Scalable games

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Abstract

We establish a link between games of complete information and games of incomplete information that facilitate the characterization of equilibria in the incomplete information game. In particular we show that many all pay auctions are closely related to stochastic contest success functions. This relationship is used to solve for equilibria in all pay auctions and to provide foundations for a number of contest succes functions.

1 Introduction

Consider firms participating in a procurement auction. While each firm knows its own cost of delivering the product or good, it does not know whether this cost is likely to be higher or lower than that of its competitors. This stems from the fact that a firm doesn't know the composition of common factors - such as the cost of construction material - and firm specific factors such as the skill of workers of its competitors.

The situation described above is an example of a game where players face maximal rank uncertainty. The term maximal rank uncertainty is used to describe situations, where despite knowing his type, a player has no information about whether his type is likely to be higher or lower than that of his opponents. Other examples of games with this property include

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auctions, where players do not know the distribution from which valuations are drawn and form beliefs centered around their own valuation. The formation of asset price bubbles studied in Abreu & Brunnermeier (2003) is an example of such a setting considered in the literature, where players learn about the existence of a bubble, but do not know whether they are among the first or the last players to find out.

In this paper we propose a general framework to study a certain type of game with maximal rank uncertainty. The two key assumptions made are that (i) the information structure satisfies an invariance property and (ii) the payoff function is scalable. The first assumption is equivalent to assuming that players know the shape of the distribution of types, but - after observing their type - have no information about what quantile of the distribution they were drawn from. This ensures that a player's type does not provide the player with information about his rank and guarantees that players face maximal rank uncertainty.

The second assumption - scalability of the payoff function - captures a number of settings where the structure of the game remains unchanged if all variables are scaled by a constant. In particular this holds when utility functions are homogeneous of degree α, or are additively invariant. This captures a large family of games including (i) models with quadratic utility, (ii) auctions and procurement contests, (iii) certain public good problems and many others. A game satisfying both of the key assumptions above is referred to as a scalable game. From now on we will use the abbreviation LPG to mean scalable game.

In this paper we primarily focus on the Nash equilibria of scalable games where players play according to linear strategies. The main contribution of this paper is to show that equilibria of an LPG can be found by studying a related complete information game. This relationship is illustrated by studying a certain all-pay auction under uncertainty and showing that this game has the same equilibria as a Tullock contest introduced by Tullock (1980). In this contest, a single prize is allocated according to a stochastic allocation rule, where a player's probability of winning is given by the proportion of his effort relative to the sum of efforts across all players. This result helps characterise equilibria of all-pay auctions when priors are possibly asymmetric, which in general is a difficult task. It also highlights a close relationship between certain classes of games, such as the all-pay auction under incomplete information and the Tullock contest. In the application section links between other games are established.

1.1 Related Literature

The proposed class of games has close links with the literature on global games introduced by Carlsson & Van Damme (1993) and Morris & Shin (2002). As in certain global games, players face uncertainty about the state of the world θ which is drawn from a diffuse prior. Moreover each player does not observe θ but instead receives a partially informative signal t_i about the state of the world, where $t_i = \theta + z_i$ and z_i can be interpreted as a noise term. However two main differences with global games is that (i) in this paper there are not necessarily dominance regions and (ii) in this paper a player's signal typically enters his payoff function directly. Above all the focus of this paper lies on the characterization of equilibria rather than equilibrium selection as is common in the global games literature.

The framework proposed also has close ties with the literature on quadratic utility models as considered in Vives (1988), Myatt & Wallace (2012) among others. In these games the state of the world is unknown and players receive a noisy signal of the state. As in this paper the signal can be interpreted to be a players type and the type may enter a players payoff function directly. On the one hand, scalable games make stronger distributional assumptions on the state and the signals: the information structure in a quadratic utility model is affine, satisfying the assumption that $E[\theta|t_i] = \alpha t_i + \beta$; in scalable games $E[\theta|t_i] = t_i + \beta$ and the shape of the distribution is known. On the other hand, scalable games make weaker assumptions on the payoff function. While the payoff function in most quadratic utility models depend on the actions of others only through the aggregate, the payoff assumption in this paper is substantially weaker and allows for a much wider range of applications.

1.2 Rank uncertainity

One of the main properties of a scalable game is that all players face maximal rank uncertainty. We now illustrate the concept of *rank uncertainty* using an example. Consider a game with a number of players each of whom may be in one of three positions in the distribution low, medium or high denoted by $\{L, M, H\}$. In the standard independent private value case, players know the distribution from which types are drawn. Hence upon observing his type a player knows whether his position is low, medium or high relative to that of others. As an example consider the case where types are drawn from a set $\{-1, 0, 1\}$ each with equal probability. This situation is depicted in Figure 1.

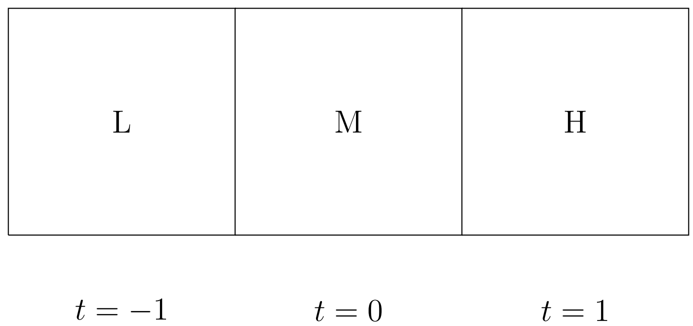

Figure 1: Independent Types

Players who have type $t = -1$ know that they are drawn from the low part of the distribution, those with a type $t = 0$ know they are drawn from the middle of the distribution, while types in $t = 1$ are high types. In this game of independent types players have complete rank information.

In order to introduce rank uncertainty consider the case where first the state θ is drawn from the set $\{-1, 0, 1\}$ with equal probability. Secondly - conditional on θ - the players' types are drawn from the set $\{\theta - 1, \theta, \theta + 1\}$ with equal probability: $t \in \{-2, -1, 0, 1, 2\}$. This situation is depicted in Figure 2.

In this case note that types on the extreme of the type space, $t = -2$ (or $t = 2$), still possess full rank information. This is because they can infer that the state is $\theta = -1$ (or $\theta = 1$) and can hence deduce that their position in the distribution is low (or high). On the other hand all other types face some rank uncertainty. In particular players with type $t = 0$ do not know whether $\theta = \{-1, 0, 1\}$ and hence do not know whether their position in the distribution is

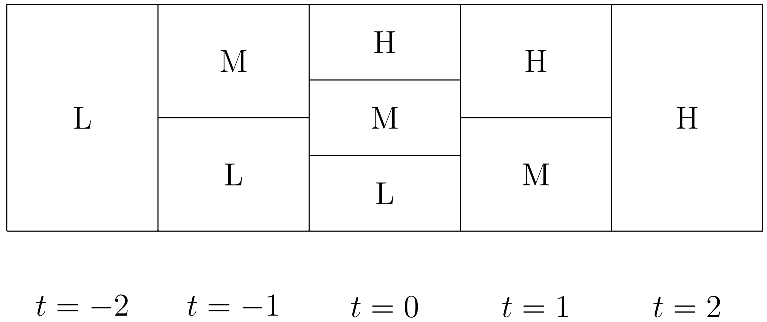

Figure 2: Correlated Types

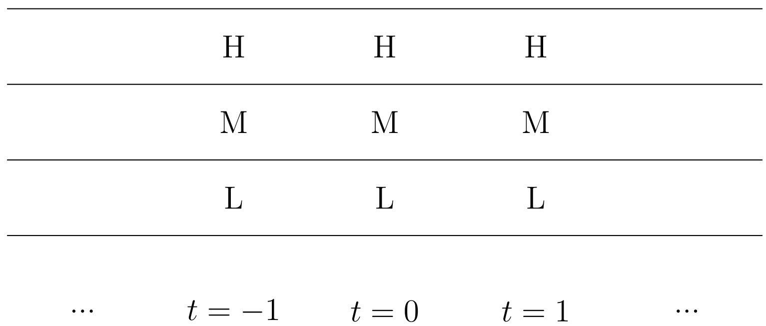

Figure 3: Maximal rank uncertainty

low, medium or high relative to that of others. We say that these players face maximal rank uncertainty, since they have no idea of their position in the distribution.

Increasing the number of values that θ can take increases the proportion of players who face maximal rank uncertainty. In the case of a diffuse prior θ takes any integer with equal likelihood. In this case upon observing his type no player receives information about whether his position in the distribution is low, medium or high and hence has no rank information. This is illustrated in Figure 3.

