Sequential Bargaining with the Global Games Information Structure

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Abstract

This paper studies an infinite-horizon bilateral bargaining model with alternating offers and private correlated values. The correlation of values is given by a global games style information structure: players' types are positively correlated with the underlying fundamental and values are given by strictly increasing functions of types. The paper analyzes two classes of equilibria: common screening equilibria and segmentation equilibria. In common screening equilibria, both parties make offers to screen the opponent's type and all types of either party follow the same path of offers. In segmentation equilibria, types partially separate themselves by the initial offer. These equilibria classes have drastically different trade dynamics and efficiency properties. Equilibrium behavior under infrequent offers is examined by numerical simulations, and limits of equilibria as both the time between offers vanishes and the correlation of values becomes nearly perfect are characterized.

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1 Introduction

Bargaining is an important feature of many economic transactions, and differences in information about preferences are particularly important in determining equilibrium efficiency and trade dynamics. The bargaining game with infinite horizon and one-sided incomplete information was extensively studied and by now is well understood.[1] When the seller's cost is commonly known and the buyer's valuation is private information, the ability of the seller to extract profits beyond the competitive level depends crucially on the support of the distribution of the buyer's valuation. When there is a gap between the seller cost and lowest buyer valuation, the seller loses all price discriminatory power as offers become frequent (see Fudenberg, Levine, and Tirole (1985), Gul, Sonnenschein, and Wilson (1986), Grossman and Perry (1986), Gul and Sonnenschein (1988)). This is the manifestation of Coasian forces. The inability of the seller to commit to future price offers drives seller offers down to the lowest buyer valuation, ensuring the competitive outcome.[2] In the case of no gap, a folk-theorem type of result obtains and a variety of outcomes are sustainable in equilibrium. In particular, both an outcome close to the static monopoly outcome and the competitive outcome are feasible (see Ausubel and Deneckere (1989a,b), Ausubel and Deneckere (1992a)).

In a recent work, Deneckere and Liang (2006) explored the case of interdependent values in a model with one-sided incomplete information. In their model, a fundamental determines values of both parties, but only one party is informed about the fundamental, while the other party holds prior beliefs about it that are commonly known. The equilibrium exhibits interesting dynamics with long periods of almost no trade interrupted by bursts of trade. This model was further studied in Fuchs and Skrzypacz (2010), Gerardi, Hörner and Maestri (2013), and Fuchs and Skrzypacz (2013).[3]

The literature on bargaining with two-sided incomplete information has thus far focused exclusively on the case of independent private values. Cramton (1984), Cho (1990), and Ausubel and Deneckere (1992) investigated the relationship between two-sided uncertainty and efficiency.[4,5] These papers restrict offers to one side which is shown in Ausubel

[1] Most of the results in the literature as well as in this paper are obtained in the limit of frequent offers. The qualifier "in the frequent-offer limit" is omitted in the description of the results.

[2] I refer to the sale of the good at the price equal to the lowest valuation of the buyer as the competitive outcome and to the sale at the optimal monopoly price as the static monopoly outcome.

[3] An earlier analysis of this model is given in Vincent (1989).

[4] See Section 7 for an overview of these results.

[5] See also Fudenberg and Tirole (1983) for the analysis of the model with two bargaining rounds, and Chatterjee and Samuelson (1987) for a neat characterization of the bargaining dynamics under additional restriction of type and action space to only two types and two offers. Watson (1998) studies a model

and Deneckere (1993) to be ex-ante efficient for a variety of welfare weights. The current paper is the first to study a bargaining model with two-sided incomplete information and private correlated values. The correlation of values spans a variety of environments that are intermediate between perfectly correlated and independent values.

Specifically, I consider a bargaining game in which values are determined by a *global games information structure,* as in Morris and Shin (1998). An unobserved fundamental ω is drawn from interval $[0,1]$, and buyer and seller types are related to the fundamental by $b=\omega+\eta_B$ and $s=\omega+\eta_S$, respectively. Random variables η_B and η_S are conditionally independent draws from distributions with support $\left[-\frac{\eta}{2},\frac{\eta}{2}\right]$ where $\eta>0$ is the *individual uncertainty parameter.* Types are mapped into values by strictly increasing functions $v(b)$ and $c(s)$ so that for any realization of ω,η_B and η_S, there are strict gains from trade. As the individual uncertainty parameter η varies from 0 to 1, the model spans environments ranging from perfectly correlated to independent values. Players negotiate the price of trade in a standard infinite-horizon bargaining game with alternating offers.

The information structure described above captures essential features of many economically significant negotiations. For example, in over-the-counter markets for corporate bonds, the price that the trader is willing to pay or accept for a particular bond depends on characteristics of the bond such as yield, maturity, credit rating, additional features like provisions and covenants, as well as the trader's portfolio strategy and hedging needs.[6] Moreover, evaluating risks associated with the bond is a complicated task that requires expertise on the side of the trader and so, traders differ in their evaluation of a particular bond. Although traders ultimately differ in their valuation of particular bonds, public trading data and use of related modeling techniques implies a degree of correlation in their assessments.

Another example is the inter-dealer market for used cars.[7] The dealer's value of a particular car is determined by characteristics of the car as well as the current state of the dealer's inventory and the preferences of the dealer's customer base. The evaluation of the car characteristics depends on the familiarity of the dealer with a particular car brand, previous experience of holding the car in stock or results of the pre-sale test drive. Despite heterogeneity in values, the fact that dealers are experienced in the car evaluation

with two-sided private information about discount factors.

[6]See Saunders, Srinivasan, and Walter (2002) for a detailed field study of the OTC market for corporate bonds.

[7]See Larsen (2013) for a comprehensive empirical analysis of the wholesale used-auto industry. On this market, every year 15 million cars are sold in the United States, and in about 20% of the cases prices are determined by the over-the-phone alternating-offer bargaining.

implies that their values are positively correlated.

In these examples, the significant idiosyncratic component in preferences resulting from the dealer's business specifics justifies the assumption of private values. At the same time, the fact that evaluation is done by professionals ensures a positive correlation in values. Moreover, although some of allocations on these markets happen through auctions, many trades are conducted through alternating-offer bargaining.

I analyze two tractable classes of equilibria of the described bargaining game. Section 3 characterizes a class of *common screening equilibria (CSEs)*. In these equilibria, both parties make price offers to screen the type of the opponent, and all types of either party follow the same path of price offers. In other words, types pool on price offers and separate by the time at which they accept the opponent's price offer. Section 5 analyzes a different class of *segmentation equilibria,* in which types partially separate themselves into several segments by the initial price offer. Together, these two equilibria classes describe a variety of possible equilibrium trade dynamics and sources of inefficiency.

To characterize frequent-offer limits of CSEs, I first introduce a related game which I refer to as a concession game. The concession game is a continuous-time counterpart of the described bargaining game except that players take price-offer paths as given and only choose the time at which they accept the opponent's offer. Competitive equilibria of the concession game consist of a pair of price-offer paths and a pair of acceptance strategies such that each player type chooses an optimal acceptance time given the price paths and the strategy of the opponent. I restrict the analysis to competitive equilibria in monotone (acceptance) strategies. Monotone strategies can be described at any time by a threshold type of the buyer and the seller. All buyer types above the threshold type of the buyer and all seller types below the threshold type of the seller accept the current price offer of the opponent. In Theorem 1, incentives of threshold types are described by a system of differential equations giving the relationship between monotone acceptance strategies and the dynamics of price offers. Threshold types are indifferent between accepting the current offer of the opponent, and marginally delaying the acceptance.

In general, the ability to choose price-offer paths puts additional restrictions on price paths and acceptance strategies in CSEs. However, as Theorem 2 shows, in the frequent-offer limit such restrictions on the CSE's equilibrium behavior are minimal. In particular, at any time players prefer to continue bargaining rather than stay out or make an unfavorable price offer that is guaranteed to be accepted by any type of opponent in any equilibrium.

The CSE characterization provides a realistic trade dynamics and uncovers new sources of bargaining inefficiency. The trade dynamics in a CSE can be described as a two-sided screening process. The screening policy is common for all types on each side, but it is not profitable to the same degree to all of them. A rejection of an offer does not lead to reduction in uncertainty about the opponent for all types but only a subset of types. This subset of types could be small if the individual uncertainty is small at the onset. The common screening policy "prioritizes" higher seller types and lower buyer types. At the beginning of the game, highest seller types and lowest buyer types benefit from the common screening policy as only their offers are accepted with positive probability. As the game proceeds, the common screening policy becomes efficient for types in the middle of the type range. It is possible that a wide range of types of the seller at the bottom and types of the buyer at the top never benefit from following the common screening policy, and with probability one they accept some price offer of the opponent.

Equilibrium behavior also provides a new perspective on how bargaining postures are formed. In CSEs, both parties start the negotiation by claiming that their values are at the extremes, even though such postures are commonly known to be unreasonable when the individual uncertainty is small.[8] Over time the common screening policy narrows down the range of types possible in the game. As bargaining continues, parties make concessions and moderate their demands to more reasonable levels.

There are three sources of inefficiency in CSEs. First, the surplus is dissipated through the channel analogous to the standard monopoly deadweight loss. In order to efficiently screen player types, each offer is targeted at a particular group of types and the allocation is delayed for the rest of the types. The second source of inefficiency is signaling costs. Higher seller types and lower buyer types prefer to reject the offer of the opponent and continue screening to signal their value and convince the opponent to accept their screening offer. These inefficiencies were already present in the model with independent private values.[9]

The third source of inefficiency arises from common screening. Since players use common screening policies, a significant time could pass until the common screening policy becomes efficient for buyer and seller types in the middle of the type range. This source of inefficiency is novel and was not present in the model with independent types. When types are independent, decreasing uncertainty about gains from trade reduces the inefficiency from the first two sources. In contrast, in CSEs, types in the middle of the type

[8] That is, it is common knowledge that at most one of the parties is telling the truth when claiming that his/her value is at the extreme.

[9] See Ausubel and Deneckere (1992b) for an excellent analysis of these two sources of inefficiency in the model with independent values.

range go through a routine of offers in the common screening policy that are guaranteed to be rejected by the opponent. This could result in a significant CSE inefficiency, even when the individual uncertainty is small.

In Section 4, I consider an extreme case when the only source of inefficiency is the common-screening inefficiency. Theorem 3 characterizes CSE limit outcomes when both the individual uncertainty and the time between rounds vanish. The characterization is in terms of static *common screening mechanisms (CSMs)*. In CSMs, types are divided into strong and weak types, and only weak types truthfully reveal themselves. Terms of trade are determined solely by the announcement of the weak type. Despite vanishingly small individual uncertainty, the equilibrium behavior still exhibits rich two-sided screening dynamics. This is in contrast with the complete information game analyzed in Rubinstein (1982), in which trade is immediate with equal split of the gains from trade in the limit of frequent offers.

The possibility of a variety of outcomes with non-trivial predictions about delay is an attractive feature for using the described bargaining model as a component of a more general economic model. As the individual uncertainty is vanishingly small, it allows the modeler to abstract from the uncertainty outside the bargaining part of the model, while the bargaining part still exhibits interesting two-sided screening dynamics.

Theorem 3 shows that small individual uncertainty of players is not sufficient for the efficient trade, as long as there is large common uncertainty, that is, the range of types in the game is large. One way to reduce common uncertainty is by exogenously dividing the market into several segments. In the examples described above, rating agencies assign credit ratings to corporate bonds, and used-car dealers trade through platforms with the pre-sale inspection and established quality standards. These features divide the market into several segments. The market for corporate bonds is divided into prime, investment grade and non-investment grade bonds. In used-car markets, luxury lots are traded separately from regular lots.

In Section 5, I show that such segmentation happens endogenously in segmentation equilibria. As opposed to CSEs, in segmentation equilibria constructed in Theorem 4, there is no inefficiency of common screening, and the waste is only due to the standard deadweight loss and signaling costs. For small individual uncertainty, in segmentation equilibria most of the types trade shortly after the start of bargaining. However, bargaining may take an arbitrarily long time for types at the boundaries of the segments. In the equilibrium, such types build reputation for belonging to a segment with a more favorable price by delaying trade and insisting on that price.

Segmentation equilibria allow us to draw a connection between the complete information bargaining game studied in Rubinstein (1982) and the model in this paper. In Theorem 5, I construct a sequence of segmentation equilibria with vanishing individual uncertainty and time between bargaining rounds that approximates Rubinstein (1982)'s immediate equal split of the realized surplus. Along this sequence, the number of segments increases and the definition of segments becomes finer.

For a wide range of types, the common screening policy in CSE can bring zero expected payoff for an extended time. Hence, it is important to understand what prevents players from deviating from this policy. Section 6 constructs the *punishing equilibrium* in which the punishing side holds optimistic beliefs. For example, in the seller punishing equilibrium, any type b buyer puts probability one on the seller type $\max\{0, b-\eta\}$. In the frequent-offer limit, the utility of the deviator in the punishing equilibrium is independent of the individual uncertainty parameter η, and the utility of all types of the deviator is equal to the lowest utility achievable in any equilibrium (Theorem 7). Therefore, detectable deviations from the CSE's equilibrium path can be punished equally harshly (in the frequent-offer limit) for all levels of individual uncertainty. I further demonstrate by numerical simulations that when offers are infrequent, the amount of individual uncertainty restricts the severity of the punishing equilibrium.

The structure of the paper is as follows. Section 2 describes the bargaining game and the information structure. Section 3 characterization the limit equilibrium behavior in CSEs in terms of competitive equilibria in the concession game. Section 4 characterizes double limits of CSE outcomes in terms of equilibrium outcomes of CSMs. Section 5 analyzes segmentation equilibria. Punishing equilibria play a key role in the results of this paper, and I study them in Section 6. Section 7 relates the paper to the existing literature. Section 8 concludes and gives directions for future research. To maintain continuity of the argument, all proofs are relegated to the Appendix.

2 The Model

This section introduces an infinite-horizon bargaining model with alternating offers and private correlated values. A buyer and a seller meet to trade one unit of a good.[10] The seller's type $s \in [0,1]$ and the buyer's type $b \in [0,1]$ are jointly uniformly distributed on $SB \equiv \{(s,b) \in [0,1]^2 : \max\{0, s-\eta\} \leq b \leq \min\{1, s+\eta\}\}$. The individual uncertainty

[10]Female pronouns are used to refer to the seller and male pronouns are used to refer to the buyer.

parameter $\eta \in (0,1)$ controls the degree of correlation of types. Types are almost perfectly correlated when $\eta \approx 0$ and close to independent when $\eta \approx 1$.[11,12]

The valuation of the good of a type b buyer is $v(b)$, and the cost of selling the good to a type s seller is $c(s)$, where $v : [0,1] \to \mathbb{R}$ and $c : [0,1] \to \mathbb{R}$ are strictly increasing, differentiable functions with derivatives bounded from below by some positive constant and bounded from above by $\ell > 0$.[13] Let $\xi \equiv \min_{(s,b)\in SB} \{v(b) - c(s)\}$ be the minimal gains from trade possible in the game. Assume $\xi > 0$.[14] The gains from trade are positive for any buyer and seller type, but the size of the gains from trade is not common knowledge due to the imperfect correlation of types. As a result, players are uncertain about how much the opponent gains from trade at a particular price and have incentives to pretend that the gains from trade are small to get a better price. However, for $\eta < 1$ the profitability of such pretense is potentially limited, as the opponent may detect that certain low gains are not possible.

Additionally, I impose the following mild technical condition on valuation and cost functions. A function $f(x)$ on a compact set X is *regular* if it is smooth and there exists $D > 0$ such that $\frac{1}{l!}\frac{d^l f(x)}{dx^l} < D$ for all $l \in \mathbb{N}$ and all $x \in X$. This condition is slightly stronger than the analyticity, and many functions used in applications are regular (for example, all polynomial functions are regular).[15,16]

Assumption R. *Functions $v(b)$ and $c(s)$ are regular.*

One can interpret the type space as follows. There is a *fundamental* ω drawn from $[0,1]$ and the buyer and the seller types $b = \omega + \eta_B$ and $s = \omega + \eta_S$, respectively, where η_b

[11] Formally, the correlation coefficient between b and s is decreasing in η, and it is equal to 1 for $\eta = 0$ and to 0 for $\eta = 1$.

[12] To focus on the novel features of the model, the extreme cases $\eta = 0$ and $\eta = 1$ are left out from the analysis. The case $\eta = 0$ has been studied in Rubinstein (1982). The analysis of the case $\eta = 1$ is simpler than the general case $\eta \in (0,1)$, but requires a separate treatment in proofs. All results of the paper carry to this case.

[13] When types are independent, it is a standard result that types can be taken to be uniformly distributed on the unit interval without loss of generality. For any distribution of values, there is a transformation of the valuation and cost functions that preserves the distribution of values and changes the distribution of types into uniform on the unit interval. With correlated types this result is no longer true as no such transformation is guaranteed to preserves the correlation structure. In this paper, I consider a general class of valuation and cost functions, but restrict the distribution of types to uniform. Relaxing this assumption is left for the future research.

[14] Observe, that this assumption does not preclude the possibility that $c(1) < v(0)$, and there is no trade at a single price that gives non-negative utility to all types.

[15] For analytic function $f(x)$ on a compact set there exists $D > 0$ such that $\frac{1}{l!}\frac{d^l f(x)}{dx^l} < D^l$ for all $l \in \mathbb{N}$.

[16] Assumption R is used in the proof of Lemma 4 and all subsequent theorems which apply this theorem. It is not needed for results about the punishing equilibria in Section 6 and the necessity part of Theorem 2.

and η_s are independent conditional on ω and have support $[-\frac{\eta}{2}, \frac{\eta}{2}] \cap [-\omega, 1-\omega]$.[17] This type of information structure is commonly used in the global games literature (see Morris and Shin (1998) and Morris and Shin (2003) for a survey). The information structure used in this paper is referred to as a *global games information structure* primarily to distinguish it from a different kind of interdependence in values studied in Deneckere and Liang (2006).

Given their types, players hold prior beliefs about their opponent's type. Prior beliefs of seller type s are uniform on the interval $B_s \equiv [b_s^\alpha, b_s^\omega]$ where $b_s^\alpha \equiv \max\{0, s-\eta\}$ and $b_s^\omega \equiv \min\{1, s+\eta\}$. Analogously, prior beliefs of buyer type b are uniform on the interval $S_b \equiv [s_b^\alpha, s_b^\omega]$ where $s_b^\alpha \equiv \max\{0, b-\eta\}$ and $s_b^\omega \equiv \min\{1, b+\eta\}$. Players' types and their priors are illustrated on Figure 1.

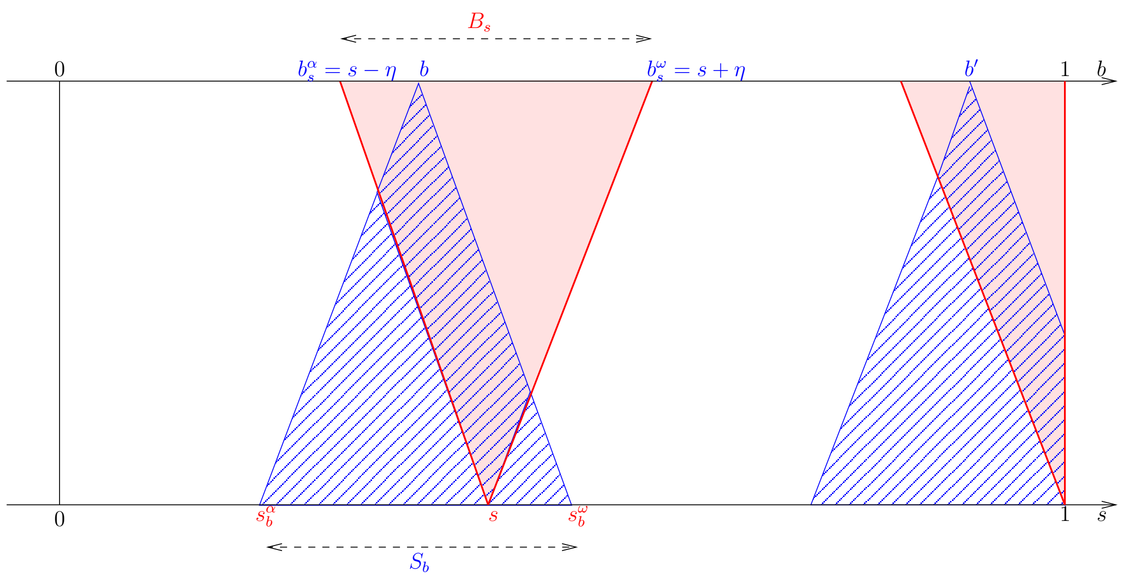


Figure 1: **Types and beliefs.** Red (filled) triangles capture the support of seller beliefs and blue (dashed) triangles capture the support of buyer beliefs. Seller type s puts probability one on buyer types in B_s, and buyer type b puts probability one of sellers in S_b. The support of the buyer and seller beliefs has length at most 2η and is truncated from below at 0 and from above at 1.

Bargaining occurs in rounds $n \in \mathbb{N}$, and the length of a time interval between bargaining rounds is $\Delta > 0$. Players discount the future at the common discount rate $r > 0$. The seller is active in odd rounds, and the buyer is active in even rounds. An active player can either accept the last offer of the opponent or make a counter-offer. Once a price offer is accepted, the game ends and payoffs are determined. An outcome $(N\Delta, p)$

[17] The joint distribution of ω, η_b and η_s is such that (s, b) is distributed uniformly on SB.

consists of the time of trade $N\Delta \leq \infty$ (where N is the round of trade) and the price of trade p. The utility of type b buyer is $e^{-r(N-1)\Delta}(v(b)-p)$ and the utility of type s seller is $e^{-r(N-1)\Delta}(p-c(s))$.[18]

In any round n by which trade has not happened, a history h^n is a sequence of rejected price offers up to round $n-1$. A (pure) behavioral strategy of the buyer $\sigma_b^n : [0,1]\times\mathbb{R}^{n-1} \to \mathbb{R}\cup\{accept\}$ is a measurable function which for any history h^n gives the acceptance decision or a counter-offer of buyer type b. The posterior beliefs $\mu_b^n : [0,1] \times \mathbb{R}^{n-1} \to \Delta(S)$ of the buyer is a measurable function that maps any buyer type b and any history h^n into a probability distribution over seller types. The behavioral strategy σ_s^n and the posterior beliefs μ_s^n are defined analogously for the seller.

A *sequential equilibrium,* which I further refer to simply as *equilibrium*, is pair of strategy profiles (σ_b^n, σ_s^n) and beliefs (μ_b^n, μ_s^n) that satisfy sequential rationality and consistency.[19] Sequential rationality requires that after any history, players best respond to the strategy of the opponent given their posterior beliefs. Consistency implies that beliefs are updated by Bayes rule whenever possible and, in addition, $\mu_b^n \in \Delta(S_b)$ and $\mu_s^n \in \Delta(B_s)$ for any history h^n. The latter requirement implies that the global games information structure is common knowledge among players. Both on and off the equilibrium path, players put positive probability only on types of the opponent that lie in the support of their priors, players are certain that their opponent also puts positive probability only on a subset of the support of his/her prior beliefs, and the regress continues indefinitely.

3 Common Screening Equilibria

This section characterizes the dynamics of CSE frequent-offer limits. The approach is to temporarily turn to a related game, referred to as a concession game in which players take price paths as given. The main result of this section (Theorem 2) relates CSE frequent-

[18] By convention, if the trade does not happen in a finite number of rounds, $N = \infty$ and both players get payoff zero.

[19] It is standard in the bargaining literature to restrict attention to equilibria in pure strategies with the reservation that mixing is possible off the equilibrium path (see Gul, Sonnenschein and Wilson (1986), and Fudenberg, Levine, and Tirole (1985) for a discussion of mixing off the equilibrium path). In this paper mixing could be necessary only for seller type 0 and buyer type 1 off the equilibrium path of the punishing equilibrium analyzed in Section 6. With minor adjustments the results in this paper could be formulated to incorporate this possibility. Instead, for notational convenience this possibility is assumed away in Section 6 (this assumption is not vacuous as verified by the numerical example in the end of Section 6).

offer limits to competitive equilibria of the concession game (characterized in Theorem 1). At the end of this section, I highlight main steps of the proof of the main result. In particular, Lemma 4 is at the heart of all equilibria constructions in this paper.

Concession game

The concession game is defined as follows. Types of the buyer and the seller are drawn uniformly from SB as in Section 2. There is a continuous path of buyer price offers $q_t^B : t \mapsto q_t^B$ and a continuous path of seller price offers $q_t^S : t \mapsto q_t^S$, Players take as given paths of price offers and choose the time at which they accept the opponent's offer. Outcome (T^c, q^c) consists of the time $T^c \in \bar{\mathbb{R}}_+$ and the price q^c at which trade happens.[20] Given outcome (T^c, q^c), the utility of buyer type b is $e^{-rT^c}(v(b) - q^c)$, and the utility of seller type s is $e^{-rT^c}(q^c - c(s))$.

Strategies are acceptance times $t_B^*(b)$ and $t_S^*(s)$ for each type b buyer and type s seller, respectively. For any types b, s and strategies $t_B^*(b), t_S^*(s)$, the outcome is determined by $T^c = \min\{t_B^*(b), t_S^*(s)\}$, and $q^c = q^B_{t_B^*(b)}$ if $t_B^*(b) \leq t_S^*(s)$ and $q^c = q^S_{t_S^*(s)}$ if $t_S^*(s) < t_B^*(b)$.[21] I assume that price paths are continuously differentiable, $q_t^S \geq q_t^B$ for all $t \geq 0$, and additionally[22]

$$c^{-1}(q_\infty^B) - v^{-1}(q_\infty^S) \geq \eta. \tag{1}$$

Condition (1) guarantees that all gains from trade can be eventually realized through one of the players accepting the opponent's offer.[23] Observe that the concession game is static, even though payoffs are determined by a dynamic procedure. I define a competitive equilibrium of the game as follows.

Definition 1. *A* competitive equilibrium *of the concession game is a tuple* $(t_B^*(b), t_S^*(s), q_t^B, q_t^S)$ *such that given price paths* q_t^S *and* q_t^B *and the strategy of the opponent (given by* $t_S^*(s)$ *or* $t_B^*(b)$*), players choose acceptance times* $t_B^*(b)$ *and* $t_S^*(s)$ *optimally.*

The definition of the competitive equilibrium is in the spirit of the Walrasian equilibrium. Each player takes price paths as given and chooses the acceptance time to maximize

[20] I use notation $\mathbb{R}_+ \equiv [0, \infty)$ for a set of positive reals, and $\bar{\mathbb{R}}_+ \equiv \mathbb{R}_+ \cup \{\infty\}$.

[21] In equilibria that I analyze, players assign probability zero to ties, and the tie-breaking rule could be specified arbitrarily.

[22] Define $x_\infty \equiv \lim_{t\to\infty} x_t$ whenever the limit exists.

[23] To see this, notice that the set of types that get negative payoffs from accepting any opponent's offer is a subset of $[0, v^{-1}(q_\infty^S)]$ for the buyer and a subset of $[c^{-1}(q_\infty^B), 1]$ for the seller. By $c^{-1}(q_\infty^B) - v^{-1}(q_\infty^S) \geq \eta$, $\left([c^{-1}(q_\infty^B), 1] \times [0, v^{-1}(q_\infty^S)]\right) \cap SB = \emptyset$ giving the desired conclusion.

his/her expected utility. However, unlike in the standard general equilibrium theory, in the concession game players' preferences are interdependent. If the buyer chooses an earlier acceptance time, then for a fixed strategy of the seller, it is more likely that bargaining will end earlier and that it will end by the acceptance of the seller price offer. This increases the expected utility of the seller and could give the seller additional incentives to delay the acceptance. Because of the preference interdependence, in general, finding a competitive equilibrium in the concession game is a difficult task. To circumvent this difficulty, I restrict the analysis of competitive equilibria to monotone strategies. The restriction to monotone strategies is common in Bayesian games with a continuum of types.[24]

Definition 2. *Acceptance strategies* $t^*_B(b)$ *and* $t^*_S(s)$ *are* monotone *if there exist processes* $b^*_t : t \mapsto b^*_t$ *and* $s^*_t : t \mapsto s^*_t$ *such that*

1. $t^*_B(b) \equiv \inf\{t : b^*_t = b\}$ *and* $t^*_S(s) \equiv \inf\{t : s^*_t = s\}$,[25]
2. *for some* $T_B, T_S \in \bar{\mathbb{R}}_+$, b^*_t *is strictly decreasing for* $0 \le t \le T_B$ *and constant for* $t \ge T_B$, *and* s^*_t *is strictly increasing for* $0 \le t \le T_S$ *and constant for* $t \ge T_S$.

Say that b^*_t *and* s^*_t *are* smooth monotone strategies *if, additionally,* b^*_t *and* s^*_t *are continuously differentiable.*

Monotone strategies specify the lowest type b^*_t of the buyer and the highest type s^*_t of the seller remaining in the game at time t. I use $t^*_B(b)$ and b^*_t interchangeably to refer to the monotone strategy of the buyer, and analogously, I use both $t^*_S(s)$ and s^*_t for the monotone strategy of the seller. The strict monotonicity of b^*_t and s^*_t implies that there are no periods with no acceptance until times T_B and T_S, respectively, when players stop accepting the opponent's offers. During "quiet" periods, price offers that are not accepted can be specified arbitrarily, as long as no types choose to accept them. The focus of this section is on the relationship between the dynamics of price paths and acceptance strategies and so, the strict monotonicity is necessary to pin down such relationship. I also make the following assumption about the monotonicity of price paths.

Condition M. *Seller price path* q^S_t *is decreasing, and buyer price path* q^B_t *is increasing.*

[24] Pure-strategy equilibria in monotone strategies were studied by Athey (2001), McAdams (2003), Reny (2011) in the context of auctions and by Van Zandt and Vives (2007) in the context of games with strategic complementarities.

[25] By convention, $\inf \emptyset = \infty$.

The monotonicity of offers required by Condition M is fairly natural, and it reflects the fact that over time parties converge in their demands. In the characterization of CSEs, I will need a stronger version of Condition M, which requires that price paths are strictly monotone up to some time after which they remain constant.

Condition M′. *There exists $\hat{T} \in \bar{\mathbb{R}}_+$ such that seller price path q_t^S is strictly decreasing on $[0, \hat{T}]$, and buyer price path q_t^B is strictly increasing on $[0, \hat{T}]$, and price paths are constant after $\hat{T}$.*

The next theorem characterizes competitive equilibria in smooth monotone strategies.

Theorem 1. *Consider a competitive equilibrium of the concession game in smooth monotone strategies described by $(b_t^*, s_t^*, q_t^B, q_t^S)$. Then the following conditions hold.*

1. *There exists a time $T \in \bar{\mathbb{R}}_+$ such that*

$$b_T^* = b_{s_T^*}^\alpha \text{ and } q_T^B \leq q_T^S \text{ with equality if } T < \infty. \tag{2}$$

2. *For all $t \in [0, T)$,*

$$r\left(v(b_t^*) - q_t^S\right) = \lambda_t^S \left(q_t^S - q_t^B\right) - \dot{q}_t^S, \tag{3}$$

$$r\left(q_t^B - c(s_t^*)\right) = \lambda_t^B \left(q_t^S - q_t^B\right) + \dot{q}_t^B; \tag{4}$$

where $\lambda_t^B \equiv -\frac{\dot{b}_t^*}{b_t^* - b_{s_t^*}^\alpha} \mathbb{1}\left\{b_{s_t^*}^\omega \geq b_t^*\right\}$ *and* $\lambda_t^S \equiv \frac{\dot{s}_t^*}{s_{b_t^*}^\omega - s_t^*} \mathbb{1}\left\{s_{b_t^*}^\alpha \leq s_t^*\right\}$.

Conversely, consider a tuple of smooth monotone strategies and price paths $(b_t^*, s_t^*, q_t^B, q_t^S)$ that satisfies condition M, and conditions (2), (3), (4). Then $(b_t^*, s_t^*, q_t^B, q_t^S)$ is a competitive equilibrium of the concession game.

