^{\mu_E;\lambda}(\mu_E)} > \psi_0(\mu_E,\mu_E;\lambda) + \alpha_0(\mu_E,\mu_E)$$

Looking at higher iterations of $T^{\mu_E;\lambda}$ we obtain:

$$\begin{pmatrix} \overline{G}_t^{\mu_E;\lambda}(\cdot|H) \\ \overline{G}_t^{\mu_E;\lambda}(\cdot|L) \end{pmatrix}(x) = \begin{pmatrix} \overline{G}_{t-1}^{\mu_E;\lambda}(\cdot|H) \\ \overline{G}_{t-1}^{\mu_E;\lambda}(\cdot|L) \end{pmatrix}(x) + \alpha_{t-1}(x,\mu_E)\begin{pmatrix} \frac{1}{m^H-\eta} \\ -\frac{1}{1-m^H+\eta} \end{pmatrix} + \beta_{t-1}(x,\mu_E)\begin{pmatrix} \frac{x}{m^H-\eta} \\ \frac{1-x}{1-m^H+\eta} \end{pmatrix}$$

where

$$\alpha_t(x,\mu_E) \equiv -\int_y^{\tilde{\mu}(x,0)} \alpha_{t-1}(\mu,\mu_E)\Bigg((1-\lambda+\lambda\theta-x)f_H\left(\tilde{r}(x,\mu)\right) + (x-\lambda\theta)f_L\left(\tilde{r}(x,\mu)\right)\frac{\partial\tilde{r}(x,\mu)}{\partial\mu}d\mu\Bigg) > 0$$

and

$$\beta_t(x,\mu_E) \equiv \int_y^{\tilde{\mu}(x,0)}\Bigg(\alpha_{t-1}(\mu,\mu_E)\left(f_L\left(\tilde{r}(x,\mu)\right) - f_H\left(\tilde{r}(x,\mu)\right)\right) - \beta_{t-1}(\mu,\mu_E)\left(\mu f_H\left(\tilde{r}(x,\mu)\right) + (1-\mu)f_L\left(\tilde{r}(x,\mu)\right)\right)\Bigg)\frac{\partial\tilde{r}(x,\mu)}{\partial\mu}d\mu$$

for all $t > 0$. Thus, we conclude that for all the elements of the sequence, $\psi_t\left(\mu_E, \mu_E; \lambda\right)$ can be written as:

$$\psi_t\left(\mu_E, \mu_E; \lambda\right) = \frac{\psi_{t-1}\left(\mu_E, \mu_E; \lambda\right) \overline{G}_{t-1}^{\mu_E;\lambda}\left(\mu_E\right) + \alpha_{t-1}\left(\mu_E, \mu_E\right) + \mu_E \beta_{t-1}\left(\mu_E, \mu_E\right)}{\overline{G}_{t-1}^{\mu_E;\lambda}\left(\mu_E\right) + \beta_{t-1}\left(\mu_E, \mu_E\right)}$$

Since $\psi_0\left(\mu_E, \mu_E; \lambda\right) = \mu_E$, recursive substitution yields:

$$\psi_t\left(\mu_E, \mu_E; \lambda\right) = \psi_0\left(\mu_E, \mu_E; \lambda\right) + \frac{\sum_{j=0}^{t-1} \alpha_j\left(\mu_E, \mu_E\right)}{\overline{G}_t^{\mu_E;\lambda}\left(\mu_E\right)} > \psi_0\left(\mu_E, \mu_E; \lambda\right) + \alpha_0\left(\mu_E, \mu_E\right)$$

Hence, we conclude that for all $t > 0$, $\psi_t\left(\mu_E, \mu_E; \lambda\right) > \psi_0\left(\mu_E, \mu_E; \lambda\right) + \alpha_0\left(\mu_E, \mu_E\right)$, which implies $\lim_{t\to\infty} \psi_t\left(\mu_E, \mu_E; \lambda\right) > \psi_0\left(\mu_E, \mu_E; \lambda\right)$. Therefore, the entry-level reputation obtained whith $\lambda > \lambda_0$ is strictly larger than $\mu_E$.

□