1.3 Structure of the paper

The remainder of the paper is structured as follows. In section two we present the model and formally introduce the class of scalable games. Section three provides the analysis of scalable games and establishes the link between scalable games and the corresponding game in complete information. Using this link, applications studying the relationship between different all pay auctions and contests with stochastic allocation rules are presented in section four. Section five concludes.

2 Model

2.1 Environment

The first element of an environment is a domain $\mathbb{D}$, from which (i) the state θ is drawn (ii) types $(t_1, ..., t_n)$ are drawn and (iii) actions $(a_1, ..., a_n)$ are chosen. We assume that $\mathbb{D} \subseteq \mathbb{R}$ and that $\mathbb{D} = (\underline{D}, \overline{D})$ is an open interval.[1]

The second element of an environment is a finite set of players $I = \{1, ..., n\}$, each of whom have a type $t_i \in T_i = \mathbb{D}$ and choose actions $a_i \in A_i = \mathbb{D}$. We use $\mathbf{t} = (t_1, \ldots, t_n)$ and $\mathbf{a} = (a_1, \ldots, a_n)$ to denote the vector of types and the vector of actions respectively. Moreover $\mathbf{v_{-i}}$ is used to mean a vector excluding the i'th element.

The third element of an environment is a strictly increasing differentiable function G which G is a mapping from $\mathbb{D}$ to $\mathbb{R}$. The function G captures how likely a particular state $\theta \in \Theta = \mathbb{D}$is. For instance if $G(3) - G(2) > G(2) - G(1)$, then - conditional on the event of $\theta \in [1, 3]$ - it is more likely that the state $\theta \in [2, 3]$ than the state $\theta \in [1, 2]$. It is further assumed that G is a bijection and hence G is not a cumulative distribution function.[2]

This completes the description of an environment $\{\mathbb{D}, I, G\}$, which can be used as the foundation of many games with an improper prior $g(\theta) = G'(\theta)$ where the state is drawn from some domain $\mathbb{D}$. The special case of a uniform improper prior where $G(\theta) = \theta$ and $g(\theta) = 1$, has been used in the global games and auction literature.

We now outline some additional notation. First define $0_G = G^{-1}(0)$ and note that $0_G \in \mathbb{D}$. Secondly define $a \oplus_G b = G^{-1}\Big(G(a) + G(b)\Big)$ and $a \ominus_G b = G^{-1}\Big(G(a) - G(b)\Big)$. Two special

[1] We allow for the case where $\underline{D} = -\infty$ or $\overline{D} = \infty$

[2] Note that G is related to a standard cdf in the same way as a standard improper prior is related to a standard pdf.

cases deserve consideration. When $G(\theta) = \theta$, then $a \oplus_G b = a + b$ and $a \oplus_G b = a - b$. Moreover when $G(\theta) = \ln \theta$, then $a \oplus_G b = a \times b$ and $a \ominus_G b = a \div b$. These two cases both play an important role in applications, so in order to avoid repetition we consider the general case.[3]

2.2 Scalable information structure

We now define a scalable information structure which is composed of an environment $\{\mathbb{D}, I, G\}$ and a set of conditional distributions $(F_i)_{i \in I}$. The conditional distribution associated with player i is given by $F_i : T_i \times \Theta \mapsto [0, 1]$, where $F_i(t_i|\theta)$ captures the probability that - given the state is θ - the type of player i is less than or equal to t_i. It is assumed throughout that $F_i(t_i|\theta)$ is differentiable with respect to t_i, with derivative $f_i(t_i|\theta)$. We assume that t_i and t_j are conditionally independent on θ whenever $i \neq j$. With this in mind, we define a scalable information structure as follows:

Assumption 1 (Scalable information structure)**.** *The conditional distribution function F_i is scalable with respect to the environment $\{I, \mathbb{D}, G\}$ if and only if*

$$F_i(t_i|\theta) = F_i(t_i \oplus_G k|\theta \oplus_G k)$$

This assumption captures the fact that conditional beliefs have a similar shape when θ is changed. When $a \oplus_G b = a + b$ this assumption implies that conditional beliefs are additively invariant: that is to say players know the shape of the distribution but not their position in it. For instance this holds when players know that the distribution is uniform over the interval $[\theta - 1, \theta + 1]$, but do not necessarily know the value of the state θ. This is illustrated in Figure 4.

Meanwhile when $a \oplus_G b = a \times b$ this assumption implies that conditional beliefs are homogenous of degree 0. For instance this holds when players know that the distribution is

[3]Note that the set $\mathbb{D}$ combined with the operation $\oplus_G$ forms a commutative group. This ensures that $(a \oplus_G b) \oplus_G c = a \oplus_G (b \oplus_G c)$. Moreover it is easy to check that $(a \oplus_G b) \ominus_G c = a \oplus_G (b \ominus_G c)$. This ensures standard addition and subtraction can be used.

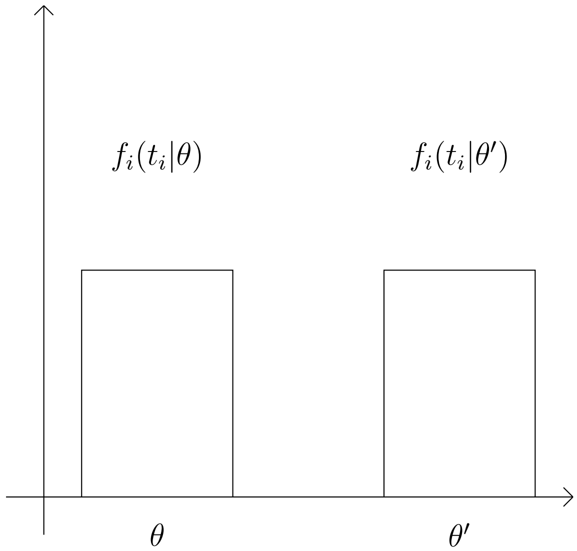

Figure 4: Uniform

uniform over the interval $[0, 2\theta]$, but do not necessarily know the value of the median θ. This is illustrated in Figure 5.

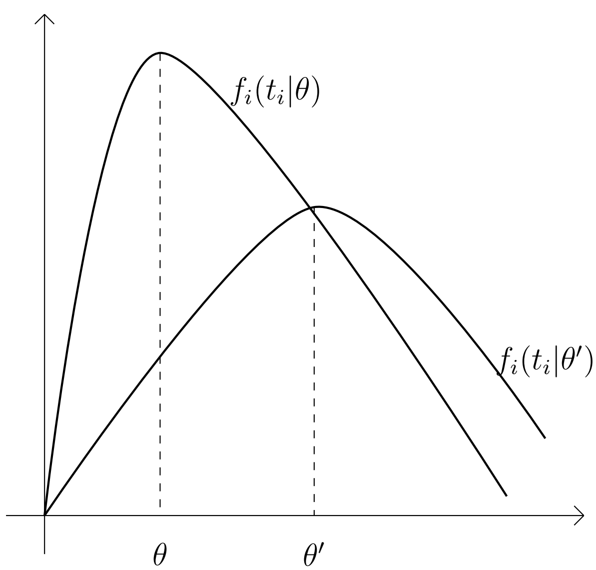

Figure 5: Homogeneity of degree zero

2.3 Scalable payoff structure

We now define a scalable payoff structure which is composed of an environment $\{I, \mathbb{D}, G\}$ and a set of payoff functions $(u_i)_{i \in I}$. The payoff function of player i is given by $u_i : (A_j)_{j \in I} \times \Theta \times T_i \mapsto \Re$ and maps (i) the actions $(a_1, ..., a_n)$ of all players, (ii) the state θ and (iii) the type t_i of player i to a payoff. Informally a payoff function is scalable if and only if, when the inputs of u_i are scaled the corresponding payoff is scaled in a similar way. For instance an auction without entry costs is a scalable environment since scaling the valuation and the bids of all players leaves the payoffs agents receive unchanged except for a scaling factor. However an auction with entry costs is not a scalable environment, since scaling the valuation and bids of all players changes the burden of the entry cost relative to the potential reward.