Theorem 1 justifies the validity of the first-order approach for the analysis of competitive equilibria in smooth monotone strategies. To see this, consider the problem of type b buyer. Suppose that the seller uses smooth monotone strategy s_t^* and price paths are given by q_t^B and q_t^S. Let $F_t^S(b) \equiv \frac{\max\left\{\min\left\{s_t^*, s_b^\omega\right\} - s_b^\alpha, 0\right\}}{s_b^\omega - s_b^\alpha}$ be the CDF of the seller's acceptance time evaluated by type b buyer and $f_t^S(b)$ be the corresponding density function. Type b buyer maximizes his expected utility,

$$u^B(t, b) = \int_0^t e^{-ru}\left(v(b) - q_u^B\right) f_u^S(b) du + (1 - F_t^S(b)) e^{-rt}\left(v(b) - q_t^S\right) \to \max_{t \in \bar{\mathbb{R}}_+},$$

and the first order condition for his problem is

$$r(v(b) - q_t^S) = \frac{f_t^S(b)}{1 - F_t^S(b)} \left(q_t^S - q_t^B\right) - \dot{q}_t^S. \tag{5}$$

Condition (3) is the first-order condition (5) evaluated at $b = b_t^*$. It describes the incentives of the threshold type of the buyer. The buyer balances the cost due to discounting (left-hand side), and the benefit from the possible concession of the seller (the first term on the right-hand side) and from the change in the seller price offer (the second term on the right-hand side). Function $u^S(t, s)$ and the problem of type s seller are defined and analyzed analogously.

The first-order condition in (5) is only a necessary condition for optimality of the monotone strategy b_t^*. The following property of expected utility is key in proving that the first-order condition is also sufficient. Function $u^B(t, b)$ satisfies the *strict single-crossing property in* (t, b) if for $t < t'$ and $b < b'$, $u^B(t, b) > u^B(t', b)$ implies that $u^B(t, b') > u^B(t', b')$, and $u^B(t, b) \geq u^B(t', b)$ implies that $u^B(t, b') > u^B(t', b')$ (see Milgrom and Shannon (1994)).

Lemma 1. *Suppose that* q_t^B *and* q_t^S *satisfy* (1) *and condition M. If* s_t^* *is a monotone seller strategy, then* $u^B(t, b)$ *on* $TB \equiv \{(t, b) : b \in [0, 1], t \in [0, t_S^*(s_b^\omega)]\}$ *satisfies the strict single-crossing property. If* b_t^* *is a monotone buyer strategy, then* $u^S(t, s)$ *on* $TS \equiv \{(t, s) : s \in [0, 1], t \in [0, t_B^*(b_s^\alpha)]\}$ *satisfies the strict single-crossing property.*

Observe that for any b, $u^B(t, b)$ is constant for $t > t_S^*(s_b^\omega)$, as buyer type b expects that the seller accepts by time $t_S^*(s_b^\omega)$ with probability one. Hence, the restriction to sets TB and TS is necessary to guarantee strict inequalities in the definition of the strict single crossing property.

There are several implications of the strict single crossing property of expected utilities. By Theorem 4′ in Milgrom and Shannon (1994), any best-reply $t_B^*(b)$ to a monotone seller strategy s_t^* is a weakly decreasing function. This is weaker than saying that a best-reply $t_B^*(b)$ is itself a monotone strategy (let alone a smooth monotone strategy), as $t_B^*(b)$ need not be strictly decreasing. In this case, it is possible that the solution to (3) gives only a local maximum or even a minimum. The next lemma rules out this possibility.

Lemma 2. *Suppose that price paths* q_t^B *and* q_t^S *satisfy* (1) *and condition M. If* b_t^* *is a smooth monotone strategy in the concession game that satisfies* (3) *for some smooth monotone strategy of the seller* s_t^**, then* b_t^* *is a best-reply to* s_t^**.*

Theorem 1 reduces the the analysis of competitive equilibria of the concession game to the mathematical problem of solving a system of ordinary differential equations. For given monotone strategies b_t^* and s_t^* such that for some $T < \infty$, $b_{s_t^*}^{\alpha} = b_t^*$, for all $t \geq T$, system $(3)-(4)$ is linear in q_t^B and q_t^S, and by the Picard-Lindelöf theorem, it has a unique solution satisfying $q_T^B = q_T^S = q$ for some $q \in (c(s_T^*), v(b_T^*))$. To guarantee that strategies b_t^* and s_t^* are indeed optimal, one needs to verify that this solution gives monotone price paths q_t^B and q_t^S.

In the next subsection, I show that under additional restrictions, competitive equilibria described in Theorem 1 can be obtained as equilibria limits in the bargaining game where players are not restricted in their price offers. The restrictions are formulated in terms of continuation utilities of players, and it is useful to denote by $\mathcal{U}_t^B(b)$ and $\mathcal{U}_t^S(s)$ the continuation utilities at time t of buyer type b and seller type s, respectively, in a competitive equilibrium of the concession game. More precisely, for $t \leq t_B^*(b)$, let $\mathcal{U}_t^B(b) \equiv u^B(t_B^*(b), b) - u^B(t, b)$, and analogously, for $t \leq t_S^*(s)$, let $\mathcal{U}_t^S(s) \equiv u^S(t_S^*(s), s) - u^S(t, s)$.

Characterization of CSE

Competitive equilibria in the concession game are appealing because of their analytic tractability. However, the assumption that price paths are fixed seems far from innocuous at first sight. Next, I present the central result of this section justifying this assumption. Even if players are allowed to change their price offers, there are equilibria in the bargaining game, in which they choose not to do so and follow a given paths of offers. I first define the class of CSEs.

Definition 3. Common screening equilibria (CSEs) *are equilibria of the bargaining game in which on-path equilibrium strategies are described by the tuple* (b_n, s_n, p_n^B, p_n^S) *which satisfies the following properties.*

1. *A path of seller offers p_n^S changes only in odd rounds, and in any odd round n, all seller types that do not accept the buyer's offer make counter-offer p_n^S.*[26] *All buyer types follow a sequence of offers p_n^B, which changes only in even rounds.*

2. *Sequence p_n^S is (weakly) decreasing, and sequence p_n^B is (weakly) increasing.*

3. *There is a non-increasing sequence of threshold buyer types b_n and a non-decreasing sequence of threshold seller types s_n. In even rounds, all remaining buyer types above*

[26] In the text, I refer to a sequence $\{x_n\}_{n=1}^{\infty}$ by its member x_n and to a continuous time process $\{x_t\}_{t\geq 0}$ by its member x_t.

b_n accept the seller's offer p^S_{n-1}, and in odd rounds all remaining seller types below s_n accept buyer's offer p^B_{n-1}, so long as there were no deviations from price paths p^S_n and p^B_n in the past.

4. $c^{-1}(p^B_\infty) - v^{-1}(p^S_\infty) \geq \eta$.

A CSE is active *if on the equilibrium path a positive mass of remaining buyer or seller types accepts the opponent's offer in every round up to some $\bar{N} \leq \infty$.*

In a CSE, both sides screen the opponent's type and all types on either side use a common screening policy, i.e. they follow the same sequence of offers. In CSEs, both price paths and acceptance strategies are monotone. The property that higher buyer types accept the seller's offer earlier than lower types (and the reverse for the seller) is referred in the bargaining literature as a *skimming property.* The skimming property greatly simplifies the Bayesian updating of beliefs. In any round n, the posterior beliefs of any remaining type b buyer is a truncation of the uniform distribution on S_b at the bottom at s_n, and symmetrically, the beliefs of any remaining type s seller is a truncation of the uniform distribution on B_s at the top at b_n.[27]

Subsequently, I define the limit of CSEs as the round length Δ converges to zero. First, I extend strategies in the discrete-time game to continuous time. For any sequence of real numbers $\{f_n\}_{n\in\mathbb{N}}$, say that a function f_t is an *extension of $\{f_n\}_{n\in\mathbb{N}}$ to a continuous domain* if $f_t|_{t=n\Delta} = f_n$ for all $n \in \mathbb{N}$ and f_t is linear on each interval $[(n-1)\Delta, n\Delta]$.[28] To distinguish CSE on-path strategies b_n, s_n, p^B_n, p^S_n from their extensions b_t, s_t, p^B_t, p^S_t, respectively, I use time index t instead of round index n, whenever I refer to the extensions. Additionally, since the characterization of CSE limits is in terms of competitive equilibria of the concession game, with a slight abuse of notation, I use the same notation for the

[27] The role of the monotonicity restrictions in CSEs is similar to that in the analysis of the concession game. In the Appendix, I show that counterparts of Lemmas 1 and 2 hold for CSEs in the bargaining game. Lemma 9 demonstrates that if price paths are monotone and the seller uses a strategy satisfying the skimming property, then the expected utility of the buyer satisfies the strict single-crossing property. Moreover, for any strategy s_n of the seller that satisfies the skimming property, if a decreasing sequence b_n of threshold buyer types is such that for any $n < \bar{N}$, threshold types are indifferent between accepting the current offer and the offer in the next round, then b_n is the best-response to s_n (see Lemma 11 in the Appendix). This guarantees global optimality of the on-path strategies in the construction of the CSE equilibrium path. I additionally require that in CSEs an analogue of condition (1) holds. This way I focus on the inefficiencies that arise due to timing of the acceptance, but not because some of the gains from trade are not realized.

[28] Perhaps, it is more natural to define the extension of f_n to continuous domain to be a right-continuous function that coincides with f_n at times $t = n\Delta$. The results of the paper carry through for this definition with only slight modifications of the proofs.

CSE limit as for strategies in the concession game. The following definition formalizes the notion of convergence.

Definition 4. *A sequence* $(b_t^\Delta, s_t^\Delta, p_t^{B\Delta}, p_t^{S\Delta})$ *of CSEs indexed by* $\Delta \to 0$ *has a* smooth limit *if*

1. *processes* $b_t^\Delta, s_t^\Delta, p_t^{B\Delta}, p_t^{S\Delta}$ *converge pointwise to continuously differentiable limit processes* $b_t^*, s_t^*, q_t^B, q_t^S$*, respectively;*

2. $T = \limsup_{\Delta\to 0} T_\Delta$*, where* $T \equiv \inf\{t \geq 0 : b_{t'}^* = b_t^* \text{ and } s_{t'}^* = s_t^* \text{ for all } t' \geq t\}$ *and* $T_\Delta \equiv \inf\{t \geq 0 : b_{t'}^\Delta = b_t^\Delta \text{ and } s_{t'}^\Delta = s_t^\Delta \text{ for all } t' \geq t\}$*;*

3. $b_T^* = \lim_{\Delta\to 0} b_{T_\Delta}^\Delta$ *and* $s_T^* = \lim_{\Delta\to 0} s_{T_\Delta}^\Delta$.

The tuple $(b_t^*, s_t^*, q_t^B, q_t^S)$ *is called the* smooth limit *of the sequence.*

Condition 1 in Definition 4 implies that in the limit, no positive mass of types accepts the opponent's price offer in any arbitrarily short interval of time, and moreover prices do not change drastically. Condition 2 guarantees that the limit preserves information about when the trade ends with certainty. Condition 3 ensures that the sets of accepting types $(b_{T_\Delta}^\Delta, 1]$ and $[0, s_{T_\Delta}^\Delta)$ do not collapse in the limit.[29]

In contrast to the concession game, in the bargaining game players choose price offers that they make. This puts additional restrictions on price paths and acceptance strategies. The next lemma gives weak restrictions on equilibrium price offers in the bargaining game.

Lemma 3. *In any equilibrium and after any history,*

1. *any buyer's offer above* $\frac{c(1)+e^{-r\Delta}v(1)}{1+e^{-r\Delta}}$ *is accepted by the seller, and the buyer never accepts any offer higher than* $\frac{v(1)+e^{-r\Delta}c(1)}{1+e^{-r\Delta}}$*;*

2. *any seller's offer below* $\frac{v(0)+e^{-r\Delta}c(0)}{1+e^{-r\Delta}}$ *is accepted by the buyer, and the seller never accepts any offer lower than* $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$.

[29] The following two examples clarify the difference between conditions 2 and 3 in Definition 4. In both examples suppose that $s_n^\Delta = 0$ for all $n \in \mathbb{N}$ and $\Delta > 0$ and so, $s_t^* = 0$ for all $t \geq 0$. In the first example, suppose that for some $T' > 0$, $b_n^\Delta = \frac{T'-n\Delta}{T'} + \frac{n\Delta}{T'}\Delta$, for $n \in \mathbb{N} \cap \left[0, \frac{T'}{\Delta}\right)$, and $b_n^\Delta = \Delta e^{-(n\Delta - T')}$, for $n \in \mathbb{N} \cap \left[\frac{T'}{\Delta}, \infty\right)$. Then $T_\Delta = \infty$, but for all $t \geq T'$, $b_t^\Delta \to 0$ as $\Delta \to 0$ and so, $T = T'$. Hence, condition 2 is not satisfied, however, $b_{T_\Delta}^\Delta = b_T^* = 0$ and condition 3 holds.

In the second example, suppose that $b_n^\Delta = \frac{1}{2}\left(1 + e^{-n\Delta}\right)$, for $n \in \mathbb{N} \cap \left[0, \Delta^{-2}\right)$, and $b_n^\Delta = \frac{1}{2}\left(1 + e^{-\Delta^{-1}}\right) e^{-n\Delta + \Delta^{-2}}$, for $n \in \mathbb{N} \cap [\Delta^{-2}, \infty)$. Then $b_{T_\Delta}^\Delta = 0$, but for all $t \geq 0$, $b_t^\Delta \to \frac{1}{2}\left(1 + e^{-t}\right)$ as $\Delta \to 0$ and so, $b_T^* = \frac{1}{2}$. Hence, condition 3 is not satisfied, however, $T_\Delta = T = \infty$ and condition 2 holds.

The bounds in Lemma 3 rely on the fact that it is common knowledge among players that valuations belong to the interval $[v(0), v(1)]$ and costs belong to the interval $[c(0), c(1)]$. The interpretation is as follows. Suppose that the seller manages to convince the buyer that he has the highest possible costs, $c(1)$, and the buyer's valuation turns out to be $v(1)$, thus maximizing the size of the surplus. Then the outcome would be as in the unique subgame perfect equilibrium of the complete information game with valuation $v(1)$ and cost $c(1)$ analyzed by Rubinstein (1982). In such an equilibrium, the seller makes offer $\frac{v(1)+e^{-r\Delta}c(1)}{1+e^{-r\Delta}}$ and rejects any offer below $\frac{c(1)+e^{-r\Delta}v(1)}{1+e^{-r\Delta}}$, and the buyer makes offer $\frac{c(1)+e^{-r\Delta}v(1)}{1+e^{-r\Delta}}$ and rejects any offer above $\frac{v(1)+e^{-r\Delta}c(1)}{1+e^{-r\Delta}}$. By Lemma 3, the seller cannot get a higher payoff than in the scenario described. Moreover, Lemma 3 implies that the buyer always has the option to trade immediately at price $\frac{v(1)+e^{-r\Delta}c(1)}{1+e^{-r\Delta}}$ by admitting that he has the highest valuation $v(1)$ and recognizing that the seller has the highest costs $c(1)$.

Lemma 3 together with the fact that players can always reject any offer implies that in the frequent-offer limit, seller type s gets at least her *reservation utility* $\max\left\{\frac{v(0)+c(0)}{2} - c(s), 0\right\}$, and theorem *reservation utility* of buyer type b is $\max\left\{v(b) - \frac{v(1)+c(1)}{2}, 0\right\}$. This translates into the following restriction on the utilities that players get in the competitive equilibrium in the concession game. For all $t \in [0, T)$ and all b and s,

$$\mathcal{U}_t^B(b) \geq \max\left\{v(b) - \frac{v(1)+c(1)}{2}, 0\right\}, \tag{6}$$

$$\mathcal{U}_t^S(s) \geq \max\left\{\frac{v(0)+c(0)}{2} - c(s), 0\right\}. \tag{7}$$

The next theorem shows that in the limit of frequent offers, conditions (6) and (7) are the only restrictions that the ability to choose price offers puts on the equilibrium price paths and acceptance strategies. In particular, it establishes that under additional generic conditions on equilibrium strategies, the sets of active CSE smooth limits and of competitive equilibria in smooth monotone strategies coincide.

Theorem 2. *Consider a sequence of active CSEs indexed by $\Delta \to 0$ with a smooth limit. Then the smooth limit of the sequence constitutes a competitive equilibrium in the concession game, and in addition, satisfies conditions (6) and (7).*

Conversely, consider a competitive equilibrium of the concession game in smooth monotone strategies $(b_t^*, s_t^*, q_t^B, q_t^S)$ satisfying Condition M′ and date T specified in condition (2)*. In addition, suppose that $b_\infty^* \in (0,1)$, $s_\infty^* \in (0,1)$, $c(s_T^*) < q_T^B \leq q_T^S < v(b_T^*)$, and strict versions of inequalities (6) and (7) hold. Then there exists a sequence of active CSEs* *indexed by* $\Delta \to 0$ *with a smooth limit* $(b_t^*, s_t^*, q_t^B, q_t^S)$.

Theorem 2 sheds light on the limit dynamics of trade and sources of inefficiency in CSEs. In CSEs, players simultaneously screen each other's types. There are three categories of types determined endogenously: *weak, strong,* and *pliable*. Strong types never accept the opponent's offer. In contrast, offers of weak types are never accepted. Hence, strong types are the screening types and weak types are the screened types. The behavior of pliable types is ambiguous. With positive probability, their offers are accepted by the opponent, but if not they will eventually give in and accept the opponent's offer. By the skimming property, these categories are ordered as follows. There exists time θ such that $b_\theta^* = b_{s_\theta^*}^\omega$. At time θ, threshold types start putting positive probability on the acceptance of their price offer by the opponent. Highest buyer types $[b_\theta^*, 1]$ are weak types, lowest buyer types $[0, b_T^*]$ are strong types and pliable types are in the interval (b_T^*, b_θ^*). The categories of seller types are ordered in the opposite way: weak types are in $[s_T^*, 1]$, pliable types are in (s_θ^*, s_T^*) and strong types are in $[0, s_\theta^*]$. Since all types use a common screening policy, the profitability of the policy is different for different types. In particular, the common screening policy is most profitable for strong types, while weak types do not benefit from the screening policy at all.

Equilibrium conditions allow us to understand sources of inefficiency in the bargaining model with a global games information structure. Two standard sources of inefficiency are reflected in conditions (3) and (4). For example, consider equation (3), which describes the evolution of threshold buyer types. A faster decrease in seller price offers q_t^S leads to higher b_t^* and creates an inefficient delay. This is the standard deadweight loss from screening. If the seller were not discriminating, then q_t^S would not change and this would lead to a lower b_t^*, hence, faster trade.

To see the inefficiency due to signaling, consider the likelihood λ_t^S that the buyer's offer is accepted. In equation (3), an increase in λ_t^S results in higher threshold buyer type b_t^*. By delaying trade, the buyer signals the seller that his valuation is low and further delay could be costly for the seller. The stronger the impact of such a signal on the seller's behavior (higher λ_t^S), the higher the incentives of the buyer to signal by inefficiently delaying trade.

The fact that there is a "pecking order" of types, and it could take a long time until the common screening policy becomes efficient for the types in the middle of the type range creates a new source of bargaining inefficiency. To see this effect, observe that seller type s expects positive profit from her screening offers only after time t when $b_t^* \leq b_s^\omega$, and buyer types in the support of her beliefs start accepting seller's screening offers. Suppose type s is such that time t when $b_t^* \leq b_s^\omega$ is finite. Until this time, seller type s follows the common screening path q_t^S, even though she knows that such offers are rejected with certainty. As a result, the delay for seller type s is increased by the amount of time it takes to screen buyer types above b_s^ω.

Theorem 2 has an important empirical implication. Irrespective of individual uncertainty, bargaining may start from offers that are far from the equal division of the realized surplus and trade can be significantly delayed. An important predictor of the spread of offers is the range of values commonly known, while the individual knowledge of the players might matter very little.

Proof Sketch of Theorem 2

I next describe the main methodological contribution of this paper. To show that a competitive equilibrium $(b_t^*, s_t^*, q_t^B, q_t^S)$ satisfying conditions of Theorem 2 can be obtained as a smooth limit of the sequence of CSEs, I construct a sequence of CSEs in grim trigger strategies. Equilibria in grim trigger strategies contain two ingredients: the main path and the punishment path. Players start the game by following the main path and continue to follow it unless a detectable deviation occurs. Detectable deviations from the CSE equilibrium path trigger the punishment, and players switch to the punishing path for the deviating side given by punishing equilibria analyzed in detail in Section 6. By Theorem 7, as $\Delta \to 0$, the utility of any type of the deviator in the punishing equilibrium converges uniformly to the lowest utility possible in the equilibrium which in conjunction with the strict versions of inequalities (6) and (7) allows us to support the main path. In this subsection, I focus on the steps in the construction of the main path.

The construction of the main path is based on the approximation of differential equations (3) and (4) by difference equations. For $T < \infty$, there is an approximating sequence of strategies $(b_t^\Delta, s_t^\Delta, p_t^{B\Delta}, p_t^{S\Delta})$ that converges uniformly to $(b_t^*, s_t^*, q_t^B, q_t^S)$ as $\Delta \to 0$. By the uniform convergence, deviations from the main path can be deterred by the threat of switching to the punishing path.

The key in the construction is to guarantee that it extends to the case $T = \infty$, since by condition (2), competitive equilibria of the concession game need not end in finite time. The difficulty is that it is no longer possible to construct a uniform approximation of competitive equilibrium strategies as for $T < \infty$. To circumvent this difficulty, Lemma 4 constructs particular continuation CSEs with $T = \infty$, in which on the equilibrium path, price offers are constant over time and the mass of the remaining types could be arbitrarily small. Given this result, an equilibrium in which negotiation continues indefinitely is approximated with an equilibrium in which after a certain time T' price offers become constant. For times before T', a uniform approximation of $(b_t^*, s_t^*, q_t^B, q_t^S)$ is available, and I can proceed as in the case $T < \infty$.

Lemma 4. *Consider $b_0 \in (0, 1-\eta]$, $s_0 \in [b_0 - \eta, b_0 + \eta) \cap [\eta, 1)$, P^B, P^S that satisfy*

$$\max\left\{c\left(s_{b_0}^{\omega}\right), \frac{v(0)+c(0)}{2}\right\} < P^B < P^S < \min\left\{v\left(b_{s_0}^{\alpha}\right), \frac{v(1)+c(1)}{2}\right\}, \tag{8}$$

Then for all Δ sufficiently small, there exists an active continuation CSE such that

1. *b_0 and s_0 are the highest buyer type and the lowest seller type, respectively, remaining in the game,*
2. *$p_n^B = P^B$ and $p_n^S = P^S$ for all $n \in \mathbb{N}$,*
3. *$\max\{b_{n-1} - b_n, s_n - s_{n-1}\} < \Delta C$ for all $n \in \mathbb{N}$, where C is a constant independent of Δ.*

Lemma 4 constructs a continuation CSE that starts from the moment when only buyer types below b_0 and seller types above s_0 remain in the game. There are two price offers P^B and P^S that are made on the equilibrium path, and each player decides whether to accept the less favorable offer of the opponent, or delay the acceptance in the hope that the opponent will accept earlier.[30] In every round, a positive mass of types of the active player accepts. Condition (8) ensures that for all types remaining in the game the utility from accepting the opponent's offer exceeds their reservation utility.[31] Given that there is

[30]The equilibrium constructed in Lemma 4 is similar to equilibria in war of attrition game. See Fudenberg and Tirole (1991) for a survey of the literature on the war of attrition. Abreu and Gul (2000) establish a connection between reputational bargaining and the war of attrition. Krishna and Morgan (1997) analyze the war of attrition with affiliated values as an auction form, in which the winning bidder pays the highest loosing bid and loosing bidders pay their bids. The literature on the war of attrition has a different payoff structure and is mostly formulated in continuous time, so I was not able to build on the techniques used in this literature.

[31]To understand the requirement on b_0 and s_0 in Lemma 4, observe that in the sufficiency part of Theorem 2, it holds $b_\infty^* \in (0,1)$ and $s_\infty^* \in (0,1)$, and together with condition (2), this implies $b_\infty^* \in (0, 1-\eta)$ and $s_\infty^* \in (\eta, 1)$. I use Lemma 4 to construct a continuation CSE, in which the remaining buyer types are below b_0 and the remaining seller types are above s_0, and b_0 and s_0 are close to b_∞^* and s_∞^*, respectively. Therefore, I place the restriction $b_0 \in (0, 1-\eta]$ and $s_0 \in [\eta, 1)$. The requirement $s_0 \in [b_0 - \eta, b_0 + \eta)$ guarantees that starting from the first round in the continuation equilibrium both sides assign positive probability to the acceptance of their offer in the next round. This makes the concession continuous in the limit $\Delta \to 0$ with no mass positive mass of types accepting in any instant of time, and in particular, implies that the bound on $\max\{b_{n-1} - b_n, s_n - s_{n-1}\}$ in Lemma 4 holds.

a positive difference in players' payoffs from trading at P^B or P^S, bargaining necessarily continues indefinitely.[32] The last property of the continuation equilibrium constructed in Lemma 4, guarantees that in the limit concession happens continuously.[33]

It should be mentioned that the equilibrium construction in Lemma 4 is significantly harder for the case of global games information structure compared to the case of independent values ($\eta = 1$). In this paper, under the skimming property, buyer types with higher valuations also put lower probability on the acceptance by the seller, and the reverse for the seller. Players have additional incentives to accept the opponent's offer earlier, which makes it harder to guarantee that bargaining continues indefinitely. For intuition of differences in the analysis, compare the incentives of the threshold types in the case $\eta = 1$ and $\eta < 1$. To keep the threshold buyer type indifferent between accepting and rejecting the current offer, the probability of the seller acceptance in the next round should be sufficiently high. When types are independent ($\eta = 1$), it is possible to vary this probability from 0 to 1 by varying the threshold seller type in the next round. In this case, for any initial choice of b_2, it is possible to construct recursively the subsequent thresholds.

However, when types are interdependent ($\eta < 1$), there is an upper bound on how high the probability of seller acceptance evaluated by buyer type b can be. This comes from the fact that for the buyer type b, all seller types in the interval $[s_\infty, s_b^\omega]$ never accept the buyer price offer. In this case, if the construction starts from an arbitrary choice of the first threshold types b_2 and proceeds recursively, it can happen that in some round n there is no threshold type of the active player in round $n+1$ that makes the threshold type of the active player in round n indifferent between the acceptance and delay. Nevertheless, Lemma 4 establishes that it is possible to find an initial threshold type b_2 so that the recursive construction of thresholds is possible.

4 Common Screening Mechanisms

The previous section, shows that in addition to the standard sources of bargaining inefficiency, such as deadweight loss from screening and signaling costs, in CSEs surplus is dissipated through the common-screening inefficiency. In this section, I show that even if

[32]Otherwise, for sufficiently small Δ, players would prefer to marginally delay the acceptance before the final date. This would give a discontinuous gain in the payoff, making the acceptance at times close to the final date suboptimal.

[33]Indeed, in an interval of length Δ at most mass ΔC of types concedes and so, the speed of acceptance is bounded above by C.

the individual uncertainty of players is vanishingly small, efficiency is not guaranteed, and in fact, a variety of equilibria with screening dynamics is possible. Theorem 3 characterizes outcomes of CSEs as both Δ and η converge to zero in terms of CSMs.

I start by defining interim CSE outcomes and describing CSMs. For any CSE and buyer type b, define the discounted probability of allocation by $P^B(b) \equiv \mathbb{E}\left[e^{-r\Delta N}|S_b, \sigma_b\right]$ and the discounted transfer by $X^B(b) \equiv \mathbb{E}\left[e^{-r\Delta N}p|S_b, \sigma_b\right]$.[34] Functions $P^S(s)$ and $X^S(s)$ for the seller are defined analogously. Tuple (P^B, X^B, P^S, X^S) determines the *interim outcome* of the game, that is, the expected outcome of each player after the type of the player is realized, but before the type of the opponent is known.

The characterization of almost-sure limits of interim CSE outcomes that I will present in Theorem 3 is given in terms of truthful equilibria in a class of static mechanisms.[35] Consider the following mechanism design problem. The values are determined by a commonly known fundamental $\omega \in [0, 1]$ that is not observed by the mechanism designer. A mechanism specifies a set of messages that each player sends, and a mapping from messages into outcomes that consists of a probability of trade and a transfer from the buyer to the seller.

Definition 5. *A* common screening mechanism (CSM) *is a game in which both players announce simultaneously the fundamental. There is a threshold $\omega^* \in [0, 1]$ such that the outcome is determined by the buyer announcement if both announcements lie to the right of ω^*, the outcome is determined by the seller announcement if both announcements lie to the left of ω^*. Outcome $(\bar{P}(\omega^*), \bar{X}(\omega^*))$ is implemented if both players announce ω^*, and the outcome $(0, 0)$ is implemented otherwise. The mapping $(\bar{P}(\omega), \bar{X}(\omega))$ from fundamentals to outcomes satisfies the following conditions.*

1. *For all $\omega, \omega' > \omega^*$,*

$$\bar{P}(\omega)v(\omega) - \bar{X}(\omega) \geq \bar{P}(\omega')v(\omega) - \bar{X}(\omega') \tag{9}$$

   and

$$\bar{P}(\omega)v(\omega) - \bar{X}(\omega) \geq \max\left\{0, v(\omega) - \frac{v(1) + c(1)}{2}\right\}. \tag{10}$$

[34] The expectations are taken conditional on the event that buyer type b follows the equilibrium strategy σ_b, a seller type is drawn from a uniform distribution on S_b and the seller follows the equilibrium strategy σ_s.

[35] Ausubel and Deneckere (1989a) and Ausubel, Cramton and Deneckere (2001) used mechanism-design approach to characterize frequent-offer limits of equilibria in bargaining models with one-sided incomplete information and offers by uninformed side.

2. *For all* $\omega, \omega' < \omega^*$,

$$\bar{X}(\omega) - \bar{P}(\omega)c(\omega) \geq \bar{X}(\omega') - \bar{P}(\omega')c(\omega) \tag{11}$$

and

$$\bar{X}(\omega) - \bar{P}(\omega)c(\omega) \geq \max\left\{0, \frac{v(0)+c(0)}{2} - c(\omega)\right\}. \tag{12}$$

3. *Left and right limits of* $\bar{X}(\omega)$ *exist at* ω^* *and* $\bar{X}(\omega^*+0) \geq \bar{X}(\omega^*-0)$.

Conditions (9) and (11) are standard incentive compatibility constraints that are required to hold only for the party that determines the outcome. Conditions (10) and (12) are individual rationality constraints, adjusted for the fact that by Lemma 3, in the bargaining game the equilibrium price of trade should lie in the interval $\left[\frac{v(0)+c(0)}{2}, \frac{v(1)+c(1)}{2}\right]$ in the limit of frequent offers. Observe that if the inequality in condition 3 of the definition is strict, then the seller prefers that the mechanism designer implements an outcome for ω slightly above ω^*, rather than slightly below ω^*, and the buyer preferences are reverse. Intuitively, by condition 3, fundamentals above ω^* correspond to terms of trade more favorable for the seller, and fundamentals below ω^* are more favorable for the buyer.

It is a simple observation that for any equilibrium of a CSM, there is an equivalent CSM in which players announce the fundamental truthfully in equilibrium. Since values are common knowledge among players, probabilities of allocation $\bar{P}^B(b)$ and $\bar{P}^S(s)$ and transfers $\bar{X}^B(b)$ and $\bar{X}^S(s)$ in the truthful equilibrium of any CSM are determined by the outcome $(\bar{P}(\omega), \bar{X}(\omega))$ with $\bar{P}^B(\omega) = \bar{P}^S(\omega) = \bar{P}(\omega)$ and $\bar{X}^B(\omega) = \bar{X}^S(\omega) = \bar{X}(\omega)$. I will refer to the tuple $(\bar{P}^B(b), \bar{P}^S(s), \bar{X}^B(b), \bar{X}^S(s))$ as the outcome of the truthful equilibrium of the CSM.

Even though players have common knowledge about the fundamental, the mechanism designer needs to extract this information to implement the CSM outcome. For this purpose, the mechanism designer screens buyer types for high fundamentals (above ω^*) and seller types for low fundamentals (below ω^*). This parallels the dynamics in CSEs where strong player types screen weak types of the opponent. The exact relation between CSM outcomes and CSE interim outcomes is given in the following theorem.

