Cheap Talk with Transfers

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Abstract

This paper tries to extend the Crawford-Sobel's model on cheap talk (1982, *Econometrica* **50**, 1431-1451) assuming that the sender's private information is endogenously learned with a costly effort. The receiver can reward the sender's undertaking through a monetary transfer. This analysis is conducted in a setting without commitment and with limited liability. Two different cases are treated: the overt and the covert effort. It results that in both situations there exists an equilibrium in which information transmission is possible, even without a monetary transfer.

Jel Classification: C72, D82

1 Introduction

There are circumstances in which two parts are involved in a deal, but only one of them has some relevant information that affects the welfare of both. In some cases the only way in which the informed part can reveal what she knows is through a costless message. That is, she can not produce evidences, or the listener is not able to understand the information content. Consider the lawyer - defendant relationship. Usually a lawyer avoids all technical details about laws and procedures and she prefers to summarize her knowledge with some recommendations. Or again, consider a financial broker facing a potential investor. There are a lot of technical details that the financial broker keeps in mind but when she makes the offer, she highlights only elements that can be easily understood by the counterpart: in general a person with a poor financial literacy. The unverifiability of the message can depend by the qualities of the listener of by the considered framework. In the above examples the final outcome of the deal depends only by the uninformed part's action: follow the advice or not, invest or not. Hence, the role played by the message is to induce the listener to choose a particular option instead of another one. From a game theoretic point of view, this kind of framework falls in the so called "cheap talk" games. Any message sent by one of the parts is cheap, in the sense that it does not affect the outcomes directly but it influences the counterpart's behavior. Starting from the seminal work of Crawford and Sobel (1982) the literature on cheap talk has extremely grown. Different development paths have been followed, i.e. letting the use of costless and costly messages (Austen-Smith and Banks, 2000; Kartik, 2007) in order to augment the communication structure; or increasing the dimension of the information and the number of informed parties (Battaglini, 1999). There are many valuable contributions to this literature, i.e. some scholars devoted their efforts to make a comparison between the cheap talk setting and the delegation scheme as in Homlström (1984). Among many others, deserve to be mentioned Ottaviani (2000) and Xie (2013). They make comparisons

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with the cheap talk setting and the delegation schemes, analyzing also the case in which the listener could be naive[1] or the informed part's knowledge is not so accurate. The extreme point of this approach is to consider a consultant that *ex ante* is totally uninformed. In this case she[2] must endogenously learn the information and then, she chooses how much of it to share with the other part. In both the previously cited works, the learning process is costly but there is no monetary transfer between the involved parts. The new contribution of our work is to introduce the possibility of a reward for the expert's effort. This kind of setting is a more likely representation of what happens in reality. There are two involved parties, a principal and an expert. *Ex ante* both are uninformed but only the expert has the qualities which let her to acquire some information. The learning process depends by a costly effort. We can interpret it as the time spent to collect data or the cost of making experiments and researches. The principal can not credibly commit to a plan of transfers, in order to refund the expert for the effort. To justify this assumption consider this situation: both sides signed a contract in the past for a counseling, during their relationship one unexpected task must be done by the expert and it is costly. They cannot renegotiate the payment or this decision does not depend by the will of the principal. Finally, from this task depends the final outcome of the contract. We will analyze two possible cases: in the first one the effort is observable by the principal and in the second one it is not. In both situations we will characterize the equilibria for the expert's and the principal's "take-it-or-leave-it" offer about a possible monetary transfer. It turns out that in either cases there exist an equilibrium in which information transmission is possible even without any transfer. The key role is played by the information surplus that the expert can obtain following the equilibrium strategy, with respect to the profitable deviation. The remainder of the article is organized as follows: in Section 2 we present the model. In Section 3 we analyze the case in which the effort is observable, we establish the existence and the uniqueness of the optimal effort, and we give a full characterization of the equilibria. In Section 4 we repeat the analysis when the effort is not observable. Section 5 concludes. In the Appendix are relegated some computations and some proofs.

2 The Model

A principal – from now on the receiver – has the right to choose a project $a \in [0,1]$ and its outcome depends by the unknown state of the world $\theta \in \Theta = [0,1]$. In order to make his choice, the receiver is advised by an expert – from now on the sender. The sender can make a costly effort in order to infer about the true value of θ. This inference produces a signal $s \in \mathcal{S} = [0,1]$ whose accuracy depends by the effort level. Only the sender can observe the signal's realization. The only way with which the sender can report the signal value is through a costless and unverifiable message $m \in \mathcal{M} = [0,1]$. Both sender and receiver share the same *prior* uniform distribution on $\theta \sim \mathrm{U}([0,1])$. The outcome of the undertaken project a affects one as well the other, $u^{\mathrm{s}}(\cdot)$ is the sender's utility function and $u^{\mathrm{r}}(\cdot)$ is the receiver's one

$$u^{\mathrm{s}}(a,\theta,b) = -(a-(\theta+b))^2 - c(p) + t$$

$$u^{\mathrm{r}}(a,\theta) = -(a-\theta)^2 - t$$

where $b \in \mathbb{R}_{++}$ is bias among their preferences, the function $c(p) : [0,1] \to \mathbb{R}_+$ describes the sender's costly effort and $t \in \mathbb{R}$ is a monetary transfer between them. Given a state $\hat{\theta}$ the sender's preferred project is $a^{\mathrm{s}}(\hat{\theta}) = a(\hat{\theta}, b) = \hat{\theta} + b$, while the receiver's one is $a^{\mathrm{r}}(\hat{\theta}) = a(\hat{\theta}, 0) = \hat{\theta}$. We make the following assumption

Assumption 2.1. *The function $c(p)$ is continuous, strictly increasing and convex, moreover $c(0) = 0$.*

[1]For naive listener we mean an agent that blindly trust the message, without taking in account any kind of strategic behavior from the counterpart.

[2]From now on we will use the pronoun she for the sender and he for the receiver.

The bias b, the functions $u^{\mathrm{s}}(\cdot), u^{\mathrm{r}}(\cdot)$ and the distribution of the signals s are common knowledge. With probability p the sender observes the true state of the world, that is the realization of s coincides with the true value $\theta \in \Theta$. The complementary event corresponds to the observation of a totally noisy and non informative signal, drawn from a uniform distribution on $[0,1]$. Given the assumption (2.1) for every effort level $c(p)$ there is a unique probability p and without any effort, the posterior belief coincides with the prior. Between the true value of θ and the probability p there is no correlation. An higher effort does not imply an higher state of the world, so the receiver can not infer on it, even if he observes the effort level. Unless $p = 0$ an higher realization of the signal s corresponds to an higher posterior estimate of θ. For a given pair $(\hat{\theta}, p) \in \mathcal{S} \times [0,1]$ the expected value of the true state of the world is

$$\mathbb{E}_p[\theta | s = \hat{\theta}] = p \cdot \hat{\theta} + (1-p) \cdot \frac{1}{2}$$

with

$$\mathbb{E}_p[\theta | s] \in \left[(1-p)\frac{1}{2}, (1+p)\frac{1}{2}\right]$$

In this case the sender's preferred action is such that

$$\begin{aligned} a(\hat{\theta}, b, p) &= \arg\max_{a \in [0,1]} \mathbb{E}_p[u^s(a, \theta, b) | s = \hat{\theta}] \\ &= \arg\max_{a \in [0,1]} -\left[p \cdot (a - (\hat{\theta} + b))^2 + (1-p) \cdot \int_0^1 (a - (x+b))^2 dx + t - c(p)\right] \\ &= p \cdot \hat{\theta} + (1-p) \cdot \frac{1}{2} + b \end{aligned}$$

while the receiver's one is

$$\begin{aligned} a(\hat{\theta}, 0, p) &= \arg\max_{a \in [0,1]} \mathbb{E}_p[u^r(a, \theta) | s = \hat{\theta}] \\ &= \arg\max_{a \in [0,1]} -\left[p \cdot (a - \hat{\theta})^2 + (1-p) \cdot \int_0^1 (a - x)^2 dx - t\right] \\ &= p \cdot \hat{\theta} + (1-p) \cdot \frac{1}{2} \end{aligned}$$

Notice that when $p = 1$, $a(\hat{\theta}, b, 1) = a^{\mathrm{s}}(\hat{\theta})$ and $a(\hat{\theta}, 0, 1) = a^{\mathrm{r}}(\hat{\theta})$ as in the standard cheap talk setting, where the sender is perfectly informed about the true state of the world. We assume limited liability for the sender that is, $t \in \mathbb{R}_+$ and the receiver can not credibly commit to a plan of transfers. Moreover, the monetary transfer t occurs before the effort $c(p)$ and at the time of the signal's realization both are sunk costs. Once the sender learned her type that is, after she has observed the signal, both play a standard cheap talk game.

