Marriage Games *

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Abstract

I study a game in which a finite number of men and women look for future spouses via bilateral search. The central question is whether equilibrium marriage outcomes are stable matchings when search frictions are negligible. The answer is No in general. For any stable matching there is an equilibrium leading to it almost surely. However, for some markets there are equilibria that lead to unstable matchings. A restriction to Markov strategies *or* to marriage markets with aligned preferences does not help. It rules out equilibria in which a particular unstable matching almost surely arises. However, unstable—and even Pareto-dominated—matchings still arise with positive probability under those two restrictions, even if combined. Finally I suggest a pro-stability result: If players on one side of the marriage market share the same preference ordering, then all equilibria are outcome equivalent and stable.

Keywords: Two Sided Matching, Marriage Problem, Stable Matchings

1 Introduction

Overview

The *stable matching* is an important solution concept for cooperative two-sided matching problems. Many centralized markets are designed to implement stable matchings because of their normative values.[1] On the other hand, whether

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[1] For a summary of the theory of stable matchings, see Roth and Sotomayor (1990). Roth (2008) surveys applications in designing two-sided matching markets.

decentralized matching markets would produce outcomes that coincide with stable matchings remains unclear. This paper contributes to the positive theory of stable matching by studying a decentralized two-sided matching market game featuring bilateral search, and analyzing whether its equilibrium outcome matchings are stable matchings of the initial market when search frictions are negligible.

The game of interest is a decentralized version of the marriage problem studied by Gale and Shapley (1962), from which I inherit the interpretation that the game represents the situation in which unmarried men and women search for their future spouses in a "marketplace". The game starts with an initial marriage market à la Gale-Shapley, which is composed of a finite number of men and women with heterogeneous preferences regarding potential spouses. In every period of the game a random pair consisting of a man and a woman meet each other. During the meeting, each decides whether to accept the person he or she is presently seeing as the future spouse. If both accept each other, they marry consequently and exit the market. Otherwise, they separate and return to the market to continue searching. Perfect information is assumed for the game. The intrinsic search frictions of the market are parametrized as a common discount factor that diminishes the value of a future marriage. The game ends, if ever, when everyone has married. The marriage pattern, that is, a record of who has married whom, of an outcome of the game corresponds to a matching of the initial marriage market. I analyze game outcomes in terms of the corresponding matchings, focusing on their stability or lack thereof.

My analysis focuses on equilibria of games in which search frictions are sufficiently small so that strategic effects due to impatience is minimized. Here I summarize the main results of this paper, all with the condition that search frictions are sufficiently small. First, for any stable matching of the initial marriage market, there is a subgame perfect equilibrium leading to that matching almost surely (Proposition 4.1). In other words, in such an equilibrium every player expects to marry the person who is assigned to him or her by that stable matching with probability one. This result establishes that the set of stable matchings is a subset of the set of matchings that could arise in equilibria. It turns out that the latter set is strictly larger in general. For instance, for some marriage markets an unstable matching may also arise almost surely in a subgame perfect equilibrium (Example 1). To enforce an unstable matching as such, a scheme that simultaneously punishes the player who initiates a pairwise blocking attempt and rewards the player who refuses to cooperate in that attempt is required (Proposition 4.3). There are two sufficient conditions that each of which independently rules out subgame perfect equilibria in which a particular unstable matching arises *almost surely*: 1. If players use strategies that only condition on the realized sequence of marriages (Proposition 4.4). 2. If the preferences of the players are aligned (the meaning of which is to be made clear) (Proposition 4.5). However, the two conditions, even combined together, are not sufficient to rule out subgame perfect equilibria in which unstable matchings arise with positive probability; some of the probable matchings may even be Pareto-dominated

(Examples 2 and 3). In addition to ex-post matchings being inefficient, significant loss of efficiency due to delay may also occur in equilibrium even when search frictions are made arbitrarily small. Delay may be due to players inefficiently using the search process as a public randomization device (Example 4), or to them waging a war of attrition (Example 5). The paper finally ends with a uniqueness result that is pro-stability: If there is a preference ordering commonly agreed upon by all players on one side of the marriage market about the desirability of players on the other, then all subgame perfect equilibria are outcome-equivalent, stable, and efficient (Proposition 4.8).

The main message of the paper is that outcome matchings of a decentralized two-sided matching market could very well be unstable and/or inefficient even when information is perfect and search frictions are minimal. This finding is in contrast with what has been conjectured on this matter,[2] and some earlier papers that confirm stability of outcome matchings in decentralized two-sided matching markets with structures that differ from the one I consider in this paper (see the next literature review subsection for more detailed discussions). In the least, this paper suggests that the stability of outcome matchings are sensitive of the market structure. In addition to their theoretical values, results in this paper may find some practical use. For example, some results may justify centralization of markets that share similar features (for example, one-shot entry, costly re-entry, random bilateral searching, etc.) with the one considered in this paper if the stability or efficiency of those markets are desirable. On the other hand, the uniqueness result (Proposition 4.8), which is implied by the existence of a common preference ordering on one side of the market, may suggest that centralization might do little good for markets in which this condition is satisfied.

Literature

This paper contributes to two strands of literature. The first strand studies implementation of stable outcomes of coalitional games in the counterpart non-cooperative decentralized games. Most of the work[3] in this strand focuses on exchange economies with money and questions whether the Walrasian price can be supported in equilibria of the corresponding market games. The present paper, among a few ones to be discussed in more detail, joins that discussion by looking at implementation of stable outcomes of matching problems without transferable utility. The second strand of literature that the present paper contributes to concerns noncooperative marriage market games, including research that studies implementability of stable matchings in revelation games[4] and generalized deferred acceptance processes.[5]

[2] Roth and Sotomayor (1990), page 245.

[3] Surveyed in Osborne and Rubinstein (1990) and Gale (2000).

[4] Surveyed in Chapter 4 of Roth and Sotomayor (1990).

[5] See Diamantoudi et al. (2007), Pais (2008), Niederle and Yariv (2009), Haeringer and Wooders (2011), and Bloch and Diamantoudi (2011).

The two strands intertwine in a few papers to which the present paper is most related. These papers consider decentralized matching games in which players look for bilateral partnerships (marriages) in a marketplace where finding coincidence of mutual want is costly due to search frictions, and is subject to whim of luck due to randomness in the search process. Among these papers, McNamara and Collins (1990), Burdett and Coles (1997), Eeckhout (1999), Bloch and Ryder (2000) and Smith (2006) assume that the underlying marriage market admits a unique stable matching that is positively assortative, and confirm that equilibrium outcomes retain some extent of assorting. Adachi (2003) and Lauermann and Nöldeke (2013) consider general marriage markets. Adachi (2003) restricts attention to market steady states in which the stock of active players is exogenous and confirms convergence of equilibrium outcomes to stability as search frictions vanish. Lauermann and Nöldeke (2013) considers endogenous steady states and finds that the limit outcomes as search frictions vanish are stable if and only if the underlying market has a unique stable matching.

The model considered in this paper entertains general marriage markets as in Adachi (2003) and Lauermann and Nöldeke (2013). It also features a nonstationary search environment: as players keep marrying and leaving, the market is ever evolving and the prospect of search is ever changing. The present model also assumes that each player is an "atomic" individual whose action, for instance the decision to marry and exit, may change the market composition significantly. All of the papers mentioned in the last paragraph look at stationary (steady state) equilibria in stationary environments with a continuum of nameless players. Nonstationarity and finiteness make the present model qualitatively very different. In terms of modeling, the present model can be seen as the nontransferable utility version of the models considered in Rubinstein and Wolinsky (1990) and Gale and Sabourian (2006). As we shall see, some interesting yet complex strategic interactions that capture elements of real life search situations are analyzed in the present model, whereas they do not occur or cannot be easily discussed in the context of stationary and continuum models. For example, in the present model players may cooperate to deter the formation of a blocking pair (Example 1); marriage decisions may not be regret-free (Example 2); mutual "misunderstanding" may lead to Pareto-dominated outcomes (Example 3). As a result, the set of matchings that can occur in equilibria in the present model is in general different from that in stationary and continuum models. For example, uniqueness of stable matchings in the underlying marriage market, a condition that guarantees equilibrium outcomes to be stable in Lauermann and Nöldeke (2013), is no longer sufficient for stability in the present model.

The layout of the paper is the following: Section 2 introduces the game. Section 3 sets up an analytic framework by relating the game to the marriage problem studied in Gale and Shapley (1962). Section 4 delivers all the analysis. Section 5 concludes. Section 6, the Appendix, contains all the lengthy proofs.

2 The Game

The Marriage Market

The players of the game we presently consider are unmarried men and women who search for future spouses among the other players. Use M to denote the set of the male players and W that of the female ones. By taking part in the game, a man may find a wife from W, or he may end up single. All men's preferences over lotteries of these marital outcomes are represented by $u : M \times (W \cup \{s\}) \mapsto \mathbb{R}$ (s denotes the alternative of staying single) where $u(m, \cdot)$ has the interpretation as man m's Bernoulli utility function over $W \cup \{s\}$. Similarly use $v : (M \cup \{s\}) \times W \mapsto \mathbb{R}$ to represent all women's preferences with the interpretation that $v(\cdot, w)$ is woman w's Bernoulli utility function over $M \cup \{s\}$.

Assume the following throughout the paper:

A1 Strict preferences: $u(m, \cdot)$ is one-to-one for any $m \in M$; $v(\cdot, w)$ is one-to-one for any $w \in W$.

A2 Normalization: $u(m, s) = 0$ for any $m \in M$; $v(s, w) = 0$ for any $w \in W$. Woman w is said to be **acceptable** to m if $u(m, w) > 0$, and vice versa if $v(m, w) > 0$.

A3 Finite market: $|M| < \infty$ and $|W| < \infty$.

The tuple (M, W, u, v) represents a **marriage market** that constitutes the primitive of the cooperative marriage problem studied by Gale and Shapley (1962) and the consequent literature. Use script $\mathcal{A}$ as a shorthand for a generic marriage market (M, W, u, v). In the present model a marriage market $\mathcal{A}$ is embedded in a dynamic noncooperative game.

Searching

A game starts on day one[6] ($t = 1$) with a marriage market $\mathcal{A}$, called the **initial marriage market** of the game, and rolls indefinitely into the future ($t = 2, 3, ...$). On any given day, a random pair consisting of a man and a woman, say m and w, meet each other. As they meet, m moves first, who chooses between accepting and rejecting w. If m rejects w then the pair separate and both return to the market, marking the end of the day. If m accepts w, it is then w's turn to choose between accepting and rejecting m. If w accepts m, they marry immediately and exit the game for good; otherwise they separate and return to the market. The day ends after w's action.

The game ends once everyone is married. Man m receives a one-time payoff of $u(m, w)$ from marrying woman w today. Likewise woman w receives a one-

[6] A "day" in the game does not correspond to a natural day. It should be understood as what in the dynamic games convention is termed a *stage* or *period*.

time payoff of $v(m, w)$ from marrying m today. For any player, the value of a marriage delayed by t days is discounted by δ^t, where the common discount factor δ is meant to capture the overall search frictions. A terminal history may be infinite because there is no definite deadline for the game to end. If someone is unable to marry after a finite terminal history, or after any finite subhistory of an infinite terminal history, he or she is said to be **single** in this outcome. Being single is of zero value to a player regardless of whether it is realized sooner or later. To avoid confusion, in this paper I reserve the term *single* to exclusively refer to this situation. In contrast, if a player has not married after a nonterminal history, he or she is said to be **unmarried** then.

The game structure, all players' being rational (that is, they maximize expected utility) and all past actions are common knowledge. Observability of all past actions is a very strong assumption. One may want to discuss game outcomes when history is partially observable at best. To entertain such analysis, I will introduce two equilibria selection criteria that differ by the amount of information allowed to be used in strategies. This point will be revisited later.

Some Terminology and Notations

Histories: A history is a sequence of $(player, action)$ pairs that specifies in chronological order who has taken what action. Use H to denote the set of all histories, and Z the set of all *terminal* histories. Note that I do not explicitly model the random nature of pairwise meetings as an exogenous mixed strategy used by the third person "Nature". For two histories h and h', denote $h < h'$ if h is a subhistory of h' (formally, h and h' agree on the first n entries where n is the length of h).

Submarkets: For a marriage market $\mathcal{A} = (M, W, u, v)$, a marriage market $\mathcal{A}' = (M', W', u', v')$ is a **submarket** of $\mathcal{A}$ if $M' \subset M$, $W' \subset W$, $u'(m, w) = u(m, w)$ and $v'(m, w) = v(m, w)$ for all $(m, w) \in M' \times W'$; in this case, abuse notation and denote u' as u and v' as v. Let $2^{\mathcal{A}}$ denote the set of all submarkets of $\mathcal{A}$. Obviously, after any history of a game the remaining marriage market consisting of players who have not exited is a submarket of the initial marriage market. For a finite history h, let $\mathcal{A}(h) = (M(h), W(h), u, v)$ denote the **remaining marriage market** after h, where $M(h)$ is the set of men unmarried after h and $W(h)$ that of women.

