The Optimal Sequence of Costly Mechanisms

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Abstract

An impatient, risk-neutral monopolist must sell one unit of an indivisible good within a fixed number of periods while one-period-lived buyers with independent private values arrive over time. In each period, the seller can either run a reserve price auction while incurring a cost or post a price without the cost. We characterize the optimal sequence of mechanisms that maximizes the seller's expected profits. When there is an infinite number of periods, repeatedly running auctions with the same reserve price or posting a constant price is optimal. When there is a finite number of periods, the optimal sequence is a sequence of declining prices, a sequence of auctions with declining reserve prices converging to the static optimal monopoly reserve price, or a sequence of prices followed by a sequence of auctions. Most interestingly, a sequence of auctions before a sequence of posted prices is never optimal. The mechanism sequence of posted prices followed by auctions remains the general optimal sequence under various extensions of the basic setting and resembles a Buy-It-Now option.

Keywords: dynamic mechanism design, operational cost, reserve price auction, posted price, Buy-It-Now
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1 Introduction

Although auctions are desirable selling institutions to extract surpluses from buyers with private information and dispersed valuations, they are usually more costly to set up than alternative mechanisms, especially the simple and traditional posted prices. In comparison with a posted price mechanism, an auction has costs involved with setting reserve price, advertising to and aggregating bidders to an auction house, reviewing bids and communicating with buyers (or contractors in procurement setting), and determining payments. Furthermore in large competitive markets where the buyers can search and choose among sales of closely substitutable goods, running an auction is associated with an implicit opportunity cost of missing out on the buyers who love posted prices for their certainty, transparency, immediacy and/or convenience and this cost is enough to alter many sellers' mechanism choices and the landscape of eBay, a website initially known for auctions but now one mainly consisting of posted prices[1].

The costs associated with auctions are prevalent, significant, numerically relevant, and investigated as theoretical primitives in the literature. Einav et al. (2013) shows that there is a significant favor for posted price over auction by both sellers and buyers on eBay because of convenience and immediacy simple posted prices provide. Numerically we show that a cost of at most 20% of the average value is enough to make a seller switch mechanism and is only 1-2% in dynamic settings. The cost of setting up an auction is a considerable factor as well: Wang (1993) models the auction costs as those associated with storage and displaying. Satterthwaite and Shneyerov (2007, 2008) model the auction costs separately as the cost of participation and the cost of paying attention to search and bid. McAfee and McMillan (1988) assume there is a communication cost between seller and buyer: a posted price is cost-effective because there is only one buyer who indicates his interest and takes the good whereas an auction requires more buyers to submit bids and in procurement setting, the costs of reviewing bids and related information are involved as well.

In this paper, we take the auction cost into account and study the problem of a seller who repeatedly chooses between costly auctions and costless posted prices. An impatient, risk-neutral monopolist seller must sell one unit of an indivisible storable good within a fixed number of periods and in each period, she either runs a reserve price auction incurring a per-period cost or posts a fixed price for free. One-period-lived buyers with independent private values enter the market according to the same process each period. What is the sequence of mechanisms that maximizes the seller's expected profit?

[1]In January 2002, more than 90% of the active listings on eBay are auctions, but by late 2012, only 10% of the active listings are auctions, and the rest are simple posted prices and hybrid Buy-It-Now formats. Einav et al. (2013) attribute the declined uses of auctions mainly to the change in consumer preferences over the mechanisms and the decreased seller margin.

The answer depends on the number of periods the seller lives. When there is one period, Myerson's optimal auction is chosen whenever the revenue gain from an auction outweighs its cost. When there is an infinite number of periods, it is optimal to either run an auction with a constant reserve price or post a constant price. More interestingly, when there is a finite number of periods, the optimal sequence takes a particular form - a sequence of downwardly adjusted prices followed by a sequence of auctions with declining reserve prices. It is possible that the seller only posts a sequence of prices or runs a sequence of auctions if the auction cost is too high or too low, but in particular, auctioning the good off before posting prices is never optimal.

In the static setting, Myerson (1981) shows the expected revenue maximizing mechanism is a reserve price auction with the optimal reserve price determined by the virtual utility curve. If the operational cost is low, the seller stills runs the optimal auction; if it is high, the seller posts a fixed price determined by a posted price marginal revenue curve similar to that of an auction. There is a cutoff cost such that the monopolist is indifferent between the optimal auction and the optimal posted price when the operational cost equals the cutoff cost. Whereas in a static setting the seller only faces tradeoff between gain in expected revenue and auction cost, in dynamic settings, there are additional intricate considerations of inter-temporal tradeoffs such as paying auction costs earlier versus later, and the effects of continuation payoffs on current mechanism decision.

When the horizon is infinite, there is no exogenous deadline to sell the good but a cost of delay that incentivizes early sale, the problem the seller faces in each period is identical. As a result of the stationary environment, the seller runs the same mechanism repeatedly in each period. While high cost seller posts a constant price, low cost seller runs auctions with a constant reserve price. The optimal reserve price is inversely related to the operational cost: a lower cost results in a longer delay in the sale of the good. Moreover, for a fixed operational cost, it is more likely for a more patient seller to post price. The optimal auction can be approximately implemented by an ascending auction with the deadline determined by the time elapsed after the latest bid, and in continuous time, a constant fixed price is always optimal.

When there is a finite number of periods, the seller's optimal sequence of mechanisms depends on the auction cost but takes a certain form: a sequence of posted prices followed by a sequence of auctions. Prices are adjusted downward until a cutoff period and auctions are run each period afterwards, with the reserve prices declining to the static optimal reserve price in the last period. When the auction cost is sufficiently low, the dynamic pricing phase does not exist: a sequence of auctions is optimal. When the auction cost is sufficiently high, the sequential auction phase does not exist: the posted prices decline over time. In essence, auctions are used later when they are more costly - if too costly, they may be abandoned altogether. Note in particular it is never optimal for any seller to run auctions then post prices in this setting.

A Buy-It-Now option[2] resembles the optimal mechanism sequence in that prices are posted before bids are submitted. Auction cost is an alternative explanation to risk aversion and impatience for the rise of such format (Budish and Takeyama, 2001; Mathews, 2004).

The resulting optimal sequence of mechanisms holds in more general settings when the seller may randomly exit, the buyers enter stochastically and have sequential outside options. These extensions help to identify the key condition that results in the resulting price-auction mechanism sequence: the optimal continuation values decrease over time. If a seller dies with gradually increasing probabilities, continuation payoff is discounted more heavily, the mechanism form does not change. If buyers' arrival is stochastic instead of deterministic, the buyers' expected utilities are affected, but the seller's optimal mechanism form is not. If the number of buyers changes, the result does not necessarily hold. Furthermore, if buyers have sequential outside options independent of the current mechanism choice, the mechanism will not change but because buyers' willingnesses to pay decrease, posted prices are more favored. We show in various situations when the condition of monotone optimal continuation values is not satisfied, the optimal sequence may be a patternless combination of prices and auctions, in particular when the buyers may strategically time their purchases. Finally, we also solve the sequence of optimal mechanisms of a buyer seeking procurement contracts.

This paper is motivated by the dynamic mechanism design literature, or more precisely the so-called single leg, multi-period problem in revenue management literature. In the earlier literature in revenue management[3] for airlines and hotels, prices are dynamically adjusted over time to yield more revenues. As the technology advances, potential buyers dynamically arrive over time and may strategically time their purchases (e.g. advance purchases, last-minute deals), variants of auctions are proven to be expected revenue maximizing[4] (Board and Skrzypacz, 2010; Li, 2009; Said, 2012; Bergemann and Välimäki, 2010). This paper considers the situation in which a use of the new technology is considerably costly and suggests that the uses of superior mechanisms in later periods when the good is near expiration may be more appropriate.

Some optimal mechanisms look similar to ours. McAfee and McMillan (1988) and Riley and Zeckhauser (1983) derive similar results as ours in the infinite horizon - a constant posted price when buyers arrive sequentially and/or costs of communications are high. Board and Skrzypacz (2010) characterize the optimal mechanism when buyers are forward looking. The optimal mechanism sequence is in a way similar to ours - posted prices followed by a fire sale

[2] The Buy-It-Now option is introduced by eBay in 2002. It allows a seller to post a price at which a buyer may get the good immediately. If some buyer bids, the price option (the Buy-It-Now button) may disappear.

