A unifying model of strategic network formation

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EXTENDED ABSTRACT:

There are two benchmark models of strategic network formation: Jackson and Wolinsky's (1996) model and Bala and Goyal's (2000) model ("J&W's model" and "B&G's model" in what follows). In J&W's model a link between two players forms only if both agree on forming it, while in B&G's model any player can unilaterally form links with others. B&G's model has two variants: in the *one-way flow* model the flow through a link runs toward a player only if he/she supports it, while in the *two-way flow* model the flow runs in both directions even if only one player supports it. Each of these models has been extended separately in different directions. In this paper instead we provide an extension *between* these models, namely, a model that integrates these three basic benchmark models of strategic network formation as particular extreme cases.

In J&W's model and in B&G's two-way flow model, with or without decay, the flow through a link is *symmetric*, i.e. the same in both directions, while in B&G's one-way flow model the flow through links is *entirely asymmetric* (only towards the player that supports it) unless it is supported by both. In real-world there are situations where things are not that extreme, but, say, intermediate between those described in these three benchmark models. It may be the case that a link can be created unilaterally, but that sort of link works (unidirectionally or bidirectionally) worse than a link supported by both players. With this motivation in mind, we explore in two previous papers intermediate models between pairs of benchmark models. In Olaizola and Valenciano (2014a) we introduce an "asymmetric flow" model which bridges the gap between B&G's one-way and two-way flow models, which in this light appear as extreme particular cases. In a similar spirit, in Olaizola and Valenciano (2014b) we introduce an intermediate model which bridges the gap between J&W's connections model and B&G's two-way flow model. In both cases we study Nash, strict Nash and pairwise stability, efficiency and dynamics. A similar extension can be done to bridge the gap from J&W's connections model to B&G's one-way flow model[1].

In other words, if each of these three benchmark models, i.e. J&W's model and B&G's one-way and two-way flow models, is seen as one of the three vertices of a triangle, then the intermediate models mentioned above cover its three sides. In this paper we present and study a new hybrid model of strategic network formation which corresponds to *any point within this triangle*. This is achieved by assuming that when a link is supported by both players (strong link) flow is perfect in both directions, while

[1]Sketched in a preliminary draft in Olaizola and Valenciano (2014c).

when it is supported *by only one* (weak link) the flow toward the player supporting it suffers a decay β, and in the opposite direction the decay is α. We also assume $\alpha \leq \beta$. Figure 1 represents the flows through a strong link (i.e. supported by both players) and through a weak link, only supported by player i.

$$i\bullet \underset{\xrightarrow{1}}{\overset{\xleftarrow{1}}{\longleftrightarrow}} \bullet j \qquad\qquad i\bullet \underset{\xrightarrow{\alpha}}{\overset{\xleftarrow{\beta}}{\longleftarrow}} \bullet j$$

A strong link A weak link

Figure 1

A detailed specification over this "triangle" is useful to convey the basics of the model.

The vertices: When $\alpha = \beta = 0$, this is J&W's model without decay: flow occurs only through strong links. When $\alpha = 0$ and $\beta = 1$, this is B&G's one-way flow model without decay: flow through a link occurs only towards the player supporting it. When $\alpha = \beta = 1$, this is B&G's two-way flow model without decay: the support of only one player is enough for the flow to run in both directions.

The sides: While when $\beta = 1$ and $0 < \alpha < 1$, this is the intermediate model studied in Olaizola and Valenciano (2014a) which has B&G's one-way flow model ($\alpha = 0$) and two-way flow ($\alpha = 1$) model as extreme cases . While $0 < \alpha = \beta < 1$ corresponds to the intermediate model studied in Olaizola and Valenciano (2014b), whose extreme cases are J&W's model ($\alpha = \beta = 0$) and B&G's two-way flow model ($\alpha = \beta = 1$). A similar intermediate model between J&W's and B&G's one-way flow model can be obtained for $\alpha = 0$ and $0 < \beta < 1$, which yields J&W's when $\beta = 0$ and B&G's when $\beta = 1$.

In this work we consider the three-parameter (α, β and *cost* c) model assuming link-formation as described above. That is, we explore the *interior* of the triangle, i.e. the intermediate models where $0 < \alpha < \beta < 1$. More precisely, as there are three parameters involved, we explore the *interior of a three-dimensional region.* Figure 2 represents this region: the three vertical segments correspond to the three vertices of the above mentioned "triangle", i.e. the three benchmark models for $0 < c < 1$: the segment standing on the origin corresponds to J&W's model, the other two correspond to B&G's models[2]; while the three rectangles standing on the sides of the triangle correspond to the three intermediate models mentioned above. Our objective is to explore the models corresponding to configurations of values of the parameters within the triangular prism whose edges represent the three benchmark models. In this paper we study Nash, strict Nash and pairwise stability, efficiency and dynamics for such models.

[2] The reader may wonder about the meaning of the segment standing on $(\alpha, \beta, c) = (1, 0, 0)$, outside the triangle. This means that flow through a weak link only occurs from the player that supports it towards the other. This makes sense, for instance, in advertising, and the situation is entirely symmetric w.r.t. that of $(\alpha, \beta, c) = (0, 1, 0)$, i.e. B&G's one-way flow model.

In order to complete this extended abstract, we briefly specify the model in full detail.