The Contact Function

Recall that on any given day a random pair of a man and a woman meet each other. This randomness is modeled by a **contact function** as the following: Given a game with initial marriage market $\mathcal{A}$, an associated contact function is a mapping $C : M \times W \times 2^{\mathcal{A}} \mapsto [0, 1]$, where $C(m, w, \mathcal{A}')$ is the probability that (m, w) will be the pair to meet on a day at the beginning of which the remaining marriage market is $\mathcal{A}'$. The game rules thus require that for any $\mathcal{A}' = (M', W', u, v) \in 2^{\mathcal{A}}$,

B1 $C(m, w, \mathcal{A}') = 0$ if $m \notin M'$ or $w \notin W'$: Only unmarried people meet.

B2 $\sum_{M' \times W'} C(m, w, \mathcal{A}') = 1$: A pair must meet.

In addition, throughout the paper impose the following assumption:

B3 There exists some strictly positive number $\epsilon(C)$ such that $C(m, w, \mathcal{A}') \geq \epsilon(C)$ for all $(m, w, \mathcal{A}') \in M \times W \times 2^{\mathcal{A}}$ satisfying $(m, w) \in M' \times W'$. Thus every unmarried man has a "good" (strictly positive) chance of seeing every unmarried woman, and vice versa. Throughout the paper, in the context where there is no ambiguity about which C is being discussed, ϵ is reserved as a shorthand for $\epsilon(C)$.

Note that by the definition of contact function it is implicitly assumed that the meeting probabilities on a given day is determined by what the remaining marriage market is at the beginning of that day. This implies that the remaining marriage market is the only payoff-relevant variable of the game. A contact function C is said to be **equal-opportunity** if for any $\mathcal{A}' = (M', W', u, v) \in 2^{\mathcal{A}}$, $C(m, w, \mathcal{A}') = 1/(|M'||N'|)$ for any $(m, w) \in M' \times W'$.

Thus far I have completed introducing all elements of the game: an initial marriage market $\mathcal{A}$, an associated contact function C, and a common discount factor δ. Use tuple $(\mathcal{A}, C, \delta)$ to denote a game of this sort, from now on referred to as the **marriage game.**

3 From Equilibria to Matchings

Equilibria

The most lenient solution concept considered in this paper is that of **subgame perfect equilibrium**.

In addition, to accommodate more stringent information settings, introduce the "private-dinner" condition as an equilibrium selection criterion: A subgame perfect equilibrium is said to be a **private-dinner equilibrium** if everyone's strategy after any history only depends on the sequence of previously married couples ordered chronologically. The private-dinner condition concerns the information setting in which players can only observe who has married and the order of the realized marriages, but nothing else. In particular, it rules out the situation that someone's strategy depends on, for example, who has said "No" in a past meeting in which he did not take part: one cannot peep into other people's dating scene, thus the namesake "private-dinner".

A subgame perfect equilibrium is a **Markov equilibrium** if everyone's strategy after any history only depends on the remaining marriage market. Clearly, the Markov condition implies the private-dinner condition, but not vice versa. Markov equilibria are compatible with the more strict information setting in which players have no memory of the past and can only observe the current market.

Matchings

In the cooperative marriage problem from Gale and Shapley (1962), a **matching** of a marriage market is a scheme that pairs people into couples, formalized as a function $\mu : M \cup W \mapsto M \cup W \cup \{s\}$ such that $\mu(m) \in W \cup \{s\}$ if $m \in M$, $\mu(w) \in M \cup \{s\}$ if $w \in W$, and $\mu(\mu(i)) = i$ for any $i \in M \cup W$ such that $\mu(i) \neq s$. A matching μ is **unstable** if there exists some individual i for whom $\mu(i)$ is unacceptable, in which case μ is said to be **individually blocked** by i; or if there exists some pair (m, w) such that $u(m, w) > u(m, \mu(m))$ and $v(m, w) > v(\mu(w), w)$, in which case μ is said to be **pairwise blocked** by (m, w). μ is **stable** if it is not unstable.

Let $\mathcal{M}(\mathcal{A})$ denote the set of all matchings of marriage market $\mathcal{A}$ and $\mathcal{S}(\mathcal{A})$ the set of all stable matchings. Gale and Shapley (1962) proves that $\mathcal{S}(\mathcal{A})$ is nonempty for any $\mathcal{A}$. Moreover, they also show that for any $\mathcal{A}$ there exists a **man-optimal stable matching** $\mu^M \in \mathcal{S}(\mathcal{A})$ that is commonly agreed by all men as the best stable matching, or formally $\mu^M = \text{argmax}_{\mu \in \mathcal{S}(\mathcal{A})} u(m, \mu(m))$ for any $m \in M$. Analogously there also exists a **woman-optimal stable matching** μ^W.

A marriage game can be seen as a mechanism that implements paring schemes for the initial marriage market: The **outcome matching** of a terminal history $h \in Z$ is a mapping $\mu_h : M \cup W \mapsto M \cup W \cup \{s\}$ such that

- If h is finite: $\mu_h(i)$ is the person who has married player i after some subhistory of h.
- If h is infinite: For any player i who has married after some finite subhistory of h, $\mu_h(i)$ is the person who has married i. Otherwise $\mu_h(i) = s$.

Obviously μ_h is a matching of the initial marriage market for any $h \in Z$.

Given a marriage game, a strategy profile σ and the contact function jointly induce a probability measure λ_σ on 2^Z, which in turn induces a probability mass function p_σ on the set of all matchings of the initial marriage market. Formally $p_\sigma(\mu) := \int_{h \in Z} \mathbf{1}_{\mu_h = \mu} d\lambda_\sigma$ where $\mathbf{1}$ is the indicator function that takes value 1 if the statement written as its subscript is true, and 0 otherwise. If $p_\sigma(\mu) = 1$, we say that σ **enforces** μ. σ enforcing μ means that μ arises as the eventual marriage pattern with probability one if all players follow σ.

Trivial Markets

A marriage market is said to be **trivial** if it does not have a pair of a man and a woman who are mutually acceptable to each other. An **empty matching** is a matching under which everyone is single.

Lemma 3.1. *If the initial marriage market of a marriage game is trivial then any of its subgame perfect equilibria enforces the empty matching.*

Proof. Fix a marriage game $(\mathcal{A}, C, \delta)$ where $\mathcal{A}$ is trivial. Everyone secures a minimax payoff of zero by not accepting anyone. Hence in a subgame perfect equilibrium a woman rejects every unacceptable man with certainty after any history because her doing otherwise implies a contradictory negative continuation payoff. Therefore in any subgame perfect equilibrium a man marries an acceptable woman with zero probability, because triviality of $\mathcal{A}$ guarantees that acceptability of a woman to a man implies unacceptability of the man to the woman. If a man marries with positive probability with a woman in some subgame perfect equilibrium, then that woman is unacceptable and the man's equilibrium payoff is negative, a contradiction. Therefore no man marries with positive probability in any subgame perfect equilibrium, immediately implying the Lemma. □

Lemma 3.1 implies that in equilibrium a game reaches a virtual end once the remaining marriage market becomes trivial, since no one will be able to marry from then on. Hence call a *terminal* history h an **impasse** if there exists a subhistory $h' < h$ such that $\mathcal{A}(h')$ is trivial.

Near-Frictionless

The main purpose of the paper is to inspect the effects that the dynamic searching environment has on equilibrium outcome matchings, in particular with regard to their *stability*. Two features are rooted in the game environment, namely decentralization and search frictions. When search frictions are large, they become a dominant factor in determining game outcomes: obviously, when δ is sufficiently close to zero, any marriage game has an essentially unique subgame perfect equilibrium in which every mutually acceptable pair of a man and a woman marry instantly if they meet. In this paper, I would like to focus attention on the effects of decentralization by minimizing those due to search frictions. To prepare for such near-frictionless analysis, allow me to introduce more terminology: A **game environment** is the tuple $(\mathcal{A}, C)$, which includes everything about a marriage game except search frictions. A strategy profile σ is a **limit equilibrium** of the game environment $(\mathcal{A}, C)$ if there exists some $\underline{\delta} < 1$ such that σ is a subgame perfect equilibrium of the marriage game $(\mathcal{A}, C, \delta)$ for any $\delta > \underline{\delta}$. Hence limit equilibria are robust locally at $\delta = 1$.

4 Analysis

Enforcing Stable Matchings

The first main result establishes that any stable matching of the initial marriage market can be enforced by a subgame perfect equilibrium if search frictions are sufficiently small.

Proposition 4.1. *Given any game environment* $(\mathcal{A}, C)$*, if* μ *is a stable matching of* $\mathcal{A}$*, then* $(\mathcal{A}, C)$ *has a limit equilibrium* σ *that enforces* μ*.*

The proof, provided in Section 6.1, constructs a cutoff strategy profile that recommends a player, say i, to accept the one he or she is seeing, say j, if and only if j is weakly preferred to $\mu(i)$[7]. That μ is a stable matching of the initial marriage market implies mutual acceptance is reached by i and j if and only if $j = \mu(i)$. The strategy profile is incentive compatible when search frictions are sufficiently small because the continuation payoff after any history for i (assumed to be male without loss of generality) approximates $u(i, j)$ and the continuation payoff for j approximates $v(i, j)$. Observe that these types of stable-matching enforcing strategy profiles are Markov. They will also be useful in the construction of many other types of equilibria to be investigated, and hence deserve a dedicated name: Given a marriage market $\mathcal{A}$ and a stable matching μ of $\mathcal{A}$, call the strategy profile σ constructed in the proof of Proposition 4.1 that enforces μ a $\mu-$**cutoff strategy profile**.

Enforcing Unstable Matchings

Can unstable matchings be enforced in a limit equilibrium? Recall that an unstable matching is one that is blocked by an individual, or a pair of a man and a woman. It is straightforward that enforcing an unstable matching which is blocked by an individual is impossible because that individual achieves at least his minimax payoff, which is zero, in any subgame perfect equilibrium. However, it is possible to enforce an unstable matching that is not blocked by any individual in a limit equilibrium, as the next example shows.

EXAMPLE 1. **Reward-and-punishment**

Consider the marriage market $\mathcal{A}$ having three men and three women, whose

[7] This description applies on equilibrium path. Off-equilibrium strategies are similar, but the "reference matching" is no longer μ, but instead is a stable matching of the remaining marriage market

preferences are represented by the following lists:

$$
\begin{aligned}
P(m_1) &= w_2, w_3, w_1, & \qquad P(w_1) &= m_1, m_2, m_3, \\
P(m_2) &= w_3, w_1, w_2, & \qquad P(w_2) &= m_2, m_3, m_1, \\
P(m_3) &= w_1, w_2, w_3, & \qquad P(w_3) &= m_3, m_2, m_1.
\end{aligned}
$$

The lists are induced from (u, v) via the following rule: For any man m, $P(m) = w_{i_1}, ..., w_{i_n}$ if and only if $u(m, w_{i_1}) > ... > u(m, w_{i_n}) > 0$. The list for each woman is induced from v analogously. Note that unacceptable alternatives are omitted in the lists.

Let C be an arbitrary contact function associated with $\mathcal{A}$. I will show that the matching μ:

$$\mu(m_1) = w_3, \quad \mu(m_2) = w_1, \quad \mu(m_3) = w_2$$

is enforced in a limit equilibrium. Observe that μ is unstable because (m_2, w_3) block it.

The candidate strategy profile σ is described by an automaton with three states given below:

q_0 : q_0 is the initial state. The following diagram represents each player's strategy in q_0. To read: Should they meet, the player before the colon accepts the player(s) after the colon and rejects everyone else.

$$
\begin{array}{lll}
m_1 : w_3, & \qquad m_2 : w_1, & \qquad m_3 : \textit{none}, \\
w_1 : m_1, m_2, & \qquad w_2 : m_2, & \qquad w_3 : m_1, m_3.
\end{array}
$$

The transition rules are:

$$
q_0 \longrightarrow \begin{cases}
q_1 & \text{If } (m_1, w_3) \text{ or } (m_2, w_1) \text{ marry} \\
 & \\
q_2 & \text{(1) } w_1 \text{ rejects } m_3\text{, or} \\
 & \text{(2) } w_2 \text{ rejects } m_1 \text{ or } m_3\text{, or} \\
 & \text{(3) } w_3 \text{ rejects } m_2\text{, or} \\
 & \text{(4) A pair other than } (m_1, w_3) \text{ or } (m_2, w_1) \text{ marry} \\
 & \\
q_0 & \text{Otherwise}
\end{cases}
$$

q_1 : In q_1, everyone follows the $\mu-$cutoff strategy. q_1 is absorbing.

q_2 : Let h denote the first history along the current gameplay the state of which is q_2. In q_2, everyone follows the μ^h-cutoff strategy, where μ^h is the woman-optimal stable matching of the remaining marriage market $\mathcal{A}(h)$. q_2 is absorbing.

