# Forming coalitions through R&D networks in oligopoly

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**Abstract**

In markets which are dominated by a relatively small amount of firms which can form bilateral (cost reducing) R&D agreements we often observe that firms end up forming R&D coalitions. The main contribution of this paper is to show that when we introduce *farsightedness* through the concept of *indirect dominance* we can support a particular network of *two asymmetric groups of firms* as a von Neumann Morgenstern Farsightedly Stable Set. This particular network consists of a large group of connected firms and a small group of connected firms and, interestingly, coincides with the equilibrium partition in Bloch's endogenous coalition formation game (1995). Introducing farsightedness thus allows us to better explain empirically observed network structures. In addition we show that that neither pairwise stable networks (Goyal and Joshi, 2003), nor efficient networks (Westbrock, 2010) can be a singleton farsightedly stable set. Efficient networks can thus not be sustained, on their own, as a farsighted standard of behavior: forward looking firms cannot fully internalize the negative externalities they impose on each other through network formation.

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## 1 Introduction

In markets which are dominated by a relatively small amount of firms which can form bilateral (cost reducing) R&D agreements we often observe that firms end up forming R&D alliances. With an alliance we mean a subset of firms who

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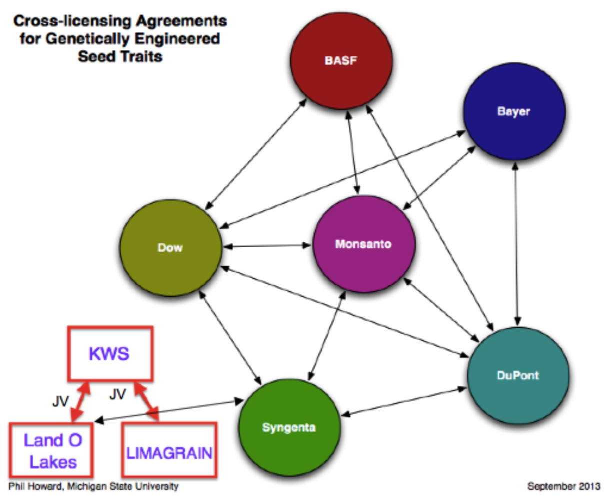
systematically form bilateral links with the other members of this subgroup (insiders) and very few links with firms not belonging to this subset (outsiders). In network terms, we observe that there is a tendency toward the formation of bilateral R&D networks displaying complete components.

Two examples make clear what we have in mind. The first is the successful attempt, as documented in Bekkers et al. (2002) of Motorola in the eighties to create a group of 5 dominant firms in the GSM industry through strategically forming bilateral R&D links with these firms and refusing R&D agreements with outsiders. That is, Motorola (and its competitors) formed its network taking into consideration future moves of competitors (insiders and outsiders) in order to influence the market structure and end up with the insiders dominating the GSM market.

The seeds industry provides another striking example: over the last decade or so, six of the largest nine firms have formed an (almost) complete network of bilateral R&D cross-license agreements while to a great extent leaving three other players out of this network. These outsiders also started to form bilateral R&D agreements, through joint ventures. The following picture, taken and adjusted from Howard (2013), summarizes the situation in 2013:




These two examples make clear that 1) R&D network formation often leads to the existence of strategic asymmetric alliances and 2) these alliances (coalitions) are formed by firms who take into account the impact of the formation of a bilateral link on the future formation of links between all firms.

So far the literature on network formation, mainly based on myopic link formation (Jackson and Wolinsky, 1996), has not been able to predict asymmetric alliance formation in oligopoly models. The conclusions obtained for network formation in cost reducing oligopolies (e.g. Goyal and Joshi (2003), Goyal and Moraga (2001)) are that when linking costs are low, then any two firms who are not linked to one another would like to form that link. As a consequence, any pairwise stable network is such that all participating agents are linked to eachother. When linking costs become important, only the complete network or a group dominant network, in which a large group of agents is completely linked while all other agents, if any, have no links, can arise in equilibrium.

We point out that these conclusions are largely at odds with empirically observed network behavior in many modes of competition. In line with the two examples above, the empirical literature on R&D networks tends to emphasize that observed cooperative networks of competitors often times display the property that they are asymmetric and display clusters of cooperating agents. The above models, on the other hand, predict that observed networks should either be complete or only the set of participating rivals should be completely linked.

Interestingly, models of R&D driven *coalition formation* between rivals do predict asymmetric network structures (e.g. Bloch (1995)). However, these models assume that components are formed in a multilateral way and benefits fully spill over to all agents who belong to any component. Nonetheless, the data show that even when links are formed in a bilateral way, often different clusters of bilaterally linked agents are observed.

The main goal and contribution of this paper is to show that when we introduce *farsightedness* through the concept of *indirect dominance* we can suppport *asymmetric groups of firms* as candidate stable networks in the Cournot oligopoly, even if links are formed bilaterally and not through coalitions. We do so by establishing the existence of a von Neumann Morgenstern Farsightedly Stable Set of networks which consist of a unique (up to a permutation) network which consist of a large group of fully connected agents and a small group of agents. The latter group is either fully linked. Thus, the large cluster manages get a cost advantage over the smaller group but has to accept the presence of a smaller competing cluster which, through forming bilateral links manages to 'stay in the game'. Introducing farsightedness thus allows us to use network formation models in order to obtain predictions generated by coalition formation models (ill fittted in a network environment) and by doing so, allow us to generate more realistic predictions: it allows us to better explain empirically observed network structures.

In addition, assuming farsighted agents and thereby giving up the myopic concept of pairwise stability is a myopic stability concept seems realistic in the study of R&D cooperation in oligopolies: firms have to anticipate the effect of a link on the future formation of the R&D network. The concept of indirect dominance (Harsany 1974 and Chwe 1994) takes this into account: network A indirectly dominates network B if there exists a sequence of networks that implements network A from network B in which at any point of that sequence all agents who 'move' do so because their payoff is higher in the end network (network B) than their current situation. It is easy to see that no network indirectly dominates all other networks and is not indirectly dominated by some network. That is, replacing the dominance concept used in pairwise stability, direct dominance, to indirect dominates yields an empty solution concept: the farsighted core is empty.

This result motivates the use of the concept of stable set (vNM 1944), based on indirect dominance (Chwe 1994). The stable set is the set of networks which is both internally stable - no network of the set indirectly dominates another network of the set - and externally stable - every network outside the set is indirectly dominates by a network belonging to the set. The farsightedly stable set can then be interpreted as a standard of behavior when agents are farsighted. In this paper we will restrict our attention to stable sets which yield a unique prediction: they contain one particular network (up to all permutations).