[3] Talluri and van Ryzin (2004) is a comprehensive and excellent treatment on the subject of revenue management.

[4] Bergemann and Said (2011) is a brief survey on the subject of dynamic auctions.

by auction in the last period. However, the particular sequence form is mainly driven by continuity of time; in discrete time, a sequence of reserve price auctions is still expected revenue maximizing.

The paper is organized as follows. Section 2 introduces the setting and solves the seller's static problem. Section 3 solves the seller's problem in infinite horizon and Section 4 in finite horizon. Section 5 extends the basic setting, identifies the key condition that guarantees the results in previous sections, and concludes.

2 Preliminaries

A (female) monopolist wants to sell one unit of an indivisible good for which she has zero consumption value. She lives for T periods and discounts each period by $\delta \in [0,1)$. T can be finite or infinite. In each period t, n one-period-lived (male) buyers enter the market and the seller chooses between two classes of selling mechanisms: a costly reserve price auction and a costless posted price mechanism. Each buyer has a private value v independently drawn from the identical value distribution $F(\cdot)$ with positive density $f(\cdot)$ on support $[0,1]$; assume $F(\cdot)$ has monotone inverse hazard rate, i.e., $(1-F(\cdot))/f(\cdot)$ decreases in its argument. All the agents are risk-neutral and have quasi-linear preferences in transfers. Except for the private values the buyers each have, everything else is common knowledge.

The two classes of mechanisms differ in their procedures and operational costs. In a posted price mechanism $P(p)$, the seller posts a fixed price p at the beginning of a period and the buyers who are willing to pay more than p have equal chances of receiving the good[5]. A second-price auction[6] $A(r)$ with reserve price r, on the other hand, asks each buyer for a sealed bid and at the end of the period assigns the good to the buyer with the highest bid if it is above r and the winner pays the bigger of the reserve price and the second highest bid. A reserve price auction costs $C \geq 0$ to run and posting a price is free.

Let m denote a mechanism and M the set of all feasible mechanisms. Let $R(m)$ denote the ex-ante expected revenue of mechanism m and $\pi(m)$ its ex-ante expected profit in the period. Any posted price mechanism's expected profit equals its expected revenue, but any auction's expected profit is its expected revenue less the cost. Let $s(m)$ represent the ex-ante expected probability that the seller sells the good using m. Specifically, $s(A(r)) = 1-F^n(r)$ and $s(P(p)) = 1-F^n(p)$; since the probability of sale does not depend on the form of the mechanism but only

[5]Alternatively and equivalently, consider it as a dynamic process in which the buyers sequentially enter the market and each decide immediately whether to purchase or not.

[6]By the Revenue Equivalence Theorem, from the seller's perspective, first-price auction generates the same expected revenue. Although buyers would behave differently, their expected utilities are the same across all standard auctions thus they are indifferent.

on the reserve price or fixed price, we can write $s(r)$ or $s(p)$ whenever convenient. Furthermore, let $\not{s}(\cdot)$ denote the probability that the good is not sold: $\not{s}(\cdot) = 1 - s(\cdot)$. $(m_\tau, \cdots, m_T) \in M^{T-\tau+1}$ is a **mechanism sequence** of the monopolist who runs m_t in period t if the good has not been sold by the end of period $t-1$. Let $\Pi_\tau(m_\tau, \cdots, m_T)$ denote the seller's discounted expected payoff at the beginning of period τ if she runs the mechanism sequence $(m_\tau, \cdots, m_T)$ from period τ to T, in particular,

$$\Pi_\tau(m_\tau, \cdots, m_T) = \sum_{t=\tau}^{T} \delta^{t-\tau} \left[\prod_{t'=\tau}^{t-1} \not{s}(m_{t'}) \right] \pi(m_t).$$

The seller's problem is to choose the **optimal** mechanism sequence $\mathbf{m}^* \equiv (m_1, \cdots, m_T)$ so that $\Pi_\tau(m_\tau, \cdots, m_T)$ is maximized for any period τ in which she can choose a mechanism,

$$\forall \tau < T: \quad \max_{(m_1, \cdots, m_T) \in M^T} \quad \Pi_\tau(m_\tau, \cdots, m_T).$$

Since the buyer arrival process is known and there is no learning by the seller, the seller essentially chooses a sequence of contingent mechanisms at the beginning of the first period, to be executed in each period if the good has not been sold.

In the remainder of the section, we characterize the optimal mechanism when $T = 1$ as a building block for the cases of infinite and finite horizons. The expected revenue $R(A(r))$ for any reserve price r auction is

$$R(A(r)) = \int_r^1 \mathrm{MR}^A(v)\, dF^n(v)$$

and by Myerson (1981), the expected revenue maximizing mechanism among all direct revelation mechanisms is implemented by a standard reserve price auction, with the optimal reserve price r^* determined by equating the marginal revenue to the opportunity cost of the seller, $\mathrm{MR}^A(r^*) \equiv r^* - [1 - F(r^*)] / f(r^*) = 0$. Posting price p generates expected revenue $p\left[1 - F^n(p)\right]$ which can be rewritten as

$$R(P(p)) = \int_p^1 \left[v - \frac{1 - F^n(v)}{(F^n(v))'} \right] dF^n(v).$$

Similar to auction's marginal revenue curve (Bulow and Roberts, 1989), posted price mechanism's marginal revenue curve is defined as

$$\mathrm{MR}^P(v) = v - \frac{1 - F^n(v)}{(F^n(v))'}.$$

When $n = 1$, the two marginal revenue curves coincide. The optimal posted price[7] p^* is determined by equating $\mathrm{MR}^P\left(p^*\right)$ to 0. Posted price's marginal revenue curve is always smaller than that of auction, so the optimal price posted is always higher than the optimal reserve price. As a consequence, the optimal posted price's expected revenue is smaller. Figure 1 provides a graphical illustration of the marginal revenue curves and the revenues of the two mechanism classes.

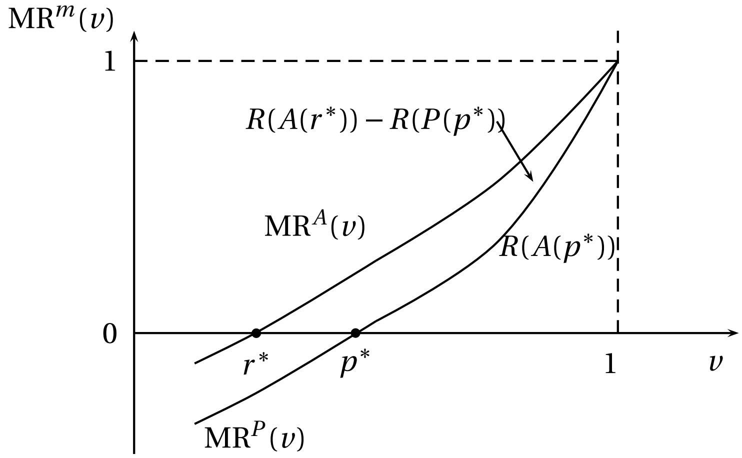

Figure 1: Marginal Revenue Curves of Posted Price and Reserve Price Auction.

Although the expected revenue-maximizing reserve price auction always generates more expected revenue, if there is a sufficiently high cost of operating auctions, it may be more desirable to post a price. There is a cutoff cost $C^* \in [0,1]$ such that the seller with such operational cost is indifferent between the optimal auction and the optimal posted price mechanism. Running auction is more appealing for the monopolist if her cost is lower than the cutoff cost, and posting price is more appealing otherwise. Lemma 1 characterizes the monopolist's optimal mechanism.