3 Equilibria under Overt Effort

In this part we assume that the sender's effort was observable, then for each $c(p)$ we can identify a proper subgame. What happens before is given, so both tries to maximize their continuation value of the game. Due to the preferences misalignment, the sender will never use a strictly monotonic message strategy $\sigma(s \,|\, p) : \mathcal{S} \to M$ otherwise the receiver, after hearing the message, could apply the inverse function $\sigma^{-1}(m \,|\, p)$ and implement the action $a(\sigma^{-1}(m \,|\, p), 0, p)$ obtaining the best possible outcome for him. So, she will introduce some noise in the message strategy. It is easy to see that for all $(\theta, p) \in \Theta \times [0,1]$ and $b \in \mathbb{R}_{++}$

$$a(\theta, b, p) \neq a(\theta, 0, p)$$

so the following Lemma holds

Lemma 3.1 (Crawford and Sobel (1982)). *If $a(\theta, b, p) \neq a(\theta, 0, p)$ for all $\theta \in \Theta$, then there exists an $\varepsilon > 0$ such that if a' and a'' are actions induced in equilibrium, $|a' - a''| \geq \varepsilon$. Further, the set of actions induced in equilibrium is finite.*

from Crawford and Sobel (1982) we know that the equilibrium will be formed by a finite partition of Θ

$$([\theta_0, \theta_1], [\theta_1, \theta_2], \ldots, [\theta_{i-1}, \theta_i], \ldots, [\theta_{N-2}, \theta_{N-1}], [\theta_{N-1}, \theta_N])$$

with $\theta_0 = 0$ and $\theta_N = 1$. For all $\theta', \theta'' \in \Theta$ with $\theta' < \theta''$ let us define

$$a([\theta', \theta''], 0, p) = \arg\max_{a \in [0,1]} -\left[p \cdot \frac{1}{\theta'' - \theta'} \cdot \int_{\theta'}^{\theta''} (a - x)^2 dx + (1 - p) \cdot \int_0^1 (a - y)^2 dy\right] \tag{1}$$

and for $\theta' = \theta'' = \theta$

$$a([\theta', \theta''], 0, p) = a(\theta, 0, p) = \arg\max_{a \in [0,1]} -\left[p \cdot (a - \theta)^2 + (1 - p) \cdot \int_0^1 (a - y)^2 dy\right] \tag{2}$$

the functions (1) and (2) represent the receiver's best responses, when the sender randomizes uniformly over the interval $[\theta', \theta'']$ or when she reports the signal's realization truthfully. It results

$$a([\theta', \theta''], 0, p) = p \cdot \frac{\theta' + \theta''}{2} + (1 - p) \cdot \frac{1}{2} \tag{3}$$

and for $p \to 1$

$$\lim_{p \to 1} a([\theta', \theta''], 0, p) = \frac{\theta' + \theta''}{2}$$

the best response coincides with Crawford and Sobel (1982), for $p \to 0$

$$\lim_{p \to 0} a([\theta', \theta''], 0, p) = \frac{1}{2}$$

any message sent by the sender is totally uninformative. The receiver takes the best project with respect to his own prior probability distribution. In order to characterize the equilibrium, the so called arbitrage condition (equation (A) in Crawford and Sobel (1982)) must holds

$$u^s(a([\theta_i, \theta_{i+1}], 0, p), \theta_i, b) - u^s(a([\theta_{i-1}, \theta_i], 0, p), \theta_i, b) = 0 \tag{4}$$

for $i = 1, 2, \ldots, N - 1$ where $u^s(a([\theta_i, \theta_{i+1}], 0, p), \theta_i, b)$ is defined as

$$-\left[p \cdot \left(p \cdot \frac{\theta_i + \theta_{i+1}}{2} + (1 - p) \cdot \frac{1}{2} - (\theta_i + b)\right)^2 + (1 - p) \cdot \int_0^1 \left(p \cdot \frac{\theta_i + \theta_{i+1}}{2} + (1 - p) \cdot \frac{1}{2} - (y + b)\right)^2 dy\right]$$

Condition (4) makes the sender of type θ_i indifferent between any messages in $[\theta_{i-1}, \theta_i]$ and $[\theta_i, \theta_{i+1}]$. As in Ottaviani (2000) the solution of (4) is represented by this second order difference equation

$$\theta_{i+1} \cdot (2p \cdot \theta_i - p \cdot \theta_{i+1} + 4b) = \theta_{i-1} \cdot (2p \cdot \theta_i - p \cdot \theta_{i-1} + 4b)$$

solved by

$$\theta_{i+1} - \theta_i = \theta_i - \theta_{i-1} + \Delta \quad , \quad \Delta = 4 \cdot \frac{b}{p} \tag{5}$$

The intervals $[\theta_{i-1}, \theta_i]$ are of increasing length, with Δ as the increase of the step size. Except for the presence of the parameter p, the equilibrium equation has the same structure as in the case of a perfectly informed sender. Whenever $p > 0$ the monotonicity condition (M) defined in Crawford and Sobel (1982) holds. This result will be useful when we will discuss the uniqueness of the partition. Set $\theta_0 = 0$, then

$$\theta_i = i\theta_1 + 2i(i - 1)\frac{b}{p}$$

for $i = 0, 1, 2, \ldots, N$ and

$$N = N(b,p) = \left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 2\frac{p}{b}} \right\rceil \tag{6}$$

where $\lceil x \rceil$ is the smallest integer greater than x. For $p \to 1$ this model coincides with the standard cheap talk and for $p \to 0$ the value of $N(b,p)$ becomes 1, which corresponds to the babbling equilibrium. Since $\theta_N = 1$ we can compute

$$1 = N\theta_1 + 2N(N-1)\frac{b}{p}$$

that is

$$\theta_1 = \frac{1}{N}\left[1 - 2N(N-1)\frac{b}{p}\right]$$

then

$$\theta_i = \frac{i}{N} - 2i(N-i)\frac{b}{p} \tag{7}$$

for $i = 0, 1, 2, \ldots, N$ and

$$\theta_i - \theta_{i-1} = \frac{1}{N} + 2(2i - 1 - N)\frac{b}{p} \tag{8}$$

for $i = 1, \ldots, N$. When $p \leq 4b$ the only possible equilibrium is the uninformative one. The sender's expected utility in the cheap talk subgame, when she reports $\theta \in [\theta_{i-1}, \theta_i]$ is

$$\mathrm{EU}^s([\theta_{i-1}, \theta_i], b, p) = -\left[p\left(\frac{\theta_{i-1}+\theta_i}{2} - (\theta + b)\right)^2 + (1-p)\int_0^1 \left(\frac{\theta_{i-1}+\theta_i}{2} - (y+b)\right)^2 dy\right] =$$

$$-\left[p\left(\frac{\theta_{i-1}+\theta_i}{2} - (\theta + b)\right)^2 + (1-p)\left(\frac{1}{3} - \frac{\theta_{i-1}+\theta_i}{2} + b + \left(\frac{\theta_{i-1}+\theta_i}{2} - b\right)^2\right)\right] \tag{9}$$

and the *ex ante* expected utility[3] is

$$\mathrm{V}^{\mathrm{s}}(b,p) = \sum_{i=1}^{N} \mathrm{EU}^s([\theta_{i-1}, \theta_i], b, p) = -(1-p)\frac{1}{6} - (2p-1)\left(\frac{1}{12N^2} + \frac{b^2(N^2-1)}{3p^2}\right) - b^2$$

while the receiver's one is

$$\mathrm{V}^{\mathrm{r}}(b,p) = -(1-p)\frac{1}{6} - (2p-1)\left(\frac{1}{12N^2} + \frac{b^2(N^2-1)}{3p^2}\right)$$

It is easy to see that when $p \to 0$

$$\lim_{p \to 0} \mathrm{V}^{\mathrm{s}}(b,p) = -\left(\frac{1}{12} + b^2\right) = \mathrm{V}^{\mathrm{s}}(b,0)$$

$$\lim_{p \to 0} \mathrm{V}^{\mathrm{r}}(b,p) = -\frac{1}{12} = \mathrm{V}^{\mathrm{r}}(b,0)$$

the utilities correspond with the babbling outcome and when $p \to 1$

$$\lim_{p \to 1} \mathrm{V}^{\mathrm{s}}(b,p) = -\left(\frac{1}{12N^2} + \frac{b^2(N^2-1)}{3}\right) - b^2 = \mathrm{V}^{\mathrm{s}}(b,1)$$

$$\lim_{p \to 1} \mathrm{V}^{\mathrm{r}}(b,p) = -\left(\frac{1}{12N^2} + \frac{b^2(N^2-1)}{3}\right) = \mathrm{V}^{\mathrm{r}}(b,1)$$

they are as in Crawford and Sobel (1982).

[3] See the Appendix for the detailed computation.