To the best of our knowledge, we are the first to study farsightedly stable sets in R&D cost reducing oligopolies. In a Cournot setting we first establish the **existence** of the farsighted stable set: a network containing two clusters, one grouping about 80% of the firms, another about 20% of the firms is always a farsightedly stable set. What is more, this particular network is equivalent to the equilibrium partition in Bloch's endogenous coalition formation game (1995). This result is not totally accidental since the concept of Bloch incorporates forwardlooking behavior when firms form a coalition as they take into account which other coalitions will form in equilibrium. What is surprising though is that this network of two asymmetric coalitions and its permutations indirectly dominate *all other networks,* not just all other coalition structures.

What we thus show, is that when firms are farsighted and form links strategically we can obtain a stable network structure which is equivalent to a partition structure of differently sized networks, yielding an empirically relevant prediction.

But how do other networks predictions porposed in the literature fare? Can they be supported as a singleton farsightedly stable set (up to a permutation)? We show that neither pairwise stable networks (Goyal and Joshi, 2003), or efficient networks (Westbrock, 2010) can be a singleton farsightedly stable set. The latter result means in particular that efficient networks can never be sustained on their own as a farsighted standard of behavior: forward looking firms cannot fully internalize the negative externalities they impose on eachother through network formation.

To the best of our knowledge only Roketskiy (2012) has studied the von Neumann Morgenstern Stable set in network formation models between competitors. In a stylized model of a contest in which the payoffs of the agents only depend on the distribution of links and not on a subsequent competition stage he also shows the existence of a singleton farsightedly stable set (up to a permutation) consisting of a network displaying a small and a large component. We show that his results do not immediately translate to oligopoly competition. First, the (socially) efficient network is not the complete network and can never be, on itself, a farsighted stable set. Second, if there are at least five firms, the complete network can never be a farsightedly stable set. Third, we show that only one (asymmetric) alliance network can be a singleton farsightedly stable set and that this network is equivalent to the the sequential equilibrium association structure in the coalition formation game proposed by Bloch (1995).

The paper is organized as follows. In Section 2 we present the network formation stage. In Section 3 we describe the oligopoly model given the R&D collaboration network structure and present our main results. Section 4 concludes.

# 2 Network Formation

## 2.1 Networks

We now describe the two stage game. In the first stage, bilateral collaboration links are formed among agents. In the second stage, the agents engage in Cournot competition (other forms of competition to be developed). We consider a finite set of ex-ante identical agents, $N = \{1, \ldots, n\}$ with $n > 3$.

They are connected through bilateral collaboration links in some network. These network relationships are reciprocal and the network is modeled as a non-directed graph. Agents are the nodes in the graph and links indicate bilateral relationships between agents. A network $g$ is a list of pairs of agents who are linked. Relationships between agents are captured by the binary variables $g_{ij} \in \{0, 1\}$ which denote a relationships between agent $i$ and $j$. In particular, if there exists a link between agents $i$ and $j$ then $g_{ij}$ takes the value of 1, and of 0 otherwise. As a consequence, the set of agents and the relationships between them define a network $g$ while the set of all possible networks is $\mathbb{G}$. Let $N_i(g)$ be the set of agents that have a link with player $i$ given some network $g$ and let $\eta_i(g) = |N_i(g)|$ be the *degree* of agent $i$ : the number of agents linked with $i$. Denote $\bar{\eta}(g) = \sum_{i \in N} \frac{\eta_i(g)}{n}$ as the average degree of network $g$.

We say that there exists a *path* between agents $i$ and $j$ if either $g_{ij} = 1$ or if there exists a sequence of $l$ distinct players $\{k_1, k_2, \ldots, k_l\}$ such that $g_{ik_1} = g_{k_1 k_2} = \ldots = g_{k_l j} = 1$. A network $g$ is *connected* if there exists a path between any two agents $i$ and $j$. Network $\hat{g} \subset g$ is said to be a component of network $g$ if for all $i$, $j$, $i \neq j$ belonging to $\hat{g}$, the there exist a path between $i$ and $j$ and for $i \in \hat{g}$ and $j \in g$, if $g_{ij} = 1$ then $j \in \hat{g}$. A component $\hat{g} \subset g$ is *complete* if $g_{ij} = 1$ for all $i, j \in \hat{g}$.

These are all the basic features which allow us to describe some network structures we will refer to later on. In particular,

- the *complete network* $g^N$ is characterized by $\eta_i(g^N) = n - 1$ for all $i \in N$,
- the *empty network* $g^0$ is characterized by $\eta_i(g^0) = 0$ for all $i \in N$,
- a network $g^\kappa$ is a *dominant group network* of size $\kappa$ when the component $\hat{g}^\kappa = \{i \in N, \eta_i(\hat{g}^\kappa) > 0\} \subsetneq N$ is complete and all $j \notin \hat{g}^\kappa$ have no links: $\eta_j(g^\kappa) = 0$.
- A network $g^{L, N \setminus L}$ is $two-group\ network$ is characterized by a two complete components: one of size $L$ where $L \geq N/2$ and the other of size $N - L$.
- A network $g^x$ is an *interlinked star network* when it induces a degree partition such that the firms with the highest degree (*the center*) are

connected to all firms with positive degree and the firms with the lowest positive degree are connected only to the center.

To simplify notation, $g + ij$ means that the link $g_{ij}$ is added to the network $g$ while, $g - ij$ corresponds to the network $g$ without the link $g_{ij}$.