Lemma 1. *Suppose $T = 1$. The monopolist's optimal mechanism is $A(r^*)$ if $C < C^*$, $P\left(p^*\right)$ if $C > C^*$, and either $A(r^*)$ or $P\left(p^*\right)$ if $C = C^*$, where r^* and p^* are the unique solutions to $MR^A(r^*) = MR^P\left(p^*\right) = 0$, and $C^* = R(A(r^*)) - R\left(P\left(p^*\right)\right)$.*

Numerical results show that a relatively small operational cost is enough to make the seller switch from the optimal auction to the optimal posted price. Figure 2 shows C^* for $n = 1, \cdots, 100$ when $v \sim \mathrm{Unif}[0,1]$ and n varies. An operational cost of 0.04 to 0.10, or about 5-15% of the

[7] The solution p^* exists and is unique because

$$\frac{1 - F^n(v)}{[F^n(v)]'} = \frac{F^{n-1} + \cdots + 1}{F^{n-1}(v)} \cdot \frac{1 - F(v)}{f(v)}$$

is positive when $v = 0$ and 0 when $v = 1$, so that $\mathrm{MR}^P(0) < 0$ and $\mathrm{MR}^P(1) = 1 > 0$, and is decreasing in v.

expected revenue, is sufficient to make the seller post price in the static setting. We will see from subsequent sections that roughly speaking the optimal auction becomes more easily undesirable relative to posted price in the dynamic setting when $T > 1$ as the monopolist holds monopoly power longer and has more opportunities to sell.

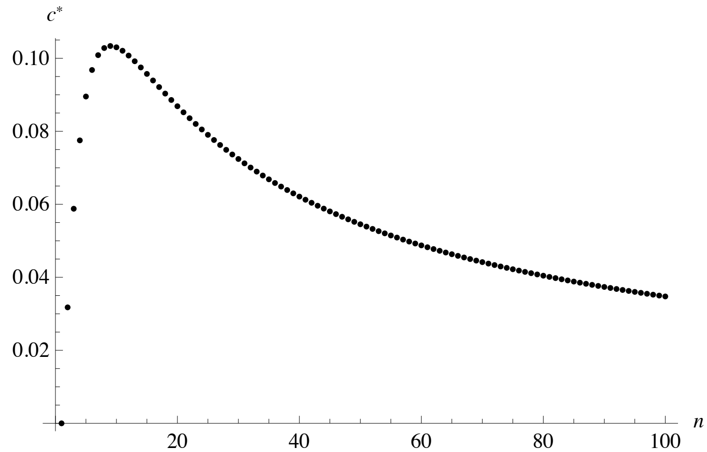

Figure 2: C^* for $n = 1, \cdots, 100$ when $v \sim \text{Unif}[0,1]$.

Evident from Figure 2, the number of buyers in the market plays a complicated role in relative attractiveness of the two different selling mechanisms. When $n = 1$, equilibrium outcomes of the two optimal mechanisms are both equivalent to that of the ultimatum bargaining with the seller making a take-it-or-leave-it offer to the sole buyer, and consequentially they generate the same expected revenue. Therefore, the optimal mechanism choice solely depends on whether C is positive or negative: $C^* = 0$. If $n > 1$, C^* is strictly positive because the marginal revenue curves are strictly different. The difference between the expected revenues of the two optimal mechanisms does not monotonically increase in the number of buyers. Rather it is inverse-U shaped: auctions are more advantageous when the number of buyers is moderately high, but can be more closely matched by a simple posted price when there are only a few buyers or when there are many. We see that for $n < 10$, auction is significantly increasingly better, but as $n > 10$, winners are paying not much less than their values in an auction and an optimally posted fixed price performs relatively well. Although the speed of convergence to $C^* = 0$ is relatively slow (when $n = 500$, $C^* \approx 0.010$; when $n = 1000$, $C^* \approx 0.0059$; when $n = 5000$, $C^* \approx 0.0015$; when $n = 10000$, $C^* \approx 0.00082$), when the number of buyers becomes large, the difference in the expected revenues between the optimal auction and the optimal posted price becomes insignificant.

3 Infinite Horizon

Unlike goods with expirations such as airline tickets and hotel rooms or seasonal clothes that might fall out of favor in three months, many goods such as cellphones, stamps and books do not lose value and do not have fixed deadlines to be sold, but nonetheless, the seller has incentive to sell the good and realize the payment as soon as possible. In this section, the optimal sequence of mechanisms when in the infinite horizon is characterized. The solution turns out to be relatively simple and straightforward: the optimal mechanism sequence is an infinite repetition of the same mechanism. Different from the static setting, the optimal reserve price depends on not only the buyer value distribution but also the number of buyers per period and crucially the operational cost.

Although it is an infinite horizon problem, the monopolist faces the same environment in each period. Therefore, she only needs to solve a static problem in a stationary setting, and her optimal mechanism sequence is either an infinitely repeated auction or an infinitely repeated posted price - running a reserve price auction until some buyers bid, or posting a constant price until some buyers buy at the price. We will first calculate respectively the optimal constant price and the optimal constant reserve price and then compare to see which infinite mechanism sequence generates more expected profits. Finally, we verify that the better of the two is indeed the expected revenue maximizing mechanism in each individual period.

First, we solve the optimal fixed price p^*_∞. Let Π^P_∞ denote the continuation value of posting p^*_∞ in each period. Together, p^*_∞ and Π^P_∞ must simultaneously satisfy the following equations,

$$\begin{aligned} \mathrm{MR}^P\left(p^*_\infty\right) &= \delta\Pi^P_\infty, && (1)\\ \Pi^P_\infty &= \pi\left(P\left(p^*_\infty\right)\right) + \delta \not{s}\left(p^*_\infty\right)\Pi^P_\infty, && (2) \end{aligned}$$

where (1) equates the posted price marginal revenue to the discounted continuation value, and (2) specifies the continuation value in terms of the optimal choice in a period and the optimal continuation value. Rearranging (2) and plugging it into (1), p^*_∞ is uniquely determined[8] by

$$\frac{1-\delta}{1-\delta\not{s}\left(p^*_\infty\right)}p^*_\infty - \frac{1-F^n\left(p^*_\infty\right)}{\left(F^n\left(p^*_\infty\right)\right)'} = 0. \qquad (3)$$

Similarly, we can calculate the optimal reserve price $r^*_\infty(C)$ for a cost C seller. Let $\Pi^A_\infty(C)$ denote the continuation value of running $A\left(r^*_\infty(C)\right)$ each period. Together, r^*_∞ and Π^A_∞ must

[8]The first term is increasing in and the second term is decreasing in p^*_∞, and when $p^*_\infty = 0$, the LHS is negative $(= -(1-F^n(0))/(F^n(0))')$, but when $p^*_\infty = 1$, the LHS is positive $(= 1)$.

simultaneously satisfy the following two equations,

$$
\begin{aligned}
\mathrm{MR}^{A}\left(r_{\infty}^{*}(C)\right) &= \delta \Pi_{\infty}^{A}(C), &(4)\\
\Pi_{\infty}^{A}(C) &= \pi\left(A\left(r_{\infty}^{*}(C)\right)\right)+\delta \not{s}\left(r_{\infty}^{*}(C)\right) \Pi_{\infty}^{A}(C). &(5)
\end{aligned}
$$

Substitute (5) into (4) and rearrange,

$$
\frac{1-\delta}{1-\delta \not{s}\left(r_{\infty}^{*}(C)\right)} r_{\infty}^{*}(C)=\frac{1-F\left(r_{\infty}^{*}(C)\right)}{f\left(r_{\infty}^{*}(C)\right)}+\frac{\delta\left[R\left(A\left(r_{\infty}^{*}(C)\right)\right)-R\left(P\left(r_{\infty}^{*}(C)\right)\right)-C\right]}{1-\delta \not{s}\left(r_{\infty}^{*}(C)\right)}. \tag{6}
$$

The LHS is increasing and the RHS is decreasing in $r_{\infty}^{*}(C)$; the LHS is 0 and the RHS is $1/f(0)+\delta[R(A(0))-C]/(1-\delta)$ when $r_{\infty}^{*}(C)=0$ and the LHS is 1 and the RHS is $-\delta C$ when $r_{\infty}^{*}(C)=1$. Therefore, if $C \leq R(A(0))$, $r_{\infty}^{*}(C)$ is uniquely determined by the system of equations. If $C > R(A(0))$, the monopolist chooses a posted price for sure because for any r, $A(r)$ generates strictly less expected profit than $P(r)$, because

$$
\pi(A(r))-\pi(P(r)) \leq \pi(A(0))-\pi(P(0))=R(A(0))-C<0
$$

where the first inequality can be seen from Figure 1: as the reserve price increases, the difference between the expected revenues decreases because the difference between the marginal revenue curves is always positive.