Existence and Uniqueness of the Optimal Effort

In order to understand how the previous part of the game is played, we must provide the existence and the uniqueness of an optimal effort for the sender. The first issue regards the continuity of the *ex ante* utilities. Notice that $\mathrm{V}^{\mathrm{s}}(b,p)$ and $\mathrm{V}^{\mathrm{r}}(b,p)$ differ only by the constant b^2. Now let $\mathcal{P} \equiv (p_0, p_1, \ldots, p_N)$ be a partition of the $[0,1]$ interval, with $p_0 = 0$ and

$$-\frac{1}{2} + \frac{1}{2}\sqrt{1 + 2\frac{p_i}{b}} = i$$

it is easy to see that $N(b, p_0) = 1$ and $N(b, p_i)$ jumps from i to $i+1$ at $p = p_i$. The following Lemma guarantee the continuity of the two functions

Lemma 3.2. *$V^s(b,p)$ and $V^r(b,p)$ are continuous in p.*

From (6) the number of induced actions in equilibrium depends by the parameters (b,p). More precisely in order to have a N sized maximal equilibrium, these conditions must hold

$$\left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 2\frac{p}{b}} \right\rceil > N - 1 \qquad \text{and} \qquad \left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 2\frac{p}{b}} \right\rceil < N$$

that is

$$\frac{1}{2N(N+1)} < \frac{b}{p} < \frac{1}{2N(N-1)} \tag{10}$$

for $N \geq 2$. Now let us consider the first derivative of $\mathrm{V}^{\mathrm{s}}(b,p)$ with respect to $p \in (p_{i-1}, p_i)$ that is, for the values in which N is constant. When $N \geq 2$

$$\frac{d\,\mathrm{V}^{\mathrm{s}}(b,p)}{dp} = \frac{1}{6}\left(\frac{N^2-1}{N^2}\right) - 2 \cdot \frac{b^2(N^2-1)}{3p^2}\left(\frac{1-p}{p}\right)$$

in a N sized maximal equilibrium

$$\frac{d\,\mathrm{V}^{\mathrm{s}}(b,p)}{dp} > \frac{1}{6}\left(\frac{N^2-1}{N^2}\right) - \frac{2}{3} \cdot \left(\frac{1}{2N(N-1)}\right)^2 \cdot (N^2-1) \cdot \left(\frac{1-p}{p}\right)$$

the right hand side of the above inequality is positive if and only if

$$\frac{1-p}{p} < (N-1)^2$$

Since we are looking for the most informative equilibrium – when $N \geq 2$ – the above inequality is true for

$$p > \frac{1}{(N-1)^2 + 1}$$

Now let us consider the second derivative

$$\frac{d^2\,\mathrm{V}^{\mathrm{s}}(b,p)}{dp^2} = 2 \cdot \frac{b^2(N^2-1)}{3p^2} \cdot \left(\frac{3-2p}{p^2}\right) > 0 \qquad \text{for} \quad N \geq 2$$

so we can state the following Lemma

Lemma 3.3. *Let $p \in (p_{i-1}, p_i)$ with $i \geq 2$, the functions $V^s(b,p)$ and $V^r(b,p)$ are strictly convex in p. Moreover $V^s(b,p)$ and $V^r(b,p)$ are strictly increasing for $p \in (p_{i-1}, p_i)$ if and only if*

$$p > \frac{1}{(N-1)^2 + 1}$$

where N is the maximal number of induced actions in equilibrium.

Let us consider the following maximization problem

$$\max_{p\in[0,1]} \mathrm{V^s}(b,p) - c(p) + t \tag{P1}$$

it represents the optimal choice of the effort level $c(p)$ for the sender, before she undertakes it. From Lemma 3.2 we know that $\mathrm{V^s}(b,p)$ is continuous in p and since we assume that also $c(p)$ is continuous in p – even in the points in which the number N of intervals makes a jump – we can state the following result

Lemma 3.4. *The maximization problem (P1) admits at least one solution.*

the proof is a direct application of the Weierstrass Extreme-Value Theorem. To get rid of the multiple solutions of the maximization problem (P1) we can: 1) Assume that *ex ante* the sender will choose the effort corresponding to this probability

$$\hat{p} = \max\left\{p' \in [0,1] : p' \in \arg\max_{p\in[0,1]} \mathrm{V^s}(b,p) - c(p) + t\right\} \tag{11}$$

between the probabilities that give her the same *ex ante* maximal expected utility, for which she is indifferent, she chooses the highest one. Or 2) we can assume that the function $c(p)$ is such that

$$\frac{d^2c(p)}{dp^2} > \frac{\partial^2\,\mathrm{V^s}(b,p)}{\partial p^2}$$

for all $p \in [0,1]$.

Uniqueness of the Partition Equilibrium

From Lemma 3 in Crawford and Sobel (1982) we know that for any given values of $(b,p) \in \mathbb{R}_{++} \times [0,1]$, if $1 \le k \le N(b,p)$ there is exactly one partition equilibrium of size k. The main issue is that *ex ante* there are multiple possibile equilibria. More precisely one for each k between 1 and $N(b,p)$. To rule out this circumstance we need a refinement. It is well known that all refinements in the signalling games fail to reduce the number of equilibria in cheap talk games. For this reason we will use the "no incentive to separate" (NITS) condition stated in Chen and Kartik and Sobel (2008). Differently by the "neologism proof" refinement, proposed by Farrell (1993), it adapts perfectly to the uniform quadratic model. An equilibrium satisfies the NITS if the sender of the lowest type prefers the project induced in equilibrium, rather than the best one that the receiver would take if he knew the sender's true type.

Definition 3.5 (Chen and Kartik and Sobel (2008))**.** *An equilibrium satisfies the no incentive to separate (NITS) condition if*

$$u^s(a([0,\theta_1],0,p),0,b) \ge u^s(a(0,p),0,b)$$

It means that this condition must hold

$$-\left[(1-p)\cdot\int_0^1\left(p\frac{\theta_1}{2}+(1-p)\frac{1}{2}-x-b\right)^2 dx + p\cdot\left(\frac{\theta_1}{2}+(1-p)\frac{1}{2}-b\right)^2\right] \ge$$
$$-\left[(1-p)\cdot\int_0^1\left((1-p)\frac{1}{2}-x-b\right)^2 dx + p\cdot\left((1-p)\frac{1}{2}-b\right)^2\right]$$

for all $(\theta,p) \in [0,1]\times(0,1]$, and it is true if and only if

$$p\cdot\theta_1 \le 4b \tag{12}$$

since we know that

$$\theta_1 = \frac{1}{N}\left[1 - 2N(N-1)\frac{b}{p}\right] < 1$$

inequality (12) holds if and only if

$$\frac{p}{b} \leq 2N(N+1) \qquad \Longleftrightarrow \qquad \frac{b}{p} \geq \frac{1}{2N(N+1)}$$

but from (10) we know that it is always strictly satisfied. As we observed previously, condition (M) from Crawford and Sobel (1982) holds for equation (5) so we can use this result

Proposition 3.6 (Chen and Kartik and Sobel (2008))**.** *If condition (M) is satisfied, only the unique equilibrium with the maximum number of induced actions satisfies NITS.*

to establish that the only equilibrium played in the subgame is the more informative, the one in which the number of induced action is maximal. Now since for all $p \in [0,1]$, $\mathrm{V}^{\mathrm{s}}(b,p)$ and $\mathrm{V}^{\mathrm{r}}(b,p)$ differs up to a constant, it results that

$$\mathrm{V}^{\mathrm{s}}(b,p) - \mathrm{V}^{\mathrm{s}}(b,0) = \mathrm{V}^{\mathrm{r}}(b,p) - \mathrm{V}^{\mathrm{r}}(b,0) = \delta(b,p) \geq 0$$

where $\mathrm{V}^{\mathrm{s}}(b,0)$ and $\mathrm{V}^{\mathrm{r}}(b,0)$ are respectively the outcomes of the babbling equilibrium for sender and receiver. Let $c(\hat{p})$ be the optimal effort level under the most informative equilibrium, as defined in (11). Let

$$\delta(b,\hat{p}) = \frac{1}{12} - (1-\hat{p})\frac{1}{6} - (2\hat{p}-1)\left(\frac{1}{12N^2} - \frac{b^2(N^2-1)}{3\hat{p}^2}\right) \geq 0$$

be the information surplus obtained when the sender acts optimally, that is when she maximizes her *ex ante* expected utility taking in account that in the subgame she will induces the maximum partition equilibria.

Sender's Take-it-or-leave-it - Γ_{S}

Now we will analyze the game in which the sender makes a "take-it-or-leave-it" offer to the receiver.