Proposition 4.2. *In Example 1, σ is a limit equilibrium of $(\mathcal{A}, C)$ and enforces μ.*

□

The proof of Proposition 4.2 is in Section 6.2. To enforce the unstable matching μ, the blocking pair (m_2, w_3) has to be deterred from marrying each other should they meet. In σ the deterrance is provided by a credible reward-punishment scheme: If m_2 initiates to block μ by accepting w_3, w_3 will be rewarded with a more preferred man, m_3, if she does the "right" thing, that is rejecting m_2; whereas m_2 will be punished by having to marry a less preferred woman, w_2, than the $\mu-$arranged w_1. Since this scheme is supported by a subgame perfect equilibrium (the μ^W-cutoff strategy profile where μ^W is the woman-optimal stable matching of $\mathcal{A}$), it forms a credible threat to m_2, persuading him not to accept w_3 in the first place. The scheme, however, has one wrinkle, which is that the "reward" m_3 and "punishment" w_2 should not have married and exited before one of the participants of the blocking pair does, otherwise either the reward or the punishment becomes a vain promise. Hence there also must be a similar reward-punishment scheme to deter (m_3, w_2) from marrying too early. In our example, this latter scheme is also conveniently supported by the μ^W-cutoff strategy profile.

The next proposition suggests that reward-punishment is an indispensable part of any subgame perfect equilibrium that implements an unstable matching.

Proposition 4.3. *Given any marriage game $(\mathcal{A}, C, \delta)$, let Π be the set of subgame perfect equilibrium payoff vectors. If there is a subgame perfect equilibrium σ that enforces an unstable matching μ of $\mathcal{A}$, then:*

1. *μ is individually rational.*
2. *For any pair (m, w) blocking μ, there exists $\pi = (\pi_i) \in \Pi$, where π_i denotes the payoff to player i, such that $\delta\pi_m \leq u(m, \mu(m))$ and $\delta\pi_w \geq v(m, w)$.*

Proof. Fix a marriage game $(\mathcal{A}, C, \delta)$ and a subgame perfect equilibrium σ that enforces an unstable matching μ. μ must be individually rational because everyone's minimax payoff is zero. Part 2 is proved by contradiction. Suppose for some pair (m, w) blocking μ there does not exist $\pi \in \Pi$ such that $\delta\pi_m \leq u(m, \mu(m))$ and $\delta\pi_w \geq v(m, w)$. This situation is broken into two cases:

1. "No reward": $\delta\pi_w < v(m, w)$ for any $\pi \in \mathbf{\Pi}$. It is optimal for w to accept m because her continuation payoff in any subgame perfect equilibrium is less than $v(m, w)$. Then according to σ, m has to reject w should they meet on the first day (which happens with positive probability by Assumption B3), because otherwise there is a positive probability that (m, w) marry, contradicting the premise that σ enforces μ. However, if m follows σ by rejecting w on the first day, his continuation payoff is at most $\delta u(m, \mu(m))$ because following σ ensures that he will marry no other

than $\mu(m)$. Since (m,w) block μ, $\delta u(m,\mu(m)) < u(m,\mu(m)) < u(m,w)$. Therefore m has incentive to deviate from σ by accepting w on the first day, a contradiction.

2. "No punishment": for any $\pi \in \Pi$, $\delta\pi_w \geq v(m,w)$ implies $\delta\pi_m > u(m,\mu(m))$. If on the first day (m,w) meet and m accepts w, w must reject m with positive probability, otherwise a similar contradiction is reached as in the "no-reward" case. If w rejects m with positive probability, then the equilibrium payoff vector of the consequent subgame must be π such that $\delta\pi_w \geq v(m,w)$, which implies $\delta\pi_m > u(m,\mu(m))$. Hence if m meets w on the first day, his expected payoff from accepting w is strictly larger than $u(m,\mu(m))$ because either acceptance or rejection by w leads to a subgame in which m's expected payoff is strictly larger than $u(m,\mu(m))$). It follows that m will accept w in this case and consequently marry some woman other than $\mu(m)$ with positive probability. This contradicts the premise that σ enforces μ.

□

In addition to confirming that credible reward-punishment schemes must be employed if an unstable matching is to be enforced, Proposition 4.3 also provides necessary conditions on the set of equilibrium enforceable matchings. For example, an immediate corollary is that an unstable matching cannot be enforced in equilibrium if there exists a blocking pair $(Bob, Alice)$ such that Bob is Alice's first choice.

To punish the man of the blocking pair who initiates a blocking attempt and reward the woman who refuses cooperate with him, the other players need to know the actions taken during the meeting between the blocking pair and condition future strategies on that. A private-dinner equilibrium disallows conditioning strategies on such knowledge. The next proposition establishes that, indeed, no private-dinner equilibrium enforces an unstable matching.

Proposition 4.4. *Any matching enforced by a private-dinner equilibrium is a stable matching of the initial marriage market.*

Proof. Prove by contradiction. Given a marriage game $(\mathcal{A}, C, \delta)$, suppose there exists a private-dinner equilibrium σ that enforces an unstable matching μ of $\mathcal{A}$. By Proposition 4.3 μ is individually rational. Therefore μ is blocked by some pair (m,w). Suppose w meets m on the first day, and w is accepted by m. She obtains a payoff of $v(m,w)$ if she also accepts m. If she rejects m, the state variable of the private-dinner equilibrium, which is the sequence of realized marriages, remains the same (the empty sequence), and hence following σ in the resulting subgame gives her a payoff no larger than $\delta v(\mu(w),w)$ since σ enforces μ in the subgame. (m,w) being a blocking pair of μ implies that $v(m,w) > \delta v(\mu(w),w)$ and therefore it is optimal for w to accept m. Given this, if m meets w on the first day, he secures a payoff of $u(m,w)$ by accepting her, whereas rejecting her gives him no more than $\delta u(m, \mu(m))$ because the state variable remains unchanged if m rejects w. $u(m, \mu(m))$ is less than $u(m, w)$, therefore it is optimal for m to accept w on the first day. Since there is positive probability that m meets w on the first day, there is positive probability that m marries w in σ, contradicting the supposition that m marries $\mu(m) \neq w$ with probability one. $\square$

Proposition 4.4 identifies a condition on the *equilibrium* to ensure non-enforceability of any unstable matching. In contrast, the next proposition suggests a sufficient condition on the *marriage market* to ensure non-enforceability of any unstable matching. A marriage market $\mathcal{A}$ satisfies the **Sequential Preference Condition** if the men can be ordered as $m_1, m_2, ...$ and the women $w_1, w_2, ...$ such that for some $k \leq \min\{|M|, |N|\}$,

1. For any $i \leq k$, $u(m_i, w_i) > u(m_i, w_j)$ and $v(m_i, w_i) > v(m_j, w_i)$ for any $j > i$.
2. The submarket $(\{m_i : i > k\}, \{w_i : i > k\}, u, v)$ is trivial.

The Sequential Preference Condition is introduced in Eeckhout (2000) as a sufficient condition for a marriage market to have a unique stable matching. The unique stable matching pairs m_i to w_i for any $i \leq k$ and leaves m_i and w_i single for any $i > k$. The Sequential Preference Condition implies the uniqueness of stable matchings via the following process: m_1 and w_1 are each other's first choice and thus must marry; m_2 and w_2 are each other's first choice among $M \backslash \{m_1\}$ and $W \backslash \{w_1\}$ respectively and thus must marry as well, and so on until the remaining marriage market becomes trivial, the members in which must then stay single. A similar top-down unraveling argument is used to show the following proposition.

Proposition 4.5. *Given a marriage game $(\mathcal{A}, C, \delta)$ such that $\mathcal{A}$ satisfies the Sequential Preference Condition, if σ is a subgame perfect equilibrium that enforces a matching μ, then μ is the unique stable matching of $\mathcal{A}$.*

Proof. Suppose $(\mathcal{A}, C, \delta)$ is a marriage game such that $\mathcal{A}$ satisfies the Sequential Preference Condition, and σ is a subgame perfect equilibrium that enforces a matching μ. If $\mathcal{A}$ is trivial then Lemma 3.1 immediately implies the proposition.

Suppose $\mathcal{A}$ is not trivial. Let the men and women be ordered in the way described by the definition of the Sequential Preference Condition, and k be that particular index threshold. Since $\mathcal{A}$ is not trivial, $k \geq 1$. Hence m_1 and w_1 are each other's first choice. It is a dominant strategy for w_1 to accept m_1 after m_1 has accepted w_1, and hence it is optimal for m_1 to accept w_1 whenever they meet. Since by Assumption B3 there is a positive probability that (m_1, w_1) meet on the first day, there is consequently a positive probability, in any subgame perfect equilibrium, that (m_1, w_1) marry on the first day. Since σ enforces μ, it follows that $\mu(m_1) = w_1$. If $\mathcal{A}' := (M \backslash \{m_1\}, W \backslash \{w_1\}, u, v)$ is trivial then obviously $\mu(m_i) = s$ and $\mu(w_i) = s$ for any $i > 1$ because μ must be individually rational. Hence μ is indeed the unique stable matching of $\mathcal{A}$. If $\mathcal{A}'$ is not trivial, then using the same iterated elimination of dominated strategy argument we establish that (m_2, w_2) must marry with positive probability in the subgame which starts with $\mathcal{A}'$ as the remaining marriage market. This subgame is reached with positive probability in any subgame perfect equilibrium because (m_1, w_1) marry with positive probability on the first day. Thus, together with the premise that σ enforces μ, we conclude that $\mu(m_2) = w_2$. We can show that $\mu(m_i) = w_i$ for any $i \leq k$ by applying the same argument iteratively until the remaining marriage market shrinks to a trivial market $\mathcal{A}^{(k)}$. Since μ is individually rational, $\mu(m_i) = s$ and $\mu(w_i) = s$ for any $i > k$. Hence μ is the unique stable matching of $\mathcal{A}$. $\square$

The Sequential Preference Condition reflects some degree of alignment in players' preferences. It is sufficient to ensure that equilibrium outcomes are stable in many decentralized two-sided matching models, including those considered in Adachi (2003), Lauermann and Nöldeke (2013), and Bloch and Diamantoudi (2011). However, a later example (Example 3) will show that the Sequential Preference Condition does *not* guarantee stable, or even Pareto-efficient, equilibrium outcomes in marriage games.

Stochastic Equilibria

Up to this point we have been studying subgame perfect equilibria that enforce a particular matching, that is, lead to a matching with probability one. Some subgame perfect equilibria lead to uncertain outcomes representable as lotteries of multiple matchings. Equilibrium outcome matchings are uncertain because of mixed strategies, and/or because of the intrinsic randomness in the search process. To analyze such equilibria, I examine the stability of ex-post outcome matchings. The following two examples show that unstable matchings can arise in stochastic limit equilibria even under conditions that guarantee non-enforceability of unstable matchings in deterministic limit equilibria.

EXAMPLE 2. **Regret**

In this example I demonstrate a limit equilibrium that leads to a lottery of matchings, each of which is unstable, even though (1) the initial marriage market has a unique stable matching, and (2) the limit equilibrium satisfies the private-dinner condition. Recall that if a private-dinner equilibrium enforces a particular matching then the matching must be stable (Proposition 4.4). The present example illustrates that this "stable property" of private-dinner equilibria fails if a lottery of multiple matchings is induced instead of a particular matching.

Consider the marriage market $\mathcal{A}$ with six men and six women whose utility functions (u, v) induce the following preference lists:

$$\begin{aligned}
P(m_1) &= w_2, w_1, w_3, & P(w_1) &= m_1, m_2, m_3, \\
P(m_2) &= w_3, w_2, w_1, & P(w_2) &= m_2, m_1, m_3, \\
P(m_3) &= w_2, w_3, w_1, & P(w_3) &= m_3, m_2, m_1, \\
P(m'_1) &= w'_2, w'_1, w'_3, & P(w'_1) &= m'_1, m'_2, m'_3, \\
P(m'_2) &= w'_3, w'_2, w'_1, & P(w'_2) &= m'_2, m'_1, m'_3, \\
P(m'_3) &= w'_2, w'_3, w'_1, & P(w'_3) &= m'_3, m'_2, m'_1.
\end{aligned}$$

Moreover, assume $\frac{1}{2}v(m_2, w_2) + \frac{1}{2}v(m_3, w_2) > v(m_1, w_2)$ and $\frac{1}{2}v(m'_2, w'_2) + \frac{1}{2}v(m'_3, w'_2) > v(m'_1, w'_2)$. Observe that $\mathcal{A}$ has a unique stable matching. Let the associated contact function C is an equal-opportunity one.