Therefore, there is a cutoff cost type C_{∞}^{*}. The seller runs $A\left(r_{\infty}^{*}(C)\right)$ if $C > C_{\infty}^{*}$, runs $A\left(p_{\infty}^{*}\right)$ if $C < C_{\infty}^{*}$ and is indifferent between the two if $C = C_{\infty}^{*}$: C_{∞}^{*} must satisfy $\Pi_{\infty}^{A}\left(C_{\infty}^{*}\right)=\Pi_{\infty}^{P}$. As a result, there is an easy way to calculate C_{∞}^{*} rather than solving the system of equations (5) and (6). $r_{\infty}^{*}\left(C_{\infty}^{*}\right)$ is determined by (5),

$$
\mathrm{MR}^{A}\left(r_{\infty}^{*}\left(C_{\infty}^{*}\right)\right)=\delta \Pi_{\infty}^{P}, \tag{7}
$$

but C_{∞}^{*} must satisfy that in any period, the seller is indifferent between posting the optimal price and running the optimal auction instead, so $\Pi_{\infty}^{P}=R\left(A\left(r_{\infty}^{*}\left(C_{\infty}^{*}\right)\right)\right)-C_{\infty}^{*}+\delta \not{s}\left(r_{\infty}^{*}\left(C_{\infty}^{*}\right)\right) \Pi_{\infty}^{P}$, or

$$
C_{\infty}^{*}=R\left(A\left(r_{\infty}^{*}\left(C_{\infty}^{*}\right)\right)\right)-\left[1-\delta \not{s}\left(r_{\infty}^{*}\left(C_{\infty}^{*}\right)\right)\right] \Pi_{\infty}^{P}. \tag{8}
$$

The optimal mechanism sequence is summarized as follows.

Proposition 1. *Suppose $T=\infty$. Let p_{∞}^{*} and Π_{∞}^{P} be the unique solution to (1) and (2). Let C_{∞}^{*} be the unique solution to (7) and (8). Fix any $C \leq R(A(0))$, let $r_{\infty}^{*}(C)$ be the unique solution to (6). The seller's optimal mechanism sequence is an infinitely repeated sequence of $m_{\infty}^{*}(C)$. If $C > C_{\infty}^{*}$, then $m_{\infty}^{*}(C)=P\left(p_{\infty}^{*}\right)$, if $C < C_{\infty}^{*}$, then $m_{\infty}^{*}(C)=A\left(r_{\infty}^{*}(C)\right)$, and if $C=C_{\infty}^{*}$, $m_{\infty}^{*}(C)=$*

$P(p^*_\infty)$ *or* $= A(r^*_\infty(C^*_\infty))$. *Her expected payoff by running the optimal mechanism is* $\Pi^*_\infty(C) = \max\{\Pi^P_\infty, \Pi^A_\infty(C)\}$ *where* $\Pi^A_\infty(C)$ *is obtained from (4).*

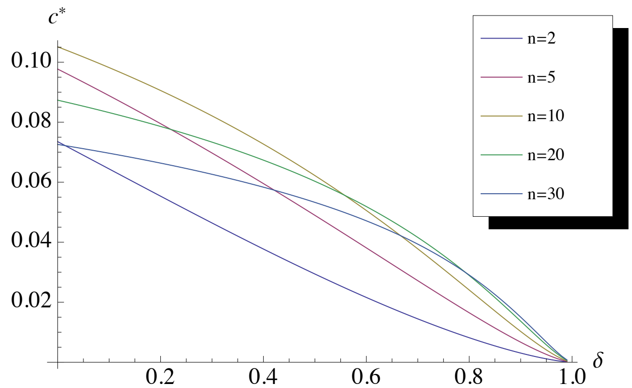

Figure 3: C^*_∞ for different n and δ when $F(\cdot) \sim \text{Unif}[0,1]$.

Figure 3 shows C^*_∞ for different combinations of n and δ when $v \sim \text{Unif}[0,1]$. For example, if $\delta = 0.95$, $n = 5$, $C^*_\infty \approx 0.0026$. Fixing any δ, the relationship between C^*_∞ and n is non-monotonic (in addition to Figure 3, Figure 2 illustrates the relationship between C^*_∞ and n when $\delta = 0$).

However, notice that fixing any n, as δ increases, i.e. when the seller becomes more patient, C^*_∞ decreases. When $\delta \to 1$, there is less discounting and because the monopolist lives forever, she essentially encounters infinitely many buyers over her life so the order statistics effect dominates and $C^*_\infty \to 0$. The monotonic relationship between C^*_∞ and δ holds in general for any $F(\cdot)$ and n. An implication of the result is that if the buyers arrive over time, as δ is small, we can think of the dynamic buyer arrival as continuous and stochastic (Poisson arrival, for example). We see that as δ is small, C^*_∞ approaches 0, so a constant price is always optimal. The proof proceeds by differentiating C^*_∞ with respect to δ in (8) and relying on the facts that as δ increases, both Π^P_∞ and p^*_∞ strictly increase, and the optimal reserve price $r^*_\infty(C^*_\infty)$ is always smaller than the posted price p^*_∞.

Proposition 2. *Fix any* $F(\cdot)$ *and* $n \geq 2$. *When* δ *increases,* C^*_∞ *strictly decreases.*

Proof to Proposition 2. See Section A in Appendix. □

4 Finite Horizon

In this section, we solve the monopolist's profit maximization problem when T is finite and fixed. We solve the problem backwards. Interestingly, the optimal mechanism sequence is always a sequence of posted prices followed by a sequence of auctions.

For any $t = 1, \cdots, T-1$, the seller's discounted sum of payoffs at period $t < T$ is her expected profit in the current period plus her discounted payoff if the good is not sold in the current period,

$$\Pi_t (m_1, \cdots, m_T) = \pi (m_{t+1}) + \delta \not{s} (m_{t+1}) \Pi_{t+1} (m_{t+1}, \cdots, m_T),$$

and the seller's payoff in period T is merely her expected profit as she does not have future selling opportunities,

$$\Pi_T (m_1, \cdots, m_T) = \pi (m_T).$$

By the Principle of Optimality, we can solve the problem backwards, namely by maximizing $\Pi_T(\cdot)$ first, and then $\Pi_{T-1}(\cdot)$ and so on.

In fact, we have already solved the period T problem, as it has the same solution as the one-period problem. We restate and generalize Lemma 1 in the frame of the T-period problem.

Lemma 2. *Suppose T is finite. The monopolist's optimal mechanism in the last period T is $m_T^*(C) = A(r_T^*)$ if $C < C_T^*$, $m_T^*(C) = P(p_T^*)$ if $C > C_T^*$, and $m_T^*(C) = A(r_T^*) = P(p_T^*)$ if $C = C_T^*$, where r_T^* and p_T^* are the unique solutions to $MR^A(r_T^*) = MR^P(p_T^*) = 0$, and $C_T^* = R(A(r_T^*)) - R(P(p_T^*))$.*

The value of the optimal policy for a cost C monopolist in period T is the maximized expected profit, or the larger of the expected profit of running the optimal auction and of posting the optimal price,

$$\Pi_T^*(C) = \max\left\{R\left(A\left(r_T^*\right)\right) - C, R\left(A\left(p_T^*\right)\right)\right\}. \tag{9}$$

Her problem in period $T-1$ is then

$$\max_{m_{T-1} \in M} \quad \pi (m_{T-1}) + \delta \not{s} (m_{T-1}) \Pi_T^* (C).$$

Her optimal auction is $A\left(r_{T-1}^*(C)\right)$ where $r_{T-1}^*(C)$ is the unique solution to $\mathrm{MR}^A\left(r_{T-1}^*(C)\right) = \delta\Pi_T^*(C)$ and her optimal posted price is $P\left(p_{T-1}^*(C)\right)$ where $p_{T-1}^*(C)$ is the unique solution to $\mathrm{MR}^P\left(p_{T-1}^*(C)\right) = \delta\Pi_T^*(C)$. The monopolist's value in period $T-1$ is then

$$\begin{aligned}\Pi_{T-1}^*(C) = \quad \max \quad & \{R\left(P\left(p_{T-1}^*(C)\right)\right) + \not{s}\left(p_{T-1}^*(C)\right)\delta\Pi_T^*(C), \\ & R\left(A\left(r_{T-1}^*(C)\right)\right) - C + \not{s}\left(r_{T-1}^*(C)\right)\delta\Pi_T^*(C)\}.\end{aligned}$$