1. The sender asks for a transfer $t \geq 0$.

2. The receiver can accept it and pay $t \geq 0$ or he can reject the requested transfer.

3. The sender undertakes the observable effort $c(p)$.

4. The sender observes the signal $s \in [0,1]$ and then she reports a message $m \in [0,1]$.

5. The receiver chooses the project $a \in [0,1]$ and then the outcomes are realized.

What it follows is a brief description of the game and of its information sets. As usual the function $\varphi : \mathcal{H} \to \{s,r\}$ identifies for each information set $h \in \mathcal{H}$ the active player, while the function $A : \mathcal{H} \to A_{\varphi(h)}(h)$ specifies the feasible actions, where $A_{\varphi(h)}(h)$ is the action set of the active player in h. The first who moves is the sender asking a transfer ($\varphi(h_1) = s, A_s(h_1) = t \in \mathbb{R}_+$). The receiver can accept and pay it or he can reject ($\varphi(h_2) = r, A_r(h_2) = \{p, np\}$). After that, the sender must choose to undertake the optimal effort or not. We can identify two information sets for the sender: h_3 after the history (t,p) and h_4 after the history (t,np), then ($\varphi(h_3) = s, A_s(h_3) = \{u, nu\}$) and ($\varphi(h_4) = s, A_s(h_4) = \{u, nu\}$). Given the actions taken in h_3 and h_4 the sender can be informed or uninformed. These situations are represented by the information sets h_5, h_6, h_7, h_8 reached after the histories

$$\begin{array}{lcl} (t,p,u) & \to & h_5 \\ (t,p,nu) & \to & h_6 \\ (t,np,u) & \to & h_7 \\ (t,np,nu) & \to & h_8 \end{array}$$

Since the receiver can observe the effort he can distinguish all of them. At this stage of the game the sender must report the signal's realization through a costless message m, so

$$\begin{array}{ccccccccc} \varphi(h_5) & = & \varphi(h_6) & = & \varphi(h_7) & = & \varphi(h_8) & = & s \\ A_s(h_5) & = & A_s(h_6) & = & A_s(h_7) & = & A_s(h_8) & = & m \in [0,1] \end{array}$$

an then for the receiver we can identify four information sets $h_9, h_{10}, h_{11}, h_{12}$ reached after these histories

$$\begin{array}{ccc} (t,p,u,m) & \rightarrow & h_9 \\ (t,p,nu,m) & \rightarrow & h_{10} \\ (t,np,u,m) & \rightarrow & h_{11} \\ (t,np,nu,m) & \rightarrow & h_{12} \end{array}$$

in which

$$\begin{array}{ccccccccc} \varphi(h_9) & = & \varphi(h_{10}) & = & \varphi(h_{11}) & = & \varphi(h_{12}) & = & r \\ A_r(h_9) & = & A_r(h_{10}) & = & A_r(h_{11}) & = & A_r(h_{12}) & = & a \in [0,1] \end{array}$$

Given this description of the game, an equilibrium would be a pair of strategies (σ^s, σ^r) with

$$\sigma^s(h) = (\sigma^s(h_1), \sigma^s(h_3), \sigma^s(h_4), \sigma^s(h_5), \sigma^s(h_6), \sigma^s(h_7), \sigma^s(h_8))$$

$$\sigma^r(h) = (\sigma^r(h_2), \sigma^r(h_9), \sigma^r(h_{10}), \sigma^r(h_{11}), \sigma^r(h_{12}))$$

and for the receiver we can define four posterior beliefs distributions $\mu_9^r, \mu_{10}^r, \mu_{11}^r, \mu_{12}^r$ over the messages received in the respective information sets $h_9, h_{10}, h_{11}, h_{12}$. For example, μ_9^r corresponds to the receiver's posterior beliefs about the quantity of information contained in a message received in h_9, in order to induce a maximal partition equilibrium in the following cheap talk subgame. Before describing the equilibria, we must discuss the reciprocals incentives to pay the transfer and make the effort. From the previous section, we know that $\delta(b,\hat{p})$ is the information surplus obtained when the sender acts optimally. Now, let us assume that $c(\hat{p}) > \delta(b,\hat{p})$, the sender will undertake the effort if and only if $\mathrm{V^s}(b,\hat{p}) - c(\hat{p}) + t \geq \mathrm{V^s}(b,0)$. This implies a transfer $t \geq c(\hat{p}) - \delta(b,\hat{p})$, but after the payment was made, the sender has a profitable deviation from undertaking the costly effort. In fact, if she undertake the effort her payoff would be $\mathrm{V^s}(b,\hat{p}) - c(\hat{p}) + t$ and if she deviates she would obtain $\mathrm{V^s}(b,0) + t$. Now notice that

$$\mathrm{V^s}(b,\hat{p}) - c(\hat{p}) + t < \mathrm{V^s}(b,0) + t$$

to see this

$$\mathrm{V^s}(b,\hat{p}) - c(\hat{p}) < \mathrm{V^s}(b,\hat{p}) - \delta(b,\hat{p}) = \mathrm{V^s}(b,0)$$

the receiver anticipates this deviation, so for him it is optimal to reject any requested transfer. In this case the only possible equilibrium is the babbling one.

Proposition 3.7. *For $c(\hat{p}) > \delta(b,\hat{p})$ this pair of strategies*

$$\sigma^s(h) = (t, nu, nu, m^b, m^b, m^b, m^b)$$

$$\sigma^r(h) = \left(np, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

with this system of beliefs

$$\mu_9^r = \mu_{10}^r = \mu_{11}^r = \mu_{12}^r = 0$$

is the only equilibrium for the extensive form game Γ_S, where $t \in \mathbb{R}_+$ and $m^b \in \Delta(\mathcal{M})$.

Now let us assume $c(\hat{p}) < \delta(b,\hat{p})$ as in the previous case the babbling equilibrium is still one of the possible equilibria of the game. There are also two other equilibria in which the information transmission is attainable. In the first one, the sender undertakes the effort even without receiving any transfer

Proposition 3.8. *For $c(\hat{p}) < \delta(b,\hat{p})$ this pair of strategies*

$$\sigma^s(h) = (t, u, u, m^a, m^b, m^a, m^b)$$

$$\sigma^r(h) = \left(np, a^m, \frac{1}{2}, a^m, \frac{1}{2}\right)$$

with this system of beliefs

$$\mu_9^r = \mu_{11}^r = 1 \quad , \quad \mu_{10}^r = \mu_{12}^r = 0$$

is an equilibrium for the extensive form game Γ_S, where $t \in \mathbb{R}_+$, $m^a \in \Delta([\theta_{i-1}, \theta_i])$ for $s \in [\theta_{i-1}, \theta_i]$, where θ_i's are determined by (7), $m^b \in \Delta(\mathcal{M})$ and $a^m = a([\theta_{i-1}, \theta_i], 0, \hat{p})$ as in (1).

in the second one, the sender undertakes the effort and she receives a transfer

Proposition 3.9. *For $c(\hat{p}) < \delta(b,\hat{p})$ this pair of strategies*

$$\sigma^s(h) = (t, u, nu, m^a, m^b, m^b, m^b)$$

$$\sigma^r(h) = \left(p, a^m, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

with this system of beliefs

$$\mu_9^r = 1 \quad , \quad \mu_{10}^r = \mu_{11}^r = \mu_{12}^r = 0$$

is an equilibrium for the extensive form game Γ_S, with $t = c(\hat{p}) - \delta(b,\hat{p})$, $m^a \in \Delta([\theta_{i-1}, \theta_i])$ for $s \in [\theta_{i-1}, \theta_i]$, where θ_i's are determined by (7), $m^b \in \Delta(\mathcal{M})$ and $a^m = a([\theta_{i-1}, \theta_i], 0, \hat{p})$ as in (1).

Receiver's Take-it-or-leave-it - $\Gamma_{\mathbf{R}}$

Now we will analyze the game in which the receiver makes a "take-it-or-leave-it" offer to the sender.

1. The receiver offers a transfer $t \geq 0$.
2. The sender can accept it and get $t \geq 0$ or she can reject the proposed transfer.
3. The sender undertakes the observable effort $c(p)$.
4. The sender observes the signal $s \in [0,1]$ and then she reports a message $m \in [0,1]$.
5. The receiver chooses the project $a \in [0,1]$ and then the outcomes are realized.

The first who moves is the receiver offering a transfer ($\varphi(h_1) = r, A_r(h_1) = t \in \mathbb{R}_+$). The sender can accept it or she can reject it ($\varphi(h_2) = s, A_s(h_2) = \{a, r\}$). After that, the sender must choose to undertake the optimal effort or not. We can identify two information sets for the sender: h_3 after the history (t, a) and h_4 after the history (t, r), then ($\varphi(h_3) = s, A_s(h_3) = \{u, nu\}$) and ($\varphi(h_4) = s, A_s(h_4) = \{u, nu\}$). Given the actions taken in h_3 and h_4 the sender can be informed or uninformed. These situations are represented by the information sets h_5, h_6, h_7, h_8 reached after the histories

$$\begin{aligned} (t,a,u) &\rightarrow h_5 \\ (t,a,nu) &\rightarrow h_6 \\ (t,r,u) &\rightarrow h_7 \\ (t,r,nu) &\rightarrow h_8 \end{aligned}$$

Since the receiver can observe the effort he can distinguish all of them. At this stage of the game the sender must report the signal's realization through a costless message m, so

$$\begin{array}{ccccccccc} \varphi(h_5) & = & \varphi(h_6) & = & \varphi(h_7) & = & \varphi(h_8) & = & s \\ A_s(h_5) & = & A_s(h_6) & = & A_s(h_7) & = & A_s(h_8) & = & m \in [0,1] \end{array}$$

an then for the receiver we can identify four information sets $h_9, h_{10}, h_{11}, h_{12}$ reached after these histories

$$\begin{array}{ccc} (t,a,u,m) & \rightarrow & h_9 \\ (t,a,nu,m) & \rightarrow & h_{10} \\ (t,r,u,m) & \rightarrow & h_{11} \\ (t,r,nu,m) & \rightarrow & h_{12} \end{array}$$

in which

$$\begin{array}{ccccccccc} \varphi(h_9) & = & \varphi(h_{10}) & = & \varphi(h_{11}) & = & \varphi(h_{12}) & = & r \\ A_r(h_9) & = & A_r(h_{10}) & = & A_r(h_{11}) & = & A_r(h_{12}) & = & a \in [0,1] \end{array}$$