We now formally introduce what it means for a payoff function to be scalable. To do this we first make two auxiliary definitions:

Definition 1.

$$\overline{u}_i(t_i) := \sup_{\mathbf{a} \in \mathbf{D}^n} \Big\{ u_i(\mathbf{a}; t_i; t_i) \Big\}$$

This is the utility that player i could achieve if the state is $\theta = t_i$ and - knowing this - player i could choose $\mathbf{a}$ and hence the actions of each of his opponents in addition to his own action. Therefore $\overline{u}_i(t_i)$ is the highest utility that player i could achieve given the state is $\theta = t_i$. We assume that $0 < \overline{u}_i(t_i) < \infty$ for all $i \in I$ and all $t_i \in \mathbf{D}$. With this measure of the maximum utility that is achievable for player i in mind, we now make the following definition:

Definition 2.

$$U_i(\mathbf{a}, \theta, t_i) := \frac{u_i(\mathbf{a}; \theta; t_i)}{\overline{u}_i(t_i)}$$

The function $U_i(\mathbf{a}, \theta, t_i)$ captures the utility level achieved by player i as a proportion of the highest utility level he could achieve when the state is $\theta = t_i$. Note that U_i measures gains and losses relative to some well-defined benchmark, and this motivates us to make the following scalability assumption directly on U_i. If $\mathbf{a} = (a_1, ..., a_n)$, define $\mathbf{a} \oplus_G k = (a_1 \oplus_G k, ..., a_n \oplus_G k)$. [4] Using this notation, formally we call a payoff function scalable if and only if it obeys the

[4] Throughout the paper, $x \oplus y$ and $x \ominus y$, are used to mean componentwise operations.

following assumption:

Assumption 2 (Scalable payoff structure)**.** *The payoff function u_i is scalable with respect to the environment $\{I, \mathbb{D}, G\}$ if and only if for all $k \in \mathbb{D}$:*

$$U_i(\mathbf{a}; \theta; t_i) = U_i(\mathbf{a} \oplus_G k; \theta \oplus_G k; t_i \oplus_G k)$$

To show that this payoff assumption can capture several economic environments, we now prove two lemmas and provide a number of examples:

Lemma 2.1. *Suppose $\mathbb{D} = \Re$ and $G(\theta) = \theta$. Moreover suppose for all $i \in I$ and for some $\alpha \in \mathbb{R}$:*

$$t_i^\alpha u_i(\mathbf{a}; \theta; t_i) = (t_i + k)^\alpha u_i(\mathbf{a} + k; \theta + k; t_i + k)$$

Then $\{\mathbb{D}, I, (u_i)_{i \in I}, G\}$ is a scalable environment

One environment that satisfies this case is a beauty contest where agents want their move a_i both to be close to the true state θ and to be close to the average move. Such a contest can be summarised by the following payoff function, where $r \in [0, 1]$ captures the relative importance of being close to the true state and being close to the average move:

$$u_i(\mathbf{a}; \theta; t_i) = 1 - (1 - r)\Big(a_i - \theta\Big)^2 - r\Big(a_i - \frac{1}{|I|}\sum_{j \in I} k_j\Big)^2$$

Note here that t_i does not directly enter the payoff function and is simply a signal player i uses to gain information about the value of θ and inform his decision. Beauty contests with similar payoff structures have been considered by Morris & Shin (2002) and Myatt & Wallace (2012).

Lemma 2.2. *Suppose $\mathbb{D} = \Re_{++}$ and $G(\theta) = \ln \theta$. Moreover suppose for all $i \in I$ and for some $\alpha \in \mathbb{R}$:*

$$t_i^\alpha u_i(\mathbf{a}; \theta; t_i) = (t_i.k)^\alpha u_i(\mathbf{a}.k; \theta.k; t_i.k)$$

Then $\{\mathbb{D}, I, (u_i)_{i\in I}, G\}$ *is a scalable environment*

This lemma shows that - by considering a suitable domain $\mathbb{D}$ and suitable distribution function G - any payoff function which is homogenous of degree α can be captured. One example that satisfies this structure is a first price auction with a combination of private values and common values. Let t_i^{β} capture the private value element of a player's valuation and $\theta^{1-\beta}$ capture the common value element of a player's valuation. Player i submits a bid a_i and if he submits the highest bid he wins the object and pays his bid. If he submits the lower bid he does not win the object and pays nothing. This is summarised by the function below, where $\beta \in [0, 1]$ captures the relative importance of private values and common values:

$$u_i(a_i, a_j; \theta; t_i) = \begin{cases} t_i^{\beta}\theta^{1-\beta} - a_i & \text{if} \quad a_i > a_j \\ 0 & \text{otherwise} \end{cases}$$

Note that a range of quadratic utility models - which are homogenous of degree two - can be captured in this setting. One example of a quadratic utility model that can be captured is a model of Cournot competition with linear demand (see for instance Vives (1988)). Here a_i captures the quantity player i produces, and θ represents a demand shock about which agents are imperfectly informed. The price is given by $\left(\theta - \sum_{j\in I} a_j\right)$ and hence the payoff of agents becomes:

$$u_i(a_i; \theta; t_i) = a_i\Big(\theta - \sum_{j\in I} a_j\Big)$$

A further application with contests is studied in detail in the application section. Having given examples of the environments that can be captured by scalable payoff functions, we now formally define a scalable game.

2.4 Scalable games

A game with an improper prior is composed of the following elements:

$$\Gamma = \{\mathbb{D}, I, G, (F_i)_{i\in I}, (u_i)_{i\in I}\}$$

We call such a game an scalable game if and only if (i) each conditional distribution function $(F_i)_{i\in I}$ forms a scalable information structure with respect to the environment $\{\mathbb{D}, I, G\}$ (Assumption 1) and (ii) each payoff function $(u_i)_{i\in I}$ forms a scalable payoff structure with respect to the environment $\{\mathbb{D}, I, G\}$ (Assumption 2).

In a scalable game players observe their type t_i but not the state θ. Having observed their type t_i, players simultaneously choose actions $a_i \in A_i = \mathbb{D}$. Payoffs are then realised and the game ends. Let $g(\theta) = G'(\theta)$, and note that $g(\theta)$ is an improper prior over the domain $\mathbb{D}$. This means that the conditional beliefs that player i holds over the state θ can be written as follows:

$$g_i(\theta|t_i) = \frac{f_i(t_i|\theta)g(\theta)}{\int_{\mathbb{D}} f_i(t_i|\theta)g(\theta)d\theta}$$

This completes the description of a scalable game.

2.5 Key property

The key property of scalable games, is that all players have Laplacian beliefs about their position in the distribution of types. This means that each player believes that his position in the distribution from which types are drawn is uniformly distributed. Hence players belief that the probability of being in the top x percent of the distribution is x percent. These beliefs are maintained when agents learn their type. Formally this property can be described as follows:

Proposition 2.3. *If Γ is a scalable game, then $F_i(t_i|\theta) = 1 - G_i(\theta|t_i)$*

This proposition says that if a type t_i is the result of a high realisation given the state θ then given the type t_i it is unlikely for the state to be below θ and the other way around. In a scalable game this property is satisfied for every type t_i. A formal proof can be found in the appendix.

2.6 Equilibrium

The equilibrium concept used in this paper is Nash equilibrium. Formally an equilibrium is defined as follows:

Definition 3. *The strategy profile $\sigma(\mathbf{t})$ is an equilibrium of the scalable game Γ, if and only if for all $i \in I$, all $t_i, t_j, a_i \in \mathbb{D}$ it holds that:*

$$\int_{\mathbb{D}^n} g_i(\theta|t_i) \prod_{j\neq i}^{n} f_j(t_j|\theta) u_i\Big(\sigma_i(t_i), \sigma_{-i}(t_{-i}); \theta; t_i\Big) dt_{-i} d\theta \;\; \geq \;\; \int_{\mathbb{D}^n} g_i(\theta|t_i) \prod_{j\neq i}^{n} f_j(t_j|\theta) u_i\Big(a_i, \sigma_{-i}(t_{-i}); \theta; t_i\Big) dt_{-i} d\theta$$

2.7 Value function

The focus of this paper lies on linear equilibria. In order to simplify notation when considering these equilibria, we introduce the value function, describing a player's expected payoffs when he has a certain type t_i and everyone plays according to a linear strategy profile $\mathbf{e}$. The exact structure of the linear strategy profile depends on the game.

Take a pure strategy profile e^* of Γ_N and the corresponding linear strategy profile $\sigma(\mathbf{t}) = \mathbf{t} \oplus_G \mathbf{e}^*$ of the original scalable game Γ. Make the following definition:

$$V_i(e_i|e^*, t_i) = \int_{\mathbf{D}} g_i(\theta|t_i) \prod_{j\neq i} f_j(t_j|\theta) u_i\Big(e_i \oplus t_i, e^*_{-i} \oplus t_{-i}; \theta; t_i\Big) d\theta \prod_{j\neq i} dt_j$$

3 Analysis

First we state the main result for general scalable games. To illustrate the result, we then consider canonical scalable games where $\mathbb{D} = \mathbb{R}$ and both the payoff function and conditional distribution are additively invariant. Thirdly we briefly show the result for a scalable game in multiplicative form where $\mathbb{D} = \mathbb{R}_{++}$ and $G(\theta) = ln(\theta)$, since this is a case relevant in many applications.

3.1 General scalable games

We now introduce the complete information game Γ_N induced by a scalable game. This game is in normal form, and hence there is no uncertainty over the types of each player. However the *ex-ante* uncertainty over types present in the scalable game is replaced with *interim* uncertainty in the payoff function. Expectations are then taken to form the complete information game.