## 2.2 Network game and (myopic) stability

A network game is a game where every agent $i \in N$ announces its intended link $s_{ij} \in \{0, 1\}$ which all other agents $j \neq i$. If $i$ wants to make a link with $j$, then $s_{ij} = 1$ and $s_{ij} = 0$ otherwise. A strategy in the network game for agent $i$ is given by $s_i = \{s_{ij}\}_{j \neq i}$, which is a $n - 1$ vector which belongs to the set of all possible strategies of agent $i$, i.e. $S_i$. We then have that $g_{ij} = 1$ if $s_{ij} = 1 = s_{ji}$ and $g_{ij} = 0$ otherwise. A strategy profile $s = \{s_i, \ldots, s_n\}$ induces a network $g(s) \in \mathbb{G}$. Once a network is formed, we assume that each agent pays a *negligible* but positive cost $c > 0$ per link formed. Given a strategy profile $s$, the payoff of agent $i$ is given by

$$\Pi_i(s_i, s_{-i}) = \pi_i(g(s)) - c \times \eta_i(g(s))$$

where, $\pi_i(g(s))$ is the agent $i$ expected gain to participate in the contest.[1] Given this framework we look for the pairwise stable networks according to the definition proposed by Jackson and Wolinsky [10], namely

[Pairwise Stability] A network $g$ is pairwise stable (PWS) if the following two conditions hold:

1. if $g_{ij} \in g \Rightarrow \Pi_i(g + ij) > \Pi_i(g)$ and $\Pi_j(g + ij) > \Pi_j(g)$
2. if $g_{ij} \notin g$ and $\Pi_i(g + ij) > \Pi_i(g) \Rightarrow \Pi_j(g + ij) < \Pi_j(g)$

Intuitively, the two conditions state that, starting form a network $g$, no one wants to delete a link and no pair of agents want to form a new link respectively. The notion of pairwise stability is a myopic stability concept: agents do not take into account the fact that the formation or deletion of a link may lead to further deviations.

## 2.3 Von Neumann-Morgenstern farsighted stability

To take the possibility of further deviations into account we use the concept of indirect dominance. The *indirect dominance* relation was first introduced by Harsanyi (1974) but was later formalized by Chwe (1994). It captures the idea that coalitions of agents can anticipate the actions of other coalitions. In other words, the indirect dominance relation captures the fact that farsighted

[1] In what follows, we we slight abuse the notation to simplify the exposition. In particular we let depend agent $i$'s payoff only on network structures and we no longer refer to its dependence on agent $i$'s strategy.

coalitions consider the end network that their deviations may lead to. A network $g'$ *indirectly dominates* $g$ if $g'$ can replace $g$ in a sequence of networks, such that at each network along the sequence all deviators are strictly better off at the end network $g'$ compared to the status-quo they face. Formally, indirect dominance is defined as follows.

A network $g$ is indirectly dominated by $g'$, or $g \ll g'$, if there exists a sequence of networks $g^0, g^1, ..., g^K$ (where $g^0 = g$ and $g^K = g'$) and a sequence of coalitions $S^0, S^1, ..., S^{K-1}$ such that for any $k \in \{1, ..., K\}$,

**(i)** $g^K \succ_i g^{k-1}\ \forall i \in S^{k-1}$, and

**(ii)** coalition $S^{k-1}$ can enforce the network $g^k$ over $g^{k-1}$.

Definition 2.3 gives us the definition of indirect dominance. The indirect dominance relation is denoted by $\ll$. Direct dominance is obtained by setting $K = 1$ in Definition 2.3. Obviously, if $g < g'$, then $g \ll g'$. Another way of introducing farsighted stability is to replace direct dominance by indirect dominance in the definition of the (pairwise stability). Diamantoudi and Xue (2003) have defined the *farsighted core* (or abstract core) as follows:

$$C(\mathbb{G}, \ll) = \{g \in \mathbb{G} \mid \nexists g' \in \mathbb{G} \text{ such that } g' \gg g\}$$

In many instances the farsighted core is empty and this has motivated the use of the vNM farsightedly stable set.

Now we give the definition of a vNM farsightedly stable set due to Chwe (1994).

A set of networks $G \subseteq \mathbb{G}$ is a vNM farsightedly stable set if

**(i)** for all $g \in G$, there does not exist $g' \in G$ such that $g' \gg g$;

**(ii)** for all $g' \notin G$ there exists $g \in G$ such that $g \gg g'$.

Definition 2.3 introduces the notion of a vNM farsightedly stable set $G(\ll)$. Part **(i)** in Definition 2.3 is the internal stability condition: no network inside the set is indirectly dominated by a network belonging to the set. Part **(ii)** is the external stability condition: any network outside the set is indirectly dominated by some network belonging to the set.

In this paper we will try to indentify farsightedly stable sets which are singletons (up to a permutation). The reason is twofold; first, identifying all farsightedly stable sets is a formidable task and second, we choose and prefer to look for a solution concept that provides a solution consisting of a uniquely predicted network (up to a permutation) structure.

## 2.4 About network formation and coalitions

Since we will show thah farsighted network formation can lead to the prediction that coalitions are formed we now define the set of all possible coalition structures, $\mathbb{C}$, simply by restriction the set of networks:

$$\mathbb{C} = \{g \in \mathbb{G} \, | \forall \hat{g} \subset g \text{ and } \forall i, j \in \hat{g} : g_{ij} = 1\} \, .$$

In principle, the set $\mathbb{C}$ can serve as the primitive of an ex ante symmetric coalition formation game with negative externalities. Such games have been studied intentively in the literature: the set $\mathbb{C}$ induces an (ex ante) symmetric association-formation game (Bloch (1995, 1996), Ray and Vohra (1997), Yi (1997)). Even if networks, in the sense described above, are more general than coalitions, we want to stress that the paper does not try to integrate network formation theory (which networks will form?) and coalition formation theory (which coalitions will form) per se. What the paper does do is to show how network formation between farsighted agents can lead to stable network structures that take the form of coalitions: even when agents do not 'decide' multilaterally we can support coalition formation through bilateral link formation.

# 3 R&D Network formation in Oligopoly

## 3.1 R&D cooperation in the linear Cournot model

We assume that $n$ firms compete à la Cournot but have the opportunity, before the competition stage, to cooperate through bilateral R&D collaboration links which jointly reduce their marginal costs. We assume as in Goyal and Joshi (2003) that the constant marginal cost of each firm is a linear function of the amount of links they have:

$$c_i(g) = \lambda - \mu \eta_i(g)$$

We assume as in Bloch (1995) that firms compete à la Cournot in a market for homogenous products with the following linear inverse demand curve:

$$p = \alpha - \sum_{i \in N} q_i \text{ where } \alpha > \lambda.$$

A network $g$ induces a vector of marginal costs $c(g)$.Assuming that all firms produce positive quantities in equilibrium[2] the Cournot equilibrium output can then be written as:

$$q_i^*(g) = \frac{1}{n+1} \left( \alpha - \lambda + n\mu\eta_i(g) - \mu \sum_{j \neq i} \eta_j(g) \right)$$

whereas profit for each firm is written as $\pi_i(g) = (q_i^*(g))^2$.