Given this description of the game, an equilibrium would be a pair of strategies (σ^s, σ^r) with

$$\sigma^s(h) = (\sigma^s(h_2), \sigma^s(h_3), \sigma^s(h_4), \sigma^s(h_5), \sigma^s(h_6), \sigma^s(h_7), \sigma^s(h_8))$$

$$\sigma^r(h) = (\sigma^r(h_1), \sigma^r(h_9), \sigma^r(h_{10}), \sigma^r(h_{11}), \sigma^r(h_{12}))$$

and for the receiver four posterior beliefs distributions $\mu_9^r, \mu_{10}^r, \mu_{11}^r, \mu_{12}^r$ over the messages received in the respective information sets $h_9, h_{10}, h_{11}, h_{12}$. The same reasoning conducted about the relation between $c(\hat{p})$ and $\delta(b,\hat{p})$ in the sender's "take-it-or-leave-it" game still apply in this one. When $c(\hat{p}) > \delta(b,\hat{p})$ the sender has an incentive to deviate, from the signal's acquisition, so the receiver will not offer any transfer. Hence the sender does not undertake any effort. The resulting equilibrium is the babbling one

Proposition 3.10. *For $c(\hat{p}) > \delta(b,\hat{p})$ this pair of strategies*

$$\sigma^s(h) = (\Delta(\{a,r\}), nu, nu, m^b, m^b, m^b, m^b)$$

$$\sigma^r(h) = \left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

with this system of beliefs

$$\mu_9^r = \mu_{10}^r = \mu_{11}^r = \mu_{12}^r = 0$$

is the only equilibrium for the extensive form game Γ_R, where $m^b \in \Delta(\mathcal{M})$.

When $c(\hat{p}) < \delta(b,\hat{p})$ as in the previous case the babbling equilibrium is one of the possible equilibria of the game. There are also two other equilibria in which the information transmission is attainable. In the first one, the sender undertakes the effort even without receiving any transfer

Proposition 3.11. *For $c(\hat{p}) < \delta(b,\hat{p})$ this pair of strategies*

$$\sigma^s(h) = (\Delta(\{a,r\}), u, u, m^a, m^b, m^a, m^b)$$

$$\sigma^r(h) = \left(0, a^m, \frac{1}{2}, a^m, \frac{1}{2}\right)$$

with this system of beliefs

$$\mu_9^r = \mu_{11}^r = 1 \quad , \quad \mu_{10}^r = \mu_{12}^r = 0$$

is an equilibrium for the extensive form game Γ_R, where $m^a \in \Delta([\theta_{i-1}, \theta_i])$ for $s \in [\theta_{i-1}, \theta_i]$, where θ_i's values are determined by (7), $m^b \in \Delta(\mathcal{M})$ and $a^m = a([\theta_{i-1}, \theta_i], 0, \hat{p})$ as in (1).

in the second one, the sender undertakes the effort and she receives a transfer

Proposition 3.12. *For $c(\hat{p}) < \delta(b, \hat{p})$ this pair of strategies*

$$\sigma^s(h) = (a, u, nu, m^a, m^b, m^b, m^b)$$

$$\sigma^r(h) = \left(t, a^m, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

with this system of beliefs

$$\mu_9^r = 1 \quad , \quad \mu_{11}^r = \mu_{10}^r = \mu_{12}^r = 0$$

is an equilibrium for the extensive form game Γ_R, with $t = c(\hat{p}) - \delta(b, \hat{p})$, $m^a \in \Delta([\theta_{i-1}, \theta_i])$ for $s \in [\theta_{i-1}, \theta_i]$, where θ_i's values are determined by (7), $m^b \in \Delta(\mathcal{M})$ and $a^m = a([\theta_{i-1}, \theta_i], 0, \hat{p})$ as in (1).

4 Equilibria under Covert Effort

In this section we assume that sender's effort is not observable. All the other elements of the game remain the same. At the time in which the sender learns her type, transfer and costly effort are sunk costs. In this setting the receiver's information sets shrink. Without the observation of the effort, the receiver must form a consistent system of beliefs about the meaningfulness of the messages $m \in [0, 1]$. We will see that they strictly depend by the optimal effort and by the incentive to deviate from the equilibrium strategy. As usual, when dealing with cheap talk games, the babbling equilibrium still remains.

Sender's incentive to deviate

Now suppose the receiver expects the sender to undertake the effort and to communicate her information inducing N distinct projects. Where $N = N(b, p)$ is the maximum number of actions induced in equilibrium. For an uninformed sender the best project is

$$a^s = \frac{1}{2} + b$$

but for a given message $m \in [\theta_{i-1}, \theta_i]$ the receiver will implement the project

$$a_i^N = a^N(\theta_{i-1}, \theta_i) = \frac{\theta_{i-1} + \theta_i}{2} = \frac{i-1}{N} + \frac{1}{2N} - 2i(N+1-i)\frac{b}{p} + (N+1)\frac{b}{p} \tag{13}$$

with $i = 1, 2, \ldots, N$. So the sender will choose a message m such that the induced action $a^N(\theta_{j-1}, \theta_j)$ solves the following problem

$$j \in \arg\min_{i \in N} \left| a^s - a^N(\theta_{i-1}, \theta_i) \right| \tag{14}$$

That is, between all the possible equilibrium actions she will choose the nearest one to a^s. From (5) we know that at each step i the difference $(\theta_i - \theta_{i-1})$ is increasing so, $(\theta_i - \theta_{i-1}) < (\theta_{i+1} - \theta_i)$ and then there are no two adjacent actions $a^N(\theta_{j-1}, \theta_j), a^N(\theta_j, \theta_{j+1})$ that solve the above minimization problem. Before providing three examples, we will discuss some properties of the action induced in this way. Suppose N odd number with $N > 3$, for $i = \frac{N+1}{2}$

$$a^N_{\left(\frac{N+1}{2}\right)} = \frac{1}{2} - \left(\frac{N^2-1}{2}\right)\frac{b}{p}$$

and for $i = \frac{N+1}{2} + 1$

$$a^N_{\left(\frac{N+1}{2}\right)+1} = \frac{1}{2} - \left(\frac{N^2-1}{2}\right)\frac{b}{p} + \left(\frac{1}{N} + 2\frac{b}{p}\right)$$

In order to simplify the computations we assume that the optimal effort is such that $\hat{p} = 1$. Evaluating the differences $a^s - a^N_{\left(\frac{N+1}{2}\right)}$ and $a^N_{\left(\frac{N+1}{2}\right)+1} - a^s$ we obtain

$$a^s - a^N_{\left(\frac{N+1}{2}\right)} < a^N_{\left(\frac{N+1}{2}\right)+1} - a^s$$

if and only if

$$\frac{1}{2N(N+1)} < b < \frac{1}{N(N^2-1)}$$

where the left inequality comes from from (10). It is easy to see that for $N > 3$ odd it is impossible, so

Lemma 4.1. *Let N odd with $N > 3$ and $\hat{p} = 1$, under the covert effort the sender will never choose any action $a^N(\theta_{i-1}, \theta_i)$ with $i = 1, 2, \ldots, \left(\frac{N+1}{2}\right)$ as a profitable deviation from the equilibrium strategy.*

Now suppose N even with $N > 4$, for $i = \frac{N}{2} + 1$

$$a^N_{\left(\frac{N}{2}+1\right)} = \frac{1}{2} + \frac{1}{2N} - \left(\frac{N^2-2}{2}\right)\frac{b}{p}$$

and for $i = \frac{N}{2} + 2$

$$a^N_{\left(\frac{N}{2}+2\right)} = \frac{1}{2} + \frac{1}{2N} - \left(\frac{N^2-2}{2}\right)\frac{b}{p} + \left(\frac{1}{N} + 4\frac{b}{p}\right)$$

As we did before let us assume $\hat{p} = 1$. Evaluating the differences $a^s - a^N_{\left(\frac{N}{2}+1\right)}$ and $a^N_{\left(\frac{N}{2}+2\right)} - a^s$ we obtain

$$a^s - a^N_{\left(\frac{N}{2}+1\right)} < a^N_{\left(\frac{N}{2}+2\right)} - a^s$$

if and only if

$$\frac{1}{2N(N+1)} < b < \frac{2}{N(N^2-4)}$$

where the left inequality comes from from (10). It is easy to see that for $N > 4$ even it is impossible, so

Lemma 4.2. *Let N even with $N > 4$ and $\hat{p} = 1$, under the covert effort the sender will never choose any action $a^N(\theta_{i-1}, \theta_i)$ with $i = 1, 2, \ldots, \left(\frac{N}{2} + 1\right)$ as a profitable deviation from the equilibrium strategy.*

We did not say anything about the cases $N = 2, 3, 4$ because in the following they will be analyzed in details.