Definition 4. *The complete information game $\Gamma_N = \{I, (A_i)_{i\in I}, (\phi_i)_{i\in I}\}$ induced by a scalable game $\Gamma_{scalablegame}$ has the following payoff function:*

$$\phi_i(e_i, e^*_{-i}) = \int_{\mathbb{D}^n} \prod_{j=1}^{n} f_j(z_j|0_G) u_i\Big(e_i, e^*_{-i} \oplus_G z_{-i} \ominus_G z_i; 0_G \ominus_G z_i; 0_G\Big) \prod_{j=1}^{n} dz_j$$

The following lemma says that this complete information game indeed describes a player's expected payoff in a scalable game, given that everyone plays according to a linear strategy profile of the form $\sigma(\mathbf{t}) = \mathbf{e} \oplus \mathbf{t}$.

Lemma 3.1. *For all $t_i \in \mathbb{D}$*

$$\phi_i(\mathbf{e}) = \frac{\overline{u}_i(0)}{\overline{u}_i(t_i)} V_i(\mathbf{e}|t_i)$$

While the general lemma is proved in the appendix, we illustrate this key lemma for the case of two player scalable game in additive form - referred to as canonical scalable game - below.

First we state the main result:

Theorem 3.2. *The strategy profile $\sigma(\mathbf{t}) = \mathbf{t} \oplus \mathbf{e}^*$ is a Nash equilibrium of a scalable game Γ, if and only if the strategy profile $\mathbf{e}^*$ is a Nash equilibrium of the corresponding complete information game Γ_N.*

Consider some linear strategy profile $\sigma^* = (\sigma_i^*)_{i\in I}$ where $\sigma_i^*(t_i) = t_i \oplus e_i^*$. Since $V_i(\mathbf{e}^*|t_i)$ captures the expected payoffs of players who play according to these linear strategies, it follows that:

$$\sigma^* \text{ is an equilibrium of } \Gamma \quad \text{iff} \quad V_i(e_i^*, e_{-i}^*|t_i) \geq V_i(\hat{e}_i, e_{-i}^*|t_i) \quad \text{for all } t_i,\ \hat{e}_i$$

From lemma 3.1 above $V_i(e_i, e_{-i}|t_i) = \frac{\overline{u}_i(t_i)}{\overline{u}_i(0)}\phi_i(e_i, e_{-i})$ for all e_i, e_{-i}. Hence:

$$\begin{aligned}
&\sigma^* \text{ is an equilibrium of } \Gamma \quad \text{iff} \quad \tfrac{\overline{u}_i(t_i)}{\overline{u}_i(0)}\phi_i(e_i^*, e_{-i}^*) \geq \tfrac{\overline{u}_i(t_i)}{\overline{u}_i(0)}\phi_i(\hat{e}_i, e_{-i}^*) \quad \text{for all } \hat{e}_i \text{ and } t_i \\
&\sigma^* \text{ is an equilibrium of } \Gamma \quad \text{iff} \quad \phi_i(e_i^*, e_{-i}^*) \geq \phi_i(\hat{e}_i, e_{-i}^*) \quad \text{for all } \hat{e}_i
\end{aligned}$$

Hence σ^* is an equilibrium of Γ if and only if e^* is an equilibrium of Γ_N.

3.2 Canonical scalable games

In this section we illustrate the result for the case of an additive or canonical scalable game Γ_0. In this example let $I = \{1, 2\}$, $\mathbb{D} = \mathbb{R}$ and $G(\theta) = \theta$. The complete information game is given as follows:

Definition 5. *The complete information game $\Gamma_N = \{I, (A_i)_{i\in I}, (\phi_i)_{i\in I}\}$ induced from a canonical scalable game Γ_0 has the following payoff function:*

$$\phi_i(\mathbf{e}) := \int_{\mathbf{z}\in\mathbb{R}^n} \left(\prod_{j=1}^n f_j(z_j|0)\right) u_i\Big(\mathbf{e} + \mathbf{z} - z_i; -z_i, 0\Big) \prod_{j=1}^n dz_j$$

To show how we can move from the scalable game to this complete information game, removing $g_i(\theta|t_i)$ from the equation, we first prove a lemma:[5]

Lemma 3.3. *If Γ_0 is a canonical scalable game, then:*

$$g_i(\theta|t_i) = f_i(t_i|\theta)$$

[5]Note that this lemma holds only for the canonical scalable game. When the scalable game is in a different form, additional terms appear in the translation. This is explained in the appendix.

Proof.

$$\begin{aligned} g_i(\theta|t_i) &= \frac{f_i(t_i|\theta)g(\theta)}{\int_{\Re} f_i(t_i|\tilde{\theta})g(\tilde{\theta})d\tilde{\theta}d\tilde{\theta}} \\ &= \frac{f_i(t_i|\theta}{\int_{\Re} f_i(t_i|\tilde{\theta}|0)d\tilde{\theta}} \\ &= \frac{f_i(t_i|\theta}{\int_{\Re} f_i(t_i-\tilde{\theta}|0)d\tilde{\theta}} \\ &= f_i(t_i|\theta) \end{aligned}$$

Moving from the first to second line appeals to the fact that $g(\theta) = 1$ for all θ in a canonical scalable game. Moving from the second to third line appeals to the fact that $F_i(t_i|\theta) = F_i(t_i - \theta|0)$ and hence $f_i(t_i|\theta) = f_i(t_i - \theta|0)$. □

Having proved this lemma, consider the following transformations: $e_i = a_i - t_i$ which will be referred to as a player's mark-up and $z_i = t_i - \theta$ which can be interpreted as a player specific shock. Suppose now that $t_i = 0$ and so the state $\theta = \theta - t_i = -z_i$. Moreover the action of any player $j \in I$ can be written as follows:

$$\begin{aligned} a_j &= e_j + t_j \\ &= e_j + (t_j - \theta) + \theta \\ &= e_j + z_j - z_i \end{aligned}$$

Substituting in these expressions leads to the following equation:

$$u_i(\mathbf{a}; \theta; 0) = u_i(\mathbf{e} + \mathbf{z} - z_i; -z_i; 0)$$

This shows that the payoff function of the scalable game can be written in terms of $\mathbf{e}$ and $\mathbf{z}$ when $t_i = 0$. To see this relationship still holds even when $t_i \neq 0$, note again that $\theta - t_i = -z_i$, while $a_j - t_i$ can be written as follows:

$$\begin{aligned} a_j - t_i &= e_j + t_j - t_i \\ &= e_j + (t_j - \theta) - (t_i - \theta) \\ &= e_j + z_j - z_i \end{aligned}$$

Now by appealing to additive invariance of the payoff function and considering the expression above, we can see that the utility function of the scalable game can be written in terms of **e** and **z**:

$$\begin{aligned} u_i(\mathbf{a}; \theta; t_i) &= u_i\Big((\mathbf{a} - \mathbf{t_i}); \theta - t_i; 0\Big) \\ &= u_i\Big(\mathbf{e} + \mathbf{z} - z_i, -z_i, 0) \end{aligned}$$

Meanwhile by appealing to the fact that $g(\theta|t_i) = f(t_i|\theta)$ and additive invariance of the conditional distribution $F(t_j|\theta)$, we can show how ex-ante uncertainty term over types of the original scalable game is related to the interim uncertainty term over payoff shocks:

$$\begin{aligned} g_i(\theta|t_i) \prod_{j \neq i} f_j(t_j \theta) &= \prod_{j=1}^{n} f_j(t_j|\theta) \\ &= \prod_{j=1}^{n} f_j(t_j - \theta|0) \\ &= \prod_{j=1}^{n} f_j(z_j|0) \end{aligned}$$

Hence by using a change of variables from $\{\mathbf{a}, \theta, \mathbf{t}\}$ to $\{\mathbf{e}, \mathbf{z}\}$ we have shown that the canonical scalable game with *ex-ante uncertainty* over types is closely related to the corresponding game of complete information with *interim uncertainty* over payoff shocks.[6] In particular there is a close correspondence when players play according to linear strategies $\sigma = (\sigma_i)_{i \in I}$ where $\sigma_i(t_i) = t_i + e_i$. To show this we first define the value function which captures the payoffs players obtain when they play according to linear strategies of this kind:

Definition 6. *The value function of player i in a canonical scalable game Γ_0 when all players play according to the linear strategy profile $\sigma(\mathbf{t}) = \mathbf{e} + \mathbf{t}$ is given as follows:*

$$V_i(\mathbf{e}|t_i) = \int_{\mathbb{R}^n} \Big(\prod_{j \neq i} f_j(t_j|\theta) \Big) g_i(\theta|t_i) u_i(e_i + t_i, e_{-i} + t_{-i}; \theta; t_i) d\theta \prod_{j \neq i} dt_j$$

The value function captures a player's expected payoff in the scalable game when all players play according to linear strategies. This game is closely related to the game of complete information. We can now prove Lemma 3.1 for the canonical scalable game.