[2]This is the case when $(\alpha - \lambda) - (n-1)(n-2)\mu > 0$.

### 3.2 Pairwise strong stability and the farsighted core

Goyal and Joshi (2003) have shown that when linking costs are negligible, the only pairwise stable network is the complete network. However, it is easy to verify that the farsighted core is empty when $n > 3$[3].

**Proposition 1** *The farsighted core is empty.*
***Proof.*** *See appendix.* ■

### 3.3 Existence of a von Neumann Morgenstern Farsightedly Stable Set

As we mentioned above, $\mathbb{C}$ can serve as the primitive of an ex ante symmetric coalition formation game with negative externalities. In Cournot competition as described above, Bloch (1995) shows that there exist a unique (up to a permutation) sequential equilibrium association structure that is given by $S^* = \{S_1^*, S_2^*\}$ where $|S_1^*| = k^*$ is the closest integer to $\frac{3n+1}{4}$ and $k^*$ maximizes the payoff of the largest association; or in network terms, the largest complete component of any bigroup network $g^{L,N\backslash L} : k^* \in \arg\max_L \pi_i(g^{L,N\backslash L})$[4].Denote $\pi^*(\mathbb{G})$ or simply $\pi^*$ as the payoff received by the members of a the 'large' component of any $g \in \mathbb{G}^*$. Likewise, denote $\pi_*(\mathbb{G})$ or simply $\pi_*$ as the payoff received by the members of a the 'small' component of any $g \in \mathbb{G}^*$.

We now present our main result: a singleton von Neumann Morgensterns farsightedly stable set always exists. We do so by proving, through a sequence of lemmas, that Bloch's solution to the coalition formation game ($S^* = \{S_1^*, S_2^*\}$ is,), is, in fact, always a singleton (up to a permutation) von Neumann Morgenstern Farsightedly Stable Set (vNMFSS). Denote $\mathbb{G}^* \subset \mathbb{G}$ as the set of all networks with coalition structure $S^* = \{S_1^*, S_2^*\}$ .

The first lemma shows the intuition behind the result of Bloch (1995). Of all bi-group networks, the one with the highest averag payoff for the large group (insiders): with coalition structure $S^* = \{S_1^*, S_2^*\}$ .

**Lemma 2** *Consider any bi-group network $g^{L,N\backslash L}$, then for all $i : \pi_i(g^{L,N\backslash L}) \leq \pi^*$.*

**Proof.** See Appendix. ■

A difficulty we need to resolve is that the $\mathbb{G}^*$ should indirectly dominate *all other networks.* That is, from any network structure, forward looking agents must be wanting to form links in order to belong to a complete component with payoff $\pi^*$ or $\pi_*$. We therefore introduce the following two lemmas:

[3] When linking costs are negligible but positive, the condition for emptiness of the farsighted core becomes $n > 2$ which always holds.

[4] It is uniquely defined whenever $n$ is even or when $n$ is uneven and $n = 5 + p4$ where $p$ is a natural number. If $n = 7 + p4$ then there are 2 solutions to the problem of finding the maximum of $\pi_i(g^{L,N\backslash L})$.

**Lemma 3** *Let $g \in \mathbb{G} \setminus \mathbb{G}^*$ such that $g$ contains a complete component $g^k$ of $k$ agents. Then there exists an agent $i \in g \smallsetminus g^k$ such that $\pi_i(g) < \pi_*$.*

***Proof.*** *See Appendix.* ■

**Lemma 4** *Let $g \in \mathbb{G} \setminus \mathbb{G}^*$ such that $g$ contains a complete component $g^k$ of $k$ agents. Then there exists an agent $i \in g \smallsetminus g^k$ such that $\pi_i(g) < \pi^*$.*

***Proof.*** *See Appendix.* ■

The significance of this lemma is if that agents, by isolating themselves into a complete component, and by doing all the other agents do not form a complete component, then some of these other agents has a payoff lower than $\pi^*$. These players would then be willing to 'cooperate' with the isolating agents towards a network belonging to $\mathbb{G}^*$. This is what we show below:

**Lemma 5** *Let $g \in \mathbb{G} \setminus \mathbb{G}^*$ such that $g$ contains a complete component $g^{k^*}$ of $k^*$ agents. Then there exists a network $g' \in \mathbb{G}^*$ such that $g' \gg g$.*
***Proof.*** *See Appendix.* ■

**Lemma 6** *Let $g \in \mathbb{G} \setminus \mathbb{G}^*$ such that $g$ contains a complete component $g^{n-k^*}$ of $n - k^*$ agents. Then there exists a network $g' \in \mathbb{G}^*$ such that $g' \gg g$.*
***Proof.*** *See Appendix.* ■

We are now ready to show that $\mathbb{G}^*$ is a vNMFSS.

**Theorem 7** *Let $\mathbb{G}^* = \left\{ g^{S,N-S} \in \mathbb{G}, |S| = k^* \in \arg\max_L \pi_i(g^{L,N\setminus L}) \right\}$. Then $\mathbb{G}^*$ is a vNMFSS.*

**Proof. internal stability:** take any $g$ and $g' \in \mathbb{G}^*$ and suppose that $g' \gg g$. Since $g'$ is a permutation of $g$, then along the farsightedly improving path from $g$ to $g'$ there must be some members of the component of size $S$ who initiated the formation of links with members of the component $N - S$. However the payoff of these members of $S$ at the time when they form their links can never be lower than they payoff they will receive at $g'$ since before these links 'between' components, agents of component $N - S$ can not create 'additional' links, thereby lowering the payoffs of agents in the component $S$.

**external stability:** take any $g \notin \mathbb{G}^*$.Let $B(g) = \{i \in N, \pi_i(g) \geq \pi^*\}$ and $W(g) = \{i \in N, \pi_i(g) < \pi^*\}$ . Denote $b(g) = |B(g)|$ and $w(g) = |W(g)|$ .Clearly: $b(g) = n - w(g)$.

Case 1: $w(g) \geq k^*$.

Step 1: $g \to g^1$. If $w(g) \geq k^*$ then let $k^*$ members of $W(g)$ form a complete component (of size $k^*$). The remaining $n-k^*$ agents then either form a complete component and we have shown that then show that there must exist a $g^1 \in \mathbb{G}^*$ such that $g^1 \gg g$. If they do not form a complete complement then we know from lemma 5 that there exists a $g^2 \in \mathbb{G}^*$ such that $g^2 \gg g^1$.