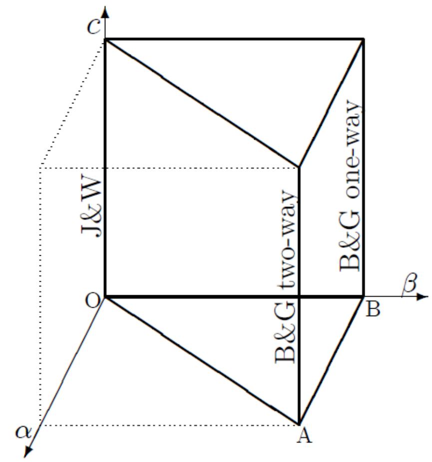

Figure 2

The model

Let $N = \{1, 2, ..., n\}$ be a set of *players*. Each player i may *intend*[3] to initiate links with other players. A map $g_i : N\backslash\{i\} \to \{0,1\}$ describes the links intended by i. We denote $g_{ij} := g_i(j)$, and $g_{ij} = 1$ ($g_{ij} = 0$) means that i *intends* (does not intend) a link with j. Thus, vector $g_i = (g_{ij})_{j \in N\backslash\{i\}} \in \{0,1\}^{N\backslash\{i\}}$ specifies the links intended by i and is referred to as a *strategy* of player i. $G_i := \{0,1\}^{N\backslash\{i\}}$ denotes the set of i's strategies and $G_N = G_1 \times G_2 \times ... \times G_n$ the set of *strategy profiles*. A strategy profile $g \in G_N$ determines a *graph* (N, Γ_g) of intended links, where $\Gamma_g := \{(i,j) \in N \times N : g_{ij} = 1\}$. Also $N^d(i;g) := \{j \in N\backslash\{i\} : g_{ij} = 1\}$, and $N(i;g)$ is the set of nodes connected with i by a path.

In the model we consider here, if g is a strategy profile, the flow through weak links in the *actual network* g^* which forms suffers a certain decay. Thus a thorough description of this actual network is achieved by means of the *actual decay's matrix* $g^* = \delta_{ij}^g$. We assume that the decay, i.e. the fraction of information that flows from a player j to i, through a link is the following[4]:

$$g^* = \delta_{ij}^g := \begin{cases} 1, \text{ if and only if } g_{ij} = g_{ji} = 1, \\ \beta, \text{ if and only if } g_{ij} = 1, \text{ and } g_{ji} = 0, \\ \alpha, \text{ if and only if } g_{ij} = 0, \text{ and } g_{ji} = 1, \end{cases}$$

[3]According to the above specification of link-formation, as far as $\alpha, \beta \in (0,1)$, any intended link is actually formed. But in the "boundary" case of J&W's model, i.e. when $\alpha = \beta = 0$, only strong links actually form.

[4]We choose to use this double notation (g_{ij}^* and δ_{ij}^g) and terminology (network/decay's matrix), the latter to emphasize that what matters is the decay associated with a link, while the first can be justified because, as we presently specify, such information can be derived from the strategy profile g.

where $0 \leq \alpha \leq \beta \leq 1$. Although other interpretations are possible, we give preference to the interpretation of links as a means for information to flow. We make the following assumptions:

1. Intending a link means a *cost:* $c_{ij} > 0$ for all $j \neq i$.
2. Player j has a particular type of information of *value* v_{ij} for player i.
3. If $\mathbf{v} = (v_{ij})_{(i,j)\in N\times N}$ and $\mathbf{c} = (c_{ij})_{(i,j)\in N\times N}$ are the matrices of value and costs, and g is the strategy profile and g^* the resulting network, the payoff of a player is given by a function

$$\Pi_i(g) = I_i(g^*, \mathbf{v}) - c_i(g, \mathbf{c}),$$

where $I_i(g^*, \mathbf{v})$ is the *information* received by i through the actual network g^*, and $c_i(g, \mathbf{c}) = \sum_{j\in N^d(i;g)} c_{ij}$ the *cost* incurred by i.

If g^* is the resulting actual network when the strategy profile is g, assume that the decay through a link from j to i of the actual network is $\delta_{ij}^g \in \{0, \alpha, \beta, 1, \}$, and the decay along a path is the product of the decays in every link of those that form the path. That is, given a path from j_0 to j_k in g, $j_0, j_1, .., j_k$, the decay through this path is given by the product $\delta^g_{j_k j_{k-1}} \delta^g_{j_{k-1} j_{k-2} j} \ldots . \delta^g_{j_1 j_0}$. Then the payoff of a player is given by

$$\Pi_i(g) = \sum_{j\in N(i;g)} \delta(i, j; g^*) v_{ij} - \sum_{j\in N^d(i;g)} c_{ij},$$

where $\delta(i, j; g^*)$ is the decay along the path in g^* from j to i for which the decay is minimal (i.e. the product of decays maximal). If $l_\alpha(p)$ denotes the number of weak links in a path p from j to i where the link is only supported by the player further to i in this path, and $l_\beta(p)$ the number of those only supported by the player closer to i, then decay along this path is $\alpha^{l_\alpha(p)} \beta^{l_\beta(p)}$. Therefore

$$\delta(i, j; g^*) := \max_{p\in p(i,j;g)} \alpha^{l_\alpha(p)} \beta^{l_\beta(p)},$$

where $p(i, j; g)$ is the set of all paths in g from j to i.

In this way a game in strategic form is specified: $(G_N, \{\Pi_i\}_{i\in N})$. In spite of the complexity of this model, assuming homogeneity in link costs and individual values across players, some of the conclusions relative to stability, efficiency and dynamics obtained for the intermediate models can be generalized in this more general context.

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