There is a cutoff cost C_{T-1}^* such that the monopolist runs $A\left(r_{T-1}^*(C)\right)$ if her cost is lower than it and runs $P\left(p_{T-1}^*(C)\right)$ otherwise,

$$\begin{aligned}C_{T-1}^* \quad = \quad & R\left(A\left(r_{T-1}^*\left(C_{T-1}^*\right)\right)\right) - R\left(P\left(p_{T-1}^*\left(C_{T-1}^*\right)\right)\right) \\ & + \left[\not{s}\left(r_{T-1}^*\left(C_{T-1}^*\right)\right) - \not{s}\left(p_{T-1}^*\left(C_{T-1}^*\right)\right)\right]\delta\Pi_T^*\left(C_{T-1}^*\right).\end{aligned}$$

The cutoff cost exists and is unique because the RHS of the equation has slope

$$0 < \left[\not{s}'\left(r_\infty^*\right) - \not{s}'\left(p_\infty^*\right)\right]\delta \frac{d\Pi_{T-1}^*\left(C_{T-1}^*\right)}{dC_{T-1}^*} \leq \delta,$$

and it increases from a positive number < 1 when $C = 0$ (running auctions for both periods if cost is 0) to a bounded number < 1 when $C = 1$ (posting prices for both periods if the cost is sufficiently high).

This procedure of solving for the optimal mechanism iterates over all periods $t \leq T-1$ and is summarized in Lemma 3.

Lemma 3. *Suppose $t \leq T-1$. A cost C monopolist's optimal mechanism is $m_t(C) = A\left(r_t^*(C)\right)$ if $C < C_t^*$, $m_t(C) = P\left(p_t^*(C)\right)$ if $C > C_t^*$, and $m_t(C) = A\left(r_t^*(C)\right) = P\left(p_t^*(C)\right)$ if $C = C_t^*$ where $r_t^*(C)$, $p_t^*(C)$, and C_t^* are determined as follows.*

Let the value of the optimal policy of cost C monopolist from period $t+1$ to period T be $\Pi_{t+1}^*(C)$. Then the optimal reserve price and optimal fixed price in period t are $r_t^*(C)$ and $p_t^*(C)$ where

$$MR^A\left(r_t^*(C)\right) = MR^P\left(p_t^*(C)\right) = \delta\Pi_{t+1}^*(C). \tag{10}$$

The cutoff cost C_t^* is uniquely determined by

$$C_t^* = R\left(A\left(r_t^*\left(C_t^*\right)\right)\right) - R\left(P\left(p_t^*\left(C_t^*\right)\right)\right) - \left[\not{s}\left(p_t^*\left(C_t^*\right)\right) - \not{s}\left(r_t^*\left(C_t^*\right)\right)\right]\delta\Pi_{t+1}^*\left(C_t^*\right). \tag{11}$$

Proof to Lemma 3. We need to show that (11) has a unique solution in each period. $\Pi_t(C)$ is continuous and almost everywhere differentiable in C. First, for any $t < T-1$, $\Pi_t^*(C) > \Pi_{t+1}^*(C)$. This holds because for any $(m_{t+1}, \cdots, m_T)$,

$$\Pi_t\left(m_{t+1}, \cdots, m_T, P\left(p_T^*\right)|C\right) \geq \Pi_{t+1}\left(m_{t+1}, \cdots, m_T|C\right),$$

that is, the expected payoff in period $t+1$ of any mechanism sequence is guaranteed in period t by running the sequence of mechanism starting periods t to $T-1$, and posting the optimal fixed price p_T^* in period T. Then,

$$\Pi_{t+1}^*(C) = \max_{\mathbf{m}\in M^{T-t}} \Pi_{t+1}(\mathbf{m}|C) < \max_{\mathbf{m}\in M^{T-t}} \Pi_t\left(\mathbf{m}, P\left(p_T^*\right)|C\right) \leq \Pi_t^*(C),$$

where the strict inequality follows from the fact that the expected profit maximizing mechanism sequence is never efficient. The RHS of (11) is increasing in C and has a slope less than 1 because

$$\begin{aligned}
\frac{d\mathrm{RHS}(C)}{dC} &= -\mathrm{MR}^{A}\left(r_t^*(C)\right)\left(F^n\left(r_t^*(C)\right)\right)' + \mathrm{MR}^{P}\left(p_t^*(C)\right)\left(F^n\left(p_t^*(C)\right)\right)' \\
&\quad - \left[\$'\left(p_t^*(C)\right) - \$'\left(r_t^*(C)\right)\right]\delta\Pi_{t+1}^*(C) - \left[\$\left(p_t^*(C)\right) - \$\left(r_t^*(C)\right)\right]\delta\frac{d\Pi_{t+1}^*(C)}{dC} \\
&= -\left[\$\left(p_t^*(C)\right) - \$\left(r_t^*(C)\right)\right]\delta\frac{d\Pi_{t+1}^*(C)}{dC} > 0
\end{aligned}$$

and $\left|\frac{d\Pi_{t+1}^*(C)}{dC}\right|$ has a maximum when $C = 0$, that is, the seller runs auction every period, and with a cost increment, she incurs a cost every period. The increment in cost, thus the reduction in profits, is

$$dC\left[1 + \delta\$(r_{t+1}) + \delta^2\$(r_{t+1})\$(r_{t+2}) + \delta^3\$(r_{t+1})\$(r_{t+2})\$(r_{t+3}) + \cdots\right]$$

which is smaller than

$$dC\left[1 + \delta\$(r_{t+1}) + \delta^2\$^2(r_{t+1}) + \cdots\right] = dC\frac{1}{1-\delta\$(r_{t+1})}.$$

Therefore,

$$\frac{d\mathrm{RHS}(C)}{dC} \le \frac{\delta\$\left(p_t^*\right) - \delta\$\left(r_t^*\right)}{1-\delta\$(r_{t+1})} \le \frac{\delta\$\left(p_t^*(C)\right) - \delta\$\left(r_t^*(C)\right)}{1-\delta\$\left(r_t^*(C)\right)} \le 1.$$

The RHS is positive when $C = 0$ and is increasing with slope less than 1, so it will intersect with the LHS= C at one point. $\square$

The value of the optimal policy in period T is defined as in (9), and in period $t \le T-1$ is

$$\Pi_t^*(C) = \max\left\{R\left(A\left(r_t^*(C)\right)\right) - C + \$\left(r_t^*(C)\right)\delta\Pi_{t+1}^*(C),\, R\left(P\left(p_t^*(C)\right)\right) + \$\left(p_t^*(C)\right)\delta\Pi_{t+1}^*(C)\right\}. \tag{12}$$

The values of the optimal policy, together with Lemmas 2 and 3, characterize the optimal mechanism sequence.

Proposition 3. *Suppose $T < \infty$. The optimal mechanism sequence is $\left(m_1^*(C), \cdots, m_T^*(C)\right)$ characterized in Lemmas 2 and 3, with $\Pi_t^*(C)$ defined by (9) and (12).*

Although Proposition 3 completely solves the seller's problem for any C, it is very non-informative: we only know that there is a cutoff cost C_t^* in each period t, and a seller chooses a reserve price auction if her cost is sufficiently small and posts price otherwise. Proposition 4 shows that it is impossible for any mechanism sequence that has an auction before a posted price to be optimal. All mechanism sequences consisting of an auction and a posted price following it are always dominated strictly by at least one alternative mechanism sequence -

dynamic pricing, sequential auctions, or posted price followed by auction, which the proof directly shows by constructing mechanisms that dominate even the expected profit maximization auction-price sequence.

Proposition 4. *An auction followed by a posted price is never optimal.*

Proof to Proposition 4. Suppose there are $T \geq 2$ periods. Without loss of generality we show that the mechanism sequence that consists of an auction in the first period and a posted price in the second period is never optimal. Suppose the seller runs the mechanism sequence $\mathbf{m} \equiv (m_3, \cdots, m_T)$ in periods 3 through T and generates expected profit $\Pi_3 \equiv \Pi_3(\mathbf{m})$; if $T = 2$, then $\Pi_3 = 0$. It suffices to show that the optimal sequence $\left(A(r_1), P(p_2), \mathbf{m}\right)$ that consists of auction-posted price sequence in the first two periods is not optimal among all mechanism sequences by showing that at least one mechanism sequence dominates it.

