Hidden stochastic games and limit equilibrium payoffs

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The study of limit equilibrium payoffs is a main issue in the literature of dynamic games, and in particular in stochastic games with finitely many states and actions. There are two main approaches.

The *uniform approach* investigates the existence of *uniform* equilibrium payoffs, that is to say, existence for each $\varepsilon > 0$ of strategy profiles which are ϵ-equilibria in any discounted games with a discount factor close enough to 1. Vieille (see [7] and [8]) proved that uniform equilibrium payoffs do exist in the 2-player case, and extending this result to the N-player case is a challenging open question. Introducing a correlation mechanism makes things simpler: Solan and Vieille [5] proved the existence of correlated uniform equilibrium payoffs for any number of players.

An alternative approach, called the *asymptotic approach*, is to study the limit of the discounted (Nash or sequential) equilibrium payoffs set, when the discount factor goes to 1.

For 2-player repeated games (stochastic games with a single state), the Folk Theorem gives that the limit of the discounted Nash equilibrium payoffs set E_δ exists and coincides with the uniform equilibrium payoffs set (and for generic payoffs, so is the limit of the discounted sequential equilibrium payoffs set E'_δ). Dutta [1] generalized the Folk theorem to stochastic games, under an ergodicity assumption made on the transition function. Fudenberg and Yamamoto [2] and Horner, Takahashi, Vieille [3] extended this result to stochastic games with imperfect monitoring, still under an ergodicity assumption made on the transition function.

In this paper we focus on 2-player stochastic games where the actions are perfectly observed.

1) Our first contribution is to prove that the set of discounted *stationary* equilibrium payoffs always converges. As a corollary, there exists a selection of E_δ which converges when δ goes to 1. This corollary can also be deduced from Theorem 5 in [4].

2) Our second contribution is to provide a simple example of a 2-player stochastic game where for all δ in $[0,1)$, E_δ and E'_δ coincide and do not converge when δ goes to 1. However, this example is not robust in many aspects, like perturbing the payoffs, or adding a correlation mechanism. Moreover, E_δ has an empty interior.

3) Then we introduce a more general model of 2-player stochastic games, already appearing in Venel [6], where players do not observe perfectly the current state, but receive a public signal on it at the beginning of each period. By analogy with hidden Markov chains, we call these games *Hidden Stochastic Games* (HSG). We are particularly interested in the subclass of HSG *with known payoffs*, where the payoff function only depends on the state through the public signal. For these games, at every period the players know the current bimatrix game they face, and can compute the realized payoffs at the end of the period. Thus players know the payoff function, but are uncertain about the transitions.

Our third contribution is to construct, for each $\varepsilon \in (0, 0.3)$, a 2-Player hidden stochastic game with the following properties:

- there are 7 states, each player has four actions, and all payoffs lie in $[0,1]$,
- the game is symmetric,
- the game has known payoffs,
- for all λ and for all initial state, the set E'_λ has full dimension,
- there exists two sequences of discount factors (γ_n) and (δ_n) which go to 1, such that for all n:

  $E_{\gamma_n} = E'_{\gamma_n}$ is the square centered in (ϵ, ϵ) with side $2\epsilon/3$, whereas $E_{\delta_n} = E'_{\delta_n}$ is the square centered in $(1-\epsilon, 1-\epsilon)$ with side $2\epsilon/3$.

Consequently, (E_δ) and (E'_δ) do not converge when δ goes to 1. Moreover, there is no selection of (E_δ) which converges, thus the set of discounted stationary equilibrium payoffs does not converge. Last, uniform equilibrium payoff do not exist. It is the first example of a symmetric dynamic game with finite action sets and state space having such properties. Moreover, adding a correlation device or perturbing the payoffs would not change the above consequences. It is thus difficult to find a reasonable way to define a limit equilibrium payoff for this game. Contrary to our first example, the construction is sophisticated and elaborates on a zero-sum example of Ziliotto [9].

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