Let μ_1 be such that $\mu_1(m_1) = w_1, \mu_1(m_2) = w_3$, $\mu_1(m_3) = w_2$, $\mu_1(m'_i) = w'_i$ for $i = 1, 2, 3$, and μ_2 be such that $\mu_2(m_i) = w_i$ for $i = 1, 2, 3$, $\mu_2(m'_1) = w'_1, \mu_2(m'_2) = w'_3$, $\mu_2(m'_3) = w'_2$. Neither μ_1 nor μ_2 is stable: (m_1, w_2) block μ_1 and (m'_1, w'_2) block μ_2. We want to use a limit equilibrium to induce a lottery of matchings such that only μ_1 and μ_2 arise with positive probability. The candidate strategy profile σ is described by an automaton with four states:

q_0: q_0 is the initial state. In this state, m_1 accepts w_1 and w_2 only, m'_1 accepts w'_1 and w'_2 only. Every other man rejects every woman. Every woman accepts her most preferred man only. The transition rules are

$$q_0 \longrightarrow \begin{cases} q_1 & \text{If } (m_1, w_1) \text{ marry} \\ q_2 & \text{If } (m'_1, w'_1) \text{ marry} \\ q_3 & \text{If some couple other than } (m_1, w_1) \text{ or } (m'_1, w'_1) \text{ marry} \\ q_0 & \text{Otherwise.} \end{cases}$$

q_1: Everyone follows the μ_1-cutoff strategy. q_1 is absorbing.

q_2: Everyone follows the μ_2-cutoff strategy. q_2 is absorbing.

q_3: Let h denote the first history along the current gameplay the state of which is q_3. In q_3, everyone follows the μ^h-cutoff strategy, where μ^h is the woman-optimal stable matching of the marriage market $\mathcal{A}(h)$. q_3 is absorbing.

Note that σ satisfies the private-dinner condition. The state turns from q_0 to q_1 or q_2 with equal probabilities. q_3 is an off-equilibrium state. Once q_1 is entered the outcome matching will almost surely be μ_1 by the proof of Proposition 4.1, because μ_1 is a stable matching of the remaining marriage market when q_1 is entered. Similarly μ_2 will be the outcome matching almost surely if q_2 is entered. Since the probability that the game always stays in q_0, which is equal to the probability that (m_1, w_1) and (m'_1, w'_1) never meet, is bounded from above by $1 - \sum_{t=0}^{\infty} 2\epsilon(1 - 2\epsilon)^t = 0$, it follows that μ_1 and μ_2 will be the outcome matching each with with probability 0.5.

Now show that σ is indeed a limit equilibrium. It suffices to check incentive compatibility in q_0 because the proof of Proposition 4.1 ensures that σ is incentive compatible in the other states if δ is sufficiently close to 1. Let $V(i,\delta)$ be the expected payoff in σ to player i evaluated at the beginning of a day when the state is q_0 and the discount factor is δ. It is easy to check that for any man m, $V(m) := \lim_{\delta\to 1}\delta V(m,\delta) = \frac{1}{2}u(m,\mu_1(m)) + \frac{1}{2}u(m,\mu_2(m))$, and for any woman w, $V(w) := \lim_{\delta\to 1}\delta V(w,\delta) = \frac{1}{2}v(\mu_1(w),w) + \frac{1}{2}v(\mu_2(w),w)$. One can verify that for each woman w, $V(w) \leq v(m,w)$ if and only if m is w's most preferred man. Since $\delta V(w,\delta)$ is continuously increasing in δ for any $w \in W$, there exists some $\underline{\delta}_W < 1$ such that for any $\delta > \underline{\delta}_W$ and $w \in W$, $\delta V(w,\delta) \leq v(m,w)$ if and only if m is w's most preferred man. Therefore σ represents the optimal strategy for each woman (to accept only the most preferred man) in q_0 if $\delta > \underline{\delta}_W$. Now consider the men. In q_0, m_1 is indifferent between accepting and rejecting w_2, w_3, and w'_i for $i = 1,2,3$ because he will be rejected anyway if he accepts w_2, and thus he expects a continuation payoff of $\delta V(m_1,\delta)$ regardless of what he does. It is optimal for m_1 to accept w_1 because he will be accepted and receive a payoff of $u(m_1,w_1)$, which is greater than $\delta V(m_1,\delta)$ for any δ. In q_0, m_2 rejects w_3, w_1 and w'_i, $i = 1,2,3$ for the same reason (indifference between accepting and rejecting) m_1 rejects w_2. m_2 rejects w_2 because he will be accepted if he accepts her and receive $u(m_2,w_2)$, which is less than his continuation payoff $\delta V(m_2,\delta)$ from rejecting her if $\delta > \underline{\delta}_{m_2}$ for some $\underline{\delta}_{m_2} < 1$ because $\delta V(m_2,\delta)$ is continuously increasing in δ and approximates $\frac{1}{2}u(m_2,w_2) + \frac{1}{2}u(m_2,w_3) > u(m_2,w_2)$ as δ tends to 1. m_3's case is analogous to that of m_2 given $\delta > \underline{\delta}_{m_3}$ for some $\underline{\delta}_{m_3} < 1$. The case of m'_i is analogous to that of m_i for $i = 1,2,3$, given $\underline{\delta} > \delta_{m'_i}$ for some $\underline{\delta}_{m'_i} < 1$. Therefore σ is optimal for every player in q_0 if $\delta > \max\{\underline{\delta}_W, \underline{\delta}_M\}$ where $\underline{\delta}_M := \max_{i=1,2,3}\{\underline{\delta}_{m_i}, \underline{\delta}'_{m_i}\}$. $\square$

Example 2 is noteworthy in that despite satisfying conditions that are intuitively favorable in supporting stable outcomes, such as the private-dinner condition and the uniqueness of stable matchings in the initial marriage market, the outcome matching is almost surely unstable. In particular, in similar dynamic marriage models (Adachi (2003), Lauermann and Nöldeke (2013)), uniqueness of stable matchings in the underlying market is a sufficient condition to guarantee stable equilibrium outcomes in near-frictionless games. Why don't a blocking pair, say (m_1, w_2), take the blocking action even there is no reward-punishment scheme encoded in σ? It turns out that at some point in the game (more precisely, in q_0) w_2 fancies a better man, m_2. Obsessed with that high-hanging fruit, w_2 will reject the less desirable m_1 after the latter accepts her. It is only after m_1 is gone, having married w_1, does w_2 realize that m_2 is impossible and m_1 is no more. She has no choice other than marrying m_3, her last choice, regretting that she has rejected m_1, as the old English saying goes, *He that will not when he may; when he will, he shall have Nay.*

EXAMPLE 3. **Co-ordination failure**

In this example I demonstrate a limit equilibrium that leads to a lottery of four matchings, three of which are unstable and Pareto dominated, even though

(1) every submarket of the initial marriage market has a unique stable matching, and (2) the limit equilibrium satisfies the Markov condition, which is even stronger than the private-dinner condition.

Consider the marriage market $\mathcal{A}$ with three men and three women. The utility functions (u, v) induce the following preference lists:

$$\begin{aligned} P(m_1) &= w_1, w_2, & P(w_1) &= m_1, m_3, \\ P(m_2) &= w_2, w_3, & P(w_2) &= m_2, m_1, \\ P(m_3) &= w_3, w_1, & P(w_3) &= m_3, m_2. \end{aligned}$$

Moreover, if w and w' are man m's most and second most preferred women respectively, then $u(m, w') > \frac{2}{3}u(m, w) + \frac{1}{6}u(m, w')$; if m and m' are woman w's most and second most preferred men respectively, then $v(m', w) > \frac{2}{3}v(m, w) + \frac{1}{6}v(m', w)$. Every submarket of $\mathcal{A}$ satisfies the Sequential Preference Condition and thus has a unique stable matching. Let the associated contact function C be an equal-opportunity one.

Consider the Markov strategy profile σ: Every player accepts either of his or her two acceptable alternatives if no one has married. After any history h after which someone has married, everyone follows the $\mu^{\mathcal{A}(h)}-$cutoff strategy where $\mu^{\mathcal{A}(h)}$ is the unique stable matching of the remaining market $\mathcal{A}(h)$.

Four matchings arise with positive probability by σ, which are

$$\begin{aligned} \mu_0: \ & m_1 \mapsto w_1, & \mu_1: \ & m_1 \mapsto w_2, & \mu_2: \ & m_1 \mapsto w_1, & \mu_3: \ & m_1 \mapsto s, \\ & m_2 \mapsto w_2, & & m_2 \mapsto s, & & m_2 \mapsto w_3, & & m_2 \mapsto w_2, \\ & m_3 \mapsto w_3, & & m_3 \mapsto w_3, & & m_3 \mapsto s, & & m_3 \mapsto w_1, \\ & & & w_1 \mapsto s, & & w_2 \mapsto s, & & w_3 \mapsto s. \end{aligned}$$

Clearly μ_0 is stable. μ_0 will be the outcome matching almost surely if on the first day any man meets his most preferred woman. Denote the latter event as E_0. μ_i, $i \in \{1, 2, 3\}$ will be the outcome matching almost surely if on the first day m_i meets his second most preferred woman, this event denoted as E_i. The conditional probability of μ_i, $i \in \{0, 1, 2, 3\}$ to be the outcome matching stays $p_\sigma(\mu_i)$ if on the first day any man meets the woman who is unacceptable to him. Hence $p_\sigma(\mu_i) = Pr(E_i) \times 1 + (1 - \sum_{j=0}^{3} Pr(E_j))p_\sigma(\mu_i)$. Since C is equal opportunity, $Pr(E_0) = 3 \times 1/9$ and $Pr(E_i) = 1/9$ for $i \in \{1, 2, 3\}$. Then $p_\sigma(\mu_0) = 1/2$ and $p_\sigma(\mu_i) = 1/6$ for $i \in \{1, 2, 3\}$. $\sum_{i=0}^{3} p_\sigma(\mu_i) = 1$ implies that no other outcome matching arises with positive probability. Observe that by following σ each player gets his or her first, second and third most preferred alternatives (the third of which is staying single) with probabilities $2/3, 1/6$ and $1/6$ respectively.

Now verify that σ is a limit equilibrium of $(\mathcal{A}, C)$. It is sufficient to check incentive compatibility in the state where no one has married, because by the proof of Proposition 4.1, σ is a limit equilibrium of subgames in states where someone has married. Suppose no one has married yet. Woman w's continuation payoff from rejecting a man is equal to $\delta V(w, \delta)$ where $V(w, \delta)$ is w's expected payoff by following σ, evaluated at the beginning of the game. It is easy to check that $\delta V(w, \delta)$ increasingly converges to $V(w) := \lim_{\delta \to 1} \delta V(w, \delta) = \frac{2}{3} v(m, w) + \frac{1}{6} v(m', w)$ where m and m' are w's most and second most preferred men respectively. Then $\delta V(w, \delta) < V(w) < v(m', w)$ where the second inequality is by assumption. Thus σ prescribes the optimal strategy (of accepting either of her most and second most preferred men) for w. A similar argument applies to any man m. Hence σ is optimal for every player in the state in which no one has married regardless of δ. □

By Proposition 4.1, the game environment in Example 3 has another limit equilibrium σ' that enforces the unique stable matching under which each player marries his or her first choice. Compared with σ', the constructed σ in Example 3 is unstable (as unstable outcome matchings arise with positive probability) and Pareto inefficient. The driving force behind the inefficiency is essentially a co-ordination failure due to self-confirmation of doubts: Despite liking each other best, m_i does not commit to waiting for w_i because w_i does not commit to waiting for m_i because m_i does not commit to waiting for w_i and so on ad infinitum. It is notable that every submarket having a unique stable matching is a sufficient condition to ensure stable equilibrium outcomes in many models I surveyed in the Introduction, in particular Adachi (2003), Lauermann and Nöldeke (2013), and Bloch and Diamantoudi (2011). Both non-stationarity (that married people are gone forever) and random search (that any sequence of meetings may occur) are jointly at work. The models in those papers lack at least one of those two features.

Delay

A subgame perfect equilibrium can have two sources of inefficiency. The first source is misallocation of resources, as illustrated by Example 3 in which all outcome matchings that arise with positive probability are weakly Pareto-dominated by the unique stable matching. The second source is delay, that is, some players search for too long before getting married. In this section I study the possibility of efficiency loss due to delay in subgame perfect equilibria even when the discount factor is arbitrarily close to one.