Proof.

$$\phi_i(e_i, e_j) \;\; = \;\; \int_{\mathbb{R}^2} f_j(z_j|0) f_i(z_i|0) u_i\Big(e_i, e_j + z_j - z_i; -z_i; 0\Big) dz_i dz_j$$

Define $\theta = -z_i$ and $t_j = z_j - z_i$. Note that $z_j = t_j - \theta$ and $z_i = -\theta$. Substituting $\{z_i, z_j\}$ for $\{t_j, \theta\}$ we obtain:

$$\begin{aligned} \phi_i(e_i, e_j) &= \int_{\mathbb{R}^2} f_j(t_j - \theta|0) f_i(-\theta|0) u_i\Big(e_i, e_j + t_j; \theta; 0\Big) d\theta dt_j \\ &= \overline{u}_i(0) \int_{\mathbb{R}^2} f_j(t_j - \theta|0) f_i(-\theta|0) U_i\Big(e_i, e_j + t_j; \theta; 0\Big) d\theta dt_j \end{aligned}$$

[6] Note we have reduced a problem with $2n+1$ variables to a problem with only $2n$ variables. Hence we should expect the new problem to be easier to solve.

Using the fact that f_i, f_j and U_i are all homogenous of degree zero in the log transform we reach:

$$\phi_i(e_i, e_j) = \overline{u}_i(0) \int_{\mathbb{R}^2} f_j(t_j + t_i | t_i + \theta) f_i(t_i | t_i + \theta) U_i\Big(e_i + t_i, e_j + t_j + t_i; \theta + t_i; t_i\Big) d\theta dt_j$$

Now substitute $\tilde{t}_j = t_j + t_i$ and $\tilde{\theta} = \theta + t_i$. Using this substitution:

$$\phi_i(e_i, e_j) = \overline{u}_i(0) \int_{\mathbb{R}^2} f_j(\tilde{t}_j | \tilde{\theta}) f_i(t_i | \tilde{\theta}) U_i\Big(e_i + t_i, e_j + \tilde{t}_j; \tilde{\theta}; t_i\Big) d\tilde{\theta} d\tilde{t}_j$$

Using the fact that $g_i(\tilde{\theta}|t_i) = f_i(t_i|\tilde{\theta})$:

$$\begin{aligned} \phi_i(e_i, e_j) &= \frac{\overline{u}_i(0)}{\overline{u}_i(t_i)} \int_{\mathbb{R}^2} f_j(\tilde{t}_j | \tilde{\theta}) g_i(\tilde{\theta} | t_i) u_i(e_i + t_i, e_j + \tilde{t}_j; \tilde{\theta}; t_i) d\tilde{\theta} d\tilde{t}_j \\ &= \frac{\overline{u}_i(0)}{\overline{u}_i(t_i)} V_i(e_i, e_j | t_i) \end{aligned}$$

□

This lemma immediately leads to the main result stated in Theorem 3.2 for the canonical scalable game. Hence this result allows us to characterise equilibria of canonical scalable games by appealing to the corresponding game of complete information.

The proof for the general case uses the same steps and can be found in the appendix.

3.2.1 Multiplicative scalable games

In many applications, the scalable game takes a multiplicative form. In these games $\mathbb{D} = \mathbb{R}_{++}$ and $G(\theta) = ln(\theta)$. Note the similarities between the structure of those complete information game associated with multiplicative scalable game compared to those associated with canonical scalable games. In this case variables are multiplied and divided rather than added and subtracted reflecting the fact that $a \oplus_G b = a.b$ in a multiplicative scalable game while $a \oplus_G b = a + b$ in a canonical scalable game. Intuition for the result could be given by performing the substitutions $z_i = \frac{t_i}{\theta}$ and $e_i = \frac{a_i}{t_i}$, much as in the case of the additive case above where $z_i = t_i - \theta$ and $e_i = a_i - t_i$. The complete information game corresponding to a scalable game in multiplicative form is given as follows:

Definition 7. *The complete information game* $\Gamma_N = \{I, (A_i)_{i\in I}, (\phi_i)_{i\in I}\}$ *induced from a multiplicative scalable game* Γ *has the following payoff function:*

$$\phi_i(\mathbf{e}) := \int_{\mathbf{z}\in\mathbb{R}^n} \left(\prod_{j=1}^{n} f_j(z_j|1)\right) u_i\Big(\frac{1}{z_i}\mathbf{e}.\mathbf{z}; \frac{1}{z_i}, 1\Big) \prod_{j=1}^{n} dz_i$$

Since this game is a special case of a scalable game, we know by Theorem 3.2 that if the strategy profile $\mathbf{e}^*$ is a Nash equilibrium of the corresponding complete information game Γ_N, then the strategy profile $\sigma(\mathbf{t}) = \mathbf{t} \oplus \mathbf{e}^*$ is a Nash equilibrium of the scalable game in multiplicative form.

4 Applications: Contests

The aim of this section is to use the theory above to establish a relationship between contests and all-pay auctions with incomplete information. We consider multiplicative scalable games capturing an all-pay auction, where players have the following utility function:

$$u_i(\mathbf{a};\theta;t_i) = \begin{cases} t_i^{\beta}\theta^{\beta-1} - a_i & \text{if} \quad a_i > a_j \text{ whenever } i \neq j \\ -a_i & \text{otherwise} \end{cases}$$

The parameter β captures the extent to which the auction is a private value auction as opposed to a common value auction. We now examine different distributional assumptions and different choices of β in order to show that the corresponding games of complete information are contest success functions which have been introduced in the literature by Yildizparlak (2013), Alcalde & Dahm (2007) and Tullock (1980). First this approach introduces a way to characterise the equilibria of certain all-pay auctions by consulting the equilibria of some complete information contest. Secondly it provides robust foundations for these contests.

4.1 Serial contest success function

We first examine the case where there are two players in a private value auction with $\beta = 1$. It is assumed that $\alpha \in (0, 1]$ and $V_2 \geq V_1 > 0$. Given these parameters the structure of uncertainty is defined as follows:

$$F_i\Big(t_i|\theta\Big) = 1 - \left(\frac{\theta V_i}{t_i}\right)^{\alpha} \quad \text{when} \quad t_i \in [V_i\theta, \infty)$$

It can easily be checked that this game satisfies assumptions 1 and 2 and is therefore a scalable game. Hence, using definition 4, the corresponding complete information game of this all pay auction is given as follows:

$$\phi_i(e_i, e_j) = \begin{cases} \left[\frac{1}{2}\left(\frac{e_iV_i}{e_jV_j}\right)^{\alpha}\right] - e_i & \text{if} \quad e_iv_i \leq e_jv_j \\ \left[1 - \frac{1}{2}\left(\frac{e_jV_j}{e_iV_i}\right)^{\alpha}\right] - e_i & \text{otherwise} \end{cases}$$

The unique pure strategy equilibrium of this complete information game[7] is given by $(e_1^*, e_2^*) = \Big(K, K\Big)$ where $K = \frac{\alpha}{2}\left(\frac{V_1}{V_2}\right)^{\alpha}$. Substituting $E_i = e_iv_i$ and multiplying the utility function of player i by a constant V_i, leads to the following expression:

$$\Phi_i(E_i, E_j) = \begin{cases} \left[\frac{1}{2}\left(\frac{E_i}{E_j}\right)^{\alpha}\right]V_i - E_i & \text{if} \quad E_i \leq E_j \\ \left[1 - \frac{1}{2}\left(\frac{E_j}{E_i}\right)^{\alpha}\right]V_i - E_i & \text{otherwise} \end{cases}$$

[7] See Alcalde & Dahm (2007) for details

This is the payoff function used in the serial contest studied by Alcalde & Dahm (2007) and shows that the serial contest is closely linked to the all-pay auction where players have incomplete information. Using Theorem 3.2 above we can now find a pure strategy equilibrium of the original all-pay auction:

Proposition 4.1. *Suppose uncertainty is distributed according to F_i^{SC} and the all-pay auction is private value with $\beta = 1$. Then there exists a pure strategy equilibrium where players play according to $\sigma_i(t_i) = Kt_i$ where $K = \frac{\alpha}{2}\left(\frac{V_1}{V_2}\right)^{\alpha}$*