Theorem 3. *For any truthful equilibrium outcome of some CSM that satisfies*

$$\bar{P}(\omega) > 0 \textit{ for all } \omega \neq \omega^* \textit{ and } \bar{P}(\omega^*+0) = \bar{P}(\omega^*-0), \tag{13}$$

there is a sequence of CSEs indexed by $(\Delta, \eta) \to (0, 0)$ *such that CSE interim outcomes* *converge to the equilibrium CSM outcome for almost all types.*

Conversely, consider a sequence of CSEs indexed by $(\Delta, \eta) \to (0,0)$. *Then the sequence of CSE interim outcomes converges over subsequence to the truthful equilibrium outcome of some CSM for almost all types.*

Condition (13) states that there cannot be unrealized gains from trade, and that probabilities of allocation near the threshold state ω^* should be similar. Theorem 3 shows that unlike in the global games literature, taking individual uncertainty to zero in the model does not lead to sharp predictions about equilibrium behavior. On the one hand, this is not surprising as the bite of the global games refinement is much weaker in the dynamic framework and when there is enough public information (see Angeletos, Hellwig and Pavan (2007) and Chassang (2010)).

On the other hand, Theorem 3 allows for qualitative predictions about delay that were not possible based on the bargaining model with complete information. For example, consider the inter-dealer market for corporate bonds. Index bonds by their normalized expected payoff $\omega \in [0,1]$. The selling side is willing to sell the bond, because of the liquidity needs or other considerations. Traders differ in their value of a particular bond both because of the specifics of their portfolio strategy and because of subjective differences in the evaluation of the bond. I consider the limit when these differences become negligible, which is a natural limit for the OTC markets on which traders have a significant expertise in evaluating the bonds.

By Theorem 3, the liquidity of an asset ω in general depends non-monotonically on ω, even if the heterogeneity in traders' values is very small. Observe that by the standard argument from the mechanism design (see Myerson (1981)), the incentive compatibility constraints (9) and (11) imply that $\bar{P}(\omega)$ is decreasing for $\omega > \omega^*$ and increasing for $\omega < \omega^*$. In particular, bonds with the highest expected payoff (ω close to one) and the lowest expected payoff (ω close to zero) are expected to be more liquid than the assets in the middle of the range of expected payoffs. Moreover, if the common uncertainty on the market about the quality of the bond is large (the range of $v(b)$ and $c(s)$ is large), then very inefficient equilibria are possible, in which the spread between the buyer and seller offer is big, and it takes a significant time to complete trades for bonds in the middle of the range.

Theorem 3 presents a simple way to take common uncertainty into account in the context of a more general model. In applied work one might be willing to incorporate the effect of uncertainty on the delay and the price. A natural way to do this is by introducing incomplete information into the model. However, this often complicates the analysis and could turn out to be intractable in the more general framework. Therefore, many models assume that the gains from trade are common knowledge and trade happens according to the Nash bargaining solution (Nash (1953)). An important drawback of this concept is that it implies that trade occurs immediately. By Theorem 3, one could assume instead that the individual uncertainty is negligible so it does not affect the decisions not directly related to the bargaining process, but at the same time, there is a non-trivial common uncertainty. Theorem 3 shows that a wide range of outcomes is possible and they have a simple and intuitive static characterization.

5 Segmentation Equilibria

In CSEs, as individual uncertainty vanishes, common-screening inefficiency becomes a dominant source of the surplus dissipation. In this section, I analyze a very different class of equilibria which I call segmentation equilibria. In this class of equilibria, common-screenig inefficiency is completely eliminated shortly after the start of bargaining. The trade dynamics and efficiency properties of these equilibria drastically differ from those of CSEs.

In *segmentation equilibria,* types partially separate by their first offers into Z segments so that after the first rounds of acceptance, both sides assign positive probability to the opponent accepting their offer in each following round. In this sense, common-screening inefficiency is eliminated after the first rounds. Each segment $z \in \{1, \cdots, Z\}$ is associated with a particular offer of the seller q_z^S and the buyer q_z^B, and on the equilibrium path only offers from $\{q_1^S, \ldots, q_Z^S\}$ and $\{q_1^B, \ldots, q_Z^B\}$ are made. The next theorem provides conditions on the number of segments and price offers in each segment, under which the construction of a segmentation equilibrium is possible.

Theorem 4. *Suppose an integer Z, an increasing sequence of offers $\{q_z^B\}_{z=1}^Z$, and increasing sequences $\{b^z\}_{z=1}^Z$ and $\{s^z\}_{z=1}^Z$ of buyer and seller types are such that $b^0 = s^0 = 0$, $b^Z = s^Z = 1$ and*

1. *$s^z = b^z - \eta$ and $c(s^z) < q_z^B < v(b^{z-1})$ for $z = \overline{1, Z-1}$,*

2. *$b^{z+1} - b^z > 4\eta$ for $z = \overline{1, Z-2}$.*

Then for sufficiently small Δ, there exists a segmentation equilibrium with Z segments and buyer price offers $\{q_1^B, \ldots, q_Z^B\}$ such that there is no almost sure upper bound on the *delay on the equilibrium path, but ex-ante probability of delay longer than two rounds is bounded above by* $\frac{4\eta(Z-1)}{2-\eta}$.

The proof of Theorem 4 generalizes the construction of Lemma 4 and condition 1 of Theorem 4 ensure that such generalization is possible. Condition 2 ensures that segments are significantly far apart, so that every player type puts positive probability on at most two opponent's offers on the equilibrium path. Given the assumption of strict gains from trade (recall $\xi > 0$), condition 1 of Theorem 4 holds for any η sufficiently small.

To understand how the segments are constructed, it is useful to consider the limiting case of low individual uncertainty $\eta \approx 0$ and frequent offers $\Delta \approx 0$. All buyer types in (b^{z-1}, b^z) and seller types in (s^{z-1}, s^z) belong to segment z. All of those types but a small $\eta-$neighborhood around boundaries trade almost immediately at the prices corresponding to the segment, q_z^B or q_z^S, and $q_z^B \approx q_z^S$. Types near boundaries of segments have incentives to delay trade and form reputation for belonging to a segment with more favorable terms of trade. These types could continue bargaining for arbitrarily long time. In the constructed segmentation equilibrium, the probability that the trade is delayed is at most $\frac{4\eta(Z-1)}{2-\eta}$, and this upper bound is increasing in both η and Z. As the individual uncertainty η increases, the mass of the types that form a reputation increases and the equilibrium becomes less efficient. For the same reason, more segments reduce the efficiency.

Theorem 4 could be reformulated to allow for the segmentation to happen over time rather than all at once in first rounds. Together with Theorem 2, this gives a rich description of possible equilibrium behavior. Intervals of gradual (common) screening, as in CSEs, are interrupted by rounds, in which remaining types split into endogenous segments and the common uncertainty is drastically reduced.

It is common in the economic literature to use the Nash bargaining solution (Nash (1953)) to make predictions about the division of the surplus.[36] In this paper, the Nash bargaining solution corresponds to equal division of the realized surplus. Rubinstein (1982) provides non-cooperative foundations for the Nash bargaining solution. He shows that the frequent-offer limit of the unique subgame perfect equilibrium of the complete information bargaining game with alternating offers is immediate trade with the division of surplus as in the Nash bargaining solution. Correspondingly, define the *Nash outcome* to be $\left(0, \frac{v(b)+c(s)}{2}\right)$. The next theorem constructs a sequence of segmentation equilibria with

[36]For example, the Nash bargaining solution was used to study the relationship between unemployment and search on labor market (see Mortensen and Pissarides (1994)), liquidity on over-the-counter markets (see Duffie, Gârleanu, and Pedersen (2005)), renegotiation in contract theory (see Tirole (1999)), equilibrium selection in repeated games (see Miller and Watson (2013)).

increasingly fine definition of segments such that outcomes of these equilibria approximate the Nash outcome.

Theorem 5. *There exists a sequence of segmentation equilibria indexed by* $(\Delta, \eta) \to (0, 0)$ *such that outcomes* $(N\Delta, p)$ *of segmentation equilibria converge in probability to the Nash outcome* $\left(0, \frac{v(b)+c(s)}{2}\right)$*, i.e. for any* $\varepsilon > 0$ *there exists a segmentation equilibirum in the sequence such that*

$$\mathbb{P}\left(N\Delta > \varepsilon, \left|p - \frac{v(b) + c(s)}{2}\right| > \varepsilon\right) < \varepsilon. \tag{14}$$

To prove Theorem 5, I apply Theorem 4 to construct segmentation equilibria with $Z \sim \frac{1}{\sqrt{\eta}}$ segments and prices $q_z^B = \frac{v(b^{z-1})+c(s^z)}{2}$. As $\eta \to 0$, the probability of any given delay is bounded from above by $\frac{4\eta(Z-1)}{2-\eta} \sim \sqrt{\eta}$ and converges to zero. The length of each segment $\sqrt{\eta}$ also converges to zero and so, q_z^B is close to the Nash division for types in each segment z.

Section 4 shows that as individual uncertainty vanishes, equilibrium outcomes in the bargaining model can differ drastically from Nash outcomes. This section establishes that reduction in the common uncertainty is crucial in achieving the Nash outcome. Observe that a public announcement restricting the range of values of both sides to $s, b \in [\underline{\omega}, \overline{\omega}]$ for some $0 < \underline{\omega} < \overline{\omega} < 1$ reduces the range of possible prices. As this announcement become more informative ($\overline{\omega} - \underline{\omega}$ decreases), by Lemma 3, the frequent-offer limit of the equilibrium outcome converges to immediate equal division of the realized surplus.[37] Combining this observation with Theorem 5, the Nash outcome is the limit of equilibrium outcomes in the bargaining game, when common uncertainty is reduced exogenously (by making $v(b)$ and $c(s)$ flatter) or endogenously (by decreasing individual uncertainty η and increasing the number of segments Z). Unlike previous models, this paper stresses the role of common uncertainty as opposed to individual uncertainty in achieving the Nash outcome as the limit of the equilibrium outcomes.

6 Punishing Equilibria

This section introduces and analyzes the *seller punishing equilibrium* which is key in deterring seller deviations from the equilibrium paths of CSEs. Since a buyer punishing equilibrium is constructed analogously, in this section, I refer to the seller punishing

[37] To see this, notice that this is equivalent to making functions $v(b)$ and $c(s)$ flatter while keeping $s, b \in [0, 1]$. By Lemma 3, the range of possible prices (in the limit of frequent offers) shrinks to that giving an immediate equal division of surplus.

equilibrium as simply the punishing equilibrium. In the punishing equilibrium, the buyer holds optimistic beliefs and puts probability one on the lowest seller type in the support of his beliefs S_b, while the seller has her prior beliefs. As a result, in the subgame following a detectable deviation from the CSE path, beliefs are not common-prior beliefs. I first describe carefully strategies in the punishing equilibrium and prove the existence in Theorem 6. I next present in Theorem 7 the crucial uncertainty invariance property of the punishing equilibrium. In the limit of frequent offers, the utility of the seller in the (seller) punishing equilibrium converges to the lowest utility possible in equilibrium. In the end of the section, I demonstrate by numerical simulations that this property does not hold for a given frequency of offers.

Description of strategies

Beliefs. The punishing equilibrium is an equilibrium of the game in which seller types hold their original beliefs, while buyer types hold *optimistic beliefs*. More precisely, buyer types put probability one on the lowest seller type in the support of their prior beliefs, i.e.

$$\mu_b^n(s_b^\alpha) = 1 \tag{15}$$

for all histories h^n with some seller detectable deviation.[38]

Beliefs described in (15) are a natural counterpart of optimistic beliefs commonly used in the bargaining literature. Since optimistic beliefs of the buyer might exclude the realized seller's type, the buyer and the seller may have different expectations regarding the path of play.[39] I refer to the path of play expected by the seller in the punishing equilibrium as the *equilibrium path of the punishing equilibrium*.

[38]Such beliefs could be justified by the following trembles in the model with a finite number of types and finite grid of price offers. Seller's and buyer's types come from $\{k/K\}_{k=1}^K$ for some integer K. Let $\eta \equiv i/K$ for some integer $i \in \{1, .., K\}$. Suppose price offers come from a discrete set $\mathtt{P}$. Seller type s trembles with probability $(1-s)^m/2$ for some integer m and conditional on trembling chooses a price offer uniformly from $\mathtt{P}$. As $m \to \infty$, the probability of tremble converges to zero. Yet, conditional on the buyer type b the probability that the tremble comes from seller type s_b^α is $\frac{(1-s_b^\alpha)^n}{(1-s_b^\alpha)^m + \sum_{s \in S_b \setminus s_b^\alpha}(1-s)^m} \to 1$ as $m \to \infty$, since $\frac{1-s}{1-s_b^\alpha} < 1$.

[39]For example, suppose that buyer type b and seller type $s \in S_b \setminus s_b^\alpha$ are realized. In the punishing equilibrium, beliefs of buyer type b assign probability one to type s_b^α. If the punishing equilibrium strategies prescribe different actions for seller types s and s_b^α, then buyer type b will observe seller's deviations from the expected path of play. In turn, seller type s knows that any buyer type b in the support of her beliefs is optimistic. The seller takes into account the fact that the buyer could perceive her action as a deviation from the equilibrium strategy.

Buyer on-path strategy. All buyer types pool on the lowest acceptable price offer $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$ (cf. Lemma 3). Buyer type b accepts any price offer less than or equal to his *willingness to pay* $P(b)$ which is left-continuous and strictly increasing in b. Since $P(b)$ is strictly increasing, for any history h^n without buyer deviations, there exists a buyer type $\beta \in [0,1]$ such that only buyer types in the interval $[0,\beta]$ remain in the game. Whenever $\beta \geq b_s^\alpha$, posterior beliefs of seller type s are uniform on $B_s \cap [0,\beta]$.

Seller on-path strategy. The seller faces the static demand function given by $P(b)$ and makes price offers to screen buyer types by their willingness to pay. Since $P(b)$ is left-continuous, it is never optimal for the seller to offer a price in $[P(b), P(b+0))$, if b is point of discontinuity of $P(b)$.[40] Indeed, alternatively the seller could offer price $P(b+0)$ and still sell the good to all buyer types above b, but at a higher price. Let $\hat{P}(b)$ be a right-continuous function that is equal to $P(b)$ in all continuity points of $P(b)$. Then the strategy of the seller could be equivalently represented as follows. Given the highest remaining buyer type β, seller type $s > 0$ chooses a cut-off buyer type $t_\beta(s)$ and allocates to all remaining buyer types above $t_\beta(s)$. To reach this goal, the seller should make offer $\hat{P}(t_\beta(s))$.[41]

The strategy of seller type 0 differs from the rest of seller types, due to the fact that a positive mass of buyer types in $[0,\eta]$ puts probability one on seller type 0. Seller type 0 (and only this seller type) accepts buyer price offer $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$, whenever the highest buyer type remaining in the game is below some $\bar{\beta} \in (0,\eta]$. Given the highest remaining buyer type $\beta \in (\bar{\beta},\eta]$, seller type 0 allocates to buyer types above $t_\beta(0)$.

Before moving on to the description of strategies off-path, I give optimality conditions that on-path strategies of the punishing equilibrium should satisfy. The problem of seller type s could be formulated recursively. Let bounded function $R_\beta(s)$ for $\beta \in [b_s^\alpha, 1]$ be the

[40] For any strictly increasing and left-continuous function $f(x)$ we use notation $f(x+0) = \lim_{x' \to x+0} f(x')$ for right limit of $f(x)$ at point x (which exists by monotonicity of $f(x)$).

[41] It might be tempting to define $P(b)$ as a right-continuous function and this way avoid the necessity to introduce auxiliary function $\hat{P}(b)$. This, however, is not possible. To see this, suppose that every seller type does not screen and allocates to buyer type b_s^α in the first round. Then

$$P(b) = \begin{cases} (1-e^{-r\Delta})v(b) + e^{-r\Delta}\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}, & \text{for } b \in [0,\eta], \\ (1-e^{-2r\Delta})v(b) + e^{-2r\Delta}P(\max\{b-2\eta, 0\}), & \text{for } b \in (\eta, 1]. \end{cases}$$

It is easy to see that such function is not right-continuous. In particular,

$$(1-e^{-r\Delta})v(\eta) + e^{-r\Delta}\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}} = P(\eta) < P(\eta+0) = (1-e^{-3r\Delta})v(\eta) + e^{-3r\Delta}\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}.$$

value function of seller type s satisfying Bellman equation[42]

$$R_\beta(s) = \sup_{b\in B_s\cap[0,\beta]} \left\{(\beta - b)(\hat{P}(b) - c(s)) + e^{-2r\Delta} R_b(s)\right\}. \tag{16}$$

Let $R_\beta(0) = e^{r\Delta}\beta\left(\frac{\delta v(0)+c(0)}{1+\delta} - c(0)\right)$ for $\beta \in [0, \bar{\beta}]$ and $R_\beta(0)$ be given by (16) for $\beta \in (\beta, \eta]$. This reflects the fact that seller type 0 accepts price offer $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$ whenever $\beta \leq \bar{\beta}$. Denote by $T_\beta(s)$ the set of maximizers of the right-hand side of (16). A seller strategy $t_\beta(s)$ is a *best-reply* to buyer strategy $P(b)$, if $t_\beta(s) = \inf T_\beta(s)$ for all s and $\beta \geq b_s^\alpha$. A special role in the analysis is played by the first cut-off buyer type chosen by seller type s, which I denote by $t(s) \equiv t_{b_s^\omega}(s)$.

For a screening strategy $t_\beta(s)$ of the seller, the willingness to pay $P(b)$ for $b \in (\eta, 1]$ is given by

$$P(b) = (1 - e^{-2r\Delta})v(b) + e^{-2r\Delta}\hat{P}(t(s_b^\alpha)) \tag{17}$$

The interpretation of (17) is the following. The expectation of buyer type b about future screening offers of the seller is determined by the screening policy of seller type s_b^α. Buyer type b in the interval $(\eta, 1]$ believes that he is the highest buyer type in the support of beliefs of seller type s_b^α. If the seller makes price offer $P(b)$, then in the next screening round, buyer type b will be the highest buyer type remaining in the game. Hence, buyer type b will expect to buy the good in the next round at price $\hat{P}(t(s_b^\alpha))$. Equation (17) states that buyer type b is just indifferent between accepting price offer $P(b)$ and getting utility $b - P(b)$, and rejecting $P(b)$ and accepting price offer $P(t(s_b^\alpha))$ in the following round of screening.

As with seller type 0, willingness to pay of buyer types in the interval $[0, \eta]$ differs from the rest of the buyer types. Both on and off the equilibrium path of the punishing equilibrium, it is determine by some strictly increasing and left-continuous function $P^0(b)$.

Strategies off-path. If the buyer makes a price offer different from $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$ or $\beta < b_s^\alpha$, seller type s switches to optimistic beliefs and assigns probability one to the highest buyer type in the support of her prior belief, i.e.

$$\mu_s^n(b_s^\omega) = 1 \tag{18}$$

[42]The value function is defined only on the set $\{(\beta, s) \in BS : b_s^\alpha \leq \beta\}$. Outside of this set, seller s detects that state β is achieved as a result of buyer deviation and switches to the optimistic belief (18) as specified below. In this case the seller's behavior is described by Lemma 5. Also observe that $R_\beta(s) = R_{b_s^\omega}(s)$ for all $\beta \geq b_s^\omega$.

for all histories h^n with both seller and buyer detectable deviations. Observe that if seller type s has optimistic beliefs after some history, then any higher seller type has optimistic beliefs as well. The following lemma describes equilibrium strategies when both players have optimistic beliefs. This result is based on the analysis of the bargaining game with complete information (Rubinstein (1982)).

Lemma 5. *Suppose that for some $\underline{b} \in [0,1]$ beliefs of buyer types above $\underline{b}$ and seller types above $s^{\alpha}_{\underline{b}}$ are described by (15) and (18). Then the following strategies are the equilibrium strategies for such buyer and seller types. After any history, buyer type b in the interval $(\underline{b}, 1]$ accepts price offer less than or equal to $\check{P}^B(b)$. Otherwise, such type makes counter-offer $\check{A}^B(b)$. After any history seller type s in the interval $(s^{\alpha}_{\underline{b}}, 0]$ accepts price offer greater than or equal to $\check{P}^S(s)$. Otherwise, such type makes counter-offer $\check{A}^S(s)$. Functions $\check{P}^B(b)$ and $\check{A}^B(b)$ are given by*

$$\check{P}^B(b) = \begin{cases} (1-e^{-r\Delta})v(b) + e^{-r\Delta}\check{P}^S(0) \\ \frac{v(b)+e^{-r\Delta}c(b-\eta)}{1+e^{-r\Delta}} \end{cases} \qquad \check{A}^B(b) = \begin{cases} \check{P}^S(0), & \text{for } b \in [0,\eta), \\ \frac{c(b-\eta)+e^{-r\Delta}v(b)}{1+e^{-r\Delta}}, & \text{for } b \in [\eta, 1]. \end{cases}$$

and functions $\check{P}^S(s)$ and $\check{A}^S(s)$ are given by

$$\check{P}^S(s) = \begin{cases} \frac{c(s)+e^{-r\Delta}v(s+\eta)}{1+e^{-r\Delta}} \\ (1-e^{-r\Delta})c(s) + e^{-r\Delta}\check{P}^B(1) \end{cases} \qquad \check{A}^S(s) = \begin{cases} \frac{v(s+\eta)+e^{-r\Delta}c(s)}{1+e^{-r\Delta}}, & \text{for } s \in [0, 1-\eta], \\ \check{P}^B(1), & \text{for } s \in (1-\eta, 1]. \end{cases}$$

Seller deviations from the equilibrium strategies in the punishing equilibrium are ignored. If buyer type b rejects a seller price offer lower than $P(b)$, then the seller detects such deviation only if $b > \beta + 2\eta$. In this case, the continuation play is as in Lemma 5. If $\beta < b \leq \beta + 2\eta$, then such deviation is not detected and buyer type b makes price offer $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$, and accepts any price offer less than $P_\beta(b) \equiv (1-e^{-2r\Delta})v(b) + e^{-2r\Delta}P(t_\beta(s^{\alpha}_{b}))$, which now depends also on the highest remaining buyer type β. This completes the description of the strategies in the punishing equilibrium.

Existence

In this subsection, I show the existence of the punishing equilibrium. The proof of the existence is constructive, and later in this section, I implement the algorithm to describe equilibrium constraints for a given frequency of offers. The key in the construction is to show that willingness to pay $P(b)$ and screening policy $t_\beta(s)$ satisfying (16) and (17) exist. The next theorem presents the result.

Theorem 6. *For all sufficiently small Δ, the seller punishing equilibrium exists.*

I sketch main steps of the construction of the punishing equilibrium. The construction is carried out starting from the bottom of the type distribution. I first analyze strategies of seller type 0 and buyer types in $[0, \eta]$ that put probability one on this seller type. This is the model with one-sided incomplete information and alternating offers, and the following result is standard in the literature (see Grossman and Perry (1986), Gul and Sonnenschein (1988)).

Lemma 6. *For all sufficiently small Δ, there exists a PBE in a game between seller type 0 and buyer types in $[0, \eta]$, in which on the equilibrium path*

1. *the buyer makes price offer $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$ and accepts seller price offers according to left-continuous and strictly increasing willingness to pay function $P^0(b)$;*
2. *there exists $\bar{\beta} \in [0, \eta]$ such that if the highest remaining buyer type is below $\bar{\beta}$, then seller type 0 accepts the buyer price offer $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$;*
3. *given the highest remaining buyer types $\beta \in (\bar{\beta}, \eta]$, seller type 0 allocates to buyer types above $t_\beta(0)$ in the current round.*

Moreover, for any $\varepsilon > 0$ the first price offer of seller type 0 does not exceed $\frac{v(0)+c(0)}{2} + \varepsilon$ for Δ sufficiently small.

One detail worth mentioning is that despite the fact that seller type 0 follows a pure strategy on the equilibrium path, out-of-the-equilibrium path the mixing might be necessary (see footnote 19). To comply with the restriction of the analysis to pure strategies, the following assumption is made.[43]

Assumption P. *Seller screening strategy in Lemma 6 is pure off the equilibrium path.*

Strategies for the rest of the types are constructed via the tâtonnement algorithm that runs as follows. Buyer types in $[0, \eta]$ put probability one on seller type 0 and have willingness to pay $P^0(b)$. By Lemma 17 in the Appendix, all seller types allocate to at least a mass $c(\eta, \Delta)$ of buyer types in the first round of screening. Hence, it is sufficient to

[43] In numerical simulations in the end of the section, I give an example of the model in which assumption P is satisfied.

know the willingness to pay of buyer types in $[0,\eta]$ to constuct the screening policy $\tau^1_\beta(s)$ of seller types in $[0, c(\eta,\Delta)]$. Moreover, buyer types in $[\eta, \eta + c(\eta,\Delta)]$ put probability one on sellers in the interval $[0, c(\eta,\Delta)]$. On Step 1 of the algorithm, screening policy $t_\beta(s)$ for seller types in $[0, c(\eta,\Delta)]$ and willingness to pay $P(b)$ for buyer types $[\eta, \eta + c(\eta,\Delta)]$ is constructed. The algorithm continues "climbing up" the types with an increment $c(\eta,\Delta)$.

Tâtonnement Algorithm

Input: Constant $c(\eta,\Delta)$ is specified in Lemma 17. Define

$$\pi^0(b) = \begin{cases} P^0(b), \text{ for } b \in [0,\eta], \\ v(b), \text{ for } b \in (\eta, 1]. \end{cases}$$

Execute Step i, i=1, . . . , I+1 where I is the smallest integer such that I$c(\eta,\Delta) \geq 1-\eta$.
Step i. Construct a best-reply $\tau^i_\beta(s)$ to $\pi^{i-1}(b)$. Construct $\pi^i(b)$ by

$$\pi^i(b) = \begin{cases} \pi^{i-1}(b), & \text{for } b \in [0, \eta + (i-1)c(\eta,\Delta)], \\ (1-e^{-2r\Delta})v(b) + e^{-2r\Delta}\hat{\pi}^{i-1}(\tau^i(s^\alpha_b)), & \text{for } b \in (\eta + (i-1)c(\eta,\Delta), \eta + ic(\eta,\Delta)], \\ v(b), & \text{for } b \in (\eta + ic(\eta,\Delta), 1]; \end{cases}$$

where $\hat{\pi}^{i-1}(b)$ denotes the right-continuous function that coincides with $\pi^{i-1}(b)$ at all continuity points of $\pi^{i-1}(b)$.
Output: $P(b) = \pi^{I+1}(b)$ **and** $t_\beta(s) = \tau^{I+1}_\beta(s)$**.**

By construction, $t_\beta(s)$ is a best-reply to $P(b)$, and it is left to verify that $P(b)$ is the optimal acceptance strategy for the buyer and it is optimal for buyer types to pool on $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$. The former is proven in Lemma 19 in the Appendix, and the argument uses the monotonicity in s of seller screening strategy $t_\beta(s)$ and the monotonicity of $P(b)$. The proof of the latter is based on the invariance property proven in the next subsection.

Invariance property

This subsection proves *the uncertainty invariance* of the punishing equilibria limits. The frequent-offer limit of the punishing equilibria gives all seller types their reservation utility level independent of η. At the same time, all optimistic buyer types expect to get the good in the first round of the seller's screening at the price that converges to the lowest (type specific) price. The former property allows us to support a wide range of equilibrium behavior in CSEs analyzed in Section 3. The latter property gives the final step in the proof of Theorem 6, as it deters deviations of the buyer from pooling on the price offer $\frac{c(0)+e^{-r\Delta}v(0)}{1+e^{-r\Delta}}$ in the punishing equilibrium. Therefore, the seller punishing equilibrium is a natural candidate for deterring deviations from the equilibrium path: it simultaneously punishes all the types of the seller as harshly as possible, and rewards all the types of buyer by the greatest amount possible.

I will now formally state the result. Consider a sequence $\Delta \to 0$ such that the punishing equilibrium exists for each Δ in the sequence. For each Δ, let $(P^{\Delta}(b), t^{\Delta}_{\beta}(s))$ be equilibrium path strategies of the punishing equilibrium for the length of rounds Δ. Then the limit of $P^{\Delta}(b)$ is given by the following theorem.

Theorem 7. *The sequence $P^{\Delta}(b)$ converges uniformly (over subsequence) to $P^*(b) = \min\left\{\frac{v(0)+c(0)}{2}, c(s^{\alpha}_{b})\right\}$ as $\Delta \to 0$.*

I refer to the result in Theorem 7 as the *uncertainty invariance property*, as the limit of $P^{\Delta}(b)$ does not depend on η. For $\eta = 1$, Theorem 7 states the Coasian property of the punishing equilibrium. As $\Delta \to 0$, seller type 0 looses all monopoly power and allocates to all buyer types at the lowest price. Surprisingly, even for small η, in the punishing equilibrium the seller gets her reservation utility in the frequent-offer limit.[44] Even though the buyer types become only marginally optimistic, the coordination of all buyer types on the optimistic beliefs creates the connection between the screening policies of different seller types. Low screening offers of seller type 0 force seller types slightly above 0 to make low price offers, as a big fraction of the buyer types that they face belongs to $[0, \eta]$ and expects almost immediate allocation at the price close to $\frac{v(0)+c(0)}{2}$ from seller type 0. This leads buyer types slightly above η to expect a low price offer and, in turn, forces a larger set of seller types to make price offer close to $\frac{v(0)+c(0)}{2}$. This way even the seller types that are significantly far from seller type 0 are forced to make low price offers.

The intuitive contagion mechanism described above is more delicate than it might seem at first sight. As players become more patient, the seller screens more thoroughly, in the sense cut-offs of the seller screening strategy become closer together. Hence, seller types slightly above seller type 0 spend an increasing number of rounds selling to buyer

[44] Observe that unlike on the CSE equilibrium path, in the punishing equilibrium all types on the punishing side benefit from coordinating on optimistic beliefs. Every type of the punishing player (subjectively) gets the highest possible utility in the frequent-offer limit.

types above η. If such time is positive in the limit, then it is possible that the limiting willingness to pay of the buyer types would be higher than $\frac{v(0)+c(0)}{2}$. In fact, this happens for the buyer types that put probability one on the seller types with costs above $\frac{v(0)+c(0)}{2}$. However, as Theorem 7 shows even though the limiting willingness to pay increases for such buyer types, it does not go above $c(s_b^\alpha)$. Given that the buyer's willingness to pay is lowest possible in the limit of frequent offers, the seller's utility approaches the reservation utility as $\Delta \to 0$. The following corollary of Theorem 7 formally states this result.

Corollary 1. *For any $\varepsilon > 0$, the continuation utility of any seller type s in the seller punishing equilibrium is at most* $\max\left\{\frac{v(0)+c(0)}{2} - c(s), 0\right\} + \varepsilon$ *for sufficiently small Δ.*

Proof Sketch of Theorem 7

In this subsection, I outline main steps of the proof of Theorem 7. The proof of Theorem 7 is broken down into three steps. On each step the limit willingness to pay function $P^*(b)$ for a separate category of buyer types is analyzed. Let s^+ be the seller type for whom $c(s^+) = \frac{v(0)+c(0)}{2}$ holds. On the first step, it is shown that for buyer types in $[0, \eta]$ and seller type 0 equilibrium behavior exhibits Coasian dynamics. Namely, as $\Delta \to 0$, the first offer of seller type 0 is close to the buyer's demand $\frac{c(0)+v(0)}{2}$. On the second step buyer types in $(\eta, b^\omega_{s^+}]$ and seller types in $(0, s^+]$ are analyzed and the last step covers the remaining types. The difference between these two cases is that seller types below s^+ have positive expected profit from the lowest buyer type in the support of their beliefs when they face limit willingness to pay function $P^*(b)$, while for seller types above s^+ such profit is zero.[45]

Step 1. The last statement in Lemma 6 states that the Coasian property holds in the game between seller type 0 and buyer types in the interval $[0, \eta]$. Hence, $P^*(b) = \frac{v(0)+c(0)}{2}$

[45] On the technical level, there is a parallel between the uncertainty invariance property of the punishing equilibrium and the Coasian property in models with one-sided incomplete information. The analysis of the flat part of function $P^*(b)$ is similar to the analysis of the gap case in the Coasian literature. In the seller punishing equilibrium, any particular seller type facing the flat part of $P^*(b)$ is guaranteed to get positive profit as in the gap case. The analysis of the increasing part of $P^*(b)$ is similar to the no-gap case, as the seller type who faces such buyer types is getting a profit close to zero. The techniques used in this paper build on and further develop the techniques in the Coasian literature (Gul, Sonnenschein and Wilson (1986), Gul and Sonnenschein (1988), Ausubel and Deneckere (1989), Ausubel and Deneckere (1992)). The difference in the analysis is most clear for seller types above s^+. Ausubel and Deneckere (1989) impose a mild technical assumption on the buyer valuation function to obtain the Uniform Coase conjecture. In this paper, the willingness to pay of buyer types in the support of beliefs of seller types above s^+ is determined endogenously from the screening policy of seller types below s^+. Therefore, assumption on the valuation function does not give much leverage and additional work is necessary to prove Theorem 7.

for $b \in [0, \eta]$.