The most serious form of delay would be that someone is never able to marry despite endless active search. This situation formally translates into an infinite terminal history h that is *not* an impasse. Recall that a terminal history is an impasse if it has a subhistory after which the remaining marriage market is trivial. If an infinite terminal history is not an impasse, then after any subhistory there is at least one pair of a man and a woman who are mutually acceptable but despite this they never marry (to each other or to other acceptable players). The next proposition rules out the possibility of this extreme form of delay. Recall that for any strategy profile σ, λ_σ denotes the probability measure on the set of terminal histories Z jointly induced by σ and the contact function. Let $Z_I := \{h \in Z : h \textit{ is an impasse}\}$ be the set of impasses, then we have:

Proposition 4.6. *Let σ be a subgame perfect equilibrium of a marriage game, and , then $\lambda_\sigma(Z_I) = 1$.*

The proof of Proposition 4.6 is in Section 6.3. Proposition 4.6 implies that if the initial marriage market is balanced, and that every player's least preferred alternative is staying single, then every player will marry almost surely in any subgame perfect equilibrium, as stated in the following corollary:

Corollary 4.7. *If a marriage game starts with an initial marriage market such that $|M| = |W|$, $u(m, w) > 0$ and $v(m, w) > 0$ for all $(m, w) \in M \times W$, then in any subgame perfect equilibrium everyone will marry with probability one.*

Proof. If the initial marriage market satisfies the premise of the present corollary then no infinite terminal history h is an impasse. Hence $\lambda_\sigma(\{h \in Z : \textit{everyone is married under } \mu_h\}) = \lambda_\sigma(\{h \in Z : h \textit{ is finite}\}) = \lambda_\sigma(Z_I) = 1$ where the last equality is due to Proposition 4.6. $\square$

A player may marry very late in an equilibrium, despite the fact that he or she will marry eventually. For a man m, efficiency loss due to delay in a subgame perfect equilibrium σ of the marriage game $(\mathcal{A}, C, \delta)$ is measured as the difference between m's expected payoff from immediate resolution of the lottery on $\mathcal{M}(\mathcal{A})$ (the set of matchings of $\mathcal{A}$) induced by σ and his equilibrium payoff in σ. Formally, this difference is

$$L(m, \sigma, \delta) := \sum_{\mu \in \mathcal{M}(\mathcal{A})} p_\sigma(\mu) u(m, \mu(m)) - V(m, \sigma, \delta)$$

where $V(m, \sigma, \delta)$ denotes m's equilibrium payoff in σ. We can define efficiency loss for each woman analogously. Obviously $L(i, \sigma, \delta) \geq 0$. Some efficiency loss is unavoidable due to the randomness of search and discounting. The notable case, then, is when efficiency loss due to delay does not disappear as search frictions vanish. This situation translates into the following condition: Fixing $(\mathcal{A}, C)$ and some $\alpha > 0$, for any $\delta < 1$ there exists a subgame perfect equilibrium $\sigma(\delta)$ of $(\mathcal{A}, C, \delta)$ such that $L(i, \sigma(\delta), \delta) > \alpha$ for some player i. Below are two examples illustrating such efficiency loss and the driving forces.

***EXAMPLE 4*. Wait and see**

Consider the game environment $(\mathcal{A}, C)$ with an initial marriage market consisting of two men and two women whose utility functions (u, v) induce the following preference lists:

$$P(m_1) = w_1, w_2, \qquad P(w_1) = m_2, m_1,$$
$$P(m_2) = w_2, w_1, \qquad P(w_2) = m_1, m_2.$$

Each player gets a payoff of 3 from marrying the most preferred player of the opposite sex and 1 from the second most preferred one. C is an equal-opportunity contact function.

Fix any $\eta \in (0,5,1)$. Given δ, choose a non-negative integer τ so that $0.5 < \delta^{\tau+1}$ and $\delta^{\tau} < \eta$. Such τ exists if δ is sufficiently close to 1. The candidate strategy profile $\sigma(\delta)$ is the following: After history h, if no one has married and the τ'th day has not been reached, $\sigma(\delta)$ recommends that each woman accepts only her most preferred man and each man rejects either of the two women. If after h no one has married and the τ'th day has been reached, and if (m_1, w_1) or (m_2, w_2) have met on the τ'th day, then $\sigma(\delta)$ recommends everyone to follow the $\mu-$cutoff strategy where $\mu(m_1) = w_1$ and $\mu(m_2) = w_2$; otherwise, if instead (m_1, w_2) or (m_2, w_1) have met on the τth day, then σ recommends everyone to follow the $\mu'-$cutoff strategy where $\mu'(m_1) = w_2$ and $\mu'(m_2) = w_1$. After someone has married, $\sigma(\delta)$ recommends the remaining man to accept the remaining woman and vice versa.

If players follow $\sigma(\delta)$, then (1) no one marries before the τth day, and (2) the pair that meet on the τth day marry on that day and the other pair marry on the $\tau+1$st day. Hence by $\sigma(\delta)$ each man has a probability 0.5 of marrying either of the two women, and vice versa. The expected payoff for a player when it is t days before the τth day is then greater than $\delta^{t+1}(0.5 \times 3 + 0.5 \times 1) = 2\delta^{t+1}$. For any $t \leq \tau$ we have $2\delta^{t+1} \geq 2\delta^{\tau+1} > 1$. Hence it is optimal for each woman to reject the second most preferred man before the τth day, and it is in turn optimal for each man to reject either women before the τth day because he expects to be rejected anyway if he accepts his most preferred woman. Starting from the τth day σ coincides with a limit equilibrium of the consequent subgame by the proof of Proposition 4.1 because both μ and μ' are stable matchings of the remaining marriage market. Hence we have shown that $\sigma(\delta)$ is a subgame perfect equilibrium of $(\mathcal{A}, C, \delta)$ if δ is sufficiently close to 1.

Efficiency loss due to delay does not go away with vanishing search frictions for any player because regardless of what δ is, a player's equilibrium payoff under $\sigma(\delta)$ is bounded from above by $\delta^{\tau}(0.5 \times 3 + 0.5 \times 1) = 2\delta^{\tau} < 2\eta < 2$ by the choice of η, whereas immediate resolution of the lottery induced by $\sigma(\delta)$ gives each player an expected payoff of $0.5 \times 3 + 0.5 \times 1 = 2$. $\square$

In Example 4, players waits to see who will be "lucky" to marry one's most preferred player of the opposite sex, the revelation of which is to happen on the τth day. As search frictions vanish, players become increasingly willing to wait longer. Efficiency loss lingers as the length of waiting grows in pace with the vanishing search frictions.

EXAMPLE 5. **War of attrition**

Use the same game environment $(\mathcal{A}, C)$ as in Example 4. Given marriage game $(\mathcal{A}, C, \delta)$, consider the strategy profile $\sigma(\delta)$ such that when no one has married, each player accepts his or her most preferred player of the opposite sex with probability 1 and the second most preferred one with probability $p(\delta) \in (0, 1)$.

After someone has married, $\sigma(\delta)$ recommends the remaining man to accept the remaining woman and vice versa. Note that $\sigma(\delta)$ is Markov.

If $\sigma(\delta)$ is a subgame perfect equilibrium, then w_1 is indifferent between accepting and rejecting m_1 to justify her randomizing. If she rejects m_1 and continues searching, each of the following four mutually exclusive events will occur with probability $p/4$: (1) (m_1, w_1) marry tomorrow, (2) (m_2, w_1) marry tomorrow, (3) (m_2, w_2) marry tomorrow then (m_1, w_1) marry the day after tomorrow, (4) (m_1, w_2) marry tomorrow then (m_2, w_1) marry the day after tomorrow. If after w_1 rejects m_1 none of the above four events occur, then no one will marry tomorrow. Hence the indifference condition for w_1 is satisfied if and only if

$$v(m_1, w_1)$$
$$= \delta\Big[\frac{p}{4}\Big(v(m_1, w_1) + v(m_2, w_1) + \delta v(m_1, w_1) + \delta v(m_2, w_1)\Big) + (1-p)v(m_1, w_1)\Big].$$

Plugging in $v(m_1, w_1) = 1$ and $v(m_2, w_1) = 3$, we get $p(\delta) = (1-\delta)/\delta^2$. The same indifference argument is the same for each of the other players. $p(\delta)$ is in $(0, 1)$ for $\delta > (\sqrt{5}-1)/2 \approx 0.618$, so $\sigma(\delta)$ is feasible and thus a subgame perfect equilibrium if δ is sufficiently close to 1.

To show that the efficiency loss remains for δ close to 1, note that the equilibrium payoff of each player in $\sigma(\delta)$ is equal to $1/\delta$, which converges to 1 as δ approaches 1, while the immediate resolution of the lottery induced by $\sigma(\delta)$ gives each player an expected payoff of $0.5 \times 3 + 0.5 \times 1 = 2$, since in equilibrium each man has probability 0.5 of marrying either of the two women and vice versa. $\square$

Example 5 is not unlike a war of attrition, because $p(\delta)$, which is the probability that a player "chickens out" by accepting the second best alternative, goes to zero as δ approaches 1.

Uniqueness

We have so far seen that a marriage game may have a large set of subgame perfect equilibria, some of which support unstable or inefficient outcomes. In this section we identify a condition, called the One-side Common Preference Condition, that implies essential uniqueness of subgame perfect equilibria in near-frictionless games.

Given any terminal history h, define $o(h) := \big(\mu_h(m), t_h(m)\big)_{m\in M}$ where $t_h(m) \in \{1, 2, ...\}$ denotes the day on which m married $\mu_h(m)$ on the equilibrium path that leads to h (if $\mu_h(m) = s$ then $t_h(m) = \infty$). Thus $o(h)$ records all the realized marriages and their timing if the terminal history is h. Denote $\mathcal{O} := \{o(h) : h \in Z\}$. Two distinct terminal histories h and h' are outcome equivalent if $o(h) = o(h')$ because a player cares only about whom the spouse is and when the marriage takes place. We say that two strategy profiles σ and σ' are **outcome equivalent** if they (jointly with the contact function) induce the same probability measure on $\mathcal{O}$.

A marriage market $\mathcal{A}$ is said to satisfy the **One-side Common Preference Condition** if either u induces for each man the same set of acceptable women W^a and the same preference ordering over W^a, or v for each woman analogously.

Proposition 4.8. *If a marriage market $\mathcal{A}$ satisfies the One-side Common Preference Condition, then for any associated contact function C there exists $\underline{\delta} < 1$ such that for any $\delta > \underline{\delta}$ all subgame perfect equilibria of the marriage game $(\mathcal{A}, C, \delta)$ are outcome equivalent.*

The proof is in Section 6.4. It is done by applying the induction principle on the number of men in the initial marriage market. To summarize the proof idea, suppose u induces for each man the same set of acceptable women W^a and the same preference ordering over W^a, and that search frictions are small. If the proposition is true for any initial marriage market with less than n men, then given a marriage game that starts with exactly n men, in any subgame perfect equilibrium the commonly most preferred woman, w_1, only accepts her most preferred man, m_1, before m_1 has married. Consequently m_1 best-responds by accepting w_1 before w_1 has married, implying that (m_1, w_1) must marry with probability one and they marry during their first meeting. Applying the same argument to the second commonly most preferred woman w_2 leads to a similar result that w_2 will marry her most preferred man among $M \backslash \{m_1\}$ with probability one and they marry during their first meeting, and so on.

If $\mathcal{A}$ satisfies the Common Preference Condition, then Proposition 4.8 implies that when search frictions are small, all subgame perfect equilibria are outcome equivalent to the $\mu-$cutoff strategy profile where μ is the unique stable matching of $\mathcal{A}$. Since the $\mu-$cutoff strategy profile enforces the unique stable matching of the initial marriage market, and its efficiency loss due to delay converges to 0 as search frictions vanish, every subgame perfect equilibrium must also be so:

Corollary 4.9. *If a marriage market $\mathcal{A}$ satisfies the One-side Common Preference Condition, then for any associated contact function C there exists $\underline{\delta} < 1$ such that for any $\delta > \underline{\delta}$ all subgame perfect equilibria of the marriage game $(\mathcal{A}, C, \delta)$ enforce the unique stable matching of $\mathcal{A}$, and whose efficiency loss due to delay converges to 0 as δ converges to 1.*

Observe that the One-side Common Preference Condition implies the Sequential Preference Condition. Proposition 4.8 shows that the One-side Common Preference Condition implies essential uniqueness of limit equilibria, whereas the Sequential Preference Condition does not (recall that the marriage market in Example 3 satisfies the Sequential Preference Condition yet the game environment has multiple limit equilibria, some of which are unstable). A higher degree of preference alignment is required to ensure that equilibrium outcome matchings are stable.

5 Conclusion

This paper studies a decentralized marriage market game and analyzes whether matchings that arise in subgame perfect equilibria are stable when search frictions are small. It is found that any stable matching can be enforced in a subgame perfect equilibrium. However, for some games there are subgame perfect equilibria that lead to unstable, or even Pareto-dominated, matchings. In addition, a significant amount of efficiency could be lost due to delay in some subgame perfect equilibria regardless of how small search frictions are. If at least one side of the market shares the same preference ordering, then when search frictions are small, all subgame perfect equilibria are outcome equivalent and enforce the unique stable matching of the initial marriage market.

6 Appendix: Proofs

6.1 Proof of Proposition 4.1

Proof. The proposition is proved by construction. Fix a marriage environment $(\mathcal{A}, C)$ and a stable matching μ of $\mathcal{A}$. The candidate strategy profile σ is state-dependent. The state of history h is a matching μ^h of the remaining marriage market $\mathcal{A}(h)$: μ^h is equal to μ on the former's domain if $\mu(i) \in M(h) \cup W(h) \cup \{s\}$ for every $i \in M(h) \cup W(h)$; otherwise μ^h is the woman-optimal stable matching of $\mathcal{A}(h)$. σ recommends i, the player who moves after h, to accept j, the player i is meeting today, if and only if j is weakly preferred to $\mu^h(i)$ by i. Thus σ is a cutoff strategy profile. Observe that μ^h is a stable matching of $\mathcal{A}(h)$ for any $h \in H$. If players follow σ starting from h, then (m, w) will marry if and only if $\mu^h(m) = w$ and as soon as they meet for the first time, because μ^h being a stable matching of $\mathcal{A}(h)$ implies that $u(m, w) \geq u(m, \mu^h(m))$ and $v(m, w) \geq v(\mu^h(w), w)$ hold simultaneously if and only if $\mu^h(m) = w$.