4.2 Tullock contest success function

Secondly we examine a case with n players $I = \{1, ..., n\}$ who participate in a sealed bid private value auction with $\beta = 1$. Associated with each player is a constant V_i and uncertainty is distributed on the interval $[0, \infty)$ according to the following distribution:

$$F_i^{TC}(t_i|\theta) = exp\Big[-\Big(\frac{V_i\theta}{t_i}\Big)^{\alpha}\Big] \quad \text{on} \quad [0, \infty)$$

Again it can easily be checked that this auction satisfies the conditions of a scalable game. The corresponding complete information game is given as follows:[8]

$$\phi_i(e_i, e_{-i}) = \frac{(V_i e_i)^{\alpha}}{\sum_{j \in I}(V_j e_j)^{\alpha}} - e_i$$

An axiomatization of this contest success function was given by Clark & Riis (1998), who interpret E_i as a contestant's absolute level of effort and the parameters V_i^{α} as a measure of how far the contest is skewed towards player i. An alternative interpretation can be reached by considering a contestant's absolute level of effort to be $E_i = V_i e_i$. Making this substitution and multiplying the utility function of player i by a constant V_i leads to the following:

$$\Phi_i(E_i, E_{-i}) = \Big[\frac{(E_i^{\alpha})}{\sum_{j \in I}(E_j)^{\alpha}}\Big]V_i - E_i$$

[8]The algebraic steps are closely related to Jia (2008) who showed how this family of contest success functions can be founded from noisy foundations. The contribution here is to show that this noise can be interpreted as uncertainty in an all-pay auction.

This shows that an all-pay auction with this structure of uncertainty is closely related to a fair Tullock contest where players have different valuations. Moreover we can use the equilibria of the Tullock contest success function to characterise equilibria of the original all-pay auction:

Proposition 4.2. *Suppose uncertainty is distributed according to F_i^{TC}, with $\alpha = 1$ and $V_i = 1$ for all i. Moreover suppose the all-pay auction is private value with $\beta = 1$ and there are n players. Then there exists a pure strategy equilibrium where players play according to $\sigma_i(t_i) = \frac{n-1}{n^2} t_i$.*

This result can be extended to settings with asymmetric uncertainty structures by solving the relevant complete information game. Such a task is normally far easier than attempting to find the equilibria of the all-pay auction directly.

4.3 Tullock contest success function with draws

Finally we examine a case with n symmetric players $I = \{1, ..., n\}$ who participate in a common value auction with $\beta = 0$. Uncertainty is distributed on the interval $[0, \infty)$ according to the following distribution:

$$F_i^{TCD}(t_i|\theta) = \exp\left[-\left(\frac{\theta}{t_i}\right)\right] \quad \text{on} \quad [0, \infty)$$

Again it can easily be checked that this game satisfies assumptions 1 and 2. Calculating the corresponding complete information game of this scalable game leads to the following:

$$\begin{aligned} \phi_i(e_i, e_{-i}) &= \left[\frac{e_i^2}{\left(\sum_{j=1}^n e_j\right)^2} \exp\left(\sum_{j=1}^n \frac{-e_j}{e_i z_i}\right)\right]_0^\infty - e_i \\ &= \frac{e_i^2}{\left(\sum_{j=1}^n e_j\right)^2} - e_i \end{aligned}$$

This complete information game is the contest success function studied by Yildizparlak (2013). The contest success function is used to model contests where ties occur with positive probabilities, such as in soccer games. The analysis provided here demonstrates that there exists a strong link between common value all pay auctions and contests with ties. This link may not seem obvious in first place and it provides additional reasons for the importance of studying contests with ties, that goes beyond straightforward applications.

5 Conclusion

In this paper we have shown that games of incomplete information with the property of maximal rank uncertainty and hence a scalable information structure one the one hand and a scalable payoff structure on the other hand are closely linked to corresponding games of complete information. In particular the equilibria of the corresponding complete information game coincide with linear equilibria of the game of incomplete information.

Considering the complete information game significantly simplifies the characterization of equilibria in the scalable game. Moreover in some cases, the game in complete information may itself be an interesting game studied in the literature. As an example of games where this is the case, we have shown that all pay auctions in scalable game form correspond to certain contest success functions, when considering a particular distribution function in the scalable game. In many cases these relationships were not previously known and the analysis in this paper provides additional foundations for the study of contest success functions such as the Tullock contest, contests with ties and serial contests.

While all the applications presented in this paper were focusing on auctions and corresponding contests, one can think of other games that may have interesting links to games in complete information. Examples include Cournot competition and certain settings of public good provision.

The analysis in this paper allows for asymmetries across players. However so far we have assumed that conditional on the state the types of players are independent. It remains an issue for future work to find out whether this condition can be relaxed. Another idea left for future work is the possibility of players receiving signals regarding their position in the distribution.

6 Appendix A: The model

6.1 Proof of Lemma 2.1

Proof. Recall that $\overline{u}_i(t_i) = \sup_{\mathbf{a}\in(A_j)_{j\in I}} \Big\{u_i(\mathbf{a}, t_i, t_i)\Big\} \in (0, \infty)$ and note that $t_i^\alpha u_i(\mathbf{a}, t_i, t_i) = (t_i+k)^\alpha u_i(\mathbf{a}+k, t_i+k, t_i+k)$ From these properties it follows that $t_i^\alpha \overline{u}_i(t_i) = (t_i+k)^\alpha \overline{u}(t_i+k)$. Hence both u_i and $\overline{u}_i$ are homogenous of degree α in the log transform. Therefore:

$$\begin{aligned}
U_i\Big(\mathbf{a}, \theta, t_i\Big) &= \frac{u_i(\mathbf{a}, \theta, t_i)}{\overline{u}_i(t_i)} \\
&= \frac{t_i^\alpha u_i(\mathbf{a}, \theta, t_i)}{t_i^\alpha \overline{u}_i(t_i)} \\
&= \frac{(t_i+k)^\alpha u_i(\mathbf{a}+k, \theta+k, t_i+k)}{(t_i+k)^\alpha \overline{u}_i(t_i+k)} \\
&= U_i\Big(\mathbf{a}+k, \theta+k, t_i+k\Big)
\end{aligned}$$

This shows that the environment is indeed scalable.

□

6.2 Proof of Lemma 2.2

Proof. Repeat the proof of the additive case above, replacing + with ×. □

Proof. Recall that $G^{-1}(0) = 0_G$. Let $\theta^{-1} = G^{-1}\Big(- G(\theta)$ and note that:

$$\theta \oplus_G \theta^{-1} = G^{-1}\Big(G(\theta) - G(\theta)\Big) = 0_G$$

Note also that:

$$t_i \oplus_G \theta^{-1} = G^{-1}\Big(G(\theta) - G(\theta)\Big) = t_i \ominus_G \theta$$

Using these two facts and scale invariance we can now show the result:

$$\begin{aligned} F_i(t_i|\theta) &= F_i(t_i \oplus_G \theta^{-1}|\theta \oplus_G \theta^{-1}) && (1) \\ &= F_i(t_i \ominus_G \theta|0_G) && (2) \\ & && (3) \end{aligned}$$

□

6.3 Proof of Proposition 2.3

Proof.

$$g_i(\theta|t_i) = \frac{f_i(t_i|\theta)g(\theta)}{\int f_i(t_i|\tilde{\theta})g(\tilde{\theta})d\tilde{\theta}}$$

By Assumption 1 $F_i(t_i|\theta) = F_i(t_i \oplus_G \theta^{-1}|\theta \oplus_G \theta^{-1}) = F_i(t_i \ominus_G \theta|0_G)$. Differentiating with respect to t_i gives $f_i(t_i|\theta) = \frac{d}{dt_i}\Big[t_i \ominus_G \theta\Big] f_i(t_i \ominus_G \theta|0_G)$.

$$g_i(\theta|t_i) = \frac{\frac{d}{dt_i}\Big[t_i \ominus_G \theta\Big] f_i(t_i \ominus_G \theta|0_G)g(\theta)}{\int \frac{d}{dt_i}\Big[t_i \ominus_G \tilde{\theta}\Big] f_i(t_i \ominus_G \tilde{\theta}|0_G)g(\tilde{\theta})d\tilde{\theta}}$$

Using the fact that $g(\theta)\frac{d}{dt_i}\Big[t_i \ominus_G \theta\Big] = -g(t_i)\frac{d}{d\theta}\Big[t_i \ominus_G \theta\Big]$ yields:

$$
\begin{aligned}
g_i(\theta|t_i) &= \frac{-g(t_i)\frac{d}{d\theta}\Big[t_i \ominus_G \theta\Big] f_i(t_i \ominus_G \theta|0_G)}{\int -g(t_i)\frac{d}{d\tilde{\theta}}\Big[t_i \ominus_G \tilde{\theta}\Big] f_i(t_i \ominus_G \tilde{\theta}|0_G)d\tilde{\theta}} \\
&= \frac{-\frac{d}{d\theta}\Big[t_i \ominus_G \theta\Big] f_i(t_i \ominus_G \theta|0_G)}{\int -\frac{d}{d\tilde{\theta}}\Big[t_i \ominus_G \tilde{\theta}\Big] f_i(t_i \ominus_G \tilde{\theta}|0_G)d\tilde{\theta}}
\end{aligned}
$$