Example 4.3. Let $N = 2$ and $\hat{p} = 1$, from (10) this implies $b \in \left(\frac{1}{12}, \frac{1}{4}\right)$ applying (7) we find

$$\theta_0 = 0 \quad , \quad \theta_1 = \frac{1}{2} - 2b \quad , \quad \theta_2 = 1$$

and from (13)

$$a^2_1 = \frac{1}{4} - b \quad , \quad a^2_2 = \frac{3}{4} - b$$

the sender would choose a^2_2 as profitable deviation and she would obtain this expected utility

$$\mathrm{V}^{\mathrm{s}}_{\mathrm{d}}(b, p) = \int_0^1 \left(\frac{3}{4} - b - (\theta + b)\right)^2 d\theta = -\left(\frac{7}{48} - b \cdot (1 - 4b)\right)$$

which is always greater than the outcome obtained with the babbling equilibrium

$$\mathrm{V}^{\mathrm{s}}(b, 0) = -\left(\frac{1}{12} + b^2\right)$$

and always lower than the outcome obtained with the maximum partition equilibrium

$$\mathrm{V}^{\mathrm{s}}(b, \hat{p}) = -\left(\frac{1}{48} + 2b^2\right)$$

then $\mathrm{V}^{\mathrm{s}}(b,0) < \mathrm{V}^{\mathrm{s}}_{\mathrm{d}}(b,p) < \mathrm{V}^{\mathrm{s}}(b,\hat{p})$. For the receiver the situation is different. If the sender deviates, his expected utility becomes

$$\mathrm{V}^{\mathrm{r}}_{\mathrm{d}}(b,p) = \int_0^1 \left(\frac{3}{4} - b - \theta\right)^2 d\theta = -\left(\frac{7}{48} - b \cdot \left(\frac{1}{2} - b\right)\right)$$

which is the worst possible outcome $\mathrm{V}^{\mathrm{r}}_{\mathrm{d}}(b,p) < \mathrm{V}^{\mathrm{r}}(b,0) < \mathrm{V}^{\mathrm{r}}(b,\hat{p})$ where

$$\mathrm{V}^{\mathrm{r}}(b,0) = -\frac{1}{12}$$

and

$$\mathrm{V}^{\mathrm{r}}(b,\hat{p}) = -\left(\frac{1}{48} + b^2\right)$$

Example 4.4. Let $N = 3$ and $\hat{p} = 1$, from (10) this implies $b \in \left(\frac{1}{24}, \frac{1}{12}\right)$ applying (7) we find

$$\theta_0 = 0 \quad , \quad \theta_1 = \frac{1}{3} - 4b \quad , \quad \theta_2 = \frac{2}{3} - 4b \quad , \quad \theta_3 = 1$$

and from (13)

$$a_1^3 = \frac{1}{6} - 2b \quad , \quad a_2^3 = \frac{1}{2} - 4b \quad , \quad a_3^3 = \frac{5}{6} - 2b$$

the sender would choose a_3^3 as profitable deviation and she would obtain this expected utility

$$\mathrm{V}^{\mathrm{s}}_{\mathrm{d}}(b,p) = \int_0^1 \left(\frac{5}{6} - 2b - (\theta + b)\right)^2 d\theta = -\left(\frac{7}{36} - 2b \cdot \left(1 - \frac{9}{2}b\right)\right)$$

which is always greater than the outcome obtained with the babbling equilibrium $\mathrm{V}^{\mathrm{s}}(b,0)$ but it is always lower than $\mathrm{V}^{\mathrm{s}}(b,\hat{p})$ with

$$\mathrm{V}^{\mathrm{s}}(b,\hat{p}) = -\left(\frac{1}{108} + \frac{11}{3}b^2\right)$$

As the case $N = 2$ for the receiver the situation is quite different $\mathrm{V}^{\mathrm{r}}_{\mathrm{d}}(b,p) < \mathrm{V}^{\mathrm{r}}(b,0) < \mathrm{V}^{\mathrm{r}}(b,\hat{p})$ where

$$\mathrm{V}^{\mathrm{r}}(b,\hat{p}) = -\left(\frac{1}{108} + \frac{8}{3}b^2\right)$$

Example 4.5. Let $N = 4$ and $\hat{p} = 1$, from (10) this implies $b \in \left(\frac{1}{80}, \frac{1}{24}\right)$ applying (7) we find

$$\theta_0 = 0 \quad , \quad \theta_1 = \frac{1}{4} - 6b \quad , \quad \theta_2 = \frac{1}{2} - 8b \quad , \quad \theta_3 = \frac{3}{4} - 6b \quad , \quad \theta_4 = 1$$

and from (13)

$$a_1^4 = \frac{1}{8} - 3b \quad , \quad a_2^4 = \frac{3}{8} - 7b \quad , \quad a_3^4 = \frac{5}{8} - 7b \quad , \quad a_4^4 = \frac{7}{8} - 3b$$

the sender would choose a_3^4 as profitable deviation and she would obtain this expected utility

$$\mathrm{V}^{\mathrm{s}}_{\mathrm{d}}(b,p) = \int_0^1 \left(\frac{5}{8} - 7b - (\theta + b)\right)^2 d\theta = -\left(\frac{49}{192} + 2b \cdot (32b - 1)\right)$$

which is always greater than the outcome obtained with the babbling equilibrium $\mathrm{V}^{\mathrm{s}}(b,0)$ but it is always lower than $\mathrm{V}^{\mathrm{s}}(b,\hat{p})$ with

$$\mathrm{V}^{\mathrm{s}}(b,\hat{p}) = -\left(\frac{1}{192} + 6b^2\right)$$

As the case $N = 3$ for the receiver the situation is quite different $\mathrm{V}^{\mathrm{r}}_{\mathrm{d}}(b,p) < \mathrm{V}^{\mathrm{r}}(b,0) < \mathrm{V}^{\mathrm{r}}(b,\hat{p})$ where

$$\mathrm{V}^{\mathrm{r}}(b,\hat{p}) = -\left(\frac{1}{192} + 5b^2\right)$$

Differently from the overt case, the receiver can form his beliefs only on the willingness of the sender to undertake the costly effort, with respect to the benefit that she will obtain deviating from the equilibrium strategy. We have defined $\delta(b,\hat{p})$ as the information surplus – under the overt effort – obtained when the sender acts optimally. We can do something similar for the covert effort case. Let $\eta^s(b,\hat{p})$ be

$$\eta^s(b,\hat{p}) = \mathrm{V}^{\mathrm{s}}(b,\hat{p}) - \mathrm{V}^{\mathrm{s}}_{\mathrm{d}}(b,p)$$

since $\mathrm{V}^{\mathrm{s}}_{\mathrm{d}}(b,p) > \mathrm{V}^{\mathrm{s}}(b,0)$ it result that $\eta^s(b,\hat{p}) < \delta(b,\hat{p})$.

Sender's Take-it-or-leave-it - $\hat{\Gamma}_{\mathbf{S}}$

1. The sender asks for a transfer $t \geq 0$.

2. The receiver can accept it and pay $t \geq 0$ or he can reject the requested transfer.

3. The sender undertakes the unobservable effort $c(p)$.

4. The sender observes the signal $s \in [0,1]$ and then she reports a message $m \in [0,1]$.

5. The receiver chooses the project $a \in [0,1]$ and then the outcomes are realized.

The first who moves is the sender asking a transfer ($\varphi(h_1) = s, A_s(h_1) = t \in \mathbb{R}_+$). The receiver can accept it and pay or he can reject it ($\varphi(h_2) = r, A_r(h_2) = \{p, np\}$). After that, the sender must choose to undertake the optimal effort or not. We can identify two information sets for the sender: h_3 after the history (t,a) and h_4 after the history (t,r), then ($\varphi(h_3) = s, A_s(h_3) = \{u, nu\}$) and ($\varphi(h_4) = s, A_s(h_4) = \{u, nu\}$). Given the actions taken in h_3 and h_4 the sender can be informed or uninformed. These situations are represented by the information sets h_5, h_6, h_7, h_8 reached after the histories

$$\begin{array}{lcl} (t,p,u) & \rightarrow & h_5 \\ (t,p,nu) & \rightarrow & h_6 \\ (t,np,u) & \rightarrow & h_7 \\ (t,np,nu) & \rightarrow & h_8 \end{array}$$

Since the receiver can not observe the effort he can not distinguish all of them. In particular he can not distinguish between h_5, h_6 and h_7, h_8. At this stage of the game the sender must report the signal's realization through a costless message m, so

$$\begin{array}{lclclclcl} \varphi(h_5) & = & \varphi(h_6) & = & \varphi(h_7) & = & \varphi(h_8) & = & s \\ A_s(h_5) & = & A_s(h_6) & = & A_s(h_7) & = & A_s(h_8) & = & m \in [0,1] \end{array}$$

an then for the receiver we can identify two information sets h_9, h_{10} reached after these histories

$$\begin{array}{lcl} \{(t,p,u,m),(t,p,nu,m)\} & \rightarrow & h_9 \\ \{(t,np,u,m),(t,np,nu,m)\} & \rightarrow & h_{10} \end{array}$$

in which

$$\begin{array}{lclcl} \varphi(h_9) & = & \varphi(h_{10}) & = & r \\ A_r(h_9) & = & A_r(h_{10}) & = & a \in [0,1] \end{array}$$

Given this description of the game, an equilibrium would be a pair of strategies (σ^s, σ^r) with

$$\sigma^s(h) = (\sigma^s(h_1), \sigma^s(h_3), \sigma^s(h_4), \sigma^s(h_5), \sigma^s(h_6), \sigma^s(h_7), \sigma^s(h_8))$$

$$\sigma^r(h) = (\sigma^r(h_2), \sigma^r(h_9), \sigma^r(h_{10}))$$

and for the receiver two posterior beliefs distributions μ_9^r, μ_{10}^r over the messages received in the respective information sets h_9, h_{10}. For example, μ_9^r corresponds to the receiver's posterior beliefs about the quantity of information contained in a message received in h_9, in order to induce a maximal partition equilibrium in the following cheap talk. Let us assume that $c(\hat{p}) > \eta^s(b, \hat{p})$, in this case the sender does not have any incentive to undertake the costly effort. The receiver knows it and he forms his beliefs accordingly. So the only possible equilibrium is the babbling one.