For any $(m, w) \in M \times W$, let

$$\underline{u}(m, w) = \begin{cases} 0 & \text{If } u(m, w) < 0 \\ \max\limits_{w' \in W \cup \{s\}} \{u(m, w') : u(m, w') < u(m, w)\} & \text{If } u(m, w) > 0. \end{cases}$$

Fix a pair (m, w). Let $\delta_m(w)$ be implicitly defined by the equation

$$\underline{u}(m, w) = \delta_m(w) \sum_{t=0}^{\infty} (\delta_m(w))^t (1 - \epsilon)^t \epsilon u(m, w). \tag{6.1}$$

If $u(m, w) < 0$, then $\underline{u}(m, w) = 0$ and hence $\delta_m(w) = 0$. If $u(m, w) > 0$, then the right hand side of equation 6.1 is strictly increasing in $\delta_m(w)$, converges to 0 as $\delta_m(w)$ converges to 0 and to $u(m, w)$ as $\delta_m(w)$ converges to 1. That $0 \leq \underline{u}(m, w) < u(m, w)$ implies a unique $\delta_m(w) \in [0, 1)$ for which equation

6.1 holds. Note that a risk-neutral person is indifferent between the following two options: (1) getting a payoff of $\underline{u}(m, w)$ immediately, and (2) conducting a sequence of i.i.d binomial trials, each with success probability of ϵ, and getting a payoff of $(\delta_m(w))^{t+1}u(m, w)$ upon the first success where t is the number of trials conducted before the first success. Let $\delta_m = \max_{w \in W}\{\delta_m(w)\}$. Find δ_w for each woman w analogously. Finally, let $\underline{\delta} = \max_{i \in M \cup W}\{\delta_i\}$. Obviously, $\underline{\delta} < 1$.

Now we check that σ is a subgame perfect equilibrium if $\delta > \underline{\delta}$. Suppose $\delta > \underline{\delta}$. If after h woman w moves, her continuation payoff V from rejecting m, the man she is seeing today, is bounded from below by $K := \delta \sum_{t=0}^{\infty} \delta^t (1-\epsilon)^t \epsilon v(\mu^h(w), w)$ because the probability that she encounters $\mu^h(w)$ is weakly greater than ϵ on each day starting tomorrow. Also $V < \delta v(\mu^h(w), w)$. By the choice of $\underline{\delta}$, $V \geq K > v(m', w)$ if m' is ranked by w below $\mu^h(w)$. Therefore, $V > v(m, w)$ if $v(\mu^h(w), w) > v(m, w)$ and $V < v(m, w)$ if $v(\mu^h(w), w) \leq v(m, w)$. This comparison justifies the optimality of the strategy prescribed by σ for w against σ_{-w}.

If after h some man m moves, his continuation payoff V' from rejecting w, the woman he is seeing today, satisfies $V' > u(m, w)$ if $u(m, \mu^h(m)) > u(m, w)$ and $V' < u(m, w)$ if $u(m, \mu^h(m)) \leq u(m, w)$. The derivation of these bounds are analogous to what we did in the last paragraph for a woman. m's expected payoff V'' from accepting w depends on who w is. If $u(m, w) < u(m, \mu^h(m))$, then $V'' \leq V'$; in particular, if w accepts m with positive probability then $V'' < V'$ since $u(m, w) < V'$ by the choice of $\underline{\delta}$. If $u(m, w) > u(m, \mu^h(m))$, $V'' = V'$ because w will reject m. If $w = \mu^h(m)$, $V'' = u(m, \mu^h(m))$ because w will accept m. This comparison justifies the optimality of the strategy prescribed for m by σ against σ_{-m}.

The above two paragraphs establish that σ is a limit equilibrium of $(\mathcal{A}, C)$. Finally we check that σ enforces μ. For any pair (m, w) such that $w = \mu(m)$, the probability that they marry eventually if σ is followed is bounded from below by $\epsilon \sum_{t=0}^{\infty} (1-\epsilon)^t = 1$. For any man m such that $\mu(m) = s$, any woman he accepts will reject him on equilibrium path. The same is true for any woman w such that $\mu(w) = s$. Therefore, every player will end up with probability one in the marital situation arranged for him or her by μ. This in turn implies that $p_\sigma(\mu) = 1$. □

6.2 Proof of Proposition 4.2

Proof. If players follow σ throughout game, the transition of states is illustrated as $q_0 \to q_1 \to \mu$. q_2 is an off-σ-path state. Observe that (m_3, w_2) are not allowed to marry in q_0; instead they have to wait until the marriage of (m_1, w_3) or (m_2, w_1) has been realized.

First we determine $\underline{\delta}$ such that σ will be shown to be a subgame perfect equilibrium of $(\mathcal{A}, C, \delta)$ for all $\delta > \underline{\delta}$. Inherit the definition of $\underline{u}(m, w)$ from the proof of Proposition 4.1. For any $(m,w) \in M \times W$ let $\delta_m(w)$ be defined so that a risk-neutral person is indifferent between the following two options: (1) getting a payoff of $\underline{u}(m,w)$ immediately, and (2) conducting a sequence of i.i.d binomial trials, each with success probability of ϵ, and getting a payoff of $(\delta_m(w))^{t+1}u(m,w)$ after the *second* success, where t is the number of trials conducted before the second success. Note that option (2) resembles the situation that (m_3, w_2) face on equilibrium path, that is, they need to wait for another couple to marry before themselves can. If $u(m,w) < 0$ then $\underline{u}(m,w) = 0$ and hence $\delta_m(w) = 0$. If $u(m,w) > 0$, then the expected payoff from option (2), which is equal to $\left[\frac{\delta_m(w)\epsilon}{1-\delta_m(w)(1-\epsilon)}\right]^2 u(m,w)$, is continuously increasing in $\delta_m(w)$, converges to 0 as $\delta_m(w)$ converges to 0 and to $u(m,w)$ as $\delta_m(w)$ converges to 1. That $0 \leq \underline{u}(m,w) < u(m,w)$ implies a unique $\delta_m(w) \in [0,1)$. Let $\delta_m = \max_{w \in W} \delta_m(w)$. Define δ_w for each woman w analogously. Finally, let $\underline{\delta} = \max_{i \in M \cup W}\{\delta_i\}$. Obviously $\underline{\delta} < 1$.

Suppose $\delta > \underline{\delta}$. We check that σ is a subgame perfect equilibrium of the marriage game $(\mathcal{A}, C, \delta)$. In states q_1 and q_2, σ coincides with the $\mu'-$cutoff strategy profile where μ' is a stable matching of the remaining marriage market when the pertained state is first entered, and by the proof of Proposition 4.1, σ is a subgame perfect equilibrium for subgames in states q_1 and q_2. It remains to check incentive compatibility in q_0.

In q_0, all women accept their respective most preferred man, which is obviously incentive compatible. w_1 is open to a second option, m_2, because accepting m_2 gives her $v(m_2, w_1)$ while rejecting him gives her at most $\delta v(m_2, w_1)$ since after this one-shot deviation the state remains q_0 and w_1 expects to marry m_2 eventually. w_3 accepts m_1 for the same reason. w_1 rejects m_3 because doing so triggers the state to become q_2 in which she expects to marry m_1 eventually and get a payoff bounded from below by $\frac{\delta\epsilon}{1-\delta(1-\epsilon)}v(m_1, w_1)$, which is greater than $v(m_3, w_1)$ due to the choice of $\underline{\delta}$. The same argument explains why it is optimal for the other women to reject the men they are supposed to reject in σ. Now consider the men. m_1 accepts w_3 because his payoff from doing so, which is $u(m_1, w_3)$ (because he will be accepted by w_3), is greater than his payoff from rejecting her, which is at most $\delta u(m_1, w_3)$ (because after the rejection the state remains q_0 and m_1 expects to marry w_3 eventually). m_1 rejects w_2, because otherwise he will be rejected, triggering the state to become q_2 in which his expected payoff is at most $\delta u(m_1, w_1)$, which is less than his expected payoff from rejecting w_2 (bounded from below by $\left[\frac{\delta\epsilon}{1-\delta(1-\epsilon)}\right]^2 u(m_1, w_3)$). The same argument explains why it is optimal for m_2 to reject w_3 and for m_3 to reject w_1 and w_2. m_1 rejects w_1, because otherwise he will be accepted by w_1 and get a payoff of $u(m_1, w_1)$, which is less than his expected payoff from rejecting w_1 (bounded from below by $\left[\frac{\delta\epsilon}{1-\delta(1-\epsilon)}\right]^2 u(m_1, w_3)$). The same argument explains why it is optimal for m_2 to reject w_2 and for m_3 to reject w_3. This completes the check for incentive compatibility.

Finally we show that σ enforces μ: for any pair (m,w) such that $\mu(m) = w$, the probability that (m, w) marry if σ is followed is bounded from below by $\left[\frac{\epsilon}{1-(1-\epsilon)}\right]^2 = 1$ (which is equal to the probability that two successes will occur in an infinite sequence of i.i.d. binomial trials with success rate ϵ). This implies that σ enforces μ.

□

6.3 Proof of Proposition 4.6

Proof. The proposition obviously holds if the initial marriage market $\mathcal{A}$ is trivial. From now on consider non-trivial initial marriage markets.

The proposition is proved by applying the induction principle on the number of men $|M|$ in the initial marriage market. Suppose $|M| = 1$ for a marriage game $(\mathcal{A}, C, \delta)$. Call the sole man m and his favorite woman w. Since $\mathcal{A}$ is non-trivial, $u(m, w) > 0$ and $v(m, w) > 0$. Obviously in any subgame perfect equilibrium w always accepts m if she is accepted by m. Given that, in any subgame perfect equilibrium m always accepts w when he meets w. Then the probability that m marries w is bounded from below by $\sum_{t=0}^{\infty} \epsilon(1-\epsilon)^t = 1$. The proposition holds in this case because after m has married the remaining marriage market is trivial.

Suppose the proposition is true for any marriage game with initial marriage market satisfying $|M| < n$ for some $n > 1$. We show that the proposition is also true if $|M| = n$. Fix a marriage game $(\mathcal{A}, C, \delta)$ such that $|M| = n$, and one of its subgame perfect equilibria σ. Introduce some notation to be used in the proof:

$H_{|h} :$ $= \{h' \in H : h < h'\}$.

$Z_{|h} :$ $= \{h' \in Z : h < h'\}$.

$\lambda_{\sigma|h} :$ The probability measure jointly induced by C and σ on $Z_{|h}$ conditional on h being reached.

$q(h, h', \sigma)$: The probability that h' will be reached conditional on h, which is a subhistory of h', being reached, and players following σ. Formally $q(h, h', \sigma) = \lambda_{\sigma|h}(Z_{|h'})$.

$g(h, \sigma)$: The probability that the game will *not* end in an impasse conditional on h being reached and players following σ. Formally $g(h, \sigma) = 1 - \lambda_{\sigma|h}(\{h' \in Z_{|h} : h' \textit{ is an impasse}\})$.