Integrating the denominator:

$$
\begin{aligned}
g_i(\theta|t_i) &= \frac{-\frac{d}{d\theta}\Big[t_i \ominus_G \theta\Big] f(t_i \ominus_G \theta|0_G)}{\Big[- F_i(t_i \ominus_G \tilde{\theta}|0_G)d\tilde{\theta}\Big]_{\theta=-\infty}^{\infty}} \\
&= -\frac{d}{d\theta}\Big[t_i \ominus_G \theta\Big] f_i(t_i \ominus_G \theta|0_G)
\end{aligned}
$$

Integrating this expression gives:

$$
\begin{aligned}
G_i(\theta|t_i) &= \int_{-\infty}^{\theta} -\frac{d}{d\tilde{\theta}}\Big[t_i \ominus_G \tilde{\theta}\Big] f_i(t_i \ominus_G \tilde{\theta}|0_G) \\
&= \Big[- F_i(t_i \ominus_G \tilde{\theta}|0_G)\Big]_{\tilde{\theta}=-\infty}^{\theta} \\
&= 1 - F_i(t_i \ominus_G \theta|0_G) \\
&= 1 - F_i(t_i|\theta)
\end{aligned}
$$

□

7 Appendix B: Analysis

7.1 Proof of Theorem 3.2 and Lemma 3.1

We now prove that Lemma 3.1 also holds for general scalable games.

Proof. Take a pure strategy profile e^* of Γ_N and the corresponding linear strategy profile $\sigma(\mathbf{t}) = \mathbf{t} \oplus_G \mathbf{e}^*$ of the original scalable game Γ.

It is immediate to see that e^* is a linear equilibrium of Γ if and only if $V_i(e_i^*|e^*, t_i) \geq V_i(\hat{e}_i|e_i^*, t_i)$ for all $\hat{e}_i$, for all t_i and for all i.

Note that when Lemma 3.1 holds for general scalable games, then Theorem 3.2 is proved.

We now aim to show that indeed $\frac{\overline{u}_i(0)}{\overline{u}_i(t_i)} V(e_i|e^*, t_i) = \phi(e_i, e_{-i}^*)$.
From the payoff assumption note that:

$$\frac{u_i\Big(e_i \oplus_G t_i, e_{-i}^* \oplus_G t_{-i}; \theta; t_i\Big)}{\overline{u}_i(t_i)} = \frac{u_i\Big(e_i, e_{-i}^* \oplus_G t_{-i} \ominus_G t_i; \theta \ominus_G t_i; 0_G\Big)}{\overline{u}_i(0_G)}$$

Using this fact and the definition of $V_i(e_i, e_{-i}|t_i)$:

$$\begin{aligned} V_i(e_i, e_{-i}|t_i) &= \int_{\mathbb{D}^n} g_i(\theta|t_i) \prod_{j\neq i} f_j(t_j|\theta) u_i\Big(e_i \oplus t_i, e_{-i}^* \oplus t_{-i}; \theta; t_i\Big) d\theta \prod_{j\neq i} dt_j \\ &= \frac{\overline{u}_i(t_i)}{\overline{u}_i(0_G)} \int_{\mathbb{D}^n} g_i(\theta|t_i) \prod_{j\neq i} f_j(t_j|\theta) u_i\Big(e_i \oplus t_i, e_{-i}^* \oplus t_{-i}; \theta; t_i\Big) d\theta \prod_{j\neq i} dt_j \end{aligned}$$

Recall from the previous proof that:

$$g_i(\theta|t_i) = -\frac{d}{d\theta}\Big[t_i \ominus_G \theta\Big] f_i(t_i \ominus_G \theta|0_G)$$

Moreover:

$$\begin{aligned} F_j(t_j|\theta) &= F_j(t_j \ominus_G \theta | 0_G) \\ f_j(t_j|\theta) &= \frac{d}{dt_j}\Big[t_j \ominus_G \theta\Big] f_j(t_j \ominus_G \theta | 0_G) \end{aligned}$$

Using these two facts we reach:

$$\begin{aligned} V_i(e_i, e_{-i}|t_i) = \quad & \frac{\overline{u}_i(t_i)}{\overline{u}_i(0_G)} \int_{\mathbb{D}^n} -\frac{d}{d\theta}\Big[t_i \ominus_G \theta\Big] \Bigg(\prod_{j \neq i} \frac{d}{dt_{j_1}}\Big[t_{j_1} \ominus_G \theta\Big]\Bigg) f_i(t_i \ominus_G \theta | 0_G) \prod_{j \neq i} f_j(t_j \ominus_G \theta | 0_G) \\ & u_i\Big(e_i, e^*_{-i} \oplus_G t_{-i} \ominus_G t_i; \theta \ominus_G t_i; 0_G\Big) d\theta \prod_{j \neq i} dt_j \end{aligned}$$

In order to do the substitution from $\{\theta, t_{j_1}, ...t_{j_{n-1}}\}$ to $\{z_i, z_{j_1}, ...z_{j_{n-1}}\}$ it is necessary to consider the following matrix:

$$M = \begin{pmatrix} \frac{dz_i}{d\theta} & \frac{dz_{j_1}}{d\theta} & ... & \frac{dz_{j_{n-1}}}{d\theta} \\ \frac{dz_i}{dt_{j_1}} & \frac{dz_{j_1}}{dt_{j_1}} & ... & \frac{dz_{j_{n-1}}}{dt_{j_1}} \\ ... & ... & ... & ... \\ \frac{dz_i}{dt_{j_{n-1}}} & \frac{dz_{j_1}}{dt_{j_{n-1}}} & ... & \frac{dz_{j_{n-1}}}{dt_{j_{n-1}}} \end{pmatrix} = \begin{pmatrix} \frac{d}{d\theta}\Big[t_i \ominus_G \theta\Big] & * & ... & * \\ 0 & \frac{d}{dt_{j_1}}\Big[t_{j_1} \ominus_G \theta\Big] & ... & 0 \\ 0 & 0 & ... & \frac{d}{dt_{j_{n-1}}}\Big[t_{j_{n-1}} \ominus_G \theta\Big] \end{pmatrix}$$

This matrix has only zero entries apart from in the first row and along the main diagonal. This means that the determinant is equal to the product of the main diagonal:

$$detM = \frac{d}{d\theta}\Big[t_i \ominus_G \theta\Big] \Bigg(\prod_{j \neq i} \frac{d}{dt_j}\Big[t_j \ominus_G \theta\Big]\Bigg)$$

By the change of variables we must divide by this expression when changing variables of integration from $\{\theta, t_{j_1}, ...t_{j_{n-1}}\}$ to $\{z_1, z_2, ..., z_n\}$. This ensures that the initial term disappears and leaves us with the expression:[9]

$$\begin{aligned} V_i(e_i, e_{-i}|t_i) &= \frac{\overline{u}_i(t_i)}{\overline{u}_i(0_G)} \int_{\mathbb{D}^n} \prod_{j=1}^{n} f_j(z_j|0_G) u_i\Big(e_i, e^*_{-i} \oplus_G z_{-i} \ominus_G z_i; 0_G \ominus_G z_i; 0_G\Big) \prod_{j=1}^{n} dz_j \\ &= \frac{\overline{u}_i(t_i)}{\overline{u}_i(0_G)} \phi(e_i, e^*_{-i}) \end{aligned}$$

Hence:

$$\phi(e_i, e^*_{-i}) = \frac{\overline{u}_i(0_G)}{\overline{u}_i(t_i)} V_i(e_i|e^*, t_i)$$

□

8 Appendix D: Applications to Contests

In this appendix we show how the three families of contests considered are indeed the complete information games associated with the relevant all-pay auction under complete information. We show this result first for the asymmetric serial contest, secondly for the asymmetric Tullock and thirdly for the Tullock with draws.