Case 2: $w(g) < k^*$.

Step 1: $g \to g^1$. Then all $w(g)$ members of $W(g)$ cut all their links. The aim of these agents is to end up in the large coalition of some $g' \in \mathbb{G}^*$ and hence to earn $\overline{\pi}^*$. They do so by 'luring' the weakest agents of $B(g)$ into joining them by making themselves stronger:

step 2: $g^1 \to g^2$. Let $\underline{\eta}(g^1) = \min_{i \in B(g)} \eta_i(g^2)$. Network $g^2$ is formed, from $g^1$, by having the agents of $W(g)$ add $m(g^1) = \min\left(w(g) - 1, \underline{\eta}(g^1)\right)$ links amongst, with the possible exception that one member of $W(g)$ has $m(g^1) - 1$ links. It is then the case that for all $i \in W(g), \pi_i(g^2) < \pi^*$ and equally, for all $j \in B(g)$ such that $\eta_j(g^2) = \eta_j(g^1) = \underline{\eta}(g^1)$ we have that $\pi_j(g^2) < \pi^*$. Let $n(g^2) = \left|\left\{j \in B(g) \text{ such that } \eta_j(g^2) = \underline{\eta}(g^1)\right\}\right|$

Step 3: $g^2 \to g^3$. If $w(g) + n(g^2) \geq k^*$ then we are in case 1. If not we repeat step 2 and obtain $n(g^3)$. After a finite amount of repetitions, say $T$, it must be that $w(g) + n(g^T) \geq k^*$ and we end up in case 1. But then there exists, as shown above $g^T \gg g$. ■

## 3.4 Further results

We have just shown that the farsighted stable set concept can single out the solution of Bloch (1995) as a singleton vNMSS. This brings about two related questions: 1) are there other 'singleton' vNMSS and if so, can we characterize them? and 2) if we cannot positively answer 1) are other common stable network (or coalition) predictions singleton vNMS sets? How about efficient network structures, can they be farsightedly stable sets? How about industry profit maximizing network structures? So far, we have not been able to characterize the set of all networks which survive farsighted stability. That being said, we have, so far not found any other network structure constituting a vNMSS. We thus have the following negative results.

### 3.4.1 No other coalition structure is a (singleton) farsightedly stable set

That is, $\mathbb{G}^*$ contains the only 'coalition structures' that arise as stable sets if agents form networks in a farsighted way. There is thus a clear link between farsighted network formation and the predictions obtain by Bloch in symmetric coalition formation games with externalities, but not with predictions obtained by other endogenous coalition formation models. This is result is summarized in the following lemma and proposition.

**Lemma 8** *Take any $g \in \mathbb{C} \setminus \mathbb{G}^*$, then any permutation $g'$ is such that $g' \gg g$.*
***Proof.*** *See Appendix.* ■

***Proposition 9*** *Take any $g \in \mathbb{C} \setminus \mathbb{G}^*$, then $\{g\}$ is not a vNMFSS.*

***Proof.*** *See Appendix.* ■

### 3.4.2 No efficient network structure is a (singleton) farsightedly stable set

Let $\mathbb{E} \subset \mathbb{G}$ be the set of efficient networks in the sense that they maximize total surplus. Westbrock (2010) shows that for any network $g$ to be efficient the it must either be regular, group dominant or an interlinked star. The following proposition show that

**Lemma 10** *Take any $g \in \mathbb{E}$, then any permutation $g'$ of $g$ is such that $g' \gg g$.*
***Proof.*** *See Appendix.* ■

**Proposition 11** *No $g \in \mathbb{E}$ can be a singleton farsightedly stable set .*
***Proof.*** *See Appendix.* ■

# 4 Conclusion

In many markets which are dominated by a relatively small amount of firms and these firms can form bilateral (cost reducing) R&D agreements we often observe that firms end up forming alliances. With an alliance we mean a subset of firms who systematically form bilateral links with the other members of this subgroup (insiders) and very few links with firms not belonging to this subset (outsiders). In network therms, we observe that there is a tendency toward the formation of bilteral R&D networks which all tend to have complete components.

The main goal and contribution of this paper is to show that when we introduce *farsightedness* through the concept of *indirect dominance* we can support *asymmetric groups of firms* as candidate stable networks in the Cournot oligopoly, even if links are formed bilaterally and not through coalitions. We do so by establishing the existence of a von Neumann Morgenstern Farsightedly Stable Set of networks which consist of a unique (up to a permutation) network which consist of a large group of fully connected agents and a small group of agents. The latter group is either fully linked. Thus, the large cluster manages get a cost advantage over the smaller group but has to accept the presence of a smaller competing cluster which, through forming bilateral links manages to 'stay in the game'. Introducing farsightedness thus allows us to reconcile network formation models and coalition formation models and by doing so, allow us to generate more realistic predictions: it allows us to better explain empirically observed network structures.

To the best of our knowledge, we are the first to study farsightedly stable sets in R&D cost reducing oligopolies. In a Cournot setting we first establish the **existence** of the farsighted stable set: a network containing two clusters, one grouping about 80% of the firms, another about 20% of the firms is always a farsightedly stable set. This particular network is equivalent to the equilibrium partition in Bloch's endogenous coalition formation game (1996). This result is not totally accidental since the concept of Bloch incorporates forwardlooking behavior when firms form a coalition as they take into account which other coalitions will form in equilibrium. What is surprising though is that this network of two asymmetric coalitions and its permutations indirectly dominate *all other networks,* not just all other coalition structures. What we thus show, is that when firms are farsighted and form links strategically we can obtain a stable network structure which is equivalent to a partition structure of differently sized networks, yielding an empirically relevant prediction.

But how do other networks predictions proposed in the literature fare? Can they be supported as a singleton farsightedly stable set (up to a permutation)? We show that neither pairwise stable networks (Goyal and Joshi, 2003), or efficient networks (Westbrock, 2010) can be a singleton farsightedly stable set. The latter result means in particular that efficient networks can never be sustained on their own as a farsighted standard of behavior: forward looking firms cannot fully internalize the negative externalities they impose on each other through network formation.

We leave the characterisation of all (singleton) farsightedly stable sets as future research.