Let h_0 denote the initial (empty) history. The goal is to show that $g(h_0, \sigma) = 0$. Suppose that $g(h_0, \sigma) > 0$. By the inductive hypothesis, $g(h, \sigma) = 0$ if $|M(h)| < n$. Hence for any history h such that $|M(h)| = n$, $g(h, \sigma)$ is the same as the probability that *no one* will be able to marry in any finite time in the subgame after h. Let $H(d, h)$ denote the set of all histories after which all of the following are satisfied: (1) a man moves, (2) the day is d days after the day of h, and (3)

$|M(h)| = n$. Hence for any d we have $g(h_0, \sigma) = \sum_{h' \in H(d,h_0)} q(h_0, h', \sigma) g(h', \sigma)$. Let $Q(d, h, \sigma) := \sum_{h' \in H(d,h)} q(h, h', \sigma)$ denote the probability that a history in $H(d, h)$ will be reached in the subgame after h. For any $d' > d$,

$$Q(d', h, \sigma) = \sum_{h' \in H(d,h)} \Big[q(h, h', \sigma) Q(d' - d, h', \sigma) \Big] \leq \sum_{h' \in H(d,h)} q(h, h', \sigma) = Q(d, h, \sigma).$$

Hence $Q(d, h, \sigma)$ is weakly decreasing in d. $g(h_0, \sigma) > 0$ implies that $Q(d, h_0, \sigma) > 0$ for any d. Let $\hat{g}(d, h_0, \sigma) := g(h_0, \sigma)/Q(d, h_0, \sigma)$ denote the probability that the game will not end in an impasse conditional on the game not having ended by the dth day. Since $Q(d, h, \sigma)$ is weakly decreasing in d, $\hat{g}(d, h_0, \sigma)$ is weakly increasing in d. Since $\hat{g}(d, h_0, \sigma)$ is bounded from above by 1, the sequence $\{\hat{g}(d, h_0, \sigma)\}_{d=1,2,\ldots}$ converges in (0,1]. Let $\bar{g} := \lim_{d \to \infty} \hat{g}(d, h_0, \sigma)$. Moreover, $\{\hat{g}(d, h_0, \sigma)\}_{d=1,2,\ldots}$ is a Cauchy sequence: for any $\eta > 0$ we can find some integer d such that $\hat{g}(d', h_0, \sigma) - \hat{g}(d, h_0, \sigma) < \eta$ for all $d' > d$, or equivalently by definition, $g(h_0, \sigma)/Q(d', h_0, \sigma) - g(h_0, \sigma)/Q(d, h_0, \sigma) < \eta$, from which we derive:

$$\begin{aligned} \frac{g(h_0, \sigma)}{Q(d', h_0, \sigma)} - \frac{g(h_0, \sigma)}{Q(d, h_0, \sigma)} < \eta \Longrightarrow 1 - \frac{Q(d', h_0, \sigma)}{Q(d, h_0, \sigma)} &< \eta \frac{Q(d', h_0, \sigma)}{g(h_0, \sigma)} \leq \frac{\eta}{g(h_0, \sigma)} \\ \Longrightarrow \frac{Q(d', h_0, \sigma)}{Q(d, h_0, \sigma)} &> 1 - \frac{\eta}{g(h_0, \sigma)}. \end{aligned} \tag{1}$$

Since $Q(d', h_0, \sigma) = \sum_{h \in H(d,h_0)} \Big[q(h_0, h, \sigma) Q(d' - d, h, \sigma) \Big]$, from inequality (1) we have

$$\sum_{h \in H(d,h_0)} \Big[\frac{q(h_0, h, \sigma)}{Q(d, h_0, \sigma)} Q(d' - d, h, \sigma) \Big] > 1 - \frac{\eta}{g(h_0, \sigma)}. \tag{2}$$

Since $\sum_{h \in H(d,h_0)} q(h_0, h, \sigma)/Q(d, h_0, \sigma) = 1$, inequality (2) implies that there exists some $h^* \in H(d, h_0)$ such that $q(h_0, h^*, \sigma) > 0$ and $Q(d' - d, h^*, \sigma) > 1 - \eta/g(h_0, \sigma)$. Since $\eta/g(h_0, \sigma)$ can be made arbitrarily close to 0 by choosing the proper η and the corresponding d, it follows that for all $d'' := d' - d \geq 1$, $Q(d'', h^*, \sigma)$ can be made arbitrarily close to 1. The next lemma shows that this is a contradiction and finishes the proof:

Lemma 6.1. *Given the inductive hypothesis, for a marriage game $(\mathcal{A}, C, \delta)$ where $\mathcal{A}$ is non-trivial, there exists some parameter pair $(\bar{d}, \bar{q})$ where $\bar{d} > 1$ and $\bar{q} < 1$, such that $d > \bar{d}$ implies $Q(d, h, \sigma) < \bar{q}$ for any history h and subgame perfect equilibrium σ.*

Proof. Pick $(\underline{d}, \bar{q})$ such that $\underline{d} > 1$, $\bar{q} < 1$, and they satisfy the following equation

$$\begin{aligned} &\Big(\bar{q} \delta^{\underline{d}} + (1 - \bar{q}) \delta \Big) \max \Big\{ \max_{M \times W} \{u(m, w)\}, \max_{M \times W} \{v(m, w)\} \Big\} \\ &< \min \Big\{ \min_{M \times W} \{u(m, w) : u(m, w) > 0\}, \min_{M \times W} \{v(m, w) : v(m, w) > 0\} \Big\}. \end{aligned} \tag{3}$$

Since $\mathcal{A}$ is non-trivial, the right hand side of (3) is strictly positive. The left hand side can be made arbitrarily close to 0 by choosing $\bar{q}$ close to 1 and $\underline{d}$ arbitrarily large. Hence a pair $(\underline{d}, \bar{q})$ satisfying inequality (3) exist. Note that the left hand side is decreasing in $\bar{q}$ and $\underline{d}$. Call a player **active** if there is someone acceptable to him or her in the initial marriage market. Given $(\underline{d}, \bar{q})$, any active player prefers the first option between the following two: (1) marrying the least preferred acceptable player today; (2) marrying the most preferred player tomorrow with probability at most $1-\bar{q}$ or after at least $\underline{d}$ days with probability at least $\bar{q}$.

Since $\mathcal{A}$ is non-trivial, there exists some pair (m, w) such that they are mutually acceptable to each other and m is w's most preferred player among men to whom w is acceptable. Call this pair a **success pair**. Consider a history h after which $|M(h)| = n$ and m moves, meeting w with whom he forms a success pair. Let h_1 denote the immediate consequent history after m rejects w, and h_2 the immediate consequent history after m accepts w. Let β be the probability that m accepts w after h, then for any $d \geq 1$,

$$\begin{aligned} Q(d, h, \sigma) &= \beta Q(d, h_2, \sigma) + (1 - \beta) Q(d, h_1, \sigma) \\ &= (1 - \beta) Q(d, h_1, \sigma) \\ &\leq Q(d, h_1, \sigma). \end{aligned}$$

Note that $Q(d, h_2, \sigma) = 0$ because if m accepts w, w will accept him with certainty since m is w's most preferred feasible man (any man ranked above m by w is infeasible because w is not acceptable to him), leading to the number of men in the consequent remaining marriage market to drop below n. Suppose (in order to lead to contradiction) that $Q(d, h, \sigma) > \bar{q}$ for some $d > \underline{d}$, then $Q(d, h_1, \sigma) > \bar{q}$. In this case, accepting w ensures m a payoff of $u(m, w)$, whereas rejecting her gives him no higher than

$$\begin{aligned} &\left(Q(d, h_1, \sigma)\delta^{\underline{d}} + (1 - Q(d, h_1, \sigma))\delta\right) \max_{w' \in W} u(m, w') \\ &< \left(\bar{q}\delta^{\underline{d}} + (1 - \bar{q})\delta\right) \max_{w' \in W} u(m, w') < u(m, w) \end{aligned}$$

due to $\bar{q} < Q(d, h, \sigma) \leq Q(d, h_1, \sigma)$ and the choice of $(\underline{d}, \bar{q})$. Therefore it is optimal for m to accept w, making $\beta = 1$ and $Q(d, h, \sigma) = 0$, which is a contradiction as we supposed that $Q(d, h, \sigma) > \bar{q}$. Call h a **success history** if after h a man moves, meeting a woman with whom he forms a success pair. We conclude that $Q(d, h, \sigma) < \bar{q}$ if h is a success history and $d > \underline{d}$.

Let $H_s(d', h)$ be the set of all histories h' in $\cup_{t=1}^{d'} H(t, h)$ such that h' is a success history and there does not exist $h'' \neq h'$ such that $h < h'' < h'$ is a success history. Hence $H_s(d', h)$ is the set of all "first" success histories within d' days after h is reached. Clearly for any distinct h' and h'' in $H_s(d', h)$, $Z_{|h'} \cap Z_{|h''} = \varnothing$. Let $H_s^C(d', h)$ be the set of histories $h' \in H(d, h)$ that has no subhistory in $H_s(d', h)$. Then

$$
\begin{aligned}
Q(d'+d,h,\sigma) &\leq \sum_{h'\in H_s(d',h)} q(h,h',\sigma)Q(d,h',\sigma) + \sum_{h'\in H_s^C(d',h)} q(h,h',\sigma)Q(d,h',\sigma) \\
&< \bar{q} \sum_{h'\in H_s(d',h)} q(h,h',\sigma) + \sum_{h'\in H_s^C(d',h)} q(h,h',\sigma) \\
&\leq \bar{q} \sum_{t=0}^{d'-1} (1-\epsilon)^t \epsilon + (1-\epsilon)^{d'}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\lim_{d'\to\infty} Q(d'+d,h,\sigma) &< \lim_{d'\to\infty} \bar{q} \sum_{t=0}^{d'-1} (1-\epsilon)^t \epsilon + (1-\epsilon)^{d'} \\
&= \bar{q},
\end{aligned}
$$

which implies that there exists some $\bar{d} > \underline{d}$ such that $Q(d,h,\sigma) < \bar{q}$ for all $d > \bar{d}$. This completes the proof the present lemma. □

□

6.4 Proof of Proposition 4.8

Proof. We prove the present proposition by applying the principle of mathematical induction on $|M|$, the number of men in the initial marriage market.

Clearly two strategy profiles σ and σ' are outcome equivalent if for any $m \in M$ one of the following is true:

1. There exists some $w \in W$ such that both σ and σ' imply m and w marry almost surely, and that they marry with certainty during their first meeting.
2. Both σ and σ' imply that m will stay single.

Call the above condition the **Outcome Equivalence Condition**.

The base case of the induction is a game whose initial marriage market $\mathcal{A}$ has only one man. In this case, $\mathcal{A}$ obviously satisfies the One-side Common Preference Condition. If $\mathcal{A}$ is trivial, then Lemma 3.1 immediately implies the present proposition. Suppose $\mathcal{A}$ is not trivial. Let m denote the only man in the initial marriage market and w his most preferred woman. Since $\mathcal{A}$ is not trivial, m and w are mutually acceptable. Accepting m is w's strictly dominant strategy when she is accepted by m. If m adopts the "safe strategy" of only accepting w and rejecting anyone else after any history, his expected payoff at the beginning of any day is bounded from below by $V(\delta) := \sum_{t=0}^{\infty} \epsilon(1 -$

$\epsilon)^t\delta^t u(m,w) = \frac{\epsilon}{1-\delta(1-\epsilon)}u(m,w)$ because his probability of meeting w is no less than ϵ on any day. Since $\delta V(\delta)$ converges to $u(m,w)$ as δ converges to 1 from below, there exists some $\underline{\delta} < 1$ such that $\delta V(\delta) > u(m,w')$ for any $w' \in W\backslash\{w\}$ if $\delta > \underline{\delta}$, which implies that the safe strategy is optimal for m if $\delta > \underline{\delta}$. Let σ be a subgame perfect equilibrium of the game with $\delta > \underline{\delta}$, then σ includes w's strategy of always accepting m and m's strategy of always accepting w and rejecting anyone else. Clearly (m,w) will marry almost surely in σ, and that they marry with certainty during their first meeting. Since σ is an arbitrarily chosen subgame perfect equilibrium, it follows that any two subgame perfect equilibria satisfy the Outcome Equivalence Condition.

Suppose the present proposition is true for any game whose initial marriage market satisfies the One-side Common Preference Condition and $|M| = n-1$. Now consider a game whose initial marriage market $\mathcal{A}$ satisfies $|M| = n$, and that u induces for each man the same set of acceptable women W^a and the same preference ordering on W^a. Hence $\mathcal{A}$ satisfies the One-side Common Preference Condition. Fix any C. Let $\mathcal{E}$ be the set of all sub-environments of $(\mathcal{A}, C)$ such that $(\mathcal{A}' = (M', W', u, v), C') \in \mathcal{E}$ implies (1) $M' \subset M$ and $|M'| = |M| - 1$, (2) $W' \subset W$ and $|W'| = |W| - 1$, and (3) C' agrees with C on the former's domain. The inductive hypothesis thus asserts that for any $(\mathcal{A}', C') \in \mathcal{E}$, there exists $\underline{\delta}_{(\mathcal{A}',C')} < 1$ such that all subgame perfect equilibria of $(\mathcal{A}', C', \delta)$ are outcome equivalent if $\delta > \underline{\delta}_{(\mathcal{A}',C')}$. Let $\underline{\delta}' := \max_{(\mathcal{A}',C') \in E}\{\underline{\delta}_{(\mathcal{A}',C')}\}$.

If $\mathcal{A}$ is trivial then Lemma 3.1 immediately implies the present proposition. Suppose $\mathcal{A}$ is not trivial, then W^a is not empty. Let $(w_1, ..., w_{|W|})$ be an ordering on W such that: (1) $w_i \in W^a$ if $i \leq |W^a|$, and (2) if $w_i \in W^a$ and $w_j \in W^a$ then $i < j$ if w_i is commonly preferred to w_j. Hence the commonly acceptable women are indexed by their ranks in the men's common preference ordering and the commonly unacceptable women have higher indices than the acceptable women. Note that the ordering may not be unique, but any two such orderings $(w_1, ..., w_{|W|})$ and $(w'_1, ..., w'_{|W|})$ agree on the first $|W^a|$ entries. The unique stable matching μ of $\mathcal{A}$ is computed using the following algorithm that iterates from $i = 1$ and goes upwards:

- $\mu(w_1) = \text{argmax}_{m \in \{s\} \cup M} v(m, w_1)$.
- $\mu(w_i) = \text{argmax}_{m \in \{s\} \cup M \backslash \cup_{j=1}^{i-1}\{\mu(w_j)\}} v(m, w_i)$ if $i \leq |W^a|$.
- $\mu(w_i) = s$ if $i > |W^a|$.