Step 2. The next lemma shows that if the limit function $P^*(b)$ is increasing at some point b, then it is equal to the reservation price of the seller type s^{α}_{b}.[46]

Lemma 7. *Suppose that for some buyer type $\hat{b} \in (0,1)$, $P^*(\hat{b}) > c(s^{\alpha}_{\hat{b}})$. Then $P^*(b)$ is constant in some open interval around $\hat{b}$, that is, there exists $\phi > 0$ such that $P^*(b)$ is constant for all $b \in (\hat{b} - \phi, \hat{b} + \phi)$.*

Lemma 7 means that function $P^*(b)$ could be increasing at point b only if buyer type b expects the seller to make the first offer close to $c(s^{\alpha}_{b})$. In other words, function $P^*(b)$ could have jumps only at points where $P^*(b) = c(s^{\alpha}_{b})$. In particular, the following is the immediate corollary of Lemma 7.

Corollary 2. *For any $b < b^{\omega}_{s^+}$, $P^*(b) = \frac{v(0)+c(0)}{2}$.*

Step 3. It is more intricate to find the limit of screening policy for seller types above s^+ for the following reason. For seller type $s < s^+$, it could be shown that a positive mass of buyer types in B_s has willingness to pay close to $\frac{v(0)+c(0)}{2}$ and so, the profit from allocating to all remaining buyer types at price close to $\frac{v(0)+c(0)}{2}$ is positive. Suppose seller type s delays trade at price $\frac{v(0)+c(0)}{2}$ to sell at price exceeding $\frac{v(0)+c(0)}{2}$. Such delay should be sufficiently large to guarantee that buyer type b^{ω}_{s} buys in the first round of screening. Then for any $\varepsilon > 0$, it is possible to construct an alternative screening policy that accelerates the trade at prices above $\frac{v(0)+c(0)}{2}$ and looses at most ε on such trades. The advantage of such policy is that it allows the seller to allocate to all buyer types with the willingness to pay $\frac{v(0)+c(0)}{2}$ sooner. Since the profit from such buyer types is strictly positive, for sufficiently small ε such alternative screening policy is preferred by the seller giving a contradiction.

The reasoning above is not valid for seller types above s^+. These seller types eventually decrease their screening offers to the level close to their costs. Hence, they could have incentives to spend a significant amount of time screening buyer types that bring them positive profit. The next lemma is key in establishing that the time seller types above s^+ screen buyer types is enough to keep $P^*(b)$ just above $c(s^{\alpha}_{b})$.

Lemma 8. *Function $P^*(b)$ is continuous.*

The next corollary of Lemmas 7 and 8 completes the proof of Theorem 7

[46] An increasing function $f(x)$ is *increasing at point x* if for all $\phi > 0$, $f(x + \phi) > f(x - \phi)$.

Corollary 3. *For any* $b \geq b^{\omega}_{s+}$, $P^*(b) = c(s^{\alpha}_b)$.

It is interesting to notice that even though buyer types $b > b^{\omega}_{s+}$ expect to get almost all the surplus from trade, they end up not trading at all with probability approaching one as $\Delta \to 0$. The reason for this is that the realized seller types is with probability one higher than s^{α}_b and so, such seller type will never make an offer close to $c(s^{\alpha}_b)$. However, buyer type b has optimistic beliefs in the punishing equilibrium, and he rejects all seller offers that are even slightly higher than $c(s^{\alpha}_b)$.

Numerical Simulations

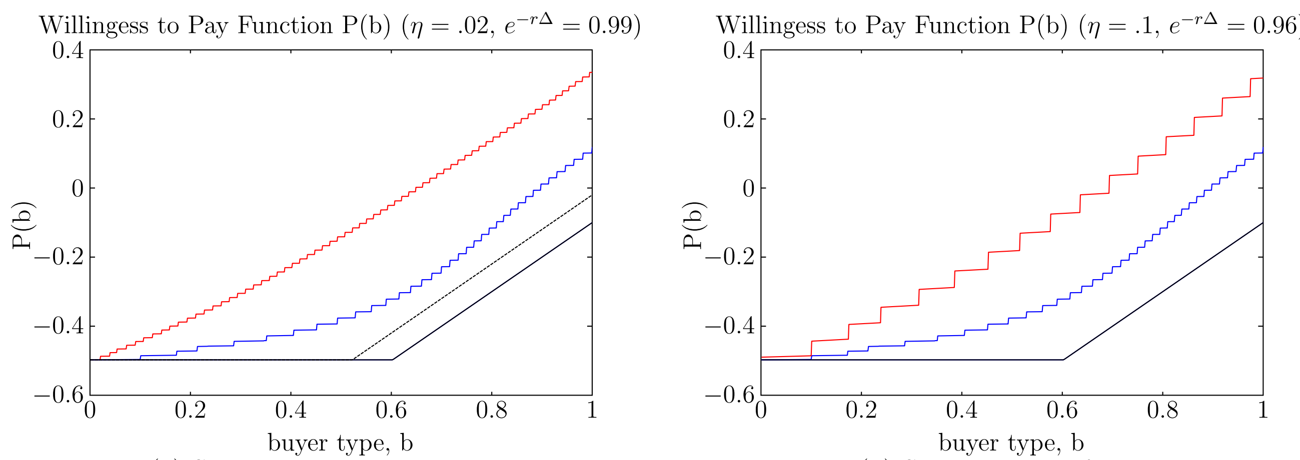
By Theorem 2, in the limit of frequent offers, the amount of individual uncertainty η affects only the equilibrium path of CSEs, but does not restrict the severity of the punishment. In this subsection, I demonstrate by numerical simulations that for a given frequency of offers, the individual uncertainty matters both on and off the equilibrium path of CSEs supported by punishing equilibria. In such equilibria, the continuation utility of players is greater than the reservation utility, and this additional constraint should be taken into account in the construction of equilibria for a given frequency of offers.

To clearly see the effect of η on the equilibrium behavior, I assume that $v(b) = b$ and $c(s) = s - 1$. In this specification, immediate trade could occur only at price 0, which gives payoff 0 to buyer type 0 and seller type 1. Therefore, if all types can guarantee strictly positive continuation utility in the punishing equilibrium, then all CSEs supported by punishing equilibria will be inefficient.

To implement the tâtonnement algorithm, I discretize the type space and make corresponding adjustments to the algorithm.[47] To illustrate the results of the simulations, I depict the willingness to pay function $P(b)$ in the seller punishing equilibrium. Recall, that $P(b)$ gives a static demand function that the seller is facing and so, the utility of each seller type s in the punishing equilibrium lies in the interval $[P(b^{\alpha}_s), P(b^{\omega}_s)]$. The higher is

[47] By Lemma 17 in the Appendix, the step of the the tâtonnement algorithm $c(\eta, \Delta)$ is of order Δ^3. Hence, the grid of the discretization should be increasingly fine as $\Delta \to 0$, which requires significant computational resources. Instead, I run the tâtonnement algorithm with the step equal to the size of the grid, and this way, I construct the candidate willingness to pay and screening policy functions. After that, I verify that the candidate screening policy of the seller is, indeed, optimal given the candidate willingness to pay function. In all simulations, this turns out to be sufficient. The tâtonnement algorithm is further simplified by the functional form assumption on $v(b)$ and $c(s)$. As shown in Lemma 24 in Appendix, for any $\eta \in (0, 1)$, there is an equilibrium in the game between seller type 0 and buyer types in B_0, in which all buyer types in B_0 pool on the offer $-\frac{1}{1+e^{-r\Delta}}$, and seller type 0 accepts this offer in the first round of the punishing equilibrium.

the willingness to pay function, the greater is the seller's expected utility in the punishing equilibrium.



(a) Small individual uncertainty (b) Small discount factor

Figure 2: Willingness to pay function $P(b)$.

Figure 3 illustrates the results of the simulation for the grid size 0.001 for three combinations of η and Δ: 1) $\eta = 0.1, e^{-r\Delta} = 0.99$, 2) $\eta = 0.02, e^{-r\Delta} = 0.99$, 3) $\eta = 0.1, e^{-r\Delta} = 0.96$. To compare the willingness to pay for different values of parameters, I take as a benchmark the case $\eta = 0.1$ and $e^{-r\Delta} = 0.99$. The willingness to pay function $P(b)$ for this combination of parameters is a blue line (middle line) on Figures 3a, 3b. Red lines (upper lines) on Figures 3a and 3b depict the willingness to pay function for cases $\eta = 0.02, e^{-r\Delta} = 0.99$, and $\eta = 0.1, e^{-r\Delta} = 0.96$, respectively. I also depict by a solid black line (lower line) function $\max\left\{-\frac{e^{-r\Delta}}{1+e^{-r\Delta}}, c(s_b^\alpha)\right\}$ for the benchmark case. This function gives a minimal price that buyer type b could pay in the punishing equilibrium. Since $\max\left\{-\frac{e^{-r\Delta}}{1+e^{-r\Delta}}, c(s_b^\alpha)\right\}$ depends on η, I also depict by a dashed line on Figure 3a the corresponding function for the case $\eta = 0.02, e^{-r\Delta} = 0.99$.

Simulations show that in the benchmark case, even though the discount factor is close to one, $P(b) > \max\left\{-\frac{e^{-r\Delta}}{1+e^{-r\Delta}}, c(s_b^\alpha)\right\}$ (see blue line on Figures 3a, 3b). Hence, the utility to highest seller types, and by symmetry, lowest buyer types exceeds their reservation utility. This implies that the delay is necessary in any CSE supported by punishing equilibria. To see that the delay could be substantial consider the following simple lower bound on the equilibrium delay. Let $r = 0.05$ and consider a CSE, in which bargaining ends at some time $T \leq \infty$ and bargaining stops at some price below 0. The latter is without loss of generality due to the symmetry of values. When $\eta = 0.1$ and $e^{-r\Delta} = 0.99$ the willingness to pay of buyer type 1 equals 0.1189, and in any active CSE he prefers to accept price offer p_1^S rather than wait until price falls below zero at some date t. Therefore, $e^{-rt} \le 1 - p_1^S$ and so, $T \ge t \ge -\frac{1}{r}\ln(1-p_1^S) \ge -\frac{1}{r}\ln 0.8811 \approx 2.53$ giving the lower bound on the equilibrium delay.

In the case when the individual uncertainty is relatively small, $\eta = 0.02, e^{-r\Delta} = 0.99$, the punishing equilibrium gives even higher utility to the punished side (red line on Figure 3a). On the one hand, this happens for an apparent reason that the lowest seller type s_b^α in the support of beliefs of buyer type b is higher for smaller η (compare dashed and solid black lines on Figure 3a). That is, even in the limit $\Delta \to 0$, the limit willingness to pay $P^*(b) = \max\left\{-\frac{1}{2}, c(s_b^\alpha)\right\}$ is higher for lower η. On the other hand, this effect is amplified when $\Delta > 0$. To see this, I consider the following pseudo-dynamics. Recall that for $b > \eta$, $P(b)$ satisfies

$$P(b) = (1 - e^{-2r\Delta})v(b) + e^{-2r\Delta}\hat{P}(t(s_b^\alpha)). \tag{19}$$

Suppose that η decreases. From equation (19), even if the seller does not screen and $t(s_b^\alpha) = \max\{0, s_b^\alpha - \eta\}$, this leads to higher $\hat{P}(t(s_b^\alpha))$, and in turn, increases $P(b)$ to some new $P_1(b)$ for all buyer types. This increase in $P(b)$ leads to higher $\hat{P}_1(t(s_b^\alpha))$, which in turn, increases $P_1(b)$ to some new $P_2(b)$. This process continues until the higher willingness to pay is reached for lower η. This increase is further amplified by the fact that the seller screens and $t(s_b^\alpha) \ge \max\{0, s_b^\alpha - \eta\}$. Steeper demand curve given by $P(b)$ gives the seller incentives to screen more finely and increases $t(s_b^\alpha)$. On Figure 3a, this is reflected in more frequent jumps in function $P(b)$ for $\eta = 0.02$ compared to the benchmark case $\eta = 0.1$. For the case of relatively small discount factor, $\eta = 0.1, e^{-r\Delta} = 0.96$, the punishing equilibrium gives higher expected utility to the punishing side because of the increased weight on the buyer valuation in (19). To understand the quantitative difference, I construct the lower bound on the equilibrium delay for these two cases as in the benchmark case. Corresponding lower bounds on the delay for cases $\eta = 0.02, e^{-r\Delta} = 0.99$ and $\eta = 0.1, e^{-r\Delta} = 0.96$ are 8.16 and 7.67, respectively, compared to 2.53 in the benchmark case. Therefore, both lower discount factor and smaller individual uncertainty lead to an increased utility of the deviator in the punishing equilibrium.

Numerical simulations demonstrate that for a given frequency of offers, the expected utility of the deviator in the punishing equilibrium is significantly greater than the reservation utility of the punished side. In particular, decreasing η leads to an increase in the utility of the punished side. From this perspective, smaller individual uncertainty refines the CSE outcomes supported by punishing equilibria for fixed length of bargaining rounds. As shown in Section 3, such refinement disappears as the length of bargaining rounds vanishes.[48]

7 Related Literature

Models with independent private values are generally known to be prone to a multiplicity of equilibria. The literature has taken the route to refine the predictions in such models by restricting the bargaining protocol to one-sided offers and putting restrictions on out-of-the-equilibrium-path beliefs. In the equilibrium analyzed in Cramton (1984), seller types initially pool on the same path of offers, but separate over time starting from the bottom of the type distribution. Cho (1990) constructs a class of separating equilibria in which all seller types separate by price offers in every round of bargaining. To eliminate optimistic conjectures, both papers require conjectures to satisfy a monotonicity condition. If a price offer higher than the equilibrium price is made, then it is believed to come from a higher type of the seller.[49] Without imposing restrictions on beliefs out-of-the-equilibrium path, Ausubel and Deneckere (1992) show that with no gap between the lowest seller and buyer types' values, optimistic conjectures can support monopoly equilibria. In the monopoly equilibria, all seller types except a small subset at the bottom of the distribution reveal themselves by offering a monopoly sales price. Such types trade in the first round and never trade after, since lowering the price would lead to the buyer switching to optimistic conjecture and imply no trade for such seller types.

Unfortunately, many interesting equilibria in the model with one-sided offers are not guaranteed to have counterparts in the model with two-sided offers. In particular, separating equilibria constructed in Cho (1990) and monopoly equilibria in Ausubel and Deneckere (1990) do not survive if both sides are allowed to make offers. In this paper, I construct a variety of equilibria, in which types separate both by the acceptance time of the opponent's offer (CSEs) and by price offers made (segmentation equilibria). I do not put restriction on beliefs off the equilibrium path except for condition G, and optimistic conjectures play an important role in the analysis. The monotonicity condition used in the previous literature seems less compelling in the environment discussed in this paper.

[48] It is an open question whether there exist different equilibria that provide a more severe punishment than the punishing equilibrium. However, optimistic conjectures received special attention in the bargaining literature, and the class of equilibria supported by punishing equilibria is interesting on its own.

[49] The intuition behind this restriction is that lower seller types are more willing to trade and, hence, they should be more likely to decrease the price. This gives higher types an opportunity to separate by making a higher price offer.

On the one hand, with heterogeneous beliefs the description of conjectures satisfying the monotonicity condition is a daunting task as one needs to carefully specify beliefs for every type of the punishing player.[50] From this point, the appeal of the optimistic conjectures lies in their simplicity. On the other hand, one might hope that by introducing correlation in the types it is possible to get a strong prediction about the bargaining outcome even without refining the equilibrium concept as in the global games literature. This paper shows that even when correlation is nearly perfect, a great variety of outcomes can be supported, which contrasts drastically with the predictions of the complete information bargaining models. Optimistic conjectures efficiently deter deviations irrespective of the amount of individual uncertainty.[51]

Another strand of literature explores models with asymmetric information and interdependent values in which a fundamental determines values, but only one side is informed about the fundamental. A comprehensive analysis of this model is given in Deneckere and Liang (2006), Fuchs and Skrzypacz (2013), and Gerardi, Hörner and Maestri (2013). The model in the current paper is complementary to both the literature on bargaining with two-sided independent private values and on bargaining with asymmetric interdependent information. It covers the applications in which both parties of the negotiation are symmetric both from the informational perspective and from the commitment perspective.[52]

The global games information structure analyzed in this paper has parallels with the information structure studied in Feinberg and Skzypacz (2006). In their model, the buyer's valuation and seller's beliefs are private information of players. The seller makes all the offers, and she could be of one two types. The seller could either be informed that the buyer's valuation is high or be uncertain about it. Under an intuitive criterion and a revelation condition, they show that the delay is unavoidable. However, if instead seller's types differ in the probability they assign to low and high valuation of the buyer, then the Coase conjecture obtains and all equilibria are efficient. Thus, the support of seller beliefs matters for the delay. The results of this paper come to the opposite conclusion. If under large uncertainty ($\eta = 1$) immediate agreement is possible ($v(0) > c(1)$), then no matter how small the support of players' beliefs is ($\eta \to 0$), there is an efficient equilibrium for sufficiently frequent offers. The key difference seems to lie in the refinement that Feinberg

[50] The task is much simpler with independent values as all types of the player have the same beliefs about the opponent's type.

[51] Of course, this reasonining is valid only for frequent offer limits but not true for a particular discount factor as shown in Section 6.

[52] On either the inter-dealer car market or the market for corporate bonds, there is no a priori reason to assume that one side is better informed or has more commitment power in the negotiation, so symmetry is a desirable property.

and Skrzypacz (2006) use in their model.[53] The implication of such refinement for the model in this paper is an interesting direction for future research.

The segmentation equilibria described in this paper are related to Abreu and Gul (2000) which is another two-sided offers and two-sided incomplete information bargaining model. Abreu and Gul (2000) build a reputational bargaining model, in which commitment types require a particular share of surplus and rational types form reputation for being commitment types. As in Abreu and Gul (2000), in segmentation equilibria types near boundaries of segments delay the trade to convince the opponent that they belong to a segment with more favorable terms of trade. Abreu and Gul (2000) show that in the unique frequent offer limit of equilibria in their model, and rational players concede with probability one by some finite time. Unlike in their model, bargaining between rational types takes infinite time. The difference stems from the fact that without commitment types, it is not possible for bargaining to end in finite time, since the utility of a rational player is discontinuous at this time and a sufficiently patient player prefers to wait past this time. This paper provides one explanation of how bargaining postures can naturally arise endogenously in the a model with uncertainty about the values, and shows that the dynamics of concession could differ from that in the reputational bargaining model.

8 Conclusion

This paper analyzes implications of a global games information structure in a standard bargaining model with alternating offers and an infinite horizon. I study two classes of equilibria with very distinct equilibrium dynamics and efficiency properties. In CSEs, both sides gradually screen the opponent's type. In such equilibria, even in the limit as the individual uncertainty vanishes, a significant delay in trade is possible for a wide range of types. At the other extreme are segmentation equilibria in which types self-select into endogenous segments by their initial price offers. For small individual uncertainty, most of the types in such equilibria trade immediately after the first offers, and only a small mass of types at boundaries of segments continues bargaining. The characterization of CSE limits is given in terms of competitive equilibria in smooth monotone strategies of the concession game (for smooth limits) and truthful equilibria of CSMs (double limits). Both of them are intuitive and analytically tractable. To support the equilibrium path, I introduce and analyze punishing equlibria with optimistic beliefs of the punishing side.

[53] This refinement allows the informed seller to convince the buyer that she possesses the information about the valuation of the buyer.

Invariance property of the punishing equilibria implies that the punished side gets the lowest utility irrespective of the level of individual uncertainty. I believe that techniques developed in this paper will be useful in the analysis of other dynamic models with correlated types. In particular, the analysis of punishing equilibria could be extended to the model with interdependent values.

Following, I describe potential avenues for future research. Results of the paper hold for general valuation and cost functions, but the distribution of types has a particular form. Weakening the assumption on the types distribution is an important question left for future research.

Another direction is to extend the model to interdependent values environment.[54] This environment is better suited for the analysis of trading on some financial markets, like over-the-counter markets for collateralized debt obligations and mortgage-backed securities. On such markets, trade is decentralized and traders' valuations of the asset depend crucially on the characteristics of the asset. Studying the role of common uncertainty in the asset liquidity is an exciting topic for future research.

This paper provides a useful benchmark for future research suggesting that to get sharper predictions additional restrictions are required. A natural development of the model is to explore the predictions of the model under the presence of outside options, as in Fuchs and Skrzypacz (2007), or to endogenize the length between bargaining rounds and use an intuitive criterion style refinement, as in Admati and Perry (1987) and Cramton (1992).

[54] By interdependent values environment I mean that the values of players are determined by an unobserved fundamental and player's get signals about the fundamental.

Appendix

In the Appendix, the proofs are presented in the order in which the results appear in the paper. We use the following additional notations. Let $\Sigma \equiv \max_{(s,b)\in SB}\{v(b) - c(s)\}$ be maximal gains from trade possible in the game. In a CSE, denote by $U_n^B(b)$ and $U_n^S(s)$ expected continuation utilities in round n of type b buyer and type s seller, respectively, and by $U_t^B(b)$ and $U_t^S(s)$ their extensions to a continuous domain. For CSE strategies b_n and s_n, denote by $n_b \equiv \inf\{n : b_n \leq b\}$ and $n_s \equiv \inf\{n : s_n \geq s\}$ rounds of acceptance of type b buyer and type s seller, respectively. We use $\delta \equiv e^{-r\Delta}$ for players' discount factor and the frequent-offer limit ($\Delta \to 0$) corresponds to the limit $\delta \to 1$. For any reals a and b, denote $a \vee b \equiv \max\{a, b\}$ and $a \wedge b \equiv \min\{a, b\}$.

To unify the notation, whenever we talk about the sequence of equilibria, we reserve index j to indicate magnitudes arising in the j's equilibrium in the sequence. In particular, we use superscript j to denote functions in the j's equilibrium, and subscript j to denote variables that we introduce in the analysis of the j's equilibrium.[55]

Proofs for Section 3

Concession Game

Proof of Lemma 1. Suppose that the acceptance strategy of the seller $t_S^*(s)$ (or alternatively s_t^*) is monotone. Consider buyer types $b < b'$, times $t < t' \leq t_S^*(s_b^\omega) \leq t_S^*(s_{b'}^\omega)$, and suppose type b buyer prefers to accept at time t rather than time t'. If $s_{t'}^* < s_b^\alpha$, then the probability that the buyer's offer is accepted before time t' is zero for both b and b' and so, buyer type b' strictly prefers to accept at time t by the single-crossing property of $e^{-rt}(v(b) - q_t^S)$. Suppose that $s_{t'}^* \geq s_b^\alpha$. Let $\varphi(b) = \frac{s_b^\omega - s_{t'}^* \wedge s_b^\omega}{s_b^\omega - s_t^* \vee s_b^\alpha}$ be the probability that type b buyer assigns to the event that the seller does not accept the buyer's offer before time t' conditional on the fact that she has not accepted by time t. Notice that $\varphi(b) < \varphi(b')$. The following two claims prove the strict single-crossing property of $u^B(t, b)$.

[55] For example, in the analysis of punishing equilibria, we consider a sequence of seller punishing equilibira as $\delta_j \to 0$. In such sequence, $(P^j(b), t_\beta^j(s))$ denote on-path equilibrium strategies in j's punishing equilibrium, and in the proof of the invariance property, we introduce types b_j,β_j,s_j,σ_j and quantities, K_j, L_j, x_{Kj}, x_{Lj}.

Claim 1. *Suppose*

$$v(b)-q_t^S \geq (1-\varphi(b)) \int_{s_t^* \vee s_b^\alpha}^{s_{t'}^* \wedge s_b^\omega} e^{-r\left(t_S^*(s)-t\right)} \left(v(b) - q_{t_S^*(s)}^B\right) \frac{ds}{s_{t'}^* \wedge s_b^\omega - s_t^* \vee s_b^\alpha} + \varphi(b) e^{-r(t'-t)} (v(b) - q_{t'}^S). \tag{20}$$

Then

$$v(b)-q_t^S > (1-\varphi(b')) \int_{s_t^* \vee s_{b'}^\alpha}^{s_{t'}^* \wedge s_{b'}^\omega} e^{-r\left(t_S^*(s)-t\right)} \left(v(b) - q_{t_S^*(s)}^B\right) \frac{ds}{s_{t'}^* \wedge s_{b'}^\omega - s_t^* \vee s_{b'}^\alpha} + \varphi(b') e^{-r(t'-t)} (v(b) - q_{t'}^S). \tag{21}$$

Proof. Choose $\tilde{s}$ so that $\frac{\tilde{s}-s_t^* \vee s_b^\alpha}{s_b^\omega - s_t^* \vee s_b^\alpha} = \frac{s_{t'}^* \vee s_{b'}^\alpha - s_t^* \vee s_{b'}^\alpha}{s_{b'}^\omega - s_t^* \vee s_{b'}^\alpha}$. Then we have the following sequence of inequalities,

$$(1-\varphi(b)) \int_{s_t^* \vee s_b^\alpha}^{s_{t'}^* \wedge s_b^\omega} e^{-r\left(t_S^*(s)-t\right)} \left(v(b) - q_{t_S^*(s)}^B\right) \frac{ds}{s_{t'}^* \wedge s_b^\omega - s_t^* \vee s_b^\alpha} + \varphi(b) e^{-r(t'-t)} \left(v(b) - q_{t'}^S\right) \geq$$

$$(1-\varphi(b')) \int_{s_t^* \vee s_b^\alpha}^{\tilde{s}} e^{-r\left(t_S^*(s)-t\right)} \left(v(b) - q_{t_S^*(s)}^B\right) \frac{ds}{s_{t'}^* \wedge s_{b'}^\omega - s_t^* \vee s_{b'}^\alpha} + \varphi(b') e^{-r(t'-t)} \left(v(b) - q_{t'}^S\right) \geq$$

$$(1-\varphi(b')) \int_{s_t^* \vee s_{b'}^\alpha}^{s_{t'}^* \wedge s_b^\omega} e^{-r\left(t_S^*(s)-t\right)} \left(v(b) - q_{t_S^*(s)}^B\right) \frac{ds}{s_{t'}^* \wedge s_{b'}^\omega - s_t^* \vee s_{b'}^\alpha} + \varphi(b') e^{-r(t'-t)} \left(v(b) - q_{t'}^S\right).$$

The first inequality follows from the $\varphi(b') > \varphi(b)$ and $q_{t_S^*(s)}^B \leq q_{t'}^B \leq q_{t'}^S$ for all $t_S^*(s) \leq t'$. To get the second inequality, observe that by monotonicity of q_t^B (condition M) and the monotonicity of $t_S^*(s)$, for all $s \leq s'$, $t_S^*(s) \leq t_S^*(s')$ and $q_{t_S^*(s)}^B \leq q_{t_S^*(s')}^B$ and so, function $e^{-rt_S^*(s)}\left(v(b) - q_{t_S^*(s)}^B\right)$ is decreasing in s. Moreover, since $s_t^* \vee s_b^\alpha < s_t^* \vee s_{b'}^\alpha$ and $\tilde{s} < s_{t'}^* \vee s_{b'}^\alpha$, the uniform distribution on $[s_t^* \vee s_{b'}^\alpha, s_{t'}^* \vee s_{b'}^\alpha]$ first-order stochastically dominates the uniform distribution on $[s_t^* \vee s_b^\alpha, \tilde{s}]$, and the inequality follows from the definition of the first-order stochastic dominance. *Q.E.D.*

Since $\varphi(b') > 0$ and $v(b)$ is strictly increasing, by substituting b' instead of b in (21), we get the strict inequality and so, type b' strictly prefers to accept at time t. By an analogous argument, we can show the following claim, which completes the proof of the strict single-crossing property of $u^B(t, b)$.

Claim 2. *Suppose*

$$v(b') - q_t^S \leq (1 - \varphi(b')) \int_{s_t^* \vee s_{b'}^\alpha}^{s_{t'}^* \wedge s_{b'}^\omega} e^{-r\left(t_S^*(s) - t\right)} \left(v(b') - q_{t_S^*(s)}^B\right) \frac{ds}{s_{t'}^* \wedge s_{b'}^\omega - s_t^* \vee s_{b'}^\alpha} + \varphi(b') e^{-r(t'-t)} (v(b') - q_{t'}^S). \tag{22}$$

Claim 3. *Then*

$$v(b) - q_t^S < (1 - \varphi(b)) \int_{s_t^* \vee s_b^\alpha}^{s_{t'}^* \wedge s_b^\omega} e^{-r\left(t_S^*(s) - t\right)} \left(v(b) - q_{t_S^*(s)}^B\right) \frac{ds}{s_{t'}^* \wedge s_b^\omega - s_t^* \vee s_b^\alpha} + \varphi(b) e^{-r(t'-t)} (v(b) - q_{t'}^S). \tag{23}$$

□

Proof of Lemma 2. The proof proceeds by the series of claims.

Claim 4. *Function $u^B(t, b)$ is a.e. continuously differentiable in t for fixed b.*

Proof. This follows from the definition of $u^B(t, b)$ and the fact that s_t^* is a smooth monotone strategy, and q_t^B and q_t^S are continuously differentiable. *Q.E.D*

Claim 5. *For any b, $t_B^*(b)$ is a local maximum of $u^B(t, b)$.*

Proof. Suppose to contradiction, there exists $\hat{b}$ such that $t_B^*(\hat{b})$ is a local minimum of $u^B(t, b)$. By Claim 4, for some $\varepsilon > 0$, $u^B(t, \hat{b})$ is increasing on $(t_B^*(\hat{b}), t_B^*(\hat{b}) + \varepsilon)$. By the strict single-crossing property, for all $b < \hat{b}$, $u^B(t, b)$ is increasing on $(t_B^*(\hat{b}), t_B^*(\hat{b}) + \varepsilon)$. Since b_t^* is a smooth monotone strategy of the buyer satisfying (3), for some $\breve{b} < \hat{b}$, $t_B^*(\breve{b}) \in (t_B^*(\hat{b}), t_B^*(\hat{b}) + \varepsilon)$, which is a contradiction to strict monotonicity of $u^B(t, \breve{b})$ on $(t_B^*(\hat{b}), t_B^*(\hat{b}) + \varepsilon)$. *Q.E.D.*

Claim 6. *For any b, $t_B^*(b)$ is a global maximum of $u^B(t, b)$.*

Proof. Consider a best-reply $\hat{t}(b)$ which is a (weakly) decreasing function by Theorem 4′ in Milgrom and Shannon (1994). Suppose to contradiction that $\hat{t}(b)$ is different from $t_B^*(b)$, i.e. there exists type $\hat{b}$ such that $\hat{t}(\hat{b}) \neq t_B^*(\hat{b})$. First, suppose that $\hat{t}(\hat{b}) > t_B^*(\hat{b})$. By Claim 4 and 5, there exists time $\hat{t} \in \left(t_B^*(\hat{b}), \hat{t}(\hat{b})\right)$ such that $\hat{t}$ is a local minimum of $u^B(t, \hat{b})$

on $\left[t_B^*(\hat{b}), \hat{t}(\hat{b})\right]$. By the strict single-crossing property, for all $b < \hat{b}$, accepting at time $\hat{t}$ is strictly worse than accepting at any time $t \in (\hat{t}, \hat{t}(\hat{b})]$. However, since b_t^* is a smooth monotone strategy and by Claim 5, $\hat{t}$ is a local maximum for some buyer $b < \hat{b}$, which gives the contradiction. The case $\hat{t}(\hat{b}) < t_B^*(\hat{b})$ is considered analogously. *Q.E.D.* □

Proof of Theorem 1. The discussion after Theorem 1 shows that conditions (3) and (4) are necessary conditions of competitive equilibria in smooth monotone strategies. Observe that by (1), eventually all gains from trade are realized and so, condition (2) is necessary.

Conversely, suppose b_t^* and s_t^* are given by (3) and (4) for the boundary condition (2). Then b_t^* and s_t^* specify acceptance strategies for all types and by Lemma 2, they are mutual best-replies and so, constitute the competitive equilibrium of the concession game. □

Preliminary Results about CSEs

The following lemma is the counterpart of Lemma 1 for the bargaining game, and its proof replicates the proof of Lemma 1.