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# 5 Appendix

**Proposition 12** *The farsighted core is empty.*
***Proof.*** *Starting from the complete network $g^N$, which is the only candidate that can belong to the farsighted core, let $n-1$ agents isolate one agent $j$ by deleting their link with this agent $j : g^N \to g^j$. The quantity produced by any agent $i \neq j$ in $g^j$ is:*

$$\begin{aligned} q_i(g^j) &= \frac{1}{n+1}\left(\alpha - \lambda + n\mu(n-2) - \mu\left[(n-2)(n-2)\right]\right) \\ &> q_i(g^N) = \frac{1}{n+1}\left(\alpha - \lambda + n\mu(n-1) - \mu\left[(n-1)(n-1)\right]\right) \\ &\Leftrightarrow (n-2)(n-n+2) > (n-1)(n-n+1) \\ &\Leftrightarrow 2(n-2) > (n-1) \\ &\Leftrightarrow n > 3 \end{aligned}$$

■

**Lemma 13** *Consider any bi-group network $g^{L,N\setminus L}$, then for all $i : \pi_i(g^{L,N\setminus L}) \leq \pi^*$.*

Proof:$\pi_i(g^{L,N\setminus L}) \leq \frac{1}{n+1}\left(\alpha - \lambda + n\mu(l-1) - \mu\left[(l-1)^2 + (n-l)(n-l-1)\right]\right) \leq \pi^*$

**Lemma 14** *Let $g \in \mathbb{G} \setminus \mathbb{G}^*$ such that $g$ contains a complete component $g^k$ of $k$ agents. Then there exists an agent $i \in g \setminus g^k$ such that $\pi_i(g) < \pi_*$.*

Proof. Suppose not, then for all $i \notin g^{k^*}$ it must be that $q_i(g) \geq q_*$ where $q_*$ is the quantity produced by an agent $i \notin g^{k^*}$ in a network $g^* \in G^*$. Consider $\underline{q}(g) = \min_{i \notin g^k} q_i(g)$. Take $i$ such that $i$ has the lowest amount of links in $g \smallsetminus g^{k^*}$.Then $\eta_i(g) \leq n-k-2$ (otherwise $g \smallsetminus g^{k^*}$would be complete and hence $g^* \in G^*$).It is then the case that $\sum_{j \neq i}^{g \smallsetminus g^{k^*}} q_j(g) \geq (n-k^*-1)q_*$ where $q_*$ is the quantity produced by an agent $i \notin g^{k^*}$ in a network $g^* \in G^*$. But then:

$$
\begin{aligned}
q_i(g) &= \frac{1}{n+1}\left(\alpha - \lambda + n\mu\eta_i(g) - \mu\left[k^*(k^*-1) + \sum_{j \neq i}^{n-k^*} \eta_j(g)\right]\right) \\
&\geq \frac{1}{n+1}\left(\alpha - \lambda + n\mu\eta_i(g) - \mu\left[k^*(k^*-1) + (n-k^*-1)^2\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{i \in g \smallsetminus g^k}^{n-k^*} q_i(g) - (n-k^*)q_* \\
> \; & 0 \\
\Leftrightarrow \; & \sum_{i \in g \smallsetminus g^k}^{n-k^*} \frac{1}{n+1}\left(\begin{array}{c} \alpha - \lambda + n\mu\eta_i(g) \\ -\mu\left[k^*(k^*-1) + \sum_{j \neq i}^{n-k^*} \eta_j(g)\right] \end{array}\right) \\
> \; & \frac{n-k^*}{n+1}\left(\begin{array}{c} \alpha - \lambda + n(n-k^*-1) \\ -\mu\left[k^*(k^*-1) + (n-k^*-1)^2\right] \end{array}\right) \\
\Leftrightarrow \; & \sum_{i \in g \smallsetminus g^k}^{n-k^*} \left(n\eta_i(g) - \sum_{j \neq i}^{n-k^*} \eta_j(g)\right) \\
> \; & (n-k^*)\left(n(n-k^*-1) - (n-k^*-1)^2\right) \\
\Leftrightarrow \; & \sum_{i \in g \smallsetminus g^k}^{n-k^*} \left(n\eta_i(g) - \sum_{j \neq i}^{n-k^*} \eta_j(g)\right) \\
> \; & (n-k^*)(n-k^*-1)(k^*+1)
\end{aligned}
$$

and we have that

$$
\begin{aligned}
& \sum_{i \in g \smallsetminus g^k}^{n-k^*} \left(n\eta_i(g) - \sum_{j \neq i}^{n-k^*} \eta_j(g)\right) \\
= \; & n \sum_{i \in g \smallsetminus g^k}^{n-k^*} \eta_i(g) - (n-k^*-1) \sum_{i \in g \smallsetminus g^k}^{n-k^*} \eta_i(g) \\
= \; & (k^*+1) \sum_{i \in g \smallsetminus g^k}^{n-k^*} \eta_j(g) \\
< \; & (k^*+1)(n-k^*)(n-k^*-1)
\end{aligned}
$$

Hence we obtain a contradiction.■

**Lemma 15** *Let $g \in \mathbb{G} \setminus \mathbb{G}^*$ such that $g$ contains a complete component $g^k$ of $k$ agents. Then there exists an agent $i \in g \smallsetminus g^k$ such that $\pi_i(g) < \pi^*$.*

Proof. Suppose not, then for all $i \in g \smallsetminus g^k$ it must be that $q_i(g) \geq q^*$.Consider $\underline{q}(g) = \min q_i(g)$. $\sum_{i \in g \smallsetminus g^k} q_i(g) \geq (n-k)\overline{q}^*$.

$$\sum_{i \in g \smallsetminus g^k}^{n-k} q_i(g) = \sum_{i \in g \smallsetminus g^k}^{n-k} \frac{1}{n+1} \left( \begin{array}{c} \alpha - \lambda + n\mu\eta_i(g) \\ -\mu \left[ k(k-1) + \sum_{j \neq i}^{n-k} \eta_j(g) \right] \end{array} \right)$$

Let $q^{n-k}$ be the quantity produced by an agent with $n-k-1$ links in a network $g^{k,n-k}$ or $g^{n-k,k}$. We know that $q^{n-k} \leq q^*$.