The optimal r_1 and p_2 are determined respectively by

$$\mathrm{MR}^P(p_2) = \delta\Pi_3$$

and

$$\mathrm{MR}^A(r_1) = \delta\Pi_2\left(P(p_2), \mathbf{m}\right) = \delta R\left(P(p_2)\right) + \delta^2 \cancel{s}(p_2)\,\Pi_3.$$

In particular, if the mechanism sequence is optimal, then $\Pi_2\left(P(p_2), \mathbf{m}\right) > \Pi_3 > 0$; otherwise, the mechanism sequence $\left(A(r_1), \mathbf{m}, A\left(r_T^*\right)\right)$ generates strictly higher expected profit.

The order between r_1 and p_2 is indeterminate, however, so we show separately the cases when $r_1 > p_2$ and when $r_1 \leq p_2$. In both cases, the mechanism sequence is dominated by at least one mechanism sequence.

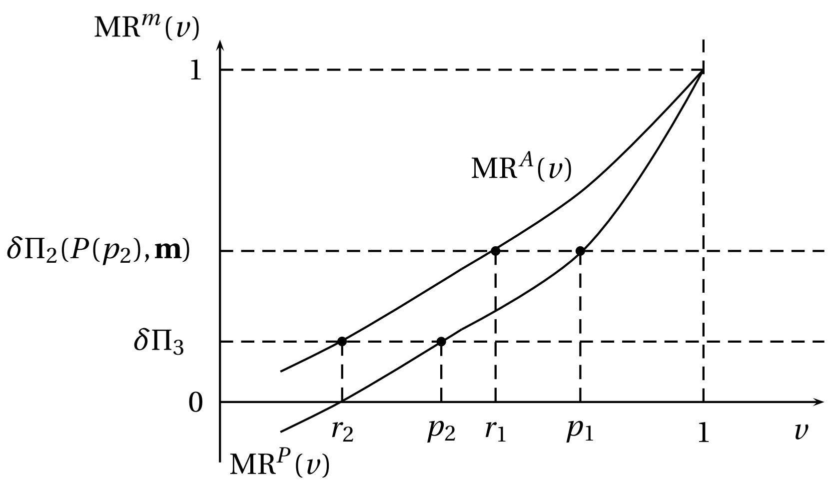

Figure 4: $r_1 > p_2$.

Case 1. $r_1 > p_2$. $\left(A(r_1), P(p_2), \mathbf{m}\right)$ is dominated by $\left(A(r_1), A(p_2), \mathbf{m}\right)$ or $(P(r_1), P(p_2), \mathbf{m})$.

If $\left(A\left(r_1\right), P\left(p_2\right), \mathbf{m}\right)$ is optimal, it is necessary that

$$\Pi_1\left(A\left(r_1\right), P\left(p_2\right), \mathbf{m}\right) \geq \max\left\{\Pi_1\left(A\left(r_1\right), A\left(p_2\right), \mathbf{m}\right), \Pi_1\left(P\left(r_1\right), P\left(p_2\right), \mathbf{m}\right)\right\}.$$

However, it cannot be true because the following two inequalities cannot hold simultaneously for any C, i.e., $\left(A\left(r_1\right), P\left(p_2\right), \mathbf{m}\right)$ cannot weakly dominate the two mechanism sequences at the same time if $r_1 > p_2$,

$$\begin{aligned}
\Pi_1\left(A\left(r_1\right), P\left(p_2\right), \mathbf{m}\right) &\geq \Pi_1\left(A\left(r_1\right), A\left(p_2\right), \mathbf{m}\right), \\
\Pi_1\left(A\left(r_1\right), P\left(p_2\right), \mathbf{m}\right) &\geq \Pi_1\left(P\left(r_1\right), P\left(p_2\right), \mathbf{m}\right).
\end{aligned}$$

Period 1's expected profits of the three mechanism sequences are respectively,

$$\begin{aligned}
\Pi_1\left(A\left(r_1\right), P\left(p_2\right), \mathbf{m}\right) &= R\left(A\left(r_1\right)\right) - C + \delta \not{s}\left(r_1\right) R\left(P\left(p_2\right)\right) + \delta^2 \not{s}\left(r_1\right) \not{s}\left(p_2\right) \Pi_3 \\
\Pi_1\left(A\left(r_1\right), A\left(p_2\right), \mathbf{m}\right) &= R\left(A\left(r_1\right)\right) - C + \delta \not{s}\left(r_1\right)\left[R\left(A\left(p_2\right)\right) - C\right] + \delta^2 \not{s}\left(r_1\right) \not{s}\left(p_2\right) \Pi_3 \\
\Pi_1\left(P\left(r_1\right), P\left(p_2\right), \mathbf{m}\right) &= R\left(P\left(r_1\right)\right) + \delta \not{s}\left(r_1\right) R\left(P\left(p_2\right)\right) + \delta^2 \not{s}\left(r_1\right) \not{s}\left(p_2\right) \Pi_3
\end{aligned}$$

Then the two inequalities are

$$\begin{aligned}
R\left(P\left(p_2\right)\right) - R\left(A\left(p_2\right)\right) + C &\geq 0 \\
R\left(A\left(r_1\right)\right) - R\left(P\left(r_1\right)\right) - C &\geq 0
\end{aligned}$$

The addition of the two inequalities implies that

$$R\left(A\left(r_1\right)\right) - R\left(P\left(r_1\right)\right) \geq R\left(P\left(p_2\right)\right) - R\left(A\left(p_2\right)\right),$$

which does not hold when $r_1 > p_2$, because

$$R\left(A\left(r\right)\right) - R\left(P\left(r\right)\right) = \int_r^1 \left[\mathrm{MR}^A\left(v\right) - \mathrm{MR}^P\left(v\right)\right] dF\left(v\right)$$

is strictly decreasing in r (evident by Figure 4). Therefore, it is impossible to have the two inequalities to hold simultaneously.

Case 2. $r_1 \leq p_2$. $\left(A\left(r_1\right), P\left(p_2\right), \mathbf{m}\right)$ is dominated by $\left(P\left(p_2\right), P\left(p_2\right), \mathbf{m}\right)$ or $\left(A\left(r_1\right), A\left(r_2\right), \mathbf{m}\right)$ where $\mathrm{MR}^A\left(r_2\right) = \delta\Pi_3$.

We arrive at a contradiction as in Case 1. The proof is aided by Figure 5. The expected

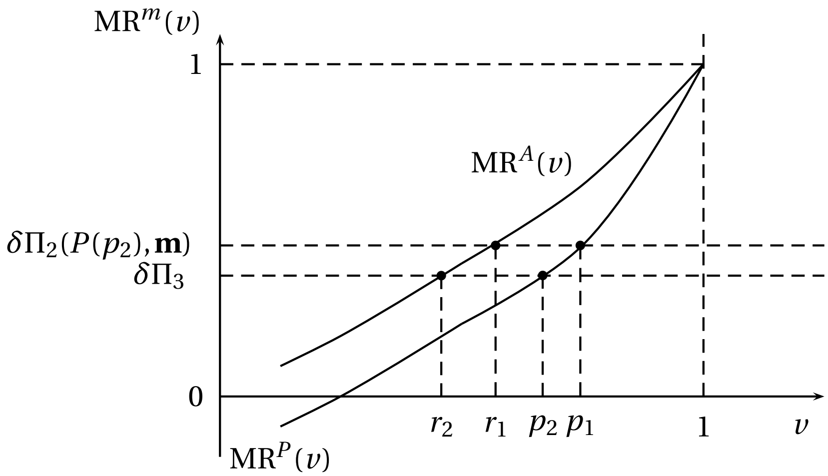

Figure 5: $r_2 < r_1 \le p_2$.

profits of the two mechanism sequences are

$$
\begin{aligned}
\Pi_1\left(P\left(p_2\right), P\left(p_2\right), \mathbf{m}\right) &= R\left(P\left(p_2\right)\right) + \delta \not s\left(p_2\right) R\left(P\left(p_2\right)\right) + \delta^2 \not s\left(p_2\right) \not s\left(p_2\right) \Pi_3 \\
\Pi_1(A(r_1), A(r_2), \mathbf{m}) &= R(A(r_1)) - C + \delta \not s(r_1)\left[R(A(r_2)) - C\right] + \delta^2 \not s\left(p_1\right) \not s(r_2) \Pi_3
\end{aligned}
$$