Let $(m_1, ..., m_{|M|})$ be an ordering on M such that $i < j$ implies $u(m, \mu(m_i)) \geq u(m, \mu(m_j))$ for any $m \in M$. Such an ordering exists because $\mu(m) \in W^a \cup \{s\}$ for any $m \in M$ and u induces the same preference ordering on $W^a \cup \{s\}$ for each man. In words, the men who are married under μ are ordered by ranks of their wives under μ in the men's common preference ordering, and the men who are single under μ have higher indices than those who are married. Let k be the number of married men under μ. Such an ordering on M may not be unique but all agree on the first k entries.

Fix the orderings $(w_1, ..., w_{|W|})$ and $(m_1, ..., m_{|M|})$. Since $\mathcal{A}$ is not trivial, $\mu(m_1) = w_i$ for some $i \leq |W^a|$. Moreover, $\text{argmax}_{M\cup\{s\}} v(m, w_j) = s$ for any $j < i$, which implies that w_j always rejects every man in any subgame perfect equilibrium. Since m_1 is w_i's most preferred alternative in $M \cup \{s\}$, accepting m_1 strictly dominates rejecting him when w_i is accepted by m_1. Consequently, in any subgame perfect equilibrium m_1 accepts w_i if they meet. Let σ be a subgame perfect equilibrium of $(\mathcal{A}, C, \delta)$ where $\delta > \underline{\delta}'$. Suppose all the other players follow σ, except w_i, who follows a "safe strategy" such that she accepts m_1 and rejects everyone else before someone has married, and then follows σ after someone has married. Her expected payoff from following this safe strategy is bounded from below by

$$\underline{V}_{safe}(\delta) := \epsilon v(m_1, w_i) + (1-\epsilon)\delta \frac{\epsilon}{1-\delta(1-\epsilon)} v(m', w_i)$$

where m' is the alternative that w_i ranks just below m_1 (note that m' can be staying single). $\underline{V}_{safe}(\delta)$ is computed by considering the worst scenario for w_i: there is probability ϵ that w_i meets (and consequently marries) m_1 today, and probability $1-\epsilon$ that m_1 marries with someone else today. If m_1 marries with someone else today, the resulting subgame becomes some $(\mathcal{A}', C', \delta)$. Clearly $(\mathcal{A}', C') \in \mathcal{E}$, and since $\delta > \underline{\delta}'$, by the inductive hypothesis all subgame perfect equilibria of the subgame $(\mathcal{A}', \mathcal{C}', \delta)$ yield the same expected payoff as the $\mu_{\mathcal{A}'}-$cutoff strategy profile where $\mu_{\mathcal{A}'}$ is the unique stable matching of $\mathcal{A}'$. The lower bound of w_i's expected payoff in the subgame $(\mathcal{A}', C', \delta)$ under the $\mu_{\mathcal{A}'}-$cutoff strategy profile is easily computed to be $\frac{\epsilon}{1-\delta(1-\epsilon)} v(m', w_i)$, thus justifying the expression of $\underline{V}_{safe}(\delta)$. Since $v(m_1, w_i) > v(m', w_i)$, there exists some δ_{w_i} such that $\delta \underline{V}_{safe}(\delta) > v(m', w_i)$ if $\delta > \delta_{w_i}$. Suppose $\delta > \max\{\delta_{w_i}, \delta'\}$. If no one has married and w_i is accepted by some man m other than m_1, then accepting m is not optimal since following the safe strategy by rejecting m ensures a continuation payoff of at least $\delta \underline{V}_{safe}(\delta) > v(m, w_i)$. Therefore, if $\delta > \max\{\delta_{w_i}, \underline{\delta}'\}$, in any subgame perfect equilibrium w_i's strategy includes that she uses the safe strategy before someone has married, that is, she only accepts m_1. Given that w_i follows the safe strategy in σ, m_1 can secure a safe expected payoff of $\frac{\epsilon}{1-\delta(1-\epsilon)} u(m_1, w_i)$ by following his own safe strategy which includes accepting only w_i before someone has married and following σ after someone has married. His safe strategy gives him at least this safe expected payoff because, (1) if he meets w_i before someone has married, he and w_i will marry, and (2) if someone manages to marry before he meets w_i, the consequent sub-environment is some $(\mathcal{A}', C') \in \mathcal{E}$ and the inductive hypothesis asserts that in the corresponding subgame σ is outcome-equivalent to the $\mu_{\mathcal{A}'}$ cutoff strategy profile where $\mu_{\mathcal{A}'}$ is the unique stable matching of $\mathcal{A}'$, and by the $\mu_{\mathcal{A}'}$ cutoff strategy profile (m_1, w_i) marry almost surely and they marry with certainty during their first meeting since $\mu_{\mathcal{A}'}(m_1) = w_i$. In either case (m_1, w_i) marry almost surely and they marry during their first meeting, thus the lower bound on the safe payoff. There exists some $\delta_{m_1} < 1$ such that $\frac{\epsilon}{1-\delta(1-\epsilon)} u(m_1, w_i) > u(m_1, w_j)$ for all $\delta > \delta_{m_1}$ and $j > i$. If $\delta > \max\{\delta_{w_i}, \delta_{m_1}, \underline{\delta}'\}$, any subgame perfect equilibrium includes that, before someone has married, m_1 accepts w_i with certainty and rejects with certainty any other woman who accepts him with positive probability. The above argument leads to the following claim.

Claim 6.2. *If σ is a subgame perfect equilibrium of the game $(\mathcal{A}, C, \delta)$ where $\delta > \max\{\delta_{w_i}, \delta_{m_1}, \underline{\delta}'\}$, then according to σ, in any subgame in which no one has married, m_1 and w_i marry each other almost surely and they marry with certainty during their first meeting.*

Proof. Fix a subgame in which no one has married. The subgame can evolve in two ways. First, the first meeting (in the subgame) between (m_1, w_i) occurs before someone has married. Second, that meeting occurs after someone has married. In the first situation, according to σ, (m_1, w_i) marry during their first meeting. In the second situation, the consequent sub-subgame after the first marriage has a game environment that is in $\mathcal{E}$, and by the inductive hypothesis σ is outcome equivalent to the unique stable cutoff strategy profile of that sub-subgame, the latter of which implies that (m_1, w_i) marry almost surely and they marry with certainty during their first meeting. Since one of the above two situations occur almost surely in the subgame (because w_i will not marry any other man before someone has married, nor m_1 any other woman), the present claim follows immediately. $\square$

Since the initial game is a subgame in which no one has married, it follows from Claim 6.2 that in any subgame perfect equilibrium, m_1 marries w_i almost surely and they marry with certainty during their first meeting.

Now consider m_2. If $\mu(m_2) = s$, then $m_l = s$ for all $l \geq 2$, implying that m_l is unacceptable to any woman in $W \backslash \{w_i\}$. It follows from Claim 6.2 that m_l, $l \geq 2$, stays single with certainty in any subgame equilibrium if $\delta > \max\{\delta_{w_i}, \delta_{m_1}, \underline{\delta}'\}$ as it is not optimal for any woman to accept an unacceptable man. Thus any two subgame perfect equilibria satisfy the Outcome Equivalence Condition, and the proof is complete.

Suppose, instead, that $\mu(m_2) = w_j \in W$. Obviously $j > i$. Then m_2 is w_j's first choice in $M \backslash \{m_1\}$. Given Claim 6.2, if $\delta > \max\{\delta_{w_i}, \delta_{m_1}, \underline{\delta}'\}$ then in any subgame equilibrium w_j accepts m_2 with certainty if she is accepted by m_2 because, even if m_1 is preferred to m_2, m_1 is "out of reach" since the probability that (m_1, w_j) marry is zero. Using a similar argument that appears in the paragraph before Claim 6.2, we can proceed to show that there exists some δ_{w_j} such that in any subgame perfect equilibrium w_j accepts the man if and only if he is weakly preferred to m_2 before someone has married if $\delta > \max\{\delta_{w_j}, \delta_{w_i}, \delta_{m_1}, \underline{\delta}'\}$, and consequently there exists some δ_{m_2} such that in any subgame perfect equilibrium and before someone has married, m_2 accepts w_j with certainty and rejects with certainty any other woman who accepts him with positive probability if $\delta > \max\{\delta_{m_2}, \delta_{w_j}, \delta_{w_i}, \delta_{m_1}, \underline{\delta}'\}$. Hence we have a claim that is analogous to Claim 6.2.

Claim 6.3. *If σ is a subgame perfect equilibrium of the game $(\mathcal{A}, C, \delta)$ where $\delta > \max\{\delta_{m_2}, \delta_{w_j}, \delta_{m_1}, \delta_{w_i}, \underline{\delta}'\}$, then according to σ, in any subgame in which no one has married, m_2 and w_j marry each other almost surely and they marry with certainty during their first meeting.*

We can apply the analogous argument iteratively to m_3, m_4, etc., and identify the corresponding δ_{m_3}, $\delta_{\mu(m_3)}$, δ_{m_4}, $\delta_{\mu(m_4)}$ ..., etc., until we hit m_k (if he exists) such that $\mu(m_k) = s$, in which case, as is already argued when considering $\mu(m_2) = s$, that for $\delta > \max\{\delta_{m_1}, ..., \delta_{m_{k-1}}, \delta_{\mu(m_1)}, ..., \delta_{\mu(m_{k-1})}, \delta'\}$, m_l stays single in any subgame perfect equilibrium for $l \geq k$. Thus any two subgame perfect equilibria satisfy the Outcome Equivalent Condition, which implies the proposition.

The proof for a market with common preference ordering on the female side, that is, v induces the same set of acceptable men M^a and the same preference ordering over $M^a \cup \{s\}$, is essentially symmetric. $\square$

References

Adachi, Hiroyuki (2003), "A search model of two-sided matching under non-transferable utility." *Journal of Economic Theory*, 113, 182 – 198.

Bloch, Francis and Effrosyni Diamantoudi (2011), "Noncooperative formation of coalitions in hedonic games." *International Journal of Game Theory*, 40, 263–280.

Bloch, Francis and Harl Ryder (2000), "Two-sided search, marriages, and matchmakers." *International Economic Review*, 41, pp. 93–115.

Burdett, Ken and Melvyn G. Coles (1997), "Marriage and class." *The Quarterly Journal of Economics*, 112, pp. 141–168.

Diamantoudi, Effrosyni, Eiichi Miyagawa, and Licun Xue (2007), "Decentralized matching: the role of commitment." *Mimeo.*

Eeckhout, Jan (1999), "Bilateral search and vertical heterogeneity." *International Economic Review*, 40, pp. 869–887.

Eeckhout, Jan (2000), "On the uniqueness of stable marriage matchings." *Economics Letters*, 69, 1–8.

Gale, David (2000), *Strategic Foundations of General Equilibrium: Dynamic Matching and Bargaining Games.* Cambridge University Press, Cambridge.

Gale, David and Lloyd Shapley (1962), "College admissions and the stability of marriage." *American Mathematical Monthly*, 69, 9–15.

Gale, Douglas and Hamid Sabourian (2006), "Markov equilibria in dynamic matching and bargaining games." *Games and Economic Behavior*, 54, 336 – 352.

Haeringer, Guillaume and Myrna Wooders (2011), "Decentralized job matching." *International Journal of Game Theory*, 40, 1–28.

Lauermann, Stephan and Georg Nöldeke (2013), "Stable marriages and search frictions." *Forthcoming: Journal of Economic Theory.*

McNamara, J. M. and E. J. Collins (1990), "The job search problem as an employer-candidate game." *Journal of Applied Probability*, 27, pp. 815–827.

Niederle, Muriel and Leeat Yariv (2009), "Decentralized matching with aligned preferences." NBER Working Papers 14840, National Bureau of Economic Research, Inc.

Osborne, Martin J. and Ariel Rubinstein (1990), *Bargaining and Markets*. Academic Press, Inc., San Diego, CA.

Pais, Joana (2008), "Incentives in decentralized random matching markets." *Games and Economic Behavior*, 64, 632 – 649.

Roth, Alvin E. (2008), "What have we learned from market design?" *Economic Journal*, 118, 285–310.

Roth, Alvin E. and Marilda A. Liveria Sotomayor (1990), *Two-Sided Matching: A Study In Game-theoretic Modeling and Analysis*. Cambridge University Press, Cambridge.

Rubinstein, Ariel and Asher Wolinsky (1990), "Decentralized trading, strategic behaviour and the walrasian outcome." *The Review of Economic Studies*, 57.

Smith, Lones (2006), "The marriage model with search frictions." *Journal of Political Economy*, 114, pp. 1124–1144.