Proposition 4.6. *For $c(\hat{p}) > \eta^s(b, \hat{p})$ this pair of strategis*

$$\sigma^s(h) = (t, nu, nu, m^b, m^b, m^b, m^b)$$

$$\sigma^r(h) = \left(np, \frac{1}{2}, \frac{1}{2}\right)$$

with this system of beliefs

$$\mu_9^r = \mu_{10}^r = 0$$

is the only equilibrium for the extensive form game $\hat{\Gamma}_S$, where $t \in \mathbb{R}_+$ and $m^b \in \Delta(\mathcal{M})$.

Now let us assume $c(\hat{p}) < \eta^s(b, \hat{p})$ as in the previous case the babbling equilibrium is still one of the possible equilibria of the game. There are also two other equilibria in which the information transmission is attainable. In the first one, the sender undertake the effort even without receiving any transfer

Proposition 4.7. *For $c(\hat{p}) < \eta^s(b, \hat{p})$ this pair of strategis*

$$\sigma^s(h) = (t, u, u, m^a, m^d, m^a, m^d)$$

$$\sigma^r(h) = (np, a^m, a^m)$$

with this system of beliefs

$$\mu_9^r = \mu_{10}^r = 1$$

is an equilibrium for the extensive form game $\hat{\Gamma}_S$, where $t \in \mathbb{R}_+$, $m^a \in \Delta([\theta_{i-1}, \theta_i])$ for $s \in [\theta_{i-1}, \theta_i]$, θ_i's values are determined by (7), $m^d \in [\theta_{j-1}, \theta_j]$ for the value of j that solves problem (14) and $a^m = a([\theta_{i-1}, \theta_i], 0, \hat{p})$ as in (1).

In the second one, the sender undertakes the effort and she receives a transfer

Proposition 4.8. *For $c(\hat{p}) < \eta^s(b, \hat{p})$ this pair of strategis*

$$\sigma^s(h) = (t, u, nu, m^a, m^d, m^d, m^d)$$

$$\sigma^r(h) = \left(p, a^m, \frac{1}{2}\right)$$

with this system of beliefs

$$\mu_9^r = 1 \quad , \quad \mu_{10}^r = 0$$

is an equilibrium for the extensive form game $\hat{\Gamma}_S$, where $t = c(\hat{p}) - \eta^s(b, \hat{p})$, $m^a \in \Delta([\theta_{i-1}, \theta_i])$ for $s \in [\theta_{i-1}, \theta_i]$, θ_i's values are determined by (7), $m^d \in [\theta_{j-1}, \theta_j]$ for the value of j that solves problem (14) and $a^m = a([\theta_{i-1}, \theta_i], 0, \hat{p})$ as in (1).

Receiver's Take-it-or-leave-it - $\hat{\Gamma}_{\mathbf{R}}$

1. The receiver offers a transfer $t \geq 0$.
2. The sender can accept it and get $t \geq 0$ or she can reject the proposed transfer.
3. The sender undertakes the unobservable effort $c(p)$.
4. The sender observes the signal $s \in [0, 1]$ and then she reports a message $m \in [0, 1]$.
5. The receiver chooses the project $a \in [0, 1]$ and then the outcomes are realized.

The first who moves is the receiver offering a transfer ($\varphi(h_1) = r, A_r(h_1) = t \in \mathbb{R}_+$). The sender can accept or she can reject ($\varphi(h_2) = s, A_s(h_2) = \{a, r\}$). After that, the sender must choose to undertake the optimal effort or not. We can identify two information sets for the sender: h_3 after the history (t, a) and h_4 after the history (t, r), then ($\varphi(h_3) = s, A_s(h_3) = \{u, nu\}$) and ($\varphi(h_4) = s, A_s(h_4) = \{u, nu\}$). Given the actions taken in h_3 and h_4 the sender can be informed or uninformed. These situations are represented by the information sets h_5, h_6, h_7, h_8 reached after the histories

$$\begin{array}{lcl} (t,a,u) & \rightarrow & h_5 \\ (t,a,nu) & \rightarrow & h_6 \\ (t,r,u) & \rightarrow & h_7 \\ (t,r,nu) & \rightarrow & h_8 \end{array}$$

Since the receiver can not observe the effort he can not distinguish all of them. In particular he can not distinguish between h_5, h_6 and h_7, h_8. At this stage of the game the sender must report the signal's realization through a costless message m, so

$$\begin{array}{ccccccccc} \varphi(h_5) & = & \varphi(h_6) & = & \varphi(h_7) & = & \varphi(h_8) & = & s \\ A_s(h_5) & = & A_s(h_6) & = & A_s(h_7) & = & A_s(h_8) & = & m \in [0,1] \end{array}$$

an then for the receiver we can identify two information sets h_9, h_{10} reached after these histories

$$\begin{array}{lcl} \{(t,a,u,m),(t,a,nu,m)\} & \rightarrow & h_9 \\ \{(t,r,u,m),(t,r,nu,m)\} & \rightarrow & h_{10} \end{array}$$

in which

$$\begin{array}{ccccc} \varphi(h_9) & = & \varphi(h_{10}) & = & r \\ A_r(h_9) & = & A_r(h_{10}) & = & a \in [0,1] \end{array}$$

Given this description of the game, an equilibrium would be a pair of strategies (σ^s, σ^r) with

$$\sigma^s(h) = (\sigma^s(h_2), \sigma^s(h_3), \sigma^s(h_4), \sigma^s(h_5), \sigma^s(h_6), \sigma^s(h_7), \sigma^s(h_8))$$

$$\sigma^r(h) = (\sigma^r(h_1), \sigma^r(h_9), \sigma^r(h_{10}))$$

and for the receiver two posterior beliefs distributions μ_9^r, μ_{10}^r over the messages received in the respective information sets h_9, h_{10}. The same reasoning conducted about the relation between $c(\hat{p})$ and $\eta^s(b, \hat{p})$ in the sender's "take-it-or-leave-it" game still apply in this one. When $c(\hat{p}) > \eta^s(b, \hat{p})$ the sender has an incentive to deviate, from the signal's acquisition, so the receiver will not offer any transfer and moreover he will not listen to any message $m \in [0,1]$. So the sender does not undertake any effort. The resulting equilibrium is the babbling one

Proposition 4.9. *For $c(\hat{p}) > \eta^s(b, \hat{p})$ this pair of strategies*

$$\sigma^s(h) = (\Delta(\{a, r\}), nu, nu, m^b, m^b, m^b, m^b)$$

$$\sigma^r(h) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$$

with this system of beliefs

$$\mu_9^r = \mu_{10}^r = 0$$

is the only equilibrium for the extensive form game $\hat{\Gamma}_R$, where $m^b \in \Delta(\mathcal{M})$.

When $c(\hat{p}) < \eta^s(b, \hat{p})$ as the previous case the babbling equilibrium is one of the possible equilibria of the game. There are also two other equilibria in which the information transmission is attainable. In the first one, the sender undertake the effort even without receiving any transfer

Proposition 4.10. *For $c(\hat{p}) < \eta^s(b,\hat{p})$ this pair of strategies*

$$\sigma^s(h) = (\Delta(\{a,r\}), u, u, m^a, m^d, m^a, m^d)$$

$$\sigma^r(h) = (0, a^m, a^m)$$

with this system of beliefs

$$\mu_9^r = \mu_{10}^r = 1$$

is an equilibrium for the extensive form game $\hat{\Gamma}_R$, where $m^a \in \Delta([\theta_{i-1},\theta_i])$ for $s \in [\theta_{i-1},\theta_i]$, θ_i's values are determined by (7), $m^d \in [\theta_{j-1},\theta_j]$ for the value of j that solves problem (14) and $a^m = a([\theta_{i-1},\theta_i],0,\hat{p})$ as in (1).

In the second one, the sender undertakes the effort and she receives a transfer

Proposition 4.11. *For $c(\hat{p}) < \eta^s(b,\hat{p})$ this pair of strategies*

$$\sigma^s(h) = (a, u, nu, m^a, m^d, m^d, m^d)$$

$$\sigma^r(h) = \left(t, a^m, \frac{1}{2}\right)$$

with this system of beliefs

$$\mu_9^r = 1 \quad , \quad \mu_{10}^r = 0$$

is an equilibrium for the extensive form game $\hat{\Gamma}_R$, where $t = c(\hat{p}) - \eta^s(b,\hat{p})$, $m^a \in \Delta([\theta_{i-1},\theta_i])$ for $s \in [\theta_{i-1},\theta_i]$, θ_i's values are determined by (7), $m^d \in [\theta_{j-1},\theta_j]$ for the value of j that solves problem (14) and $a^m = a([\theta_{i-1},\theta_i],0,\hat{p})$ as in (1).