Lemma 9. *Suppose p_n^B and p_n^S are price paths as in the definition of the CSE. If s_n satisfies the skimming property, then $U_n^B(b)$ on $NB = \left\{(n,b) : b \in [0,1], n = \overline{1, n_{s_b^\omega}}\right\}$ satisfies the strict single crossing property. Analogously, if b_n is a monotone buyer strategy, then $U_n^S(s)$ on $NS = \left\{(n,s) : s \in [0,1], n = \overline{1, n_{b_s^\alpha}}\right\}$ satisfies the strict single crossing property.*

We next state the necessary condition for the optimality of strategies b_n and s_n in the active CSE that reflects the indifference of threshold types between accepting in the current round and delaying the acceptance until the next active round.

Lemma 10. *Suppose (b_n, s_n, p_n^B, p_n^S) describe an active CSE. Then for all even $n \le \bar{N}$,*

$$v(b_n) - p_n^S = \delta\alpha_n^S \left(v(b_n) - p_{n+1}^B\right) + \delta^2 \left(1 - \alpha_n^S\right)\left(v(b_n) - p_{n+2}^S\right) \tag{24}$$

where

$$\alpha_n^S = \begin{cases} \frac{s_{n+1} - \max\{s_{n-1}, s_{b_n}^\alpha\}}{s_{b_n}^\omega - \max\{s_{n-1}, s_{b_n}^\alpha\}}, & \text{if } s_{b_n}^\alpha \le s_{n+1}, \\ 0, & \text{otherwise,} \end{cases} \tag{25}$$

and for all odd $n \le \bar{N}$,

$$p_n^B - c(s_n) = \delta\alpha_n^B \left(p_{n+1}^S - c(s_n)\right) + \delta^2 \left(1 - \alpha_{n+2}^S\right)\left(p_{n+2}^B - c(s_n)\right) \tag{26}$$

where

$$\alpha_n^B = \begin{cases} \frac{\min\{b_{n-1}, b_{s_n}^{\omega}\} - b_{n+1}}{\min\{b_{n-1}, b_{s_n}^{\omega}\} - b_{s_n}^{\alpha}}, & \text{if } b_{s_n}^{\omega} \geq b_{n+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

Proof. The left-hand side of equation (24) gives the utility of buyer type b_n from accepting the seller's offer p_n^S. The right-hand side of equation (24) gives the utility of buyer type b_n from delaying the acceptance till the next active round. Then in round $n+1$, the seller accepts the buyer price offer p_{n+1}^B with probability α_n^S (according to the beliefs of buyer type b_n) and with complementary probability in round $n+2$ buyer type b_n accepts offer p_{n+2}^S. Notice that the probability α_n^S is the probability of acceptance of the offer in the next round for the threshold type of the buyer. Condition (26) is derived by the analogous argument. □

The following lemma is the counterpart of Lemma 11 for the bargaining game. We say that a tuple (b_n, s_n, p_n^B, p_n^S) is a *common screening strategy profile* if it satisfies conditions of CSE on-path strategies.

Lemma 11. *Suppose a tuple (b_n, s_n, p_n^B, p_n^S) is a common screening strategy profile such that b_n and s_n are constant after some $\bar{N} \leq \infty$. If (24) holds for all rounds $n \leq \bar{N}$, then b_n is a best-reply to s_n. Symmetrically, if (26) holds for all rounds $n \leq \bar{N}$, then s_n is a best-reply to b_n.*

Proof. Consider a buyer type b that accepts in round $n_b \leq \bar{N}$. Consider any $n \leq \bar{\mathbb{N}}$. By (24), buyer type b_n is indifferent between accepting p_n^S and delaying the acceptance till $n+2$. By Lemma 9, all buyer types above b_n strictly prefer to accept in round n, rather than delay the acceptance until round $n+2$, and all buyer types below b_n strictly prefer to delay the acceptance until round $n+2$ to accepting in round n. For $n < n_b$, $b_n > b$ and so, buyer type b prefers to accept in round $n+2$ rather than in round n. For $n > n_b$, $b_n < b$ and so, buyer type b prefers to accept in round n rather than in round $n+2$. Therefore, n_b is an optimal acceptance time for buyer type b. □

Proof of Lemma 3. Let $\bar{p}$ be the supremum of equilibrium price offers accepted by the buyer and $\bar{p}^B$ be the supremum of equilibrium price offers made by the buyer. We show that $\bar{p} \leq \frac{v(1)+\delta c(1)}{1+\delta}$ and $\bar{p}^B \leq \delta\bar{p} + (1-\delta)c(1) \leq \frac{\delta v(1)+c(1)}{1+\delta}$, which proves the first statement of the lemma. The second statement of the lemma is a symmetric statement for the seller and is proven analogously.

Claim 7. $\bar{p}^B \leq \delta\bar{p} + (1-\delta)c(1)$.

Proof. Suppose to contradiction that this is not the case. Then for any $\gamma > 0$ there is a history such that some buyer type makes offer higher than $\bar{p}^B - \gamma/2$. Consider a deviation of this buyer type to $\bar{p}^B - \gamma$. Such price offer is accepted by the seller with probability one only if $\bar{p}^B - \gamma - c(s) > \max\{\delta(\bar{p} - c(s)), \delta^2(\bar{p}^B - c(s))\}$ for all $s \in [0, 1]$. This is indeed the case whenever $\gamma < \min\{1 - \delta^2, \bar{p}^B - \delta\bar{p} - (1 - \delta)c(1)\}$ which is possible since the right-hand side of the inequality is positive. Given that price offer $\bar{p}^B - \gamma$ is accepted buyer prefers to deviate to price offer $\bar{p}^B - \gamma$ rather than make price offer $\bar{p}^B - \gamma/2$ which is a contradiction. Therefore, in equilibrium the buyer never makes offer higher than $\delta\bar{p} + (1 - \delta)c(1)$. *Q.E.D.*

Claim 8. $\bar{p} \leq \frac{v(1)+\delta c(1)}{1+\delta}$

Proof. Suppose to contradiction that $\bar{p} > \frac{v(1)+\delta c(1)}{1+\delta}$ is accepted by the buyer. Then for any $\gamma > 0$ there is a history such that some seller type s makes price offer $\tilde{p} \in (\bar{p} - \gamma, \bar{p}]$ that is accepted by some buyer type b. Consider a deviation by the buyer to counter-offer p_d. For such deviation not to be profitable it is necessary that $p_d - c(s) \leq \max\{\delta(\bar{p} - c(s)), \delta^2(\bar{p}^B - c(s))\}$ and $\delta(v(b) - p_d) \leq v(b) - \tilde{p}$ for some s and b. If this were not the case then all sellers would prefer to accept price offer p_d (by the first inequality) and all buyer types would prefer such counter-offer to accepting $\tilde{p}$ (by the second inequality). Then $\frac{1}{\delta}(\tilde{p} - (1 - \delta)v(b))) \leq c(s) + \max\{\delta(\bar{p} - c(s)), \delta^2(\bar{p}^B - c(s))\}$ for some s and b, from which it follows that $\frac{1}{\delta}(\tilde{p} - (1 - \delta)v(1))) \leq \max\{\delta\bar{p} + (1 - \delta)c(1), \delta^2\bar{p}^B + (1 - \delta^2)c(1))\}$. The maximum in the right-hand side is equal to $\delta\bar{p} + (1 - \delta)c(1)$. Indeed, if it were not the case then $\bar{p} < \delta\bar{p}^B + (1 - \delta)c(1) \leq \delta(\delta\bar{p} + (1 - \delta)c(1)) + (1 - \delta)c(1)$ or $\bar{p} < c(1)$ which contradicts $\bar{p} > \frac{v(1)+\delta c(1)}{1+\delta}$. Hence, $\tilde{p} - (1 - \delta)v(1)) \leq \delta^2\bar{p} + \delta(1 - \delta)c(1)$ or $\frac{\tilde{p} - \delta^2\bar{p}}{1-\delta^2} \leq \frac{v(1)+\delta c(1)}{1+\delta}$. The left-hand side is greater than $\bar{p} - \frac{\gamma}{1-\delta^2} > \frac{v(1)+\delta c(1)}{1+\delta} - \frac{\gamma}{1-\delta^2}$. Since γ was chosen arbitrary we get a contradiction. *Q.E.D.* □

Proof of Theorem 2. Necessity

Consider a sequence of active CSEs indexed by $j \to \infty$ with a smooth limit and such that $\Delta_j \to 0$ as $j \to \infty$. We show that the smooth limit satisfies condition M, conditions (2), (3) and (7) (and hence, constitutes a competitive equilibrium in the concession game by Theorem 1) and conditions (6) and (7). The latter follow immediately from Lemma 3 and condition M is satisfied since p_n^B and p_n^S are monotone by the Definition 3 and so are their limits q_t^B and q_t^S.

Claim 9. *Condition (2) holds.*

Proof. Suppose to contradiction that $b_T^* > b_{s_T^*}^\alpha$. By condition 3 in the Definition 4, for any $\varepsilon \in \left(0, \frac{b_T^* - b_{s_T^*}^\alpha}{3}\right)$ and $\Delta_j > 0$ sufficiently small, $b_T^j > b_T^* - \varepsilon$ and $b_{s_T^j}^\alpha < b_{s_T^*}^\alpha + \varepsilon$. Consider seller type s_T^j and any time t. The continuation utility of seller s_T^j at time t from following equilibrium strategy is bounded above by

$$\frac{\min\left\{b_t^j, b_{s_T^j}^\omega\right\} - b_T^j}{\min\left\{b_t^j, b_{s_T^j}^\omega\right\} - b_{s_T^j}^\alpha}\left(v(b_t^j) - c(s_T^j)\right).$$

Since $\min\{b_t^j, b_{s_T^j}^\omega\} - b_{s_T^j}^\alpha \geq \min\{b_T^j - b_{s_T^j}^\alpha, \eta\} > \min\{b_T^* - b_{s_T^*}^\alpha - 2\varepsilon, \eta\} > \min\{\varepsilon, \eta\} > 0$, the upper bound converges to zero as $t \to \infty$. Analogous upper bound (converging to zero as $t \to \infty$) could be derived for buyer type b_T^j. This is is in contradiction with condition 4 in the Definition 3, requiring that over time price offers converge enough so that gains from trade could be realized through the acceptance of one of the parties.

Now suppose $T < \infty$, but $q_T^S > q_T^B$. By condition 2 of the Definition 4, for any $\varepsilon > 0$, $T_j < T + \varepsilon$. By the continuity of q_t^B and q_t^S, for ε small enough, $q_t^S - q_t^B > \frac{q_T^S - q_T^B}{2}$ for all $t \in [T - \varepsilon, T + \varepsilon]$ and so, for Δ_j sufficiently small, $p_t^{Sj} > p_t^{Bj} + \frac{q_T^S - q_T^B}{4}$ for all $t \in [T - \varepsilon, T + \varepsilon]$. Suppose buyer type $b_{T-\varepsilon}^j$ deviates by rejecting $p_{T-\varepsilon}^{Sj}$ and waiting for 2ε until the seller accepts some price offer of the buyer. Type $b_{T-\varepsilon}^j$ gets utility at least $\min_{t\in[T-\varepsilon,T+\varepsilon]} e^{-2r\varepsilon}\left(v(b_{T-\varepsilon}^j) - p_t^{Bj}\right)$. On the other hand, from following equilibrium strategy type $b_{T-\varepsilon}^j$ gets $v(b_{T-\varepsilon}^j) - p_{T-\varepsilon}^{Sj}$. For ε small enough, such deviation is profitable which is a contradiction. This proves the condition (2). *Q.E.D.*

Claim 10. *Conditions (3) and (7) hold.*

Proof. For any $t < T$, let $\tau_t \equiv 2\Delta_j \left\lfloor \frac{t}{2\Delta_j} \right\rfloor$. By Lemma 10 condition (24) holds for all even $n \leq \bar{N}$, which could be rewritten for any τ_t as follows

$$v\left(b_{\tau_t}^j\right) - p_{\tau_t}^{Sj} = e^{-r\Delta_j}\alpha_{\tau_t}^{Sj}\left(v\left(b_{\tau_t}^j\right) - p_{\tau_t+\Delta_j}^{Bj}\right) + e^{-2r\Delta_j}\left(1 - \alpha_{\tau_t}^{Sj}\right)\left(v\left(b_{\tau_t}^j\right) - p_{\tau_t+2\Delta_j}^{Sj}\right).$$

Subtracting $e^{-2r\Delta_j}\left(v\left(b_{\tau_t}^j\right) - p_{\tau_t}^{Sj}\right)$ from both sides and dividing by $2\Delta_j$, we get

$$\frac{1 - e^{-2r\Delta_j}}{2\Delta_j}\left(v\left(b_{\tau_t}^j\right) - p_{\tau_t}^{Sj}\right) = e^{-r\Delta_j}\frac{\alpha_{\tau_t}^{Sj}}{2\Delta_j}\left(v\left(b_{\tau_t}^j\right) - p_{\tau_t+\Delta_j}^{Bj}\right) - e^{-r\Delta_j}\left(v\left(b_{\tau_t}^j\right) - p_{\tau_t}^{Sj}\right) + e^{-2r\Delta_j}\frac{p_{\tau_t}^{Sj} - p_{\tau_t+2\Delta_j}^{Sj}}{2\Delta_j}.$$

Taking $\Delta_j \to 0$, we get condition (3) where convergence is guaranteed by the definition of the smooth limit and continuity of function $v(b)$. The derivation of equation (4) for buyer price offers is symmetric. *Q.E.D.*

□

Proof of Lemma 4

We reduce the problem of finding a CSE with price offers constant over time to a mathematical problem of finding a positive trajectory satisfying a particular recursive system. The following lemma is a key mathematical fact in the proof of Lemma 4.

Lemma 12. *Consider* $b_\infty \in (0, 1-\eta)$, $s_\infty = b_\infty + \eta$, P^B, P^S *that satisfy*

$$\max\left\{c\left(s_\infty\right), \frac{v(0)+c(0)}{2}\right\} < P^B < P^S < \min\left\{v\left(b_\infty\right), \frac{v(1)+c(1)}{2}\right\}, \qquad (28)$$

There exists $\bar{\delta} \in (0,1)$ *such that for all* $\delta \in (\bar{\delta}, 1)$ *there are positive trajectories* x_k *and* y_k *that satisfy recursive system*

$$\begin{cases} x_{k+1} = (1-\alpha^B(y_{k+1}))x_k - \alpha^B(y_{k+1})y_{k+1}, \\ y_{k+1} = (1-\alpha^S(x_k))y_k - \alpha^S(x_k)x_k, \\ b_\infty + x_k \le s_\infty - y_k + \eta; \end{cases} \qquad (29)$$

where $\alpha^B(y) \equiv \frac{(1-\delta^2)(P^B - c(s_\infty - y))}{\delta(P^S - c(s_\infty - y)) - \delta^2(P^B - c(s_\infty - y))}$ *and* $\alpha^S(x) \equiv \frac{(1-\delta^2)(v(b_\infty + x) - P^S)}{\delta(v(b_\infty + x) - P^B) - \delta^2(v(b_\infty + x) - P^S)}$. *Moreover, for all* $k \in \mathbb{N}$,

$$\max\{x_{k-1} - x_k, y_{k-1} - y_k\} < (1-\delta)C \qquad (30)$$

where C *is a constant that does not depend on* δ.

Proof of Lemma 12. Observe that if x_k and y_k are given for $k \ge k_0$, then by (29), we can construct x_k and y_k for $k < k_0$. The following claim show that it is sufficient to construct x_k and y_k that are positive starting from some k_0.

Claim 11. *If there are trajectories* x_k *and* y_k *satisfying* (29) *that are positive starting from some* k_0, *then* x_k *and* y_k *are positive for all* $k \in \mathbb{N}$.

Proof. By rearranging terms in the first equation of (29), $x_k = \frac{x_{k+1} + \alpha^B(y_{k+1})y_{k+1}}{1 - \alpha^B(y_{k+1})}$. Observe that $\alpha^B(y) \in (0,1)$ for $y > 0$ and so, x_k is positive, whenever x_{k+1} and y_{k+1} are positive. Analogously, it could be shown from the second equation of (29) that y_k is positive, whenever x_{k+1} and y_{k+1} are positive. *Q.E.D.*

Claim 12. *For given* x_{k_0} *and* y_{k_0}, *there is* $K(x_{k_0}, y_{k_0})$ *such that* $k_0 \le K(x_{k_0}, y_{k_0})$.

Proof. First, observe that x_k and y_k are decreasing whenever they are positive. Indeed, for all $k \in \mathbb{N}$, we have $x_{k-1} - x_k = \alpha^B(y_k)(x_{k-1} + y_k) > 0$ and similarly, $y_{k-1} - y_k > 0$. Next, from (29), for all $k \leq k_0$,

$$x_{k-1} - x_k = \alpha^B(y_k)(x_{k-1} + y_k) \geq \alpha^B(y_{k_0})(x_{k_0} + y_{k_0}) > c_1 \tag{31}$$

for some $c_1 > 0$ where we used the fact that $\alpha^B(y)$ is increasing and x_k and y_k are decreasing sequences. Suppose for any $K \in \mathbb{N}$, we could construct sequences $x_k(K)$ and $y_k(K)$ such that $x_K(K) = x_{k_0}$ and $y_K(K) = y_{k_0}$. From (31), for K sufficiently large $b_\infty + x_0(K) > s_\infty - y_0(K) + \eta$ which contradicts (29). *Q.E.D.*

Let $V^B \equiv v(b_\infty) - P^S$, $V^S \equiv P^B - c(s_\infty)$ and $\Delta P \equiv P^S - P^B$. The following claim gives the Taylor expansion of $\alpha^B(y)$ and $\alpha^S(x)$.

Claim 13. *There exists $\delta_1 \in (0,1)$ and $\varepsilon_1 > 0$ such that for all $\delta \in (\delta_1, 1)$ and all $x \in (0, \varepsilon_1)$,$y \in (0, \varepsilon_1)$,*

$$\alpha^B(y) \equiv \alpha_B - \phi_B \sum_{l=1}^{\infty} \gamma_l^B y^l, \tag{32}$$

$$\alpha^S(x) \equiv \alpha_S - \phi_S \sum_{l=1}^{\infty} \gamma_l^S x^l, \tag{33}$$

where

$$\alpha_B \equiv \frac{(1-\delta^2)V^B}{\delta(\Delta P + (1-\delta)V^B)}, \gamma_B \equiv -\frac{1-\delta}{\Delta P + (1-\delta)V^B} < 0, \phi_B \equiv \frac{(1+\delta)\Delta P}{\delta(\Delta P + (1-\delta)V^B)} > 0,$$

$$\alpha_S \equiv \frac{(1-\delta^2)V^S}{\delta(\Delta P + (1-\delta)V^S)}, \gamma_S \equiv -\frac{1-\delta}{\Delta P + (1-\delta)V^S} < 0, \phi_S \equiv \frac{(1+\delta)\Delta P}{\delta(\Delta P + (1-\delta)V^S)} > 0,$$

$$\gamma_l^B \equiv \sum_{j=1}^{l} \gamma_B^j \left(\sum_{l_1+\cdots+l_j=l} \frac{d^{l_1}c(s_\infty)/ds^{l_1}}{l_1!} \cdots \frac{d^{l_j}c(s_\infty)/ds^{l_j}}{l_j!} \right),$$

$$\gamma_l^S \equiv \sum_{z=1}^{l} \gamma_S^z \left(\sum_{l_1+\cdots+l_z=l} \frac{d^{l_1}v(b_\infty)/db^{l_1}}{l_1!} \cdots \frac{d^{l_z}v(b_\infty)/db^{l_z}}{l_z!} \right),$$

and $\gamma_l^S \leq |\gamma_S D|(1+|\gamma_S D|)^{l-1}$, $\gamma_l^B \leq |\gamma_B D|(1+|\gamma_B D|)^{l-1}$.

Proof. As $\delta \to 1$, γ_S and γ_B converge to zero and so, for δ sufficiently close to one, $|\gamma_S(v(1)-v(0))| < 1$ and $|\gamma_B(c(1)-c(0))| < 1$. Expanding $\alpha^S(x)$ into the Taylor series, we get

$$\alpha^S(x) = \alpha_S - \phi_S \sum_{z=1}^{\infty} \gamma_S^z (v(b_\infty + x) - v(b_\infty))^z,$$

Since $v(b)$ is a smooth function, expanding it into the Taylor series around b_∞, we get $v(b_\infty + x) - v(b_\infty) = \sum_{l=1}^{\infty} \frac{d^l v(b_\infty)}{db^l} \frac{x^l}{l!}$. By the regularity of $v(b)$, all derivatives $\frac{d^l v(b)/db^l}{l!}$, $l \in \mathbb{N}$ are bounded by D for some $D > 1$. Therefore, the Taylor expansion of $v(b)$ around b_∞ is an absolute convergent series, and by the Merten's theorem the z's power of it equals

$$(v(b_\infty + x) - v(b_\infty))^z = \sum_{l=z}^{\infty} x^l \left(\sum_{l_1+\dots+l_z=l} \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \dots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right)$$

and so,

$$\alpha^S(x) = \alpha_S - \phi_S \sum_{z=1}^{\infty} \gamma_S^z \sum_{l=z}^{\infty} x^l \left(\sum_{l_1+\dots+l_z=l} \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \dots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right). \tag{34}$$

Observe that

$$\sum_{z=1}^{\infty} \left| \gamma_S^z \sum_{l=z}^{\infty} x^l \sum_{l_1+\dots+l_z=l} \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \dots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right| \leq$$

$$\sum_{z=1}^{\infty} |\gamma_S|^z \sum_{l=z}^{\infty} x^l \sum_{l_1+\dots+l_z=l} \left| \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \dots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right| \leq$$

$$\sum_{z=1}^{\infty} |\gamma_S|^z \sum_{l=z}^{\infty} x^l \sum_{l_1+\dots+l_z=l} D^z = \sum_{z=1}^{\infty} |\gamma_S|^z D^z \sum_{l=z}^{\infty} x^l \binom{l-1}{z-1} = \sum_{z=1}^{\infty} \frac{(|\gamma_S| D x)^z}{(1-x)^z} < \infty$$

where the first inequality follows from the triangle inequality, the second inequality follows from the regularity of $v(b)$ and the fact that $(l_1 + \ldots + l_z)! \geq l_1! \cdot \ldots \cdot l_z!$, the first equality follows from the fact that a number of compositions of l into exactly z parts is equal to $\binom{l-1}{z-1}$, the second equality is by summing over l, and the resulting series is converging for x sufficiently small (so that $x < (1 + |\gamma_S| D)^{-1}$). Therefore, the series in

(34) is absolutely convergent, and by the Fubini's theorem, we could exchange the order of summation in (34) to get expression (32). We have the following upper bound on the absolute values of coefficients γ_l^S

$$|\gamma_l^S| \leq \sum_{z=1}^{l} |\gamma_S|^z \left(\sum_{l_1+\cdots+l_z=l} \left| \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \cdots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right| \right) \leq \sum_{z=1}^{l} |\gamma_S D|^z \binom{l-1}{z-1} = |\gamma_S D|(1+|\gamma_S D|)^{l-1} \tag{35}$$

where the first inequality is by the triangle inequality, the second inequality follows from the regularity of $v(b)$, and the equality is obtained by algebraic manipulations.The derivation of the corresponding expression for $\alpha^S(y)$ is analogous. *Q.E.D.*

System (29) has steady states $(z, -z), z \in \mathbb{R}$. By the specification of the problem we are interested only in steady state $(0, 0)$. Around this steady state the linearized system could be written in matrix form

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 1 - \alpha_B + \alpha_S \alpha_B & -\alpha_B(1 - \alpha_S) \\ -\alpha_S & 1 - \alpha_S \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}.$$

The matrix has eigenvalues 1 and $\lambda \equiv (1 - \alpha_B)(1 - \alpha_S)$. Since one of eigenvalues is equal to 1, the steady state is not stable, and we cannot conclude that in the neighborhood of the steady state the non-linear system converges to the steady state. Therefore, we find a particular trajectory that satisfies desired properties.

We conjecture that there exist μ_i^x and μ_i^y such that

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \sum_{i=1}^{\infty} \lambda^{ik} \begin{pmatrix} \lambda^{i/2} \mu_i^x \\ \mu_i^y \end{pmatrix} \tag{36}$$

is the required solution and for all $i \in \mathbb{N}$,

$$|\mu_i^x| \leq u_\delta M^i \text{ and } |\mu_i^y| \leq u_\delta M^i \tag{37}$$

for some positive M and u_δ such that

$$M < 1 < \frac{1}{\lambda(1 + u_\delta(1 + \max\{|\gamma_S|, |\gamma_B|\}D))}. \tag{38}$$

Given this conjecture, we next derive expressions for coefficients μ_i^x and μ_i^y, and then verify that for δ sufficiently close to one, upper bounds on absolute values of coefficients hold. Series (36) defining (x_k, y_k) are absolutely convergent, as they are dominated by the absolutely convergent series $u_\delta \sum_{i=1}^{\infty} \lambda^{ik} M^i$.

Plugging the solution (36) into system (29), we get

$$\begin{cases} \sum_{i=1}^{\infty} \lambda^{ik}(\mu_i^x - \mu_i^x\lambda^i - \alpha_B(\mu_i^x + \mu_i^y \lambda^{i/2})) = -\phi_B \left(\sum_{l=1}^{\infty} \gamma_l^B \left(\sum_{i=1}^{\infty} \mu_i^y \lambda^{i(k+1)} \right)^l \right) \left(\sum_{i=1}^{\infty} \lambda^{ik}(\mu_i^x + \mu_i^y \lambda^{i/2}) \right), \\ \sum_{i=1}^{\infty} \lambda^{ik}(\mu_i^y - \mu_i^y\lambda^i - \alpha_S(\mu_i^x\lambda^{i/2} + \mu_i^y)) = -\phi_S \left(\sum_{l=1}^{\infty} \gamma_l^S \left(\sum_{i=1}^{\infty} \mu_i^x \lambda^{ik} \right)^l \right) \left(\sum_{i=1}^{\infty} \lambda^{i(k+1/2)}(\mu_i^x\lambda^{i/2} + \mu_i^y) \right). \end{cases} \tag{39}$$

Consider the first equation in system (39). By the Merten's theorem, $\left(\sum_{i=1}^{\infty} \mu_i^y \lambda^{i(k+1)}\right)^l = \sum_{i=l}^{\infty} \sum_{i_1+\cdots+i_l=i} \mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y \lambda^{i(k+1)}$ and

$$\sum_{l=1}^{\infty} \gamma_l^B \left(\sum_{i=1}^{\infty} \mu_i^y \lambda^{ik} \right)^l = \sum_{l=1}^{\infty} \gamma_l^B \sum_{i=l}^{\infty} \sum_{i_1+\cdots+i_l=i} \mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y \lambda^{i(k+1)}. \tag{40}$$

The series in (40) is absolutely convergent by

$$\sum_{l=1}^{\infty}\sum_{i=l}^{\infty} \left| \lambda^{i(k+1)} \gamma_l^B \sum_{i_1+\cdots+i_l=i} \mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y \right| \le$$

$$\sum_{l=1}^{\infty}\sum_{i=l}^{\infty} \lambda^{i(k+1)} |\gamma_l^B| \sum_{i_1+\cdots+i_l=i} \left|\mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y\right| \le$$

$$\sum_{l=1}^{\infty}\sum_{i=l}^{\infty} \lambda^{i(k+1)} |\gamma_l^B| \sum_{i_1+\cdots+i_l=i} u_\delta^l M^i = \sum_{l=1}^{\infty}\sum_{i=l}^{\infty} \lambda^{i(k+1)} |\gamma_l^B| u_\delta^l M^i \binom{i-1}{l-1} \le$$

$$|\gamma_B D| \sum_{l=1}^{\infty}\sum_{i=l}^{\infty} \lambda^{i(k+1)} (1+|\gamma_B D|)^{l-1} u_\delta^l M^i \binom{i-1}{l-1} = \frac{|\gamma_B D|}{1+|\gamma_B D|} \sum_{l=1}^{\infty} (1+|\gamma_B D|)^l u_\delta^l \left(\frac{\lambda^{k+1} M}{1-\lambda^{k+1} M} \right)^l \le$$

$$\frac{|\gamma_B D|}{1+|\gamma_B D|} \sum_{l=1}^{\infty} (1+|\gamma_B D|)^l u_\delta^l \left(\frac{\lambda M}{1-\lambda M} \right)^l,$$

where the first inequality is by the triangle inequality, the second inequality follows by (37), the first equality is by the fact that the number of compositions of i into exactly l parts is $\binom{i-1}{l-1}$, the third inequality is by (35), the forth inequality is by $\lambda^{k+1} < \lambda$ and the resulting series is convergent, whenever $u_\delta(1+|\gamma_B D|)\frac{\lambda M}{1-\lambda M} < 1$ which holds by (38). Therefore, by the Fubini's theorem, exchanging the order of summation in (40), we get

$$\sum_{l=1}^{\infty}\gamma_l^B\left(\sum_{i=1}^{\infty}\mu_i^y\lambda^{ik}\right)^l = \sum_{i=1}^{\infty}\lambda^{i(k+1)}\sum_{l=1}^{i}\sum_{i_1+\cdots+i_l=i}\gamma_l^B\mu_{i_1}^y\cdot\ldots\cdot\mu_{i_l}^y.$$

By the absolute convergence of both series in the right-hand side of (39), the product in the right-hand side is equal to the Cauchy product, and so we could rewrite system (39) as follows

$$\begin{cases}\sum_{i=1}^{\infty}\lambda^{ik}\left(\mu_i^x-\mu_i^x\lambda^i-\alpha_B(\mu_i^x+\mu_i^y\lambda^{i/2})+\phi_B\sum\limits_{j=1}^{i-1}(\mu_{i-j}^x\lambda^{j/2}+\mu_{i-j}^y\lambda^{i/2})\sum\limits_{l=1}^{j}\gamma_l^B\sum\limits_{j_1+\cdots+j_l=j}\mu_{j_1}^y\cdot\ldots\cdot\mu_{j_l}^y\right)=0,\\ \sum_{i=1}^{\infty}\lambda^{ik}\left(\mu_i^y-\mu_i^y\lambda^i-\alpha_S(\mu_i^x\lambda^{i/2}+\mu_i^y)+\phi_S\sum\limits_{j=1}^{i-1}(\mu_{i-j}^x\lambda^{i/2}+\mu_{i-j}^y\lambda^{j/2})\sum\limits_{l=1}^{j}\gamma_l^S\sum\limits_{j_1+\cdots+j_l=j}\mu_{j_1}^x\cdot\ldots\cdot\mu_{j_l}^x\right)=0.\end{cases}$$

Setting all coefficient at λ^{ik} equal to zero, we get the system

$$\begin{cases}\mu_i^x-\mu_i^x\lambda^i-\alpha_B(\mu_i^x\lambda^{j/2}+\mu_i^y\lambda^{i/2})=-\phi_B\sum\limits_{j=1}^{i-1}\left((\mu_{i-j}^x\lambda^{j/2}+\mu_{i-j}^y\lambda^{i/2})\sum\limits_{l=1}^{j}\gamma_l^B\sum\limits_{j_1+\cdots+j_l=j}\mu_{j_1}^y\cdot\ldots\cdot\mu_{j_l}^y\right),\\ \mu_i^y-\mu_i^y\lambda^i-\alpha_S(\mu_i^x\lambda^{i/2}+\mu_i^y)=-\phi_S\sum\limits_{j=1}^{i-1}\left((\mu_{i-j}^x\lambda^{i/2}+\mu_{i-j}^y\lambda^{j/2})\sum\limits_{l=1}^{j}\gamma_l^S\sum\limits_{j_1+\cdots+j_l=j}\mu_{j_1}^x\cdot\ldots\cdot\mu_{j_l}^x\right).\end{cases}$$

Using notation $A_i\equiv\begin{pmatrix}1-\lambda^i-\alpha_B & -\alpha_B\lambda^{i/2}\\ -\alpha_S\lambda^{i/2} & 1-\lambda^i-\alpha_S\end{pmatrix}$, $\mu_i\equiv\begin{pmatrix}\mu_i^x\\ \mu_i^y\end{pmatrix}$, and

$$\varphi_i=\begin{pmatrix}\varphi_i^x\\ \varphi_i^y\end{pmatrix}\equiv\begin{pmatrix}-\phi_B\sum\limits_{j=1}^{i-1}\left((\mu_{i-j}^x\lambda^{j/2}+\mu_{i-j}^y\lambda^{i/2})\sum\limits_{l=1}^{j}\gamma_l^B\sum\limits_{j_1+\cdots+j_l=j}\mu_{j_1}^y\cdot\ldots\cdot\mu_{j_l}^y\right)\\ -\phi_S\sum\limits_{j=1}^{i-1}\left((\mu_{i-j}^x\lambda^{i/2}+\mu_{i-j}^y\lambda^{j/2})\sum\limits_{l=1}^{j}\gamma_l^S\sum\limits_{j_1+\cdots+j_l=j}\mu_{j_1}^x\cdot\ldots\cdot\mu_{j_l}^x\right)\end{pmatrix}\tag{41}$$

and we could write the system in matrix form as $A_i\mu_i=\varphi_i$. Since $\det(A_i)=(1-\lambda^i)(\lambda-\lambda^i)>0$, for $i\geq 2$, matrix A_i is invertible, and we could solve for all μ_i (with the exception of $i=1$)

$$\mu_i=A_i^{-1}\varphi_i.\tag{42}$$

For $i=1$, the equations are linearly dependent and the relation between μ_1^x and μ_1^y is given by

$$\mu_1^x = \mu_1^y \frac{\alpha_B}{\alpha_S}(1-\alpha_S). \tag{43}$$

Equations (42) and (43) give the desired expressions for μ_i through the parameters of the model. The next claim verifies that bounds (37) and (38) indeed hold and so, our derivation was justified.