$\left(A(r_1), P\left(p_2\right), \mathbf{m}\right)$ dominates the two mechanism sequences if and only if the following two inequalities hold simultaneously,

$$
\begin{aligned}
R(A(r_1)) - C - R\left(P\left(p_2\right)\right) - \delta\left[\not s\left(p_2\right) - \not s(r_1)\right]\Pi_2 &\geq 0 \\
R\left(P\left(p_2\right)\right) - R(A(r_2)) + C - \delta\left[\not s\left(p_2\right) - \not s(r_2)\right]\Pi_3 &\geq 0
\end{aligned}
$$

Summing the two up,

$$
R(A(r_1)) - R(A(r_2)) - \delta\left[\not s\left(p_2\right) - \not s(r_1)\right]\Pi_2 - \delta\left[\not s\left(p_2\right) - \not s(r_2)\right]\Pi_3 \geq 0
$$

Because $0 \le \Pi_3 < \Pi_2\left(P\left(p_2\right), \mathbf{m}\right)$ as argued above, $r_T^* \le r_2 < r_1$. As a result, $R\left(A\left(r_1^*\right)\right) < R\left(A\left(r_2^*\right)\right)$. Coupled with the inequality that $r_2 < r_1 \le p_2$, the inequality is strictly negative, showing the impossibility that the inequalities simultaneously hold.

□

Intuitively, it is always worth paying the operational cost later rather than earlier. In the earlier periods, there are still many more periods left so the continuation value is higher - it is not worth paying a fixed cost as it is more costly than the cost of delay. However, as time passes on and the sale opportunities diminish, the fixed operational cost is deemed more attractive relative to the risk of not selling the good. Equivalently, Proposition 5 shows that the cutoff costs are declining over periods so that there is an optimal period of switch from posted prices to auctions. If the operational cost is C, in the optimal mechanism sequence, the monopolist posts declining prices until period t^* such that $C^*_{t^*} < C \leq C^*_{t^*+1}$ (define $C^*_0 = -\infty$ and $C^*_{T+1} = \infty$) and then runs auctions from period $t^* + 1$ on. Moreover, the optimal fixed prices and the optimal reserve prices are declining over time. The cutoff period t^* can be 0 or T If the cost is too high, the auctions phase does not exist and the seller posts a sequence of declining prices; if the cost is too low, the posted prices phase does not exist and the seller runs a sequence of auctions with declining reserve prices. The proof relies on two facts: the optimal continuation value of period t is always bigger than that of period $t+1$, and when the continuation value is higher, the expected revenue difference between the optimal posted price and the optimal auction decreases.

Proposition 5. *Suppose $T < \infty$. For any $t < T-1$, $C^*_t < C^*_{t+1}$; or equivalently, $C^*_1 < \cdots < C^*_T$. Fix a cost C, then $r^*_1(C) > r^*_2(C) > \cdots > r^*_T$ and $p^*_1(C) > p^*_2(C) > \cdots > p^*_T$.*

The argument directly depends on (11) that determines the cutoff cost in each period. In (11), the LHS is increasing in Π_{t+1}, but as Lemma 3 shows, $\Pi^*_{t+1}(C) > \Pi^*_t(C)$, therefore, $C^*_t < C^*_{t+1}$ for any t, as clearly demonstrated by Figure 6. The second part of the proposition follows

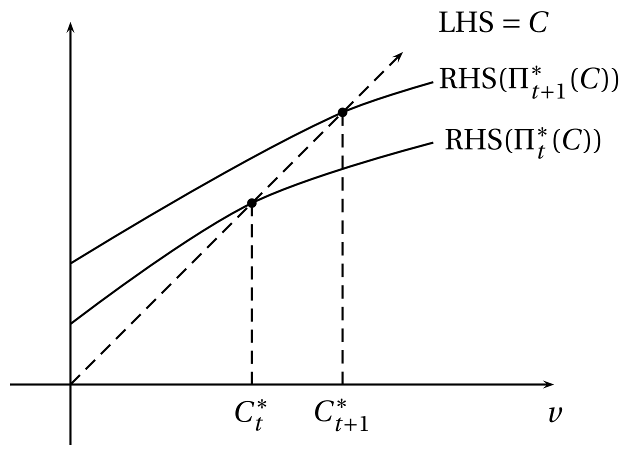

Figure 6: $C^*_t < C^*_{t+1}$.

directly from the determination of optimal posted prices and reserve prices and the facts that $\Pi^*_{t+1}(C) > \Pi^*_t(C)$ and the marginal revenue curves are increasing. Decreasing continuation values are the key to the result.

Similar to Proposition 2, we can show that in the finite horizon, as δ increases, C^*_t decreases for all t. In other words, if the seller is less impatient the seller will post prices in more periods and use auctions in fewer periods. The prices, both optimal fixed prices and reserve prices, are declining over time in equilibrium in finite horizon, as opposed to be constant in infinite horizon. Therefore, fixed deadlines are main attributors to price declines in this model and in general (Board and Skrzypacz, 2010; Dilme and Li, 2012).

There are several implications of the possible optimal mechanism sequences, especially with their approximate implementations in the real world market. We see that a seller's mechanism choice behavior depends crucially on her fixed cost of running an auction. If the cost is high, she will post prices and adjust prices downward as the deadline to sell approaches. This has been the predominant selling mechanism in many markets. However, with the development of Internet and less friction with information transmission, the cost associated with organizing auctions has been reduced and in particular on sites such as eBay where bidding can be automated and the sellers can set a variety of options to tailor their mechanisms. Nonetheless, some disadvantages associated with auctions, mentioned in the Introduction, cannot be eradicated by technological advances.

When the costs of auctions are reduced, alternative mechanism sequences than posting prices become attractive and are implementable by new eBay mechanisms, in particular, bidding with deadline and Buy-It-Now options. If the cost is very low possibly because she is an adept and/or fervent auctioneer, she always runs auctions and adjusts her reserve price downwards if no one has bought. When the buyers arrive continuously and randomly in the real world, instead of having a short duration for bidding, the seller can specify the closing time of the ascending auction to be dependent of the time of latest submission bid. If the auction cost is still a significant consideration, we show that the seller will try to sell it by posting prices first before considering auctions. On eBay, this can be very closely implemented by an Buy-It-Now (BIN) option by specifying a fixed price that can be immediately sold and a reserve price for minimum bid. Although the time of switch to auction is determined by the buyers, the essential cost-saving strategy of using prices first is captured by BIN.

5 Extensions

We relax some of the conditions and assumptions from the basic setting and show the essential results, the optimal mechanism sequence forms, continue to hold, even in a procurement setting where the contract is bought rather than sold, but may be altered in several other cases. We also suggest some interesting extensions worth pursuing.

If in each period, the buyer arrival process is stochastic instead of deterministic, the results do not change qualitatively. Suppose the number of buyers is distributed $P(n)$, then the probability of selling is $s\left(p\right)=1-\sum P\left(n\right)F^{n}\left(p\right)$. The optimal fixed price is then determined by

$$\mathrm{MR}^{P}\left(p_{t}^{*}\right)=p_{t}^{*}+\frac{s\left(p_{t}^{*}\right)}{s'\left(p_{t}^{*}\right)}=\delta\Pi_{t+1}^{*}\left(C\right).$$

A posted price mechanism's expected revenue is $R\left(P\left(p\right)\right)=ps\left(p\right)$ and an auction's expected revenue is

$$R\left(A\left(r\right)\right)=\int_{r}^{1}\mathrm{MR}^{A}\left(v\right)d\left[\sum P\left(n\right)F^{n}\left(v\right)\right].$$

Since the determination of C_{t}^{*} is not changed from (11), the results from the previous section on the form of mechanism sequence carry over. Furthermore, if the buyers have type-dependent outside options in the latter periods, their willingnesses-to-pay are depressed and the seller solves the problem with respect to the buyers' reservation values instead of their values. As long as the willingness-to-pay function $W\left(v\right)$ is a concave transformation of the value, the problem is well-behaved as

$$\mathrm{MR}^{A}\left(v\right)=v-\frac{1-F\left(v\right)}{f\left(v\right)}W'\left(v\right)$$

is still increasing and the results continue to hold.