$$\begin{array}{rl} & \sum_{i \in g \smallsetminus g^k}^{n-k} q_i(g) - \sum_{i \in g \smallsetminus g^k}^{n-k} q^{n-k} \\ = & \sum_{i \in g \smallsetminus g^k}^{n-k} q_i(g) - (n-k)q^{n-k} \\ = & \sum_{i \in g \smallsetminus g^k}^{n-k} \frac{1}{n+1} \left( n\mu\eta_i(g) - \mu \sum_{j \neq i}^{n-k} \eta_j(g) \right) \\ & -(n-k)\frac{1}{n+1} \left( \begin{array}{c} n\mu(n-k-1) \\ -\mu(n-k-1)(n-k-1 \end{array} \right)) \\ = & \frac{\mu}{n+1} \left[ \begin{array}{c} n \sum_{i \in g \smallsetminus g^k}^{n-k} \eta_i(g) - \sum_{i \in g \smallsetminus g^k}^{n-k} \sum_{j \neq i}^{n-k} \eta_j(g) \\ -(n-k)(n-k-1)(k+1) \end{array} \right] \\ = & \frac{\mu}{n+1} \left[ \begin{array}{c} n \sum_{i \in g \smallsetminus g^k}^{n-k} \eta_i(g) - (n-k-1) \sum_{i \in g \smallsetminus g^k}^{n-k} \eta_i(g) \\ -(n-k)(n-k-1)(k+1) \end{array} \right] \\ = & \frac{\mu}{n+1} \left[ (k+1) \sum_{i \in g \smallsetminus g^k}^{n-k} \eta_i(g) - (k+1)(n-k)(n-k-1) \right] \\ = & \frac{\mu}{n+1} \left[ (k+1) \left( \sum_{i \in g \smallsetminus g^k}^{n-k} \eta_i(g) - (n-k)(n-k-1) \right) \right] \\ < & 0 \text{ whenever the } n-k \text{ agents do not form a complete component.} \end{array}$$

To end the proof we know that $q^{n-k} \leq q^*$.■

**Lemma 16** *Let $g \in \mathbb{G} \setminus \mathbb{G}^*$ such that $g$ contains a complete component $g^{k^*}$ of $k^*$ agents. Then there exists a network $g' \in \mathbb{G}^*$ such that $g' \gg g$.*

*Proof:By lemma 3 there exists an agent $i \in g \smallsetminus g^{k^*}$ such that $\pi_i(g) < \underline{\pi}^*$.Let* *all these agents $i : S_1$ cut all their links in $g_1$: $g \to g_1$. Starting from $g_1$ then let the agents of $S_1$ form new links sequentially respecting that the amount of links between these agents cannot differ more than one, until one member of $g \smallsetminus (g^{k^*} \setminus S_1)$ sees her payoff drop below $\pi_* : g_1 \to g_{1t} \to ... \to g_2$. Suppose this is not the case, then the lowest payoff $\pi_k(g_2)$ of $k \in g \smallsetminus (g^{k^*} \setminus S_1)$ when $S_1$ is a complete component is larger or equal to $\pi_*$ :*

$$\begin{aligned}
\pi_k(g_2) &\geq \pi_* \\
q_k(g_2) &\geq q_* \\
q_k(g_2) &= \frac{1}{n+1}\left(\alpha - \lambda + n\mu\eta_k(g_2) - \mu\left[k^*(k^*-1) + (s_1-1)s_1 + \sum_{l}^{n-k^*-s_1} \eta_l(g_2)\right]\right)
\end{aligned}$$

But also:

$$\begin{aligned}
q_k(g_2) &\leq \frac{1}{n+1}\left(\begin{array}{c} \alpha - \lambda + n\mu\eta_k(g_2) \\ -\mu\left[k^*(k^*-1) + (s_1-1)s_1 + (n-k^*-s_1-1)\eta_k(g)\right] \end{array}\right) \\
&\leq \frac{1}{n+1}\left(\begin{array}{c} \alpha - \lambda + n\mu(n-k^*-s_1-1) \\ -\mu\left[k^*(k^*-1) + (s_1-1)s_1 + (n-k^*-s_1-1)^2\right] \end{array}\right)
\end{aligned}$$

$$\begin{aligned}
\pi_* - q_k(g_2) &\geq \pi_* - \frac{1}{n+1}\left(\begin{array}{c} \alpha - \lambda + n\mu(n-k^*-s_1-1) \\ -\mu\left[k^*(k^*-1) + (s_1-1)s_1 + (n-k^*-s_1-1)^2\right] \end{array}\right) \\
\pi_* - q_k(g_2) \geq 0 &\Leftrightarrow \\
(n-k^*-1)(k^*+1) &\geq (n-k^*-s_1-1)(k^*+s_1+1)) - (s_1-1)s_1 \\
(n-k^*-1)(k^*+1) &\geq (n-k^*-1)(k^*+s_1+1)) - s_1(k^*+s_1+1) - (s_1-1)s_1 \\
s_1(k^*+2) &\geq (n-k^*-1)s_1 \\
2k^*+3 &\geq n \text{ which holds for all } k^*(n)
\end{aligned}$$

At $g_2$ all agents in $g \smallsetminus g^{k^*}$ with lower payoff than $\pi_*$ Starting from $g_2$ then let the agents the set $S_2(\ S_1 \subsetneq S_2)$ delete all their links in $g_2$ and start to form new links sequentially respecting that the amount of links between these agents cannot differ more than one, until one member of $g \smallsetminus (g^{k^*} \setminus S_2)$ sees her payoff drop below $\pi_* : g_2 \to ... \to g_{2t} \to ... \to g_3$.

Iteratively, after a finite amount of steps, say $q$ steps, we will will obtains obtain a network $g_q$where $S_q = g \smallsetminus g^{k^*}$. In the next step all members of $g \smallsetminus g^{k^*}$ form a complete component: $g_q \to g' \in \mathbb{G}^*$. ■

**Lemma 17** *while maintaining payoffs of members of $N \setminus B(g)$ below $\pi^*$ since they will always have less links this member of $B(g)$Then there exists an agent $j \notin S_i$ and $j \in g \smallsetminus g^{k^*}$ such that $\pi_j(g_1) < \underline{\pi}^*$.*

$$
\begin{aligned}
q_i(g) &= \frac{1}{n+1}\left(\alpha-\lambda+n\mu\eta_i(g)-\mu\left[k^*(k^*-1)+\sum_{j\neq i}^{n-k^*}\eta_j(g)\right]\right) \\
&\geq \frac{1}{n+1}\left(\alpha-\lambda+n\mu\eta_i(g)-\mu\left[k^*(k^*-1)+(n-k^*-1)^2\right]\right)
\end{aligned}
$$