Claim 14. *For $M < 1$, there exists $\hat{\delta} \in (0,1)$ such that for any $\delta \in (\hat{\delta}, 1)$ there exist positive u_δ and μ_1^y such that* (38) *holds, and for μ_i defined by (42) and* (43)*,* bounds (37) *hold.*

Proof. The proof is by induction on i. Without loss of generality, we assume that

$$V^S \leq V^B \tag{44}$$

and so, $\alpha_S \leq \alpha_B, |\gamma_S| \geq |\gamma_B|, \phi_S \geq \phi_B$. Let $u_\delta \equiv \frac{u}{2}\min\{|\gamma_S|, |\gamma_B|\}$ where $u = \frac{1}{2}\min\{V^S, V^B\}$. Let us first show that for our choice of u_δ, $1 < \frac{1}{\lambda(1+u_\delta(1+\max\{|\gamma_S|,|\gamma_B|\}D))}$ for δ sufficiently close to one. To see this, observe that for δ sufficiently close to one, $\max\{|\gamma_B|, |\gamma_S|\}D < 1$ and so, $\frac{1}{\lambda(1+2u_\delta)} < \frac{1}{\lambda(1+u_\delta(1+\max\{|\gamma_S|,|\gamma_B|\}D))}$. Therefore, it is sufficient to show that $\lambda^{1/2}(1+2u_\delta) < 1$. Then

$$\lambda^{1/2}(1+2u_\delta) = \left((1-\alpha_S)(1-\alpha_B)\right)^{1/2}(1+u\min\{|\gamma_S|, |\gamma_B|\}) \leq (1-\alpha_S)(1+u|\gamma_S|).$$

Observe

$$(1-\alpha_S)(1+u|\gamma_S|) = \left(1 - \frac{(1-\delta^2)V^S}{\delta(\Delta P + (1-\delta)V^S)}\right)\left(1 + \frac{(1-\delta)u}{\Delta P + (1-\delta)V^S}\right),$$

and $\lambda^{1/2}(1+2u_\delta) < 1$ is equivalent to

$$\Delta P + (1-\delta)V^S + u(1-\delta) < \frac{\delta(\Delta P + (1-\delta)V^S)(\Delta P + (1-\delta)V^S)}{\Delta P\delta - (1-\delta)V^S},$$

or

$$u < (1+\delta)V^S \frac{\Delta P + (1-\delta)V^S}{\Delta P\delta - (1-\delta)V^S}. \tag{45}$$

As $\delta \to 1$, the right-hand side of (45) converges to $2V^S$. Since $u < V^S$, inequality (45) holds and so, $(1-\alpha_S)(1+u|\gamma_S|) < 1$ for sufficiently large δ. Hence, we have proven that (38) holds.

To prove bounds (37), Let μ_1^x and μ_1^y be defined as follows. If $\frac{\alpha_B}{\alpha_S}(1-\alpha_S) \leq 1$, then let

$\mu_1^y = u_\delta M$ and $\mu_1^x = \mu_1^y \frac{\alpha_B(1-\alpha_s)}{\alpha_S} \le \mu_1^y$, and otherwise let $\mu_1^x = u_\delta M$ and $\mu_1^y = \mu_1^x \frac{\alpha_S}{\alpha_B(1-\alpha_S)} \le \mu_1^x$. By the definition, $|\mu_1^x|$ and $|\mu_i^y|$ are less that $u_\delta M$ which proves the base of induction.

Suppose that the statement is true for all $j < i$. I show that $|\mu_i^x| < u_\delta M^i$ and $|\mu_i^y| < u_\delta M^i$. We could find closed form solution of system (42),

$$|\mu_i^x| = \frac{|(1-\lambda^i-\alpha_S)\varphi_i^x + \alpha_B \lambda^{i/2}\varphi_i^y|}{(1-\lambda^i)(\lambda-\lambda^i)} \le \frac{4\max\{1-\lambda^i, \alpha_S, \alpha_B\}\cdot \max\{|\varphi_i^x|, |\varphi_i^y|\}}{(1-\lambda^i)(\lambda-\lambda^i)}$$

and the same upper bound holds for $|\mu_i^y|$. It is sufficient to show that $\frac{4\max\{(1-\lambda^i),\alpha_S,\alpha_B\}\cdot\max\{|\varphi_i^x|,|\varphi_i^y|\}}{(1-\lambda^i)(\lambda-\lambda^i)u_\delta M^i} < 1$.

Notice that $\frac{\alpha_S}{1-\lambda^i} < \frac{\alpha_S}{1-\lambda}$ for $i \ge 2$, and by L'Hospital rule $\lim_{\delta\to 1} \frac{\alpha_S}{1-\lambda} = \lim_{\delta \to 1} \frac{\alpha_S}{\alpha_S+\alpha_B-\alpha_S\alpha_B} = \frac{V^S}{V^S+V^B} \le 1$. Hence, for sufficiently large δ and all $i \ge 2$, we have $\frac{\alpha_S}{1-\lambda^i} < 1$, and by the analogous argument, $\frac{\alpha_B}{1-\lambda^i} < 1$. Therefore, $\frac{4\max\{1-\lambda^i,\alpha_S,\alpha_B\}}{1-\lambda^i} < 5$ for sufficiently large δ and it remains to show that $\frac{\max\{|\varphi_i^x|,|\varphi_i^y|\}}{(\lambda-\lambda^i)u_\delta M^i} < \frac{1}{5}$ for sufficiently large δ.

We next show that $\frac{|\varphi_i^x|}{(\lambda-\lambda^i)u_\delta M} < \frac{1}{5}$ (by the symmetric argument $\frac{|\varphi_i^y|}{(\lambda-\lambda^i)u_\delta M} < \frac{1}{5}$). From (41)

$$\frac{|\varphi_i^x|}{\phi_B} \le \sum_{j=1}^{i-1} \lambda^{j/2} \sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} |\mu_{i-j}^x \mu_{j_1}^y \cdot \ldots \cdot \mu_{j_l}^y| + \lambda^{i/2} \sum_{j=1}^{i-1}\sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} |\mu_{i-j}^y \mu_{j_1}^y \cdot \ldots \cdot \mu_{j_l}^y| \le$$

$$\sum_{j=1}^{i-1} \lambda^{j/2} \sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} u_\delta^{l+1} M^i + \lambda^{i/2} \sum_{j=1}^{i-1}\sum_{l=1}^{j} |\gamma_l^B| \sum_{j_1+\cdots+j_l=j} u_\delta^{l+1} M^i \le$$

$$2u_\delta M^i \sum_{j=1}^{i-1} \lambda^{j/2} \sum_{l=1}^{j} |\gamma_l^B| u_\delta^l \binom{j-1}{l-1} \le$$

$$2u_\delta M^i |\gamma_B D| \sum_{j=1}^{i-1} \lambda^{j/2} \sum_{l=1}^{j} u_\delta^l (1+|\gamma_B D|)^{l-1} \binom{j-1}{l-1} \le$$

$$2u_\delta M^i |\gamma_B D| \sum_{j=1}^{i-1} \lambda^{j/2} u_\delta \left(1 + u_\delta(1+|\gamma_B D|)\right)^{j-1} \le$$

$$2u_\delta M^i |\gamma_B D| \sum_{j=1}^{i-1} \lambda^{j/2} u_\delta (1+2u_\delta)^{j-1} =$$

$$2u_\delta M^i |\gamma_B D| \frac{u_\delta \lambda^{1/2}(1-\lambda^{(i-1)/2}(1+2u_\delta)^{i-1})}{1-\lambda^{1/2}(1+2u_\delta))},$$

where the first inequality is by the triangle inequality, the second inequality is by the inductive hypothesis, the third inequality uses the fact that the number of compositions of j into exactly l parts is $\binom{j-1}{l-1}$ and that $\lambda^j > \lambda^i$ for $j < i$, the forth inequality is using a bound on $|\gamma_l^B|$, the fifth inequality is by summing over l, the sixth inequality is by $|\gamma_B D| < 1$ for sufficiently large δ, the equality is the summation over j. It remains to show that

$$2\phi_B|\gamma_B D|\frac{u_\delta\lambda^{1/2}(1-\lambda^{(i-1)/2}(1+2u_\delta)^{i-1})}{(\lambda-\lambda^i)(1-\lambda^{1/2}(1+2u_\delta))} < \frac{1}{5}. \quad (46)$$

Since the denominator in (46) is positive, (46) is equivalent to

$$\lambda - \lambda^i - 10\phi_B|\gamma_B D|\frac{u_\delta\lambda^{1/2}(1-\lambda^{(i-1)/2}(1+2u_\delta)^{i-1})}{1-\lambda^{1/2}(1+2u_\delta)} > 0. \quad (47)$$

The derivative of (47) with respect to i is equal to

$$\lambda^{i/2}\left(-\ln(\lambda)\lambda^{i/2} + 10\ln(\lambda^{1/2}(1+2u_\delta))\phi_B|\gamma_B D|\frac{u_\delta(1+2u_\delta)^{i-1}}{1-\lambda^{1/2}(1+2u_\delta)}\right).$$

Multiplication by $\lambda^{i/2}$ does not affect the sign of the derivative and so, we focus on the term in brackets. The positive (first) term in brackets is decreasing in absolute value, while the negative (second) term is increasing in absolute value. Therefore, minimum of expression (47) is either attained at $i = 2$ or $i \to \infty$. For $i = 2$, (47) is equal to

$$\lambda - \lambda^2 - 10\phi_B|\gamma_B D|u_\delta\lambda^{1/2} > 0, \quad (48)$$

whenever $u_\delta < \frac{\lambda^{1/2}(1-\lambda)}{10\phi_B|\gamma_B D|}$. By (44), $\frac{\lambda^{1/2}(1-\lambda)}{10\phi_B|\gamma_B D|} = \frac{\lambda^{1/2}(1-(1-\alpha_B)(1-\alpha_S))}{10\phi_B|\gamma_B D|} \leq \frac{\lambda^{1/2}(1-(1-\alpha_B)^2)}{10\phi_B|\gamma_B D|} \leq \frac{\alpha_B}{\phi_B|\gamma_B|} \to V^B$. Since u_δ converges to zero as $\delta \to 0$, for δ close to one, inequality (48) holds

For $i = \infty$, (47) is equal to

$$\lambda\left(1 - 10D\frac{\phi_B|\gamma_B|}{\lambda^{1/2}}\frac{u_\delta}{1-\lambda^{1/2}(1+2u_\delta)}\right). \quad (49)$$

Observe that $\lim\limits_{\delta\to 1}\frac{u_\delta}{1-\lambda^{1/2}(1+2u_\delta)} = \frac{u}{V^S+V^B-2u}$. Since $|\gamma_B| \to 0, \lambda \to 1, \phi_B \to 2$ as $\delta \to 1$, we have that (49) is positive for sufficiently large δ. *Q.E.D.*

So far we have constructed a candidate trajectories (x_k, y_k) given by (36). First, notice that by making k sufficiently large the solution approaches zero and so, the Taylor expansion in Claim 13 is justified. Second, observe that $x_k = \sum\limits_{i=1}^{\infty}\lambda^{ik}\lambda^{i/2}\mu_i^x =$

$\lambda^{k+1/2}\left(\mu_1^x + \sum_{i=2}^{\infty} \lambda^{(i-1)k}\lambda^{i/2}\mu_i^x\right)$, and for sufficiently large k , the sign of x_k is determined by μ_1^x which we could choose positive. Analogously, since μ_1^y has the same sign as μ_1^x (by the definition), y_k is positive for sufficiently large k. By Claim 11, constructed trajectory (x_k, y_k) is positive.

To show that we could bound the change in x_k and y_k by a term of order $1-\delta$, observe that

$$x_{k-1} - x_k = \alpha^B(y_k)(x_{k-1} + y_k) \leq 2\left(\alpha_B - \phi_B \sum_{z=1}^{\infty} \gamma_B^z (c(s_\infty) - c(s_\infty - y_k))^z\right) \leq$$

$$2\left(\alpha_B - \frac{\phi_B|\gamma_B|\Sigma}{1-|\gamma_B|\Sigma}\right) \sim 1-\delta$$

and so, there exists C such that $x_{k-1} - x_k < (1-\delta)C$ for all $k \in \mathbb{N}$. The analogous bound holds for $y_{k-1} - y_k$. □

Lemma 13. *Consider $b_0 \in (0, 1-\eta], s_0 \in [b_0 - \eta, b_0 + \eta) \cap [\eta, 1)$,$P^B$,$P^S$ that satisfy* (8). *There exist $\bar{\delta} \in (0,1), b_\infty \in (b_{s_0}^\alpha, b_0), s_\infty = b_\infty + \eta$ such that for all $\delta \in (\bar{\delta}, 1)$ there are positive trajectories x_k and y_k that satisfy recursive system* (29). *Moreover, for all $k \in \mathbb{N}$,* (30) *holds for some constant C that does not depend on δ.*

Proof. Fix any $b_\infty \in (b_{s_0}^\alpha, b_0)$ and $s_\infty = b_\infty + \eta$. By Lemma 12, we can construct positive trajectories x_k and y_k that satisfy (29) and(30). We next construct sequences $\hat{b}_n$ and $\hat{s}_n$ by defining $\hat{b}_{2k-1} = \hat{b}_{2k-2} = b_\infty + x_{k-1}$ and $\hat{s}_{2k} = \hat{s}_{2k-1} = s_\infty - y_k$ for $k \in \mathbb{N}$ and letting $\hat{s}_0 = \hat{b}_0 - \eta$. There exist minimal k_B and k_S such that $\hat{b}_{2k_B} < b_0$ and $\hat{s}_{2k_S-1} > s_0$. We define b_n and s_n as subsequences of $\hat{b}_n$ and $\hat{s}_n$ starting from $n_0 = 2\max\{k_B, k_S\}$. Observe that by the construction of x_k and y_k in (36),(42) and (43), any x_k and y_k are continuous in b_∞ and s_∞. Moreover, for $b_\infty = b_0$, we have that $n_0 = 2k_B$, $b_0 - b_1 = 0$, $s_1 - s_0 = 2\eta$, and at the other extreme, for $b_\infty = s_0 - \eta$, $n_0 = 2k_S$,$b_0 - b_1 = 2\eta$, $s_1 - s_0 = 0$. By the continuity, there exists b_∞ (and correspondingly, $s_\infty = b_\infty + \eta$) such that for corresponding b_n and s_n constructed as described above, we havemax$\{|b_0 - b_1|, |s_1 - s_0|\} < (1-\delta)C$. For all $n \geq 1$, $\max\{|b_{n-1} - b_n|, |s_n - s_{n-1}|\} < (1-\delta)C$ follows from the corresponding inequality for x_k and y_k. □

Proof of Lemma 4. By Lemma 13, we could construct sequences of threshold types b_n and s_n so that corresponding sequences x_k and y_k defined by $x_k = b_{2k} - b_\infty$ and $y_k = s_\infty - s_{2k-1}$ for $k \in \mathbb{N}$ satisfies (29). Since (x_k, y_k) is a positive trajectory and $\alpha^B(y) > 0$ whenever $y > 0$, from (29) it follows that $x_{k+1} - x_k = -\alpha^B(y_{k+1})(x_k + y_{k+1}) < 0$ for all $n \in \mathbb{N}$, and analogously, $y_{k+1} - y_k < 0$. Hence, b_n and s_n are monotone sequences. Since (x_k, y_k) converges to $(0,0)$, the limits of b_n and s_n are b_∞ and s_∞, respectively.

The form of functions $\alpha^B(x)$ and $\alpha^S(y)$ guarantees that equations (24) and (26) hold. Hence, threshold types are indifferent between accepting the opponent's offer in the current round and rejecting it (and accepting in the following round). By Lemma 11, this is sufficient for the optimality of acceptance strategies given by thresholds b_n and s_n. Moreover, recursive system (29) guarantees that the probability that threshold types assign to their offer being accepted in the next round is derived from the acceptance policy of the opponent. This completes the construction of the equilibrium strategies on the equilibrium path.

All deviations from acceptance strategies b_n and s_n are ignored. To deter deviations from offers P^B and P^S specify that after deviations from price offers P^B and P^S, players switch to the punishing equilibrium of the deviator. By Theorem 7, in such equilibrium the expected utility of the deviator is uniformly (over all types of the deviator) close to the reservation utility as δ converges to one. On the other hand, by following equilibrium strategy any seller type $s \leq s^\omega_{b_0}$ gets at least $P^B - c(s)$, and any buyer type $b \geq b^\alpha_{s_0}$ gets at least $v(b) - P^S$. These utilities are bounded away from the reservation utility by (8).

This proves that the constructed thresholds constitute a required continuation CSE whenever recursion (29) has a positive solution. □

Proof of Theorem 2. Sufficiency

Consider a tuple $(b^*_t, s^*_t, q^B_t, q^S_t, T)$ as in the sufficiency part of Theorem 2. For any $\tilde{\varepsilon} > 0$, choose $\tilde{t} \in \mathbb{R}_+$ such that $b^*_{\tilde{t}} < b^*_T + \tilde{\varepsilon}$ and $s^*_{\tilde{t}} > s^*_T - \tilde{\varepsilon}$. Since $b^*_\infty \in (0,1)$ and $s^*_\infty \in (0,1)$, $b^*_T = s^*_T - \eta$ by (2). Therefore, we can choose $\tilde{t}$ sufficiently large so that

$$0 < b^*_{\tilde{t}} < 1 - \eta,\ \eta < s^*_{\tilde{t}} < 1, \text{ and } s^*_{\tilde{t}} \in [b^*_{\tilde{t}} - \eta, b^*_{\tilde{t}} + \eta). \tag{50}$$

By the strict versions of (6) and (7), we have $q^S_0 < \frac{v(1)+c(1)}{2}$ and $q^B_0 > \frac{v(0)+c(0)}{2}$ and so, by Condition M′,

$$\frac{v(0)+c(0)}{2} < q^B_t \leq q^S_t < \frac{v(1)+c(1)}{2}, \tag{51}$$

for all $t \in [0,T]$. For any time t, let $N^j_t \equiv \left\lfloor \frac{t}{\Delta_j} \right\rfloor$. There are three cases to consider: 1) $T = \infty$ and $q^S_T > q^B_T$, 2) $T < \infty$ and $q^S_T = q^B_T$, 3) $T = \infty$ and $q^S_T = q^B_T$.

Case 1) $\mathbf{T = \infty}$ and $\mathbf{q_T^S > q_T^B}$. CSEs that we construct to approximate $(b_t^*, s_t^*, q_t^B, q_t^S)$ are in grim-trigger strategies. Players start the game by following the *main path,* and continue following it so long as there were no detectable deviations in the past. If one of the sides detects deviation from the main path, then the play switches to the *punishing path of the deviatior.*

Construction of the main path $(b_n^j, s_n^j, p_n^{Bj}, p_n^{Sj})$. Strategies on the main path are constructed separately for times before and after $\tilde{t}$. Since $c(s_T^*) < q_T^B < q_T^S < v(b_T^*)$, we can choose $\tilde{\varepsilon}$ small enough and $\tilde{t}$ large enough so that $v(b_T^* - \tilde{\varepsilon}) > v(b_T^*) - \tilde{\varepsilon}\ell > q_{\tilde{t}}^S$ and $c(s_T^* + \tilde{\varepsilon}) < c(s_T^*) + \tilde{\varepsilon}\ell < q_{\tilde{t}}^B$ where we use the Lipschitz continuity of $v(b)$ and $c(s)$ in the inequalities. Combining this with (51) we get

$$\min\left\{c(s_T^* + \tilde{\varepsilon}), \frac{v(0) + c(0)}{2}\right\} < q_{\tilde{t}}^B < q_{\tilde{t}}^S < \min\left\{v(b_T^* - \tilde{\varepsilon}), \frac{v(1) + c(1)}{2}\right\}. \tag{52}$$

Let $b_{N_{\tilde{t}}^j} \equiv b_{\tilde{t}}^*$, $s_{N_{\tilde{t}}^j} \equiv s_{\tilde{t}}^*$. By (50) and (52), conditions of Lemma 4 are satisfied and so, for Δ_j sufficiently small, there exists a continuation CSE for $n > N_{\tilde{t}}^j$ such that price paths are constant, $p_n^{Sj} = q_{\tilde{t}}^S$ and $p_n^{Bj} = q_{\tilde{t}}^B$, and $\max\{b_{N_{\tilde{t}}^j}^j - b_{N_{\tilde{t}}^j+2}^j, s_{N_{\tilde{t}}^j+2}^j - s_{N_{\tilde{t}}^j}^j\} < C\Delta_j$ for some $C > 0$ independent of Δ_j.

For $n \le N_{\tilde{t}}^j - 1$, construct sequences $b_n^j, s_n^j, p_n^{Sj}, p_n^{Bj}$ as follows. For any integer $n \le N_{\tilde{t}}^j - 1$, we define $b_n^j = b_{n\Delta_j}^*$ for even n and $b_n^j = b_{n-1}^j$ for odd n. Analogously, for any integer $n \le N_{\tilde{t}}^j - 1$, we define $s_n^j = s_{n\Delta_j}^*$ for odd n and $s_n^j = s_{n-1}^j$ for even n. For any integer $n \le N_{\tilde{t}}^j - 1$, we define α_n^{Sj} and α_n^{Bj} by (25) and (27). Given $b_n^j, s_n^j, \alpha_n^{Bj}, \alpha_n^{Sj}$, we construct price paths p_n^{Bj} and p_n^{Sj} starting from round $N_{\tilde{t}}^j - 1$ and proceeding backwards in time so that equations (24) and (26) are satisfied.

Convergence. Since b_t^* is continuously differentiable , function b_t^* is Lipschitz continuous with some modulus ℓ_1 on $[0, \tilde{t}]$, and without loss of generality, let $C < \ell_1$. Hence, the extension b_t^j of b_n^j to continuous domain is Lipschitz continuous with the same modulus ℓ_1 on $[0, \tilde{t}]$. Therefore, b_t^j converges to b_t^* uniformly on $[0, \tilde{t}]$ as $\Delta_j \to 0$. Analogously, extension s_t^j of s_n^j to continuous domain converges uniformly to s_t^* on $[0, \tilde{t}]$.

Writing equation (24) with $n = N_t^j$, subtracting from both sides $e^{-2r\Delta_j}(v(b_{N_t^j}^j) - p_{N_t^j}^{Sj})$

and dividing by $2\Delta_j$, we get

$$\frac{1-e^{-2r\Delta_j}}{2\Delta_j}\left(v(b^j_{N^j_t})-p^{Sj}_{N^j_t}\right)=e^{-r\Delta_j}\frac{\alpha^{Sj}_{N^j_t}}{2\Delta_j}\left((1-e^{-r\Delta_j})v(b^j_{N^j_t})-p^{Bj}_{N^j_t+1}+e^{-r\Delta_j}p^{Sj}_{N^j_t+2}\right)+e^{-2r\Delta_j}\frac{p^{Sj}_{N^j_t}-p^{Sj}_{N^j_t+2}}{2\Delta_j}. \tag{53}$$

Observe that for $n\le N^j_{\tilde t}-1$,

$$\alpha^{Sj}_n=\max\left\{\frac{s^j_{n+1}-\max\{s^j_{n-1},s^\alpha_{b^j_n}\}}{s^\omega_{b^j_n}-\max\{s^\Delta_{n-1},s^\alpha_{b^j_n}\}},0\right\}\le\frac{2\Delta_j\ell_1}{\tilde\varepsilon}, \tag{54}$$

and the same upper bound holds for α^{Bj}_n. Therefore, by (53) for all Δ_j function p^{Sj}_t is Lipschitz continuous with a common (for all Δ_j) modulus of continuity, and hence, over a subsequence p^{Sj}_n converges uniformly on $[0,\tilde t]$ to a Lipschitz continuous function $\tilde q^S_t$ with the same modulus of continuity. Taking the limit of (53) we get that $\tilde q^S_t$ satisfies equation (4). By the Picard-Lindelöf theorem the limit $\tilde q^S_t$ coincides with q^S_t. Therefore, p^{Sj}_t converges uniformly to q^S_t on $[0,\tilde t]$, and by an analogous argument, p^{Bj}_t converges uniformly on $[0,\tilde t]$ to q^B_t.

Claim 15. For $\hat T=\infty$, there exists $\underline{\Delta}>0$ such that for all $\Delta_j<\underline{\Delta}$, p^{Bj}_n and p^{Sj}_n are monotone for $n\le N^j_{\tilde t}$.

Proof. Observe that, unless $\tilde t>\hat T$ in which case price paths are constant, q^S_t is strictly decreasing on $[0,\tilde t]$. By the continuous differentiability of q^S_t there exists $\tilde\gamma>0$ such that $\dot q^S_t<-\tilde\gamma$ on $[0,\tilde t]$. By the uniform convergence of b^j_t, s^j_t,p^{Bj}_t,p^{Sj}_t, (53) implies that $\frac{p^{Sj}_{N^j_t}-p^{Sj}_{N^j_t+2}}{2\Delta_j}$ converges uniformly to $\dot q^S_t$ and so, p^S_n is decreasing for sufficiently small Δ_j. Analogously, p^B_n is increasing for sufficiently small Δ_j. *Q.E.D.*

Notice that $T_j=T=\infty$ for any CSE constructed. By the definition of $\tilde t$, $b^*_{\tilde t}\to b^*_T$ and $s^*_{\tilde t}\to s^*_T$ as $\tilde t\to T$. Therefore, as we take $\tilde\varepsilon$ to zero, and correspondingly $\tilde t$ to T, we get the desired sequence of approximating CSEs with the smooth limit $(b^*_t,s^*_t,q^B_t,q^S_t)$.

Construction of the punishing path. After deviations from the price paths of the main path, players switch to the deviator's punishing equilibrium described in Section 6. If the seller deviates from the acceptance strategy, then this deviation might remain undetected by the buyer at least for some time. Observe, however, that when this deviation occurs, buyer types below $b^\alpha_{s_t}$ detect such deviation at time t. Analogously, the buyer deviation from the acceptance strategy is detected by seller types above $s^\omega_{b_t}$ at time t. Specify that immediately after the detection, the buyer switches to the seller punishing equilibrium described in Section 6. That is, beliefs of the buyer are given by (15) and all buyer types below $b^{\alpha}_{s_t}$ pool on $\frac{c(0)+e^{-r\Delta_j}v(0)}{1+e^{-r\Delta_j}}$. Beliefs of seller types below s_t are uniform on $B_s \cap [0, b^{\alpha}_{s_t}]$ and the seller follows equilibrium strategies in the seller punishing continuation equilibrium. Strategies and beliefs after the buyer deviation from the acceptance strategy are specified symmetrically. We next show that such punishing paths deter deviations from the main path for sufficiently small Δ_j.

Claim 16. For Δ_j sufficiently small, there are no profitable deviations from the main path.

There is a difference in the analysis of the incentives to deviate from the main path of buyer types below and above $b^*_T - \tilde{\varepsilon}$. On the one hand, buyer types below $b^*_T - \tilde{\varepsilon}$ expect that with probability one, one of buyer's offers is accepted by time $\tilde{t}$. Therefore, the strategies specified after time $\tilde{t}$ do not affect their incentives to deviate. On the other hand, buyer types above $b^*_T - \tilde{\varepsilon}$ could remain in the game after time $\tilde{t}$ and so, their incentives to deviate could be affected by the way we specified the main path for $t > \tilde{t}$. The following two claims ensure that both groups of types do not have incentives to deviate.

Claim 17. *There exists $\hat{\ell}$ such that for any Δ_j, $U^{Bj}_t(b)$ and $\mathcal{U}^B_t(b)$ are Lipschitz continuous in both arguments with modulus $\hat{\ell}$. Moreover, for any $\varepsilon > 0$,*

$$\max_{t\in[0,\tilde{t}],b\in[0,b^*_T-\tilde{\varepsilon}]} |U^{Bj}_t(b) - \mathcal{U}^B_t(b)| < \varepsilon$$

for sufficiently small Δ_j.

Proof. Let n^j_b be the round, in which buyer type b accepts the seller offer if he follows the strategy b^{Δ}_n. Consider two buyer types b and b'. Observe that

$$U^{Bj}_t(b) = \mathbb{E}[e^{-r(\Delta_j N - t)}(v(b) - p)|s \in S_b \cap S_{b'}, s \geq s^j_t, n^j_b]\frac{|S_b \cap S_{b'} \cap [s^j_t, 1]|}{|S_b \cap [s^j_t, 1]|} +$$
$$\mathbb{E}[e^{-r(\Delta_j N - t)}(v(b) - p)|s \in S_b \backslash S_{b'}, s \geq s^j_t, n^j_b]\frac{|(S_b \backslash S_{b'}) \cap [s^j_t, 1]|}{|S_b \cap [s^j_t, 1]|} \geq$$

$$\mathbb{E}[e^{-r(\Delta_j N - t)}(v(b) - p)|s \in S_b \cap S_{b'}, s \geq s^j_t, n^j_{b'}]\frac{|S_b \cap S_{b'} \cap [s^j_t, 1]|}{|S_b \cap [s^j_t, 1]|} +$$
$$\mathbb{E}[e^{-r(\Delta_j N - t)}(v(b) - p)|s \in S_b \backslash S_{b'}, s \geq s^j_t, n^j_{b'}]\frac{|(S_b \backslash S_{b'}) \cap [s^j_t, 1]|}{|S_b \cap [s^j_t, 1]|} =$$

$$U_t^{Bj}(b') + \mathbb{E}[e^{-r(\Delta_j N-t)}(v(b)-v(b'))|s \in S_b \cap S_{b'}, s \geq s_t^j, n_{b'}^j]\frac{|S_b \cap S_{b'} \cap [s_t^j,1]|}{|S_{b'} \cap [s_t^j,1]|} -$$
$$\mathbb{E}[e^{-r(\Delta_j N-t)}(v(b')-p)|s \in S_{b'} \backslash S_b, s \geq s_t^j, n_{b'}^j]\frac{|S_{b'} \backslash S_b \cap [s_t^j,1]|}{|S_{b'} \cap [s_t^j,1]|} +$$
$$\mathbb{E}[e^{-r(\Delta_j N-t)}(v(b)-p)|s \in S_b \backslash S_{b'}, s \geq s_t^j, n_{b'}^j]\frac{|S_b \backslash S_{b'} \cap [s_t^j,1]|}{|S_b \cap [s_t^j,1]|} \geq$$
$$U_t^{Bj}(b') - \ell|b-b'| - \Sigma|b-b'|,$$

where the equalities are by the application of the law of total expectation to $U_t^{Bj}(b)$ and $U_t^{Bj}(b')$, the first inequality is by the fact that buyer type b prefers to accept in round n_b^j rather than in round $n_{b'}^j$, and the second inequality is by the Lipschitz continuity of $v(b)$ and the upper bound on the size of the surplus. Therefore, $U_t^{Bj}(b)$ is Lipschitz continuous in b with modulus $\ell + \Sigma$.