8.1 Asymmetric serial contest

First note that $F_i(t_i|\theta) = 1 - \left(\frac{V_i}{z_i}\right)^{\alpha}$ and the utility function can be rewritten as follows:

$$u_i\Big(a_i, a_{-i}; \theta; t_i\Big) = t_i \mathbf{1}_{\Big\{a_i > a_j \text{ for all } j \neq i\Big\}} - a_i$$

This leads to the following expression for the complete information game $\phi_i(e_i, e_{-i})$:

[9] Note there are two minus signs that cancel. One minus sign in the original expression disappears and the domain of integration is reversed.

$$\phi_i(e_i, e_{-i}) = \int_{\mathbb{R}_+^2} f_i(z_i|1) f_j(z_j|1) \mathbf{1}_{\left\{e_i z_i > e_j z_j\right\}} d\mathbf{z} - e_i$$

Integrating this expression with respect to j gives:

$$\phi_i(e_i, e_{-i}) = \int_{\mathbb{R}_+} f_i(z_i|1) F_j\Big(\frac{e_i z_i}{e_j}\Big|1\Big) dz_i - e_i$$

Case (i): If $e_i V_i \geq e_j V_j$, then player i can win whenever $z_i \in [V_i, \infty)$. Hence $\phi_i(e_i, e_{-i})$ is given by:

$$\begin{aligned}
\phi_i(e_i, e_{-i}) &= \int_{V_i}^{\infty} \frac{\alpha V_i^\alpha}{z_i^{\alpha+1}} \Big[1 - \Big(\frac{V_j e_j}{V_i z_i}\Big)^\alpha\Big] dz_i - e_i \\
&= \Big[-\Big(\frac{V_i}{z_i}\Big)^\alpha + \frac{\alpha}{2\alpha}\Big(\frac{V_i V_j e_j}{z_i z_i e_i}\Big)^\alpha\Big]_{V_i}^{\infty} \\
&= \Big[1 - \frac{1}{2}\Big(\frac{V_j e_j}{V_i e_i}\Big)^\alpha\Big] - e_i
\end{aligned}$$

Case (ii): If $e_i V_i < e_j V_j$, then player i can win whenever $z_i \in [e_j v_j / e_i, \infty)$. Hence $\phi_i(e_i, e_{-i})$ is given by:

$$\begin{aligned}
\phi_i(e_i, e_{-i}) &= \int_{\frac{e_j v_j}{e_i}}^{\infty} \frac{\alpha V_i^\alpha}{z_i^{\alpha+1}} \Big[1 - \Big(\frac{V_j e_j}{V_i z_i}\Big)^\alpha\Big] dz_i - e_i \\
&= \Big[-\Big(\frac{V_i}{z_i}\Big)^\alpha + \frac{\alpha}{2\alpha}\Big(\frac{V_i V_j e_j}{z_i z_i e_i}\Big)^\alpha\Big]_{\frac{e_j v_j}{e_i}}^{\infty} \\
&= \Big[\frac{1}{2}\Big(\frac{V_i e_i}{V_j e_j}\Big)^\alpha\Big] - e_i
\end{aligned}$$

This completes the derivation of the complete information game.

8.2 Asymmetric Tullock contest

First recall that $F_i(t_i|\theta) = \exp\left[\left(\frac{V_j\theta}{t_j}\right)^{\alpha}\right]$ and note that the utility function can be rewritten as follows:

$$u_i\Big(a_i, a_{-i}; \theta; t_i\Big) = t_i\mathbf{1}_{\Big\{a_i > a_j \quad \text{for all} \quad j \neq i\Big\}} - a_i$$

This leads to the following expression for the complete information game $\phi_i(e_i, e_{-i})$:

$$\phi_i(e_i, e_{-i}) = \int_{\mathbb{R}^n_+} f_i(z_i|1) \prod_{j\neq i} f_j(z_j|1)\mathbf{1}_{\Big\{e_i z_i > e_j z_j\Big\}} d\mathbf{z} - e_i$$

Integrating this expression with respect to all $j \neq i$ gives:

$$\phi_i(e_i, e_{-i}) \;\; = \;\; \int_{\mathbb{R}_+} f_i(z_i|1) \prod_{j\neq i} F_j\Big(\frac{e_i z_i}{e_j}\Big|1\Big) dz_i - e_i$$

Note that $f_i(z_i|1) = \frac{\alpha V_i^{\alpha}}{z_i^{\alpha+1}} F(z_i|1)$. Hence:

$$\begin{aligned}
\phi_i(e_i, e_{-i}) &= \int_{\mathbb{R}_+} \frac{\alpha V_i^{\alpha}}{z_i^{\alpha+1}} \prod_{j\in I} F_j\Big(\frac{e_i z_i}{e_j}\Big|1\Big) dz_i - e_i \\
&= \int_{\mathbb{R}_+} \frac{\alpha V_i^{\alpha}}{z_i^{\alpha+1}} \prod_{j\in I} \exp\left[\Big(\frac{v_j e_j}{z_i e_i}\Big)^{\alpha}\right] dz_i - e_i \\
&= \int_{\mathbb{R}_+} \frac{\alpha V_i^{\alpha}}{z_i^{\alpha+1}} \exp\left[\frac{1}{z_i^{\alpha}} \sum_{j\in I} \Big(\frac{V_j e_j}{e_i}\Big)^{\alpha}\right] dz_i - e_i
\end{aligned}$$

Integrating this expression gives:

$$\begin{aligned}\phi_i(e_i, e_{-i}) &= \frac{V_i^\alpha}{\sum_{j\in I}\left(\frac{V_j e_j}{e_i}\right)^\alpha} \\ &= \frac{(V_i e_i)^\alpha}{\sum_{j\in I}(V_j e_j)^\alpha}\end{aligned}$$

8.3 Tullock with draws

First note that $F_i(t_i|\theta) = F(t_i|\theta) = \exp\left(\frac{-\theta}{t_i}\right)$ and the utility function can be rewritten as follows:

$$u_i\Big(a_i, a_{-i}; \theta; t_i\Big) = \theta \mathbf{1}_{\Big\{a_i > a_j \text{ for all } j \neq i\Big\}} - a_i$$

This leads to the following expression for the complete information game $\phi_i(e_i, e_{-i})$:

$$\phi_i(e_i, e_{-i}) = \int_{\mathbb{R}_+^n} f(\mathbf{z}|1)\frac{1}{z_i}\mathbf{1}_{\Big\{e_i z_i > e_j z_j \text{ for all } j \neq i\Big\}} d\mathbf{z} - e_i$$

Integrating this expression with respect to all $j \neq i$ gives:

$$\begin{aligned}\phi_i(e_i, e_{-i}) &= \int_{\mathbb{R}_+} \frac{f(z_i|1)}{z_i}\left[\int_{\mathbb{R}_+^{n-1}} f(z_{-i}|1)\mathbf{1}_{\Big\{e_i z_i > e_j z_j \text{ for all } j \neq i\Big\}} dz_{-i}\right] dz_i - e_i \\ &= \int_{\mathbb{R}_+} \frac{f(z_i|1)}{z_i}\prod_{j\neq i} F\Big(\frac{e_i z_i}{e_j}\Big| 1\Big) dz_i - e_i\end{aligned}$$

Using the fact that $F\left(\frac{e_i z_i}{e_j}\middle| 1\right) = \exp\left(\frac{-e_j}{e_i z_i}\right)$:

$$
\begin{aligned}
\phi_i(e_i, e_{-i}) &= \int_{\mathbb{R}_+} \frac{\exp\left(\frac{-1}{z_i}\right)}{z_i^3} \prod_{j \neq i} \exp\left(\frac{-e_j}{e_i z_i}\right) dz_i - e_i \\
&= \int_{\mathbb{R}_+} \frac{1}{z_i^3} \exp\left(\sum_{j=1}^{n} \frac{-e_j}{e_i z_i}\right) dz_i - e_i \\
&= \int_{\mathbb{R}_+} \left[\frac{1}{z_i}\right] \left[\frac{1}{z_i^2} \exp\left(\sum_{j=1}^{n} \frac{-e_j}{e_i z_i}\right)\right] dz_i - e_i
\end{aligned}
$$

Integrating by parts gives:

$$
\begin{aligned}
\phi_i(e_i, e_{-i}) &= \left[\frac{1}{z_i} \frac{e_i}{\sum_{j=1}^{n} e_j} \exp\left(\sum_{j=1}^{n} \frac{-e_j}{e_i z_i}\right)\right]_0^\infty \\
&\quad + \int_{\mathbb{R}_+} \frac{1}{z_i^2} \frac{e_i}{\sum_{j=1}^{n} e_j} \exp\left(\sum_{j=1}^{n} \frac{-e_j}{e_i z_i}\right) dz_i - e_i
\end{aligned}
$$

Note that the first term is zero, while the second term can now be integrated directly:

$$
\begin{aligned}
\phi_i(e_i, e_{-i}) &= \left[\frac{e_i^2}{\left(\sum_{j=1}^{n} e_j\right)^2} \exp\left(\sum_{j=1}^{n} \frac{-e_j}{e_i z_i}\right)\right]_0^\infty - e_i \\
&= \frac{e_i^2}{\left(\sum_{j=1}^{n} e_j\right)^2} - e_i
\end{aligned}
$$

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