If the seller may survive the market with probability $\mu<1$ to the next period, the stochastic deadline may result from the seller's uncertainty when or whether her good will be out of favor or when it may be perished or banned to be sold. It has a similar effect as time discounting δ, as the expected payoff of the future periods becomes $\mu\delta\Pi_{t+1}$; by redefining $\delta'=\mu\delta$, the problem is the same as before. Moreover, even if the seller survives the market less likely over the periods, i.e. μ_{t} decreases, the results are unchanged. Because the assertion that discounted payoff in a period is still greater than that in a later period, $\mu_{t}\delta\Pi_{t}^{*}\left(C\right)>\mu_{t+1}\delta\Pi_{t+1}^{*}\left(C\right)$ for any C, decline of C_{t}^{*} over time is unaltered. However, if μ_{t} increases over time, the sequence of mechanisms does not necessarily remain - alternated uses of posted prices and auctions may occur.

Finally, we consider the problem of a procurement contract and show that the optimal mechanism sequence takes the same form. McAfee and McMillan (1988) consider the procurement problem when the communication cost is high and similar to our argument, find that the principal may search sequentially for contractors if the communication is too costly. A principal who needs a cost of 1 to complete the task, finds a contractor to finish a task for a minimum compensation. Suppose n contractors whose true costs c for the project are independently and identically distributed $G\left(\cdot\right)\in\left[0,1\right]$ arrive each period; assume $G\left(\cdot\right)$ has increasing hazard rate $\left(G\left(\cdot\right)/g\left(\cdot\right)\right)$. The principal's choice is the same as the seller in the previous sections. Her auction's expected profit with a requirement of a maximum bid r is

$$\pi(A(r)) = \int_0^r \left(1 - c - \frac{G(c)}{g(c)}\right) dG^n(c) - C.$$

The probability of sale becomes $s(r) = G^n(r)$. A posted price $P(p)$ yields expected revenue $(1-p)\,G^n(p)$.

The optimal prices are determined by

$$r_t^* + G(r_t^*)/g(r_t^*) = p_t^* + G^n(p_t^*)/\left(G^n(p_t^*)\right)' = 1 + \delta\Pi_{t+1}. \tag{13}$$

In the static setting, the optimal auction yields more expected revenue than the optimal posted price does. In a finite horizon, the counterpart of (11) is

$$C_t^* = R\left(A\left(r_t^*\left(C_t^*\right)\right)\right) - R\left(P\left(p_t^*\left(C_t^*\right)\right)\right) - \left[\not{s}\left(p_t^*\left(C_t^*\right)\right) - \not{s}\left(r_t^*\left(C_t^*\right)\right)\right]\delta\Pi_{t+1}^*\left(C_t^*\right). \tag{14}$$

Therefore, the optimal sequence of mechanisms in the procurement problem depends on the auction cost in the same way as the seller's problem.

There are at least two important extensions. The seller may have multiple copies of the identical goods so the dynamic programming problem carries one more state variable besides her age - the number of remaining objects the monopolist has. Given the relative complexity of the problem with even one good, the multiple-goods extension should be treated as a separate pursuit complementary to the current one.

Another nontrivial and interesting extension is to allow the buyers to be long-lived and strategic in choosing their purchase times and may exit randomly. Deb and Pai (2012) and Pai and Vohra (Forthcoming) study this problem in a costless environment and show that simple index rules and ironing can be optimal. Said (2012) and Li (2009) show that open ascending auctions are suitable for perishable and storable goods, respectively, when the buyers arrive over time and stay and strategically time their purchases. If the buyers live for more periods, the problem is more complicated. The optimal continuation values do not necessarily decrease over time, because new buyers arrive while old buyers stay. The price-auction sequence may not be optimal - switches between posted prices and auctions may arise as optimal mechanism sequence.

One important aspect associated with long-lived strategic buyers that is not a problem in our setting is the seller's commitment to future mechanisms. Dilme and Li (2012) consider the equilibrium non-committal price paths of the seller. However, none of the papers considers how the mechanism cost may affect the seller's mechanism choice.

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Appendix

A Proof to Proposition 2

Differentiate C_∞^* with respect to δ in (8) (let $r_\infty^* \equiv r_\infty^*\left(C_\infty^*\right)$ throughout the proof as no confusion will arise)

$$\begin{aligned}\frac{dC_\infty^*}{d\delta} &= -\mathrm{MR}^A\left(r_\infty^*\right)\not{s}'\left(r_\infty^*\right) - \left[1-\delta\not{s}\left(r_\infty^*\right)\right]\frac{d\Pi_\infty^P}{d\delta} - \left[-\not{s}\left(r_\infty^*\right) - \delta\not{s}'\left(r_\infty^*\right)\right]\Pi_\infty^P \\ &= \not{s}\left(r_\infty^*\right)\left[-\mathrm{MR}^A\left(r_\infty^*\right) + \delta\Pi_\infty^P\right] + \not{s}\left(r_\infty^*\right)\Pi_\infty^P - \left[1-\delta\not{s}\left(r_\infty^*\right)\right]\frac{d\Pi_\infty^P}{d\delta}.\end{aligned}$$

The first term is 0 by (7), so

$$\frac{dC_\infty^*}{d\delta} = \not{s}\left(r_\infty^*\right)\left[\Pi_\infty^P + \delta\frac{d\Pi_\infty^P}{d\delta}\right] - \frac{d\Pi_\infty^P}{d\delta}. \tag{15}$$

It can be shown directly from (1) and (2) that $\frac{dp_\infty^*}{d\delta}$ and $\frac{d\Pi_\infty^P}{d\delta}$ are both positive; we argue economically. When δ increases to δ', Π_∞^P can be guaranteed by posting the same price as under δ. Then by (1), p_∞^* increases because $\mathrm{MR}^P(\cdot)$ is increasing (or directly from (3)).

Differentiate (1) with respect to δ,

$$\Pi_\infty^P + \delta\frac{d\Pi_\infty^P}{d\delta} = \frac{\partial\mathrm{MR}^P\left(p_\infty^*\right)}{\partial p_\infty^*}\frac{dp_\infty^*}{d\delta} > 0.$$

By (1), (7), and $\mathrm{MR}^A(\cdot) \geq \mathrm{MR}^P(\cdot)$, $r_\infty^* \leq p_\infty^*$ and $\not{s}\left(r_\infty^*\right) \leq \not{s}\left(p_\infty^*\right)$. Therefore, in (15),

$$\frac{dC_\infty^*}{d\delta} \leq \not{s}\left(p_\infty^*\right)\left[\Pi_\infty^P + \delta\frac{d\Pi_\infty^P}{d\delta}\right] - \frac{d\Pi_\infty^P}{d\delta}.$$

Therefore, it suffices to show the RHS is negative. This is obtained by rearranging (2),

$$\left[1-\delta\not{s}\left(p_\infty^*\right)\right]\Pi_\infty^P = p_\infty^*\left[1-\not{s}\left(p_\infty^*\right)\right],$$

and differentiating it with respect to δ,

$$\frac{d\Pi_\infty^P}{d\delta}\left[1-\delta\not{s}\left(p_\infty^*\right)\right] - \Pi_\infty^P\left[\not{s}\left(p_\infty^*\right) + \not{s}'\left(p_\infty^*\right)\right] = -p_\infty^*\not{s}'\left(p_\infty^*\right) + \left[1-\not{s}\left(p_\infty^*\right)\right]\frac{dp_\infty^*}{d\delta}.$$

Rearrange, we obtain,

$$\not{s}\left(p_{\infty}^{*}\right)\left[\Pi_{\infty}^{P}+\delta\frac{d\Pi_{\infty}^{P}}{d\delta}\right]-\frac{d\Pi_{\infty}^{P}}{d\delta}=\left[\mathrm{MR}^{P}\left(p_{\infty}^{*}\right)-\Pi_{\infty}^{P}\right]\not{s}'\left(p_{\infty}^{*}\right)\frac{dp_{\infty}^{*}}{d\delta}.$$

The RHS by (1) equals

$$(\delta-1)\,\Pi_{\infty}^{P}\not{s}'\left(p_{\infty}^{*}\right)\frac{dp_{\infty}^{*}}{d\delta}<0.$$