Now for fixed b consider even integers $n < n' < n_b^j$. We have

$$U_n^{Bj}(b) = \sum_{m=n/2+1}^{n'/2-1} e^{-r\Delta_j(2m+1-n)}\frac{s_{2m+1}^j - s_{2m-1}^j}{s_b^\omega - \max\{s_b^\alpha, s_n^j\}}\left(v(b) - p_{2m}^{Bj}\right) + e^{-r\Delta_j(n'-n)}\frac{s_b^\omega - \max\{s_b^\alpha, s_{n'}^j\}}{s_b^\omega - \max\{s_b^\alpha, s_n^j\}}U_{n'}^{Bj}(b). \tag{55}$$

Notice that

$$0 < 1 - e^{-r\Delta_j(n'-n)}\frac{s_b^\omega - \max\{s_b^\alpha, s_{n'}^j\}}{s_b^\omega - \max\{s_b^\alpha, s_n^j\}} \leq 1-(1-r\Delta_j(n'-n))\left(1 - \frac{\max\{s_b^\alpha, s_{n'}^j\} - \max\{s_b^\alpha, s_n^j\}}{s_b^\omega - \max\{s_b^\alpha, s_n^j\}}\right) =$$

$$r\Delta_j(n'-n) + (1 - r\Delta_j(n'-n))\frac{s_{n'}^j - s_n^j}{s_b^\omega - \max\{s_b^\alpha, s_n^j\}} \leq r\Delta_j(n'-n) + \frac{2\Delta_j(n'-n)\ell_1}{\tilde{\varepsilon}}. \tag{56}$$

By (54) and (56), (55) implies

$$|U_n^{Bj}(b) - U_{n'}^{Bj}(b)| \leq \Sigma\left(\frac{4\Delta_j(n'-n)\ell_1}{\tilde{\varepsilon}} + r\Delta_j(n'-n)\right) \equiv \Delta_j(n'-n)\ell_2. \tag{57}$$

Since function $U_t^{Bj}(b)$ is piecewise linear and by inequality (57), its slope does not exceed ℓ_2, $U_t^{Bj}(b)$ is Lipschitz continuous in t with modulus ℓ_2. Hence, $U_t^{Bj}(b)$ is Lipschitz continuous in both arguments with modulus $\hat{\ell} \equiv \ell + \Sigma + \ell_2$. The proof of Lipschitz continuity of $\mathcal{U}_t^B(b)$ is analogous.

The sequence of functions $U_t^{Bj}(b)$ are Lipschitz continuous for all j with common modulus $\hat{\ell}$. Hence, they converge uniformly to some limit which is Lipschitz continuous with the same modulus $\hat{\ell}$. Moreover, $U^{Bj}_{N_t}(b)$ converges pointwise to $\mathcal{U}^B_t(b)$ for $b \in [0, b^*_T - \tilde{\varepsilon}]$ by construction by the dominated convergence theorem. Hence, $U^{Bj}_t(b)$ converges uniformly to $\mathcal{U}^B_t(b)$ on $t \in [0, \tilde{t}]$ and $b \in [0, b^*_T - \tilde{\varepsilon}]$. *Q.E.D.*

Claim 18. There exists $\underline{\Delta} > 0$ and $u > 0$ such that for all $\Delta_j < \underline{\Delta}$,

$$\min_{t\in[0,\tilde{t}],b\in(b^*_T-\tilde{\varepsilon},1]} U^{Bj}_t(b) - \max\left\{v(b) - \frac{v(1)+c(1)}{2}, 0\right\} > u.$$

Proof. The buyer could accept p^{Sj}_n in any even round n. Moreover, the buyer could accept seller offer $q^S_{\tilde{t}}$ in round $N^j_{\tilde{t}}$. Therefore,

$$U^{Bj}_n(b) \geq \max\{v(b) - p^{Sj}_n, e^{-r\tilde{t}}(v(b) - q^S_{\tilde{t}})\}.$$

Denote $u_1 \equiv e^{-r\tilde{t}}(v(b^*_T - \tilde{\varepsilon}) - q^S_{\tilde{t}})$, and $u_1 > 0$ by (52). For any $b > b^*_T - \tilde{\varepsilon}$, $e^{-r\tilde{t}}(v(b) - q^S_{\tilde{t}}) \geq u_1 > 0$.

We next show that for $u_2 \equiv \frac{1}{4}(\frac{v(1)+c(1)}{2} - q^S_0) > 0$ (by (51)), we have $p^{Sj}_n < \frac{v(1)+c(1)}{2} - u_2$ for Δ_j sufficiently small. By the convergence of p^{Bj}_t to q^B_t on $[0, \tilde{t}]$, there exists $\underline{\Delta} > 0$ such that for all $\Delta_j < \underline{\Delta}$, $p^{Sj}_0 < q^S_0 + u_2$ and so, $p^{Sj}_n \leq p^{Sj}_0 < \frac{v(1)+c(1)}{2} - u_2$. We complete the proof by taking $u \equiv \min\{u_1, u_2\} > 0$. *Q.E.D.*

Proof of Claim 16. By Claim 17, continuation utilities of buyer types in $[0, b^*_T - \tilde{\varepsilon}]$ from following the main path converge uniformly (in type and time) to $\mathcal{U}^B_t(b)$. By the strict version of inequality (6), there exists $\varepsilon > 0$ such that $\mathcal{U}^B_t(b) > \max\left\{v(b) - \frac{v(1)+c(1)}{2}, 0\right\} + \varepsilon$ for all b and $t \in [0, \tilde{t}]$. By Claim 18, continuation utilities of buyer types in $(b^*_T - \tilde{\varepsilon}, 1]$ from following the main path are greater than $\max\left\{v(b) - \frac{v(1)+c(1)}{2}, 0\right\}$ by at least $u > 0$, for sufficiently small Δ.

By Corollary 1, for any $\varepsilon > 0$, the continuation utility of any type of the punished player is at most ε away from the reservation utility $\max\left\{v(b) - \frac{v(1)+c(1)}{2}, 0\right\}$, for sufficiently small Δ_j. Therefore, deviations from the price paths constructed are not profitable for buyer types. By Claim 15, the constructed price paths are monotone and so, deviations from the acceptance strategies are not optimal by Lemma 11. The proof for the seller is symmetric.

Q.E.D.

Case 2) $\mathbf{T < \infty}$ and $\mathbf{q^S_T = q^B_T}$. Let $\tilde{t} = T, \tilde{\varepsilon} = 0$, and the construction of the main path for case 1 is repeated with the difference that after time T trade stops and there are no types remaining. By the analogous argument as in case 1, it could be verified that the constructed main path could be supported by the punishing path and that the strategies describing the main path converge a.e. to the corresponding limits on $[0,T]$. Moreover, for all Δ_j, $T_j = T$, and $b_T = b_T^*$, $s_T = s_T^*$ which completes the analysis of case 2.

Case 3) $\mathbf{T} = \infty$ and $\mathbf{q_T^S} = \mathbf{q_T^B}$. We first construct the following approximation of $(b_t^*, s_t^*, q_t^B, q_t^S)$. For any $\tilde{t} \in \mathbb{R}_+$, let $\hat{b}_t = b_t^* - \frac{t}{\tilde{t}}\left(b_{\tilde{t}}^* - b_\infty^*\right)$ and $\hat{s}_t = s_t^* + \frac{t}{\tilde{t}}\left(s_\infty^* - s_{\tilde{t}}^*\right)$ and construct price offers $\hat{q}_t^S$ and $\hat{q}_t^B$ satisfying (3), (4) and $\hat{q}_t^B = \hat{q}_t^S = q_T^B = q_T^S$. We could proceed as in case 2 to construct an approximating sequence of CSEs of the limit $(\hat{b}_t, \hat{s}_t, \hat{q}_t^B, \hat{q}_t^S, \tilde{t})$. By construction, as $\tilde{t} \to \infty$, $\hat{b}_t$ and $\hat{s}_t$ converge uniformly to b_t^* and s_t^*, respectively, as well as their derivatives converge uniformly to the corresponding derivatives of b_t^* and s_t^*. By Theorem 1.1 in Freidlin and Wentzell (1984) price offers $\hat{q}_t^S$ and $\hat{q}_t^B$ converge to q_t^S and q_t^B, respectively. Moreover, $T_j = \tilde{t}$ converges to $T = \infty$, and $\hat{b}_\infty = \hat{b}_{\tilde{t}} = b_\infty^*$ and $\hat{s}_\infty = \hat{s}_{\tilde{t}} = s_\infty^*$. □

Proofs for Section 4

Proof of Theorem 3. Let $\bar{U}^B(\omega) \equiv \bar{P}(\omega)v(\omega) - \bar{X}(\omega)$, $\bar{U}^S(\omega) \equiv \bar{X}(\omega) - \bar{P}(\omega)c(\omega)$, $U^B(b) \equiv P^B(b)v(b) - X^B(b)$, $U^S(s) \equiv X^S(s) - P^S(s)c(s)$. We start with a preliminary observation, which follows from the argument in Lemma 2 in Myerson (1981).

Claim 19. *Condition* (9) *is equivalent to*

$$\bar{P}(\omega) \geq \bar{P}(\omega') > 0 \tag{58}$$

and

$$\bar{U}^B(\omega) = \bar{U}^B(\omega') + \int_{\omega'}^{\omega} \bar{P}(w)dv(w), \tag{59}$$

for any $\omega \geq \omega' > \omega^*$.

Direction CSM→CSE sequence. Consider a CSM and the outcome $\mathcal{O} = (\bar{P}^B(b), \bar{P}^S(s), \bar{U}^B(b), \bar{U}^S(s))$ of the truthful equilibrium in this CSM. To prove the first statement of Theorem 3, we construct a sequence of CSEs indexed by $j \in \mathbb{N}$ with outcomes $\mathcal{O}^j = (P^{Bj}(b), P^{Sj}(s), U^{Bj}(b), U^{Sj}(s))$ such that $\mathcal{O}^j$ converges a.e. to $\mathcal{O}$, and $(\Delta_j, \eta_j) \to (0,0)$.

Consider a monotone sequence $\eta_j \to 0$. Suppose that in the CSM, $\omega^* \in (0,1)$, and without loss of generality, suppose that

$$\eta_j < \min\left\{\frac{\omega^*}{2}, \frac{1-\omega^*}{2}\right\}$$

for all $j \in \mathbb{N}$. When ω^* equals 0 or 1, the argument below is first carried for $\tilde{\omega}^*$ equal to $\tilde{\varepsilon}$ or $1-\tilde{\varepsilon}$, respectively, for some $\tilde{\varepsilon} > 0$ and then we take $\tilde{\varepsilon} \to 0$.

Define $t_b^* \equiv -\frac{1}{r}\ln \bar{P}(b)$ for $b > \omega^*$ and $t_s^* \equiv -\frac{1}{r}\ln \bar{P}(s)$ for $s < \omega^*$. By (58) in Claim 19, function $\bar{P}(b)$ is increasing in type b for $b > \omega^*$ and so, t_b^* is decreasing in b. Analogously, t_s^* is increasing in s. Consider inverse functions $b_t^* \equiv \inf\{b \in [0,1] : t_b^* \leq t\}$ and $s_t^* \equiv \sup\{s \in [0,1] : t_s^* \leq t\}$. Since $\bar{P}(\omega^*+0) = \bar{P}(\omega^*-0)$ and $\bar{P}(\omega) > 0$ for all $\omega \neq \omega^*$ (by condition (13)), we can choose $\tau_j < \infty$ to be the minimal τ_j such that $b_{\tau_j}^* - s_{\tau_j}^* \leq \eta_j$ and, in particular,

$$0 < b_{\tau_j}^* < \omega^* + \eta_j \leq 1 - \eta_j \text{ and } 1 > s_{\tau_j}^* > \omega^* - \eta_j \geq \eta_j. \tag{60}$$

Let $T \equiv t_{\omega^*}^*$ and observe that $\tau_j \to T$ as $j \to \infty$.

Construction of CSE strategies. We construct a CSE by the same scheme as in the proof of Theorem 2. Since $v(b)$ and $c(s)$ are continuous and $v(\omega^*) - c(\omega^*) \geq \xi > 0$, $v(\omega^* - \eta_j) > c(\omega^* + \eta_j)$ for sufficiently small η_j. If $\bar{X}(b_{\tau_j}^*) = \bar{X}(s_{\tau_j}^*)$, specify that at time τ_j all remaining types trade at price $\bar{X}(b_{\tau_j}^*)/\bar{P}(b_{\tau_j}^*)$ and the construction is carried as in case 2 in the proof of Theorem 2. If $\bar{X}(b_{\tau_j}^*) > \bar{X}(s_{\tau_j}^*)$, then we proceed as in case 1 in the proof of Theorem 2. We define $P^{Sj} \equiv \bar{X}(b_{\tau_j}^*)/\bar{P}(b_{\tau_j}^*) - \varepsilon_j$, $P^{Bj} \equiv \bar{X}(s_{\tau_j}^*)/\bar{P}(s_{\tau_j}^*) + \varepsilon_j$ where $\varepsilon_j \in [0, 2^{-j}]$ is small enough so that condition (8) in Lemma 4 is satisfied. By (60), for times after time τ_j, the continuation equilibrium can be constructed by Lemma 4. For the rounds before time τ_j, the acceptance functions b_t^j and s_t^j, and price offers p_t^{Bj} and p_t^{Sj} are constructed as in the proof of the sufficiency part of Theorem 2. We choose Δ_j sufficiently small so that for given η_j, the constructed main path can be supported by the punishing path in the construction in the proof of Theorem 2.

Convergence. By the construction, for $b \in (b_{\tau_j}^*, 1]$ the difference between the type b's acceptance time and t_b^* is at most $2\Delta_j$ and so, as $j \to \infty$, $P^{Bj}(b)$ converges uniformly to $\bar{P}(b)$ for such types. By the argument analogous to Claim 19, for weak types $b \in$

$(b^*_{\tau_j-2\Delta_j},1]$,

$$U^{Bj}(b) = U^{Bj}(b^*_{\tau_j-2\Delta_j}) + \int\limits_{b^*_{\tau_j-2\Delta_j}}^{b} P^{Bj}(b)dv(b).$$

As $j \to \infty$, $U^{Bj}(b^*_{\tau_j-2\Delta_j}) = e^{-r\tau_j}(v(b^*_{\tau_j-2\Delta_j}) - P^{Bj})$ converges to $\bar{U}^B(b^*_\tau)$ and so, by the dominated convergence theorem, $U^{Bj}(b)$ converges to $\bar{U}^B(b)$ for $b \in (b^*_{\tau_j},1]$. By the integral formula, $U^{Bj}(b)$ and $\bar{U}^B(b)$ are Lipschitz continuous with modulus one and so, $U^{Bj}(b)$ converges uniformly to $\bar{U}^B(b)$ on $(b^*_{\tau_j},1]$.

Now for seller types $s \in [\omega^* + 2\eta_j, 1]$,

$$P^{Sj}(s) = \frac{1}{|B_s|}\int\limits_{B_s} P^{Bj}(b)db$$

and

$$U^{Sj}(s) = \frac{1}{|B_s|}\int\limits_{B_s}(P^{Bj}(b)(v(b) - c(s)) - U^{Bj}(b))db.$$

By the monotonicity of $P^{Bj}(b)$, for $s \in [\omega^* + 2\eta_j, 1]$, $P^{Bj}(b^\alpha_s) \le P^{Sj}(s) \le P^{Bj}(b^\omega_s)$ and so, by the uniform convergence of $P^{Bj}(b)$ on $(b^*_\tau,1]$, $\bar{P}^B(s-\eta_j)-\eta_j \le P^{Sj}(s) \le \bar{P}^B(s+\eta_j)+\eta_j$ for Δ_j sufficiently small. As $\eta_j \to 0$, $P^{Sj}(s)$ converges to $\bar{P}^S(s)$ for a.e seller type above ω^*.[56] Further,

$$P^{Bj}(b^\alpha_s)(v(b^\alpha_s)-c(s))-\frac{1}{|B_s|}\int\limits_{B_s} U^{Bj}(b)db \le U^{Sj}(s) \le P^{Bj}(b^\omega_s)(v(b^\omega_s)-c(s))-\frac{1}{|B_s|}\int\limits_{B_s} U^{Bj}(b)db,$$

and by the uniform convergence of $P^{Bj}(b)$ and $U^{Bj}(b)$ on $(b^*_\tau, 1]$, for Δ_j sufficiently small,

$$\bar{P}^B(s-\eta_j)(v(s-\eta_j) - c(s)) - \frac{1}{|B_s|}\int\limits_{B_s}\bar{U}^B(b)db - \eta_j \le U^{Sj}(s),$$

$$\bar{P}^B(s+\eta_j)(v(s+\eta_j) - c(s)) - \frac{1}{|B_s|}\int\limits_{B_s}\bar{U}^B(b)db + \eta_j \ge U^{Sj}(s),$$

for all $sin[\omega^* + 2\eta_j, 1]$. As $\eta_j \to 0$, $\frac{1}{|B_s|}\int\limits_{B_s}\bar{U}^B(b)db \to \bar{U}^B(s)$ for a.e. seller type in $[\omega^* + 2\eta_j, 1]$ and so, $U^S(s)$ converges to $\bar{U}^S(s)$ for such types. The argument for types below ω^* is symmetric. Therefore, we constructed the required sequence of CSEs.

[56]The convergence is not guaranteed only at discontinuity points of $\bar{P}^S(s)$. By Claim 19, $\bar{P}^S(s)$ is monotone on $(\omega^*, 1]$ and the set of its discontinuity points is at most countable.

Direction CSE sequence→CSM. To prove the second statement, for any $j \in \mathbb{N}$, let b^j be the lowest weak buyer type, and s^j be the highest weak seller type in the CSE of the game with the length of bargaining round Δ_j and the individual uncertainty parameter η_j. Denote $w^j \equiv \frac{1}{2}(b^j + s^j) \in [0, 1]$. Then there exists $\omega^* \in [0, 1]$ such that w^j converges to ω^* over subsequence. Therefore, for any $\varepsilon_2 > 0$, far enough in the sequence all buyer types above $\min\{1, \omega^* + \varepsilon_2\}$ and all seller types below $\max\{0, \omega^* - \varepsilon_2\}$ are weak types. We consider only outcomes for these types, and we cover all the types but type ω^* by choosing ε_2 sufficiently small.

Any weak type knows at what time and at what price trade will happen, since the probability of the opponent's concession for weak types is zero. In a CSE corresponding to (Δ_j, η_j), for buyer $b > \min\{1, \omega^* + \varepsilon_2\}$, let t_b^j and p_b^j be the time and the price at which such type trades and define analogous quantities t_s^j and p_s^j for sellers $s < \max\{0, \omega^* - \varepsilon_2\}$. By the single crossing property of the payoffs, t_b^j is decreasing and t_s^j is increasing and so, p_b^j is decreasing and p_s^j is increasing. Therefore, a sequence of four monotone functions has a pointwise converging subsequence by Helly's theorem and the limits $(t_b^*, t_s^*, p_b^*, p_s^*)$ exist. For any weak buyer types b and b', buyer type b prefers accepting at time t_b^j to accepting at $t_{b'}^j$, $e^{-rt_b^j}\left(v(b) - p_b^j\right) \geq e^{-rt_{b'}^j}\left(v(b) - p_{b'}^j\right)$. Hence, in the limit $e^{-rt_b^*}\left(v(b) - p_b^*\right) \geq e^{-rt_{b'}^*}\left(v(b) - p_{b'}^*\right)$, which is condition (9) for buyer and by the same logic condition (11) obtains. Conditions (10) and (12) follow from Lemma 3. Condition 3 follows from the monotonicity of price paths in the definition of the CSE. □

Proofs for Section 5

Proof of Theorem 4. We carry the construction from the top of type distribution. Let $q_Z^B = q_Z^S$. By Lemma 4, for sufficiently small Δ, there exists a CSE with constant offers on the equilibrium path q_{Z-1}^B and q_Z^S, and acceptance strategies b_n^{Z-1} and s_n^{Z-1} such that $b_\infty^{Z-1} = b^{Z-1}$ and $s_\infty^{Z-1} = s^{Z-1}$. For all $z = 1, \ldots, Z-2$, let $\hat{s}^z \equiv b^z - \eta$ and q_z^S be such that

$$q_z^S - c(\hat{s}^z) = \delta(q_z^B - c(\hat{s}^z)). \tag{61}$$

By Lemma 4, we construct a CSE with constant offers on the equilibrium path q_{z-1}^B and q_z^S, and acceptance strategies s_n^z and b_n^z such that $b_\infty^z = b^z$ and $s_\infty^z = s^z$. Denote $\hat{b}^0 = \hat{s}^1 = 0$, $\hat{b}^{Z-1} = \hat{s}^{Z-1} = 1$, and $\hat{b}^z = b_2^z$ for $z = 1, \ldots, Z-1$.

In the first round, for $z = 2, \ldots, Z-1$ seller types in $[\hat{s}^{z-1}, \hat{s}^z]$ make offer q_z^S. In the second round, buyer types in $[\hat{b}^0, \hat{b}^1]$ reject offer q_2^S, types in $[\hat{b}^{z-1}, \hat{b}^z], z = 2, \ldots, Z-1$ accept q_z^S and make counter-offer q_z^B to q_{z+1}^S, buyer types in $[\hat{b}^{Z-1}, \hat{b}^Z]$ accept q_Z^S. After the first two rounds, the remaining types play a corresponding continuation CSE with all subsequent price offers of players equal to their initial price offers. After any detectable deviation players switch to the punishing equilibrium of the deviator.

Observe that if a seller type $\hat{s}^z, z = 2, \dots, Z-1$ makes a lower offer q_z^S, then it is accepted with probability one. Indeed, since $b^z - b^{z-1} > 4\eta$, $\hat{s}^z = b^z - \eta$ and $\hat{b}^{z-1} < b^{z-1} + 2\eta$, we have $\hat{b}^{z-1} < \hat{s}^z - \eta$ and so, all buyer types in $B_{\hat{s}^z}$ accept offer q_z^S. By (61), seller type $\hat{s}^z$ is indifferent between offering q_z^S that is accepted for sure and offering q_{z+1}^S that is rejected for sure and accepting the buyer's offer q_z^B. By the single-crossing property of the payoffs, seller types above $\hat{s}^z$ strictly prefer the acceptance of offer q_z^S by the buyer in two rounds, and seller type below $\hat{s}^z$ strictly prefer the acceptance of q_z^S in the next round. By the choice of q_z^S and q_z^B, no player prefers to deviate from the equilibrium price offers for Δ sufficiently small.

After the first two rounds the game continues only if offers q_z^S and q_{z-1}^B were made for some $z = 2, \dots, Z$. Then only buyer types are below $\hat{b}^z$ and seller types above $\hat{s}^{z-1}$ remain in the game. Such types are playing a continuation CSE constructed by Lemma 4 with offers q_z^S and q_{z-1}^B. Therefore, the probability that the game continues for longer than three rounds is at most $\frac{4\eta^2(Z-1)}{\eta(2-\eta)}$. At the same time, continuation CSEs constructed by Lemma 4 have no almost sure upper bound on the equilibrium delay and so, there is no almost sure upper bound on the equilibrium delay in the constructed segmentation equilibria. □

Proof of Theorem 5 . We apply Theorem 4 with price offers and segments defined as follows. Fix $\varepsilon > 0$ and choose $b^1 = \sqrt{\eta}$, $b^{z+1} = b^z + \sqrt{\eta}$ and $q_z^B = \frac{v(b^{z-1})+c(s^z)}{2}$. Then $Z = 1 + \left\lfloor \frac{1-\eta}{\sqrt{\eta}} \right\rfloor$. We consider only outcomes for types that trade in the first three rounds. As shown in the proof of Theorem 4, the probability of such types is at least $1 - \frac{4\eta^2(Z-1)}{\eta(2-\eta)}$ which converges to one as $\eta \to 0$, since $Z \sim \frac{1}{\sqrt{\eta}}$. Moreover, for such types, $|N\Delta| \le 2$ and $\left|p - \frac{v(b)+c(s)}{2}\right| \le \frac{1}{2}|v(b^{z-1}) - v(b)| + \frac{1}{2}|c(s^z) - c(s)| \le \frac{\ell\sqrt{\eta}}{2} \to 0$ as $\eta \to 0$. This proves, the desired convergence in probability of segmentation equilibria outcomes to the Nash outcome. □

Proofs for Section 6

Proof of Lemma 5. Consider buyer types in the interval $[\underline{b}, 1] \cap [\eta, 1]$. Buyer type b in such interval puts probability one on seller type s_b^α by (15), while seller type s_b^α puts probability one on type b by (18). By Rubinstein (1982) strategies of these two types given in Lemma

5 constitute the subgame perfect equilibrium of the complete information game with valuation $v(b)$ and cost $c(s_b^\alpha)$.

Now consider seller types $s \in [s_{\underline{b}}^\alpha, 1] \cap (1-\eta, 1]$ that put probability one on buyer type 1. Buyer type 1, in turn, puts probability one on seller $1-\eta$ is willing to pay $\check{P}^B(1)$. Since $\check{P}^B(1) > c(1)$, seller types $s \in (1-\eta, 1]$ make price offer $\check{P}^B(1)$. Moreover, they are willing to pay up to $\check{P}^S(s)$ given by $\check{P}^S(s) - c(s) = \delta\left(\check{P}^B(1) - c(s)\right)$. The argument for buyer types $b \in [\underline{b}, 1] \cap [0, \eta)$ is symmetric. □

Existence of the Punishing Equilibrium

Proof of Lemma 6. The analysis of this subgame is standard, and we only sketch the argument. We start by constructing a PBE in a game between seller type 0 and buyer types in $[0, \eta]$, in which the buyer is restricted to either accept the last seller price offer or make counter-offer $\frac{\delta v(0)+c(0)}{1+\delta}$. We use the analysis of Fudenberg, Levine, and Tirole (1985) to construct a PBE in such game described by two functions $P^0(b)$ and $t_{\beta,p}$ and $\bar{\beta} \in [0, \eta]$ such that

1. buyer type b accepts any price offer below $P^0(b)$ and makes counter-offer $\frac{\delta v(0)+c(0)}{1+\delta}$ otherwise;
2. given the highest buyer type $\beta > \bar{\beta}$ and previous price offer p, seller type 0 randomized between the lowest types of the buyer to whom she allocates in the current round according to $t_{\beta,p} \in \Delta(\mathbb{R})$;
3. for $\beta \le \bar{\beta}$, seller type 0 accepts offer $\frac{\delta v(0)+c(0)}{1+\delta}$;
4. $P^0(b)$ is strictly increasing and left-continuous.

The argument in Fudenberg, Levine, and Tirole (1985) should be slightly modified to incorporate the possibility that all buyer types pool on a particular price offer that could be accepted by seller type 0. We start by showing that for β smaller than some $\bar{\beta}$ the seller prefers to accept $\frac{\delta v(0)+c(0)}{1+\delta}$ rather than continue screening. This implies that there is a finite date after which bargaining ends with probability one by the argument analogous to Lemma 3 in Fudenberg, Levine, and Tirole (1985). We follow the steps in their proof of Proposition 1 to construct equilibrium strategies by backward induction on beliefs starting from beliefs supported by $[0, \beta], \beta < \bar{\beta}$ with the only difference that instead of asking price $v(0)$, the seller accepts price offer $\frac{\delta v(0)+c(0)}{1+\delta}$ for such beliefs. This gives the desired equilibrium in the game with restricted buyer price offers. Note that by the argument from the Theorem 3 in Gul, Sonnenschein and Wilson (1986) the Coase Conjecture holds for such game, and for any $\varepsilon > 0$, after any history the first price offer of the seller does not exceed $\frac{v(0)+c(0)}{2} + \varepsilon$ for δ sufficiently close to one.

To support the constructed equilibrium as an equilibrium in the game with unrestricted buyer price offers specify the following punishment for detectable deviations of the buyer. If the buyer deviates and makes an offer different from $\frac{\delta v(0)+c(0)}{1+\delta}$, then the seller puts probability one on the buyer type η and the game proceeds as in the unique subgame perfect equilibrium of the game with complete information with the seller cost equal $c(0)$ and the buyer valuation equal $v(\eta)$. Then trade happens immediately at a price that is close to $\frac{v(\eta)+c(0)}{2}$ for δ close to one. By the Coase Conjecture the first seller price offer is close to $\frac{v(0)+c(0)}{2}$ for δ close to one, making the deviation of the buyer non-profitable. $\square$

Lemma 14. *Suppose $t_\beta(s)$ is a best-reply to willingness to pay $P(b)$. Then $R_\beta(s)$ is non-decreasing in β , satisfying: for $0 \le \beta'' < \beta' \le 1$ we have $0 < R_{\beta'}(s) - R_{\beta''}(s) \le \Sigma(\beta' - \beta'')$ whenever $R_{\beta'}(s) > 0$, and $R_{\beta'}(s) = R_{\beta''}(s) = 0$ whenever $R_{\beta'}(s) = 0$. Moreover, $R_\beta(s)$ is Lipschitz-continuous in both β and s of modulus $\ell_R \equiv \ell + \Sigma$.*

Proof. The first part of Lemma 14 follows from Lemma A.2 in Ausubel, Deneckere (1989). To show that $R_\beta(s)$ is Lipschitz continuous consider two seller types s and s'. Let $R_\beta(s, s')$ be the value function of seller type s from following $t_\beta(s')$. Since seller type s prefers policy $t_\beta(s)$ to $t_\beta(s')$, $R_\beta(s) \ge R_\beta(s, s')$. Let $p_s^{s'}$ and $q_s^{s'}$, respectively, be discounted transfer and probability of allocation, respectively, when seller type s follows optimal policy of seller type s' (and we write p_s for p_s^s and q_s for q_s^s). Then

$$R_\beta(s, s') = p_s^{s'} - q_s^{s'} c(s) \ge p_s^{s'} - q_s^{s'} c(s') - |c(s) - c(s')| \ge$$

$$p_{s'} - q_{s'} c(s') - (\ell + \Sigma)|s - s'| = R_\beta(s') - (\ell + \Sigma)|s - s'|.$$

The first inequality is by $q_s^{s'} \in [0, 1]$. To see the second inequality consider two cases. When $s > s'$, by using $t_{s'}(\beta)$ seller type s gets the same profit from buyer types in $[b_s^\alpha, b_{s'}^\omega]$ as seller type s', but looses at most Σ from buyer types in $[b_{s'}^\alpha, b_s^\alpha]$. When $s < s'$, by using $t_{s'}(\beta)$ seller type s gets the same profit from buyer types in $[b_s^\alpha, b_s^\omega]$ as seller type s', but looses at most Σ from buyer types in $[b_s^\omega, b_{s'}^\omega]$. Hence, $|R_\beta(s) - R_\beta(s')| \le (\ell + \Sigma)|s - s'|$. $\square$

Lemma 15. *Suppose that $t_\beta(s)$ is a best-reply to willingness to pay P_b. Then $t_\beta(s)$ is non-decreasing in s and β. Moreover, for any β, $T_\beta(s)$ has a closed graph, and in particular, $t(s)$ is left-continuous in s.*

Proof. Denote current profit function of seller type s by $\pi_\beta(s,b) = (\beta - b)(P(b) - c(s))$ and constraint is $b \in B_s \cap [0,\beta]$. Since $\frac{\partial}{\partial \beta}\pi_\beta(s,b) = P(b)$ is increasing in b, function $\pi_\beta(s,b)$ is supermodular in (β,b). Since $\frac{\partial}{\partial s}\pi_\beta(s,b) = -c'(s)(\beta - b)$ is increasing in b, function $\pi_\beta(s,b)$ has increasing differences in b and s. Further, consider $b \geq b'$, $\beta \geq \beta'$, $s \geq s'$ and suppose $b' \in B_s \cap [0,\beta]$ and $b \in B_{s'} \cap [0,\beta']$. Then $b \leq \beta' \leq \beta$, $b \leq s' + \eta \leq s + \eta$, $b \geq b' \geq s - \eta$ and, therefore, $b \in B_s \cap [0,\beta]$. Analogously, we could show that $b' \in B_{s'} \cap [0,\beta']$. Hence, the constraint sets are ascending in the terminology of Hopenhayn and Prescott (1992).[57] By Proposition 2 in Hopenhayn and Prescott (1992) value function $R_\beta(s)$ has increasing differences in β and s and solution $t_\beta(s)$ is non-decreasing in s and β. By the generalization of Theorem of the Maximum in Ausubel and Deneckere (1988), for any β, $T_\beta(s)$ has a closed graph and so, $t(s)$ is left-continuous in s. $\square$

Lemma 16. *For all b we have $\pi^i(b) \geq c(s_b^\alpha) + (1-\delta^2)\xi$ and for all $s \in [-1,-\eta]$, $\Pi^i(s) > C(\eta,\delta) > 0$ with $C(\eta,\delta) \sim (1-\delta)^2$ where $\Pi^i(s)$ is the expected profit of seller s that faces demand $\pi^i(b)$.*

Proof. For all buyers b, $\pi^i(b) = (1-\delta^2)v(b) + \delta^2\hat{\pi}^{i-1}(\tau^i(s_b^\alpha)) \geq (1-\delta^2)v(b) + \delta^2 c(s_b^\alpha) \geq c(s_b^\alpha) + (1-\delta^2)\xi$. The first inequality follows from the fact that seller types in $[0, s_{i+1})$ get positive profit when best-replying to static demand given by $\pi^{i-1}(b)$ and the second inequality follows from $v(b) - c(s_b^\alpha) \geq \xi$.