**Lemma 18** *Let $g \in \mathbb{G}\setminus\mathbb{G}^*$ such that $g$ contains a complete component $g^{n-k^*}$ of $n-k^*$ agents. Then there exists a network $g' \in \mathbb{G}^*$ such that $g' \gg g$.*

*Proof:By lemma 2 there exists an agent $i \in g^{k^*}$ such that $\pi_i(g) < \pi^*$.Let all*

*these agents $i : S_1$ cut all their links in $g_1$: $g \to g_1$. Starting from $g_1$ then let the agents of $S_1$ form new links sequentially respecting that the amount of links between these agents cannot differ more than one, until one member of $g^{k^*}\setminus S_1$ sees her payoff drop below $\pi_* : g_1 \to g_{1t} \to ... \to g_2$. Suppose this is not the case, then the lowest payoff $\pi_k(g_2)$ of $k \in g^{k^*}\setminus S_1$ when $S_1$ is a complete component is larger or equal to $\pi^*$ :*

$$
\begin{aligned}
\pi_k(g_2) &\geq \pi^* \\
q_k(g_2) &\geq q^* \\
q_k(g_2) &= \frac{1}{n+1}\left(\begin{array}{c}\alpha-\lambda+n\mu\eta_k(g_2) \\ -\mu\left[(n-k^*-1)(n-k^*)+(s_1-1)s_1+\sum_{l}^{k^*-s_1-1}\eta_l(g_2)\right]\end{array}\right)
\end{aligned}
$$

But also:

$$
\begin{aligned}
q_k(g_2) &\leq \frac{1}{n+1}\left(\begin{array}{c}\alpha-\lambda+n\mu\eta_k(g_2)- \\ \mu\left[(n-k^*-1)(n-k^*)+(s_1-1)s_1+(k^*-s_1-1)\eta_k(g_2)\right]\end{array}\right) \\
&\leq \frac{1}{n+1}\left(\begin{array}{c}\alpha-\lambda+n\mu(k^*-s_1-1)- \\ \mu\left[(n-k^*-1)(n-k^*)+(s_1-1)s_1+(k^*-s_1-1)^2\right]\end{array}\right)
\end{aligned}
$$

$$
\begin{array}{rcl}
\pi^* - q_k(g_2) & \geq & \pi^* - \dfrac{1}{n+1}\left(\begin{array}{c} \alpha - \lambda + n\mu(k^* - s_1 - 1) \\ -\mu\left[(n-k^*-1)(n-k^*) + (s_1-1)s_1 + (k^*-s_1-1)^2\right] \end{array}\right) \\
\pi_* - q_k(g_2) & \geq & 0 \\
\Leftrightarrow & & \\
& \Leftrightarrow & (n(k^*-1) - (k^*-1)^2 - (n-k^*-1)(n-k^*) \\
& \geq & n(k^*-s_1-1) - (n-k^*-1)(n-k^*) - (s_1-1)s_1 - (k^*-s_1-1)^2 \\
& \Leftrightarrow & n(k^*-1) - (k^*-1)^2 \\
& \geq & n(k^*-s_1-1) - (s_1-1)s_1 - (k^*-s_1-1)^2 \\
& \Leftrightarrow & (k^*-1)(n-k^*+1) \\
& \geq & (k^*-s_1-1)(n-k^*+s_1+1) - (s_1-1)s_1 \\
& \Leftrightarrow & (k^*-1)(n-k^*+1) \\
& \geq & (k^*-s_1-1)(n-k^*+1) + s_1(k^*-s_1-1) - (s_1-1)s_1 \\
& \Leftrightarrow & s_1(n-k^*+1) \\
& \geq & (n-k^*-1)s_1 \\
& \Leftrightarrow & 2k^*+3 \geq n \text{ which holds for all } k^*(n)
\end{array}
$$

At $g_2$ all agents in $g \smallsetminus g^{k^*}$ with lower payoff than $\pi_*$ Starting from $g_2$ then let the agents the set $S_2( \ S_1 \subsetneqq S_2)$ delete all their links in $g_2$ and start to form new links sequentially respecting that the amount of links between these agents cannot differ more than one, until one member of $g \smallsetminus (g^{k^*} \setminus S_2)$ sees her payoff drop below $\pi_* : g_2 \to ... \to g_{2t} \to ... \to g_3$.

Iteratively, after a finite amount of steps, say $q$ steps, we will will obtains obtain a network $g_q$where $S_q = g \smallsetminus g^{k^*}$. In the next step all members of $g \smallsetminus g^{k^*}$ form a complete component: $g_q \to g' \in \mathbb{G}^*$. ■

**Lemma 19** *Take any $g \in \mathbb{C} \setminus \mathbb{G}^*$, then any permutation $g' \in \varphi(g)$ is such that $g' \gg g$.*
***Proof.*** *To be added.* ■

***Proposition 20*** *Take any $g \in \mathbb{C} \setminus \mathbb{G}^*$, then $\{g\}$ is not a vNMFSS.*

***Proof.*** *To be added.* ■

**Lemma 21** *Take any $g \in \mathbb{E}$, then any permutation $g'$ of $g$ is such that $g' \gg g$.*
***Proof.*** *To be added.* ■

**Proposition 22** *No $g \in \mathbb{E}$ can be a singleton farsightedly stable set .*
***Proof.***

- *If $g$ is regular then internal stability is satisfied but for external stability to be satisfied it must be that there exists a regular network $g$ such that $g \gg g'$ for all $g' \in G^*$. But that must mean that there exists a regular network with higher payoffs for all players than $\pi^*$. Since the payoff in regular networks is increasing in the amount of links then this means that the complete network should yield higher payoffs than $\pi^*$. By lemma 1, this is impossible.*

- *If $g$ is group dominant, then suppose that the dominant group has at least two elements. Then at most 1 permutation can belong to the singleton vNMSS, since all permutation indirectly dominate each other: all the singletons form links after which they lure in members of the dominant group who saw their payoff decrease. However this singleton does not indirectly dominate the network $g + ij$ where $i$ nor $j$ belong to the dominant group. Now consider $g$ such that the dominant group contains $n$ or $n-1$ members, then we know that $g$ cannot indirectly dominant $g' \in \mathbb{G}^*$*

- *If $g$ is an interlinked star then no permutation of $g$ can belong to the stable set since all permutations indirectly dominate each other. On the other hand, consider $g + ij$ where $i$ and $j$ have the lowest amount of links and $\eta_j < n - 1$.Then $g$ does not indirectly dominate $g + ij$.*

■