5 Conclusions

We found that either in the overt and covert effort case, there are equilibria in which the sender undertakes the effort even without receiving any transfer. This result is not surprisingly. The sender's payoffs depend by the action a undertaken through the number of actions induced in equilibrium. As we see, a coarsening of communication decreases the expected utilities of both. When the benefit of being informed $\delta(b,\hat{p})$ or $\eta(b,\hat{p})$ is greater than its cost $c(\hat{p})$, the sender has an unilateral incentive to undertake the costly effort. The receiver anticipates this and then he refuses any kind of transfer. For this reason there exist equilibria in which communication takes places even without a transfer. An ulterior motive for this is the total absence of receiver's commitment about the transfer structure. Due to this, the only determinant for the acquisition of the signal is the difference between its cost and the benefit obtained. If we allow the receiver to commit about a plan of transfers, the equilibria commented before disappear and the sender will be able to undertake more costly efforts. But this kind of framework is beyond the scope of this work. Here we analyzed what happens in a model in which the relations between two agents can not be regulated by a contract. It is interesting to note that under the covert effort, the benefit of being informed is always lower than the overt one. In the overt case for an off path equilibrium behavior, the receiver can threaten the sender with the babbling equilibrium, the worst possible outcome for her. In the covert case the situation is reversed. The outcome originated by a sender's deviation, induces for the receiver a payoff even worst than the babbling equilibrium. Suppose that before playing the game, the principal can choose between k different projects $y_1, y_2, \ldots, y_k$ and each of them has an optimal effort $c(\hat{p}_1), c(\hat{p}_2), \ldots, c(\hat{p}_k)$. Depending on the possibility of observing or not the effort, there will be project that he will discard a *priori.* For any project y_i such that $c(\hat{p}_i) > \delta(b,\hat{p}_i)$ we know that the only possible outcome is the babbling equilibrium.

For projects y_i such that $\delta(b,\hat{p}_i) > c(\hat{p}_i) > \eta(b,\hat{p}_i)$ we know that the only way to avoid the babbling equilibrium is to be able to observe the effort, and for projects y_i such that $\eta(b,\hat{p}_i) > c(\hat{p}_i)$ there is no difference between the overt or the covert effort. A possible extension of this work is introducing the possibility of a naive principal. That is, a principal that blindly trust the received message as in Ottaviani (2000) or the possibility of partial commitment by one of the two parts.

6 Appendix

The *ex ante* expected utility of the sender is

$$\mathrm{V^s}(b,p) = -\left[(1-p)\sum_{i=1}^{N}\int_{\theta_{i-1}}^{\theta_i}\left(\frac{1}{3}-\frac{\theta_{i-1}+\theta_i}{2}+\left(\frac{\theta_{i-1}+\theta_i}{2}-b\right)^2\right)d\theta+ \right.$$
$$\left. +p\sum_{i=1}^{N}\int_{\theta_{i-1}}^{\theta_i}\left(\frac{\theta_{i-1}+\theta_i}{2}-b-\theta\right)^2 d\theta\right]$$

consider the first summation

$$(1-p)\sum_{i=1}^{N}\int_{\theta_{i-1}}^{\theta_i}\left(\frac{1}{3}-\frac{\theta_{i-1}+\theta_i}{2}+\left(\frac{\theta_{i-1}+\theta_i}{2}-b\right)^2\right)d\theta =$$
$$(1-p)\sum_{i=1}^{N}\left[\frac{1}{3}(\theta_i-\theta_{i-1})-\frac{\theta_i^2-\theta_{i-1}^2}{2}+b(\theta_i-\theta_{i-1})+\right.$$
$$\left.+\frac{(\theta_i+\theta_{i-1})^2}{4}(\theta_i-\theta_{i-1})+b^2(\theta_i-\theta_{i-1})-b(\theta_i^2-\theta_{i-1}^2)\right]$$

notice that

$$\sum_{i=1}^{N}(\theta_i-\theta_{i-1})=1 \qquad \sum_{i=1}^{N}(\theta_i^2-\theta_{i-1}^2)=1$$

and

$$\sum_{i=1}^{N}(\theta_i+\theta_{i-1})^2\cdot(\theta_i-\theta_{i-1})=\frac{1}{3}\left[4+\frac{4b^2(1-N^2)}{p^2}-\frac{1}{N^2}\right]$$

the first summation becomes

$$(1-p)\left[\frac{1}{12}\left(2-\frac{1}{N^2}\right)+\frac{b^2(1-N^2)}{3p^2}+b^2\right]$$

now consider the second summation

$$p\sum_{i=1}^{N}\int_{\theta_{i-1}}^{\theta_i}\left(\frac{\theta_{i-1}+\theta_i}{2}-b-\theta\right)^2 d\theta =$$
$$p\sum_{i=1}^{N}\left[b^2+(\theta_i-\theta_{i-1})+\frac{\theta_i^3-\theta_{i-1}^3}{3}-\frac{(\theta_i+\theta_{i-1})^2}{4}(\theta_i-\theta_{i-1})\right]=$$
$$p\left[\frac{1}{12N^2}+\frac{b^2(N^2-1)}{3p^2}+b^2\right]$$

and then

$$\mathrm{V^s}(b,p) = -(1-p)\left[\frac{1}{12}\left(2-\frac{1}{N^2}\right)+\frac{b^2(1-N^2)}{3p^2}+b^2\right]-p\left[\frac{1}{12N^2}+\frac{b^2(N^2-1)}{3p^2}+b^2\right]=$$
$$-(1-p)\frac{1}{6}-(2p-1)\left(\frac{1}{12N^2}+\frac{b^2(N^2-1)}{3p^2}\right)-b^2$$

and

$$\mathrm{V^r}(b,p) = -(1-p)\frac{1}{6}-(2p-1)\left(\frac{1}{12N^2}+\frac{b^2(N^2-1)}{3p^2}\right)$$

The left derivative of $V(b,p)$ with respect to p is

$$\frac{d^- V(b,p)}{dp} = \lim_{h\to 0^+} \frac{V(b,p-h) - V(b,p)}{h}$$

and it resuts

$$\frac{d^- V(b,p)}{dp} = \lim_{h\to 0^+} -\frac{1}{6} - (2p-1)\frac{b^2(N^2-1)}{3p^2} \cdot \frac{2p-h}{(p-h)^2} + 2\left(\frac{1}{12N^2} - \frac{b^2(N^2-1)}{3(p-h)^2}\right) =$$

$$\frac{d^- V(b,p)}{dp} = -\frac{1}{6} \cdot \left(\frac{N^2-1}{N^2}\right) + 2 \cdot \frac{b^2(N^2-1)}{3p^2} \cdot \left(\frac{1-p}{p}\right)$$

from (10) it is easy to see that

$$\frac{b^2}{p^2}\left(\frac{1-p}{p}\right) < \frac{1}{4N^2(N-1)^2}\left(\frac{1-p}{p}\right) < \frac{1}{4N^2}$$

but then for

$$p > \frac{1}{(N-1)^2+1}$$

it results

$$\frac{d^- V(b,p)}{dp} < 0$$

Proof of Lemma 3.2

For any $p \in (p_i, p_{i+1})$ the number $N(b,p)$ is constant and so continuity is trivial. Now it must be shown that

$$\lim_{\varepsilon\to 0^+} \mathrm{V}^{\mathrm{s}}(b, p_i - \varepsilon) = \mathrm{V}^{\mathrm{s}}(b, p_i) = \lim_{\varepsilon\to 0^+} \mathrm{V}^{\mathrm{s}}(b, p_i + \varepsilon)$$

the first equality is satisfied since

$$\lim_{\varepsilon\to 0^+} N(b, p_i - \varepsilon) = N(b, p_i)$$

for the second equality notice that

$$\lim_{\varepsilon\to 0^+} N(b, p_i + \varepsilon) = N(b, p_i) + 1$$

and so

$$\lim_{\varepsilon\to 0^+} \mathrm{V}^{\mathrm{s}}(b, p_i + \varepsilon) - \mathrm{V}^{\mathrm{s}}(b, p_i) = (2p-1)\left[\frac{1+2i}{12i^2(i+1)^2} - \frac{b^2}{p^2}\frac{1+2i}{3}\right] = 0$$

where the last equality comes from

$$-\frac{1}{2} + \frac{1}{2}\sqrt{1 + 2\frac{p_i}{b}} = i$$

by definition. This complete the proof for $\mathrm{V}^{\mathrm{s}}(b, p_i - \varepsilon)$. Given that $\mathrm{V}^{\mathrm{s}}(b,p)$ and $\mathrm{V}^{\mathrm{r}}(b,p)$ differ only for a constant, the continuity of the first implies the continuity of the second and vice versa. □

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