To derive the lower bound on the profit, suppose seller type $s \in [0, 1-\eta]$ makes price offer $c(s) + (1-\delta^2)\frac{\xi}{2}$. By the lower bound on willingness to pay $P(b)$ derived above, buyer types with $\pi^i(b) > c(s) + (1-\delta^2)\frac{\xi}{2\ell}$ accept such price offer. The mass of buyer types who accept such price offer and are in the support of beliefs B_s is at least $\min\{2\eta, (1-\delta^2)\frac{\xi}{2\ell}\}$ and seller type s is guaranteed to get profit $\min\{2\eta, (1-\delta^2)\frac{\xi}{2\ell}\}(1-\delta^2)\frac{\xi}{2\ell} \equiv C(\eta,\delta)$. This minimal profit is equal to $(1-\delta^2)^2\frac{\xi^2}{4\ell}$ for δ close to one and, hence, $C(\eta,\delta) \sim (1-\delta)^2$. $\square$

Lemma 17. *For all $s \in (-1,-\eta]$, $b_s^\omega - t(s) > c(\eta,\delta)$. Moreover, $c(\eta,\delta) \sim (1-\delta)^3$ as δ goes to one.*

Proof. We make change of variable $x = b_s^\omega - b$ in the seller's problem (16). Then $\Pi^i(s) = x(\pi^i(b_s^\omega - x) - c(s)) + \delta^2\Pi^i_{b_s^\omega - x}(s) \leq x(\pi^i(b_s^\omega - x) - c(s)) + \delta^2(\Pi^i(s) + \ell_R x)$ where the inequality follows from the Lipschitz continuity of $\Pi^i(s)$ (by Lemma 14). Therefore, we get $x \geq \frac{\Pi^i(s)(1-\delta^2)}{\pi^i(b_s^\omega) - c(s) + \delta^2\ell_R} \geq \frac{C(\eta,\delta)(1-\delta^2)}{\Sigma + \ell_R}$ where we used the lower bound on $R(s)$ from Lemma 16. $\square$

[57] The case when $b' \geq b$ is checked trivially.

Lemma 18. *On each step of the tâtonnement algorithm, function $\pi^i(b)$ is left-continuous and strictly increasing.*

Proof. The proof is by induction on the step of the algorithm. For $i = 0$, the strict monotonicity of $\pi^0(b)$ follows from the strict monotonicity of $P^0(b)$ and $v(b)$, and the fact that $P^0(\eta) \le v(\eta)$. The left-continuity of $\pi^0(b)$ follows from the left-continuity of $P^0(b)$ and the continuity of $v(b)$.

Suppose by the inductive hypothesis that $\pi^{i-1}(b)$ is left-continuous and strictly increasing. For $b \in (\eta + ic(\eta, \Delta), 1]$, $\pi^i(b) = v(b)$ is strictly increasing and left-continuous. For $b \in [0, \eta + ic(\eta, \Delta)]$, $\pi^i(b)$ is a convex combination of strictly increasing $v(b)$ and $\hat{\pi}^{i-1}(\tau^i(s^\alpha_b))$. Function $\hat{\pi}^{i-1}(\tau^i(s^\alpha_b))$ is increasing, as $\hat{\pi}^{i-1}$ is increasing by the inductive hypothesis and $\tau^i(s^\alpha_b)$ is increasing by Lemma 15. Therefore, $\pi^i(b)$ is strictly increasing on $[0, \eta + ic(\eta, \Delta)]$. Moreover, $\pi^i(\eta + ic(\eta, \Delta)) \le v(\eta + ic(\eta, \Delta))$, which completes the proof of the strict monotonicity of $\pi^i(b)$.

We next show that $\hat{\pi}^{i-1}(\tau^i(s^\alpha_b))$ is left-continuous. This would imply that $\pi^i(b)$ is left-continuous on $[0, \eta + ic(\eta, \Delta)]$ as a convex combination of left-continuous functions. Suppose to contradiction that there exist $\hat{b}$ and an increasing sequence $b_j \to \hat{b}$ such that $\lim_{j\to\infty} \hat{\pi}^{i-1}(\tau^i(s^\alpha_{b_j})) < \hat{\pi}^{i-1}(\tau^i(s^\alpha_{\hat{b}}))$. Denote $s_j = s^\alpha_{b_j}$ for all $j \in \mathbb{N}$ and $\hat{s} = s^\alpha_{\hat{b}}$. By Lemma 15,

$$\lim_{j\to\infty} \tau^i(s_j) = \tau^i(s_j). \tag{62}$$

If $\hat{\pi}^{i-1}(b)$ is continuous at $\tau^i(\hat{s})$, then $\lim_{j\to\infty} \hat{\pi}^{i-1}(\tau^i(s_j)) = \hat{\pi}^{i-1}(\tau^i(\hat{s}))$, which is a contradiction. If $\hat{\pi}^{i-1}(b)$ is discontinuous at $\tau^i(\hat{s})$, then the first price offer of all seller type s_j is below $\hat{\pi}^{i-1}(\tau^i(\hat{s})) - \varepsilon$ for some $\varepsilon > 0$, while the first price offer of seller type $\hat{s}$ is equal to $\hat{\pi}^{i-1}(\tau^i(\hat{s}))$. Therefore,

$$\Pi^i(\hat{s}) = (\hat{\pi}^{i-1}(\tau^i(\hat{s})) - c(\hat{s}))(\hat{b} - \tau^i(\hat{s})) + \delta^2 \Pi^i_{\tau^i(\hat{s})}(\hat{s}) >$$

$$(\varepsilon + \hat{\pi}^{i-1}(\tau^i(s_j)) - c(\hat{s}))(\hat{b} - \tau^i(\hat{s})) + \delta^2 \Pi^i_{\tau^i(\hat{s})}(\hat{s}) = \varepsilon(\hat{b} - \tau^i(\hat{s})) + \lim_{j\to 1} \Pi^i(s_j),$$

where the equality follows from the continuity of $c(s)$ and $\Pi^i_\beta(s)$ (by Lemma 14) and (62). This contradicts the continuity of $\Pi^i(s)$ (again by Lemma 14) and so, $\hat{\pi}^{i-1}(\tau^i(s^\alpha_b))$ is left-continuous. For $b \in [0, \eta + ic(\eta, \Delta)]$, $\pi^i(b)$ is a convex combination of continuous $v(b)$ and left-continuous $\hat{\pi}^{i-1}(\tau^i(s^\alpha_b))$ and so, is left-continuous itself completing the proof of the inductive step. $\square$

Lemma 19. *Suppose $P(b)$ and $t_\beta(s)$ satisfy equations (16) and (17). Then for δ sufficiently close to one, in the (seller) punishing equilibrium on-path strategies given by $P(b)$ and $t_\beta(s)$ are optimal for the seller and the buyer.*

Proof. From the design of the algorithm the screening strategy $t_\beta(s)$ is optimal for the seller who faces the static demand given by $P(b)$. We next show that the buyer does not have incentives to deviate either from the acceptance strategy $P(b)$ or from pooling on the price offer $\frac{\delta v(0)+c(0)}{1+\delta}$.

If the highest remaining buyer type exceeds b, then buyer type b interprets the previous seller's offers as seller's deviations. In this case, buyer type b expects the seller to restart screening. From equation (17) it follows that any price offer above $P(b)$ would be rejected by buyer b. To complete the verification of optimality of the threshold strategy, we next show that prices below $P(b)$ are accepted by buyer b.

Suppose to contradiction that the seller makes price offer p which is accepted by buyer b' and rejected by buyer type b and $b > b'$. First, observe that if $b \leq \bar{\beta}$, then both types b and b' put probability one on seller type 0, and the result follows from the single crossing property of the payoffs

Next, suppose that $b' > \bar{\beta}$. Define buyer $b'' = \inf\{b : P(b) \geq p\}$. If the buyer rejects price offer p, then the highest buyer type remaining in the game is b''. Each seller type s uses screening policy $t_{b''}(s)$ after rejection. Then for all $k \in \mathbb{N}$,

$$v(b') - p \geq \delta^{2k} \left(v(b') - \hat{P}(t^{(k)}_{s^\alpha_{b'}}(b'')) \right) \tag{63}$$

and

$$v(b) - p < \delta^{2K} \left(v(b) - \hat{P}(t^{(K)}_{s^\alpha_{b}}(b'')) \right) \tag{64}$$

for some K.[58] That is, buyer type b' accepts price offer p, and buyer type b rejects such price offer and expects to accept price offer $P(t^K_{s^\alpha_b}(b''))$ from seller type s^α_b. Subtracting inequality (63) (with $k = K$) from (64), we get

$$v(b) - v(b') < \delta^{2K} \left(v(b) - v(b') - \hat{P}(t^{(K)}_{s^\alpha_{b}}(b'')) + \hat{P}(t^{(K)}_{s^\alpha_{b'}}(b'')) \right)$$

or

$$\left(1 - \delta^{2K}\right) \left(v(b) - v(b')\right) < -\delta^{2K} \left(\hat{P}(t^{(K)}_{s^\alpha_{b}}(b'')) - \hat{P}(t^{(K)}_{s^\alpha_{b'}}(b'')) \right). \tag{65}$$

[58]Notation $f^{(k)}(x)$ stands for k-superposition of function f, i.e. let $f^{(0)}(x) \equiv f(x)$ and for $k \geq 1$ $f^{(k)}(x) \equiv f(f^{(k-1)}(x))$.

The left-hand side of (65) is greater than zero, as $b > b'$. By Lemma 15, $t_{b''}^{(K)}(s_b^\alpha) \geq t_{b''}^{(K)}(s_{b'}^\alpha)$, and moreover, $P(b)$ is increasing. Hence, the right-hand side of (65) is less than zero, which gives a contradiction.

Finally, if $b' \leq \bar{\beta} < b$, then the only difference with the previous case is that now buyer b' could prefer price p not only to all the future price offers of the seller, but also to the seller's acceptance of offer $\frac{\delta v(0)+c(0)}{1+\delta}$. That is, it is possible that

$$v(b') - p \geq \delta \left(v(b') - \frac{\delta v(0) + c(0)}{1 + \delta} \right)$$

or more weakly

$$v(b') - p \geq \delta^{2K} \left(v(b') - \frac{\delta v(0) + c(0)}{1 + \delta} \right).$$

Combining this inequality with the same argument as before we get contradiction again.

The fact that buyers are better off pooling on $\frac{\delta v(0)+c(0)}{1+\delta}$ is the following claim and follows from the invariance property proven in the next section.

Claim 20. For sufficiently large δ, in the seller punishing equilibrium no buyer type prefers to deviate from pooling on offer $\frac{\delta v(0)+c(0)}{1+\delta}$.

Proof. By Theorem 7 any buyer type b above η expect to get the good in the next round buyer is active at price uniformly close to $P^*(b)$. By Lemma 5 if such buyer type deviates he trades with the seller at price close to $\frac{v(b)+c(s_b^\alpha)}{2} > P^*(b)$, hence, the deviation is not profitable for such buyer types for sufficiently large δ. Now buyer types below η expect the first price offer of the seller to be close to $\frac{v(0)+c(0)}{2}$ which is preferred to immediate trade at $\frac{v(\eta)+c(0)}{2}$, making the deviation unprofitable for such types. *Q.E.D.* □

Proof of Theorem 6. The tâtonnement algorithm converges in a finite number of steps by Lemma 17 and the resulting strategies are optimal by Lemma 19. □

Proof of the Uncertainty Invariance Property

Let $Q_\beta(s) \equiv \min\{\beta, b_s^\omega\} - \min\{\beta, b_s^\alpha\}$ be the mass of remaining buyer types in the support of beliefs of seller type s when β is the highest remaining buyer type. Consider a sequence of discount factors $\delta_j \to 1$. In the punishing equilibrium of the game with discount factor δ_j, we denote by $A^j(s)$ the first price offer of seller s.

Lemma 20. *There exist a limit point of sequences $P^j(b), t^j(s), A^j(s), R_\beta^j(s)$.*

Proof. By Lemma 14, function $R^j_\beta(s)$ is Lipschitz continuous in s and β with Lipschitz constants not exceeding 3. Hence, for (s,β) and (s',β') such that $|s-s'|+|\beta-\beta'|<\varepsilon$, $|R^j_\beta(s)-R_{\beta'}(s')|\le|R^j_\beta(s)-R^j_\beta(s')|+|R^j_\beta(s')-R^j_{\beta'}(s')|\le\ell_R(|s-s'|+|\beta-\beta'|)<\ell_R\varepsilon$. Hence, family of continuous functions $R^j_\beta(s)$ is equicontinuous and so, by the Arzela-Ascoli theorem, $R^j_\beta(s)$ converges (over subsequence) to some continuous function $R^*_\beta(s)$. Moreover, $R_\beta(s)$ converges uniformly to $R^*_\beta(s)$ as a sequence of continuous functions on a compact set that converges to a continuous function. Consider now sequences of non-decreasing functions $P^j(b)$, $t^j(s)$, $A^j(s)$. By Helly's theorem there is a subsequence along which the sequence converges to a non-decreasing limit $P^*(b)$, $t^*(s)$, $A^*(s)$ pointwise. ☐

Proof of Lemma 7. Suppose to contradiction that there exists $\hat b\in(0,1)$ with $P^*(\hat b)>c(s^\alpha_{\hat b})$, and for any $\phi>0$, $P^*(\hat b-\phi)\le P^*(\hat b)<P^*(\hat b+\phi)$. Let $\varepsilon\equiv\frac{P^*(\hat b)-c(s^\alpha_{\hat b})}{2}$. Consider some seller type $\hat s>s^\alpha_{\hat b}+\frac{\varepsilon}{4\ell}$. By the left-continuity of $P^*(b)$, we choose ϕ small enough so that $P^*(\hat b-\phi)>P^*(\hat b)-\frac{\varepsilon}{4}$ and $(\hat b-\phi,\hat b+\phi)\subset B_{\hat s}$. By the pointwise convergence of the sequence $P^j(b)$, for δ_j sufficiently large, we have $P^j(\hat b-\phi)>P^*(\hat b-\phi)-\frac{\varepsilon}{4}>P^*(\hat b)-\frac{\varepsilon}{2}>c(s^\alpha_{\hat b})+\frac{\varepsilon}{2}>c(\hat s)+\frac{\varepsilon}{4}$. There are two cases to consider: $A^*(\hat s)>P^*(\hat b)$ and $A^*(\hat s)\le P^*(\hat b)$.

Case 1) $\mathbf{A^*(\hat s)>P^*(\hat b)}$. In the proof of case 1, we restrict that δ_j is sufficiently large so that $A^j(\hat s)>\frac13P^*(\hat b)+\frac23A^*(\hat s)$ and $P^j(\hat b)<\frac23P^*(\hat b)+\frac13A^*(\hat s)$ (by pointwise convergence of $A^j(s)$ and $P^j(b)$) and so,

$$A^j(\hat s)>P^j(\hat b)+\frac13(A^*(\hat s)-P^*(\hat b)). \tag{66}$$

We show that seller type $\hat s$ prefers to deviate from the equilibrium strategy by speeding up screening of buyer types above $\hat b$ which gives a contradiction. Observe that for all δ_j sufficiently large, $R^j_{\hat b}(\hat s)\ge2\phi(P^j(\hat b-\phi)-c(\hat s))\ge\frac{\phi\varepsilon}{2}>0$.

Let $K_j\le\infty$ be the round of screening when price offer of the seller type $\hat s$ drops below $P^j(\hat b)$. Buyer type $b^\omega_{\hat s}$ prefers to purchase immediately rather than wait until price drops below $P^j(\hat b)$ and so, $v(b^\omega_{\hat s})-A^j(\hat s)\ge\delta_j^{2K_j}(v(b^\omega_{\hat s})-P^j(\hat b))$ or by (66)

$$\delta_j^{2K_j}\le\frac{v(b^\omega_{\hat s})-A^j(\hat s)}{v(b^\omega_{\hat s})-P^j(\hat b)}<\frac{v(b^\omega_{\hat s})-P^j(\hat s)-\frac13(A^*(\hat s)-P^*(\hat b))}{v(b^\omega_{\hat s})-P^j(\hat b)}=1-\frac13\frac{A^*(\hat s)-P^*(\hat b)}{v(b^\omega_{\hat s})-P^j(\hat b)}. \tag{67}$$

The right-hand side of (67) converges to a limit that is strictly less than 1 and so, $\lim_{j\to\infty}\delta_j^{2K_j}<1$.

Observe that the profit of seller type $\hat{s}$ in the equilibrium satisfies $R^j(\hat{s}) \leq \int\limits_{\hat{b}}^{b_{\hat{s}}^\omega} (P^j(b) - c(\hat{s}))db + \delta_j^{2K_j} R_{\hat{b}}^j(\hat{s})$. Consider an alternative screening policy in which for integer M_j seller type $\hat{s}$ posts price sequence $\{A_m\}_{m=1}^{M_j}$ such that $A_m = v(b_{\hat{s}}^\omega) + \frac{m}{M_j}(c(\hat{s}) - v(b_{\hat{s}}^\omega))$ and sell with probability one in M_j rounds. Moreover, the loss in profit from each sale is at most $\frac{\Sigma}{M_j}$. By the optimality of the seller's equilibrium strategy, $R^j(\hat{s}) \geq \delta_j^{2M_j} \left(\int\limits_{\underline{b}}^{b_{\hat{s}}^\omega} (P^j(b) - c(\hat{s}))db - \frac{\Sigma}{M_j} \right)$ where $\underline{b} = \inf\{b \in B : P^j(b) > c(\hat{s})\}$. Therefore,

$$\delta_j^{2M_j} \left(\int_{\underline{b}}^{b_{\hat{s}}^\omega} (P^j(b) - c(\hat{s}))db - \frac{\Sigma}{M_j} \right) \leq \int_{\hat{b}}^{b_{\hat{s}}^\omega} (P^j(b) - c(\hat{s}))db + \delta_j^{2K_j} R_{\hat{b}}^j(\hat{s})$$

or after rearranging terms

$$\delta_j^{2M_j} \left(\int_{\underline{b}}^{\hat{b}} (P^j(b) - c(\hat{s}))db - \frac{\Sigma}{M_j} \right) \leq \left(1 - \delta_j^{2M_j}\right) \int_{\hat{b}}^{b_{\hat{s}}^\omega} (P^j(b) - c(\hat{s}))db + \delta_j^{2K_j} R_{\hat{b}}^j(\hat{s}).$$

Since $R_{\hat{b}}^j(\hat{s}) \leq \int\limits_{\underline{b}}^{\hat{b}} (P^j(b) - c(\hat{s}))db$,

$$\delta_j^{2M_j} \left(R_{\hat{b}}^j(\hat{s}) - \frac{\Sigma}{M_j} \right) \leq \left(1 - \delta_j^{2M_j}\right) \int_{\hat{b}}^{b_{\hat{s}}^\omega} (P^j(b) - c(\hat{s}))db + \delta_j^{2K_j} R_{\hat{b}}^j(\hat{s}) \leq \Sigma(b_{\hat{s}}^\omega - \hat{b}) \left(1 - \delta_j^{2M_j}\right) + \delta_j^{2K_j} R_{\hat{b}}^j(\hat{s}),$$

where the last inequality follows from the fact that values are bounded. Since $R_{\hat{b}}^j(\hat{s}) \geq \frac{\phi\varepsilon}{2} > 0$,

$$\delta_j^{2K_j} \geq \delta_j^{2M_j} - \frac{1}{R_{\hat{b}}^j(\hat{s})} \left(\frac{\delta_j^{2M_j}}{M_j} + \Sigma(b_{\hat{s}}^\omega - \hat{b}) \left(1 - \delta_j^{2M_j}\right) \right) \geq \delta_j^{2M_j} - \frac{2}{\phi\varepsilon} \left(\frac{\delta_j^{2M_j}}{M_j} + \Sigma(b_{\hat{s}}^\omega - \hat{b}) \left(1 - \delta_j^{2M_j}\right) \right)$$

For each δ_j, we could choose M_j such that $\delta_j^{2M_j}$ converges to one, as $\delta_j \to 1$. Hence, from the last inequality it follows that $\delta_j^{2K_j}$ is arbitrarily close to one which contradicts (67).

Case 2) $\mathbf{A}^*(\hat{\mathbf{s}}) \leq \mathbf{P}^*(\hat{\mathbf{b}})$. Consider an alternative screening policy, in which seller type $\hat{s}$ posts price $P^j(\hat{b} + \phi)$ in the first round, then makes offer $A^j(\hat{s})$ and proceeds with the screening policy as in the equilibrium. From the optimality of the equilibrium strategy, it follows

$$(b_{\hat{s}}^\omega - t^j(\hat{s}))(A^j(\hat{s}) - c(\hat{s})) + \delta_j^2 R_{t^j(\hat{s})}^j(\hat{s}) \geq$$

$$(b^{\omega}_{\hat{s}} - \hat{b} - \phi)(P^j(\hat{b}+\phi) - c(\hat{s})) + \delta_j^2(\hat{b}+\phi - t^j(\hat{s}))(A^j(\hat{s}) - c(\hat{s})) + \delta_j^4 R_{t(\hat{s})}(\hat{s})$$

or

$$(1-\delta_j^2)\left((b^{\omega}_{\hat{s}} - t^j(\hat{s}))(A^j(\hat{s}) - c(\hat{s})) + \delta_j^2 R^j_{t^j(\hat{s})}(\hat{s})\right) \geq (b^{\omega}_{\hat{s}} - \hat{b} - \phi)\left(P^j(\hat{b}+\phi) - \delta_j^2 A^j(\hat{s}) - (1-\delta_j^2)c(\hat{s})\right) \tag{68}$$

The left-hand side of (68) goes to zero as $\delta_j \to 1$ and the right hand side of (68) converges to $(b^{\omega}_{\hat{s}} - \hat{b} - \phi)(P^*(\hat{b}+\phi) - A^*(\hat{s})) > 0$ which is a contradiction. □

Proof of Corollary 2. For any buyer type $b \in [0, b^{\omega}_{s+})$, $P^j(b) \geq \frac{v(0)+\delta c(0)}{1+\delta} > \frac{v(0)+c(0)}{2} > c(s^{\alpha}_b)$ and so, $P^*(b) > c(s^{\alpha}_b)$ for $b \in [\eta, b^{\omega}_{s+})$. Therefore, by Lemma 7, function $P^*(b)$ is constant on this interval. Since $P^*(b) = \frac{v(0)+c(0)}{2}$ for $b \in [0, \eta]$, we have $P^*(b) = \frac{v(0)+c(0)}{2}$ on $[0, b^{\omega}_{s+})$. □

Definition 8.1. A monotone function $f(x)$ on $[0,1]$ is *ε-continuous* if for any open interval $I \subset [f(0), f(1)]$ of length at least ε we have $f([0,1]) \cap I \neq \emptyset$.[59]

Lemma 21. *For any $\varepsilon > 0$ there exists $\bar{\delta} \in (0,1)$ such that for all $\delta_j > \bar{\delta}$, function $P^j(b)$ is ε-continuous, and for any seller type $s \in [0,1]$ and buyer type $\beta \in B_s$,*

$$\hat{P}^j(\beta) - \hat{P}^j(t^j_{\beta}(s)) \leq \varepsilon. \tag{69}$$

Proof. Suppose to contradiction that there exist $\varepsilon > 0$, $\underline{P}$, and $\overline{P} > \underline{P} + \varepsilon$ such that for any $b \in [0,1]$ and infinitely many js, either $P^j(b) \geq \overline{P}$ or $P^j(b) \leq \underline{P}$. Without loss of generality, take $\underline{P}$ and $\overline{P}$ such that $\overline{P} - \underline{P}$ is maximal. For any j, consider $b_j \equiv \sup\{b : P^j(b) < \underline{P}\}$. By equation (17), for any $b \in [0,1]$,

$$P^j(b) - \hat{P}^j(t(s^{\alpha}_b)) = (1-\delta_j^2)(v(b) - \hat{P}^j(t^j(s^{\alpha}_b)) \leq (1-\delta_j^2)\Sigma < \frac{\varepsilon}{2} \tag{70}$$

for δ_j sufficiently close to one. Consider buyer type $\hat{b}_j \equiv b_j + \frac{c(\eta,\delta_j)}{2}$ and $\check{b}_j \equiv b_j + \frac{c(\eta,\delta_j)}{2}$. Then

$$P^j(\hat{b}_j) - \hat{P}^j(t(s^{\alpha}_{\hat{b}_j})) > P^j(\hat{b}_j) - \hat{P}^j(\check{b}_j) > \varepsilon,$$

which gives a contradiction to (70).

[59]Observe that a monotone function $f(x)$ is continuous if and only if it is ε-continuous for all $\varepsilon > 0$. The notion ε-continuity captures the fact that jumps of an ε-continuous function cannot exceed ε.

To prove (69), observe that by Lemma 15, for any $j \in \mathbb{N}$,

$$P^j(\beta) - \hat{P}^j(t^j_\beta(s)) \leq P^j(\beta) - \hat{P}^j(t^j_\beta(s^\alpha_\beta)), \tag{71}$$

for all $b \in [0,1]$. For any $\varepsilon > 0$, choose δ_j sufficiently large so that $P^j(b)$ is $\frac{\varepsilon}{2}$-continuous. This implies that the right-hand side of (71) is less than $\frac{\varepsilon}{2}$, and moreover, there exists $\beta_j > \beta$ such that $P^j(\beta_j) - P^j(\beta) < \frac{\varepsilon}{2}$. Together with (71), this gives

$$\hat{P}^j(\beta) - \hat{P}^j(t^j_\beta(s)) \leq P^j(\beta_j) - P^j(\beta) + P^j(\beta) - \hat{P}^j(t^j_\beta(s)) < \varepsilon,$$

which proves (69). □

Lemma 22. *For any δ_j, let two converging sequences of buyer types $\{b_j\}_{j=1}^\infty$ and $\{b'_j\}_{j=1}^\infty$ be such that $P^j(b_j) - \hat{P}^j(b'_j)$ and $v(b_j) - \hat{P}^j(b'_j)$ are uniformly bounded away from zero. Then there exist a function $\gamma(\delta_j) \sim (1-\delta_j)^2$ and an integer J such that $b_j - b'_j \geq \gamma(\delta_j)$ for all $j \geq J$.*

Proof. Define sequence $t^j_l,\ l = 0, \ldots, L_j + 1$ as follows. Let $t^j_0 = b_j$ and $t^j_l = t^j(s^\alpha_{t^j_{l-1}})$ for $l = 1, \ldots, L_j + 1$ where L_j is the largest integer such that $t^j_{L_j} \geq b'_j$. By (17),we have

$$P^j(b_j) = (1-\delta_j^2)\sum_{l=0}^{L_j} \delta^{2l} v(t^j_l) + \delta_j^{2(L_j+1)} \hat{P}^j(t^j_{L_j+1}).$$

Since $\hat{P}^j(b)$ is increasing in b and $b'_j \in [t^j_{L_j+1}, t^j_{L_j}]$,

$$P^j(b_j) - \hat{P}^j(b'_j) \leq (1-\delta_j^2)\sum_{l=0}^{L_j} \delta^{2l} v(t^j_l) - (1-\delta_j^{2(L_j+1)})\hat{P}^j(b'_j) \leq (1-\delta_j^{2(L_j+1)})(v(b_j) - \hat{P}^j(b'_j)).$$

Since $P^j(b_j) - \hat{P}^j(b'_j)$ and $v(b_j) - \hat{P}^j(b'_j)$ are uniformly bounded away from zero, $1 - \delta_j^{2(L_j+1)}$ is uniformly bounded away from zero. Hence, the exists $C_1 > 0$ and an integer J_1 such that $L_j \geq -C_1/\ln\delta_j$ for all $j \geq J_1$.

By Lemma 17, there exists $C_2 > 0$ and an integer J_2 such that $t^j_{l-1} - t^j_l > C_2(1-\delta_j)^3$ for all $l \in 1, \ldots, L_j$ and all $j \geq J_2$. Hence, $b_j - b'_j = \sum_{l=1}^{L_j}(t^j_{l-1} - t^j_l) + t^j_{L_j} - b'_j \geq C_2(1-\delta_j)^3 L_j \geq -C_1C_2(1-\delta_j)^3/\ln\delta_j \sim (1-\delta_j)^2$ for $j \geq J \equiv \max\{J_1, J_2\}$. The function $\gamma(\delta_j) = -C_1C_2(1-\delta_j)^3/\ln\delta_j$ satisfies the desired properties. □

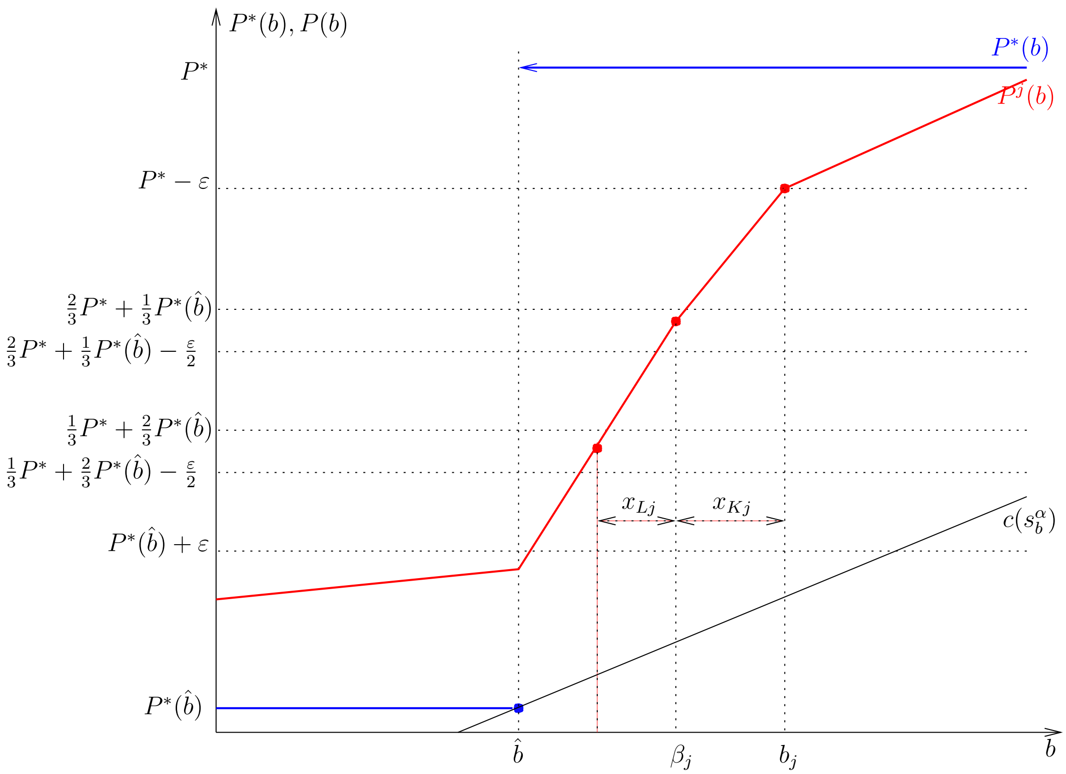


Figure 3: Illustration of the proof of Lemma 8

Proof of Lemma 8. Suppose to contradiction that there exists $\hat{b}$ such that $P^* \equiv P^*(\hat{b} + 0) > P^*(\hat{b})$ (see Figure 4 for the illustration of the proof). By Corollary 2, $\hat{b} \geq s^+ + \eta$, and by Lemma 7, $P^*(\hat{b}) = c(s^\alpha_{\hat{b}})$. Fix $\varepsilon > 0$ small enough so that $P^* - P^*(\hat{b}) > \frac{9}{2}\varepsilon$, which ensures that $P^* - \varepsilon > \frac{2}{3}P^* + \frac{1}{3}P^*(\hat{b}) > \frac{2}{3}P^* + \frac{1}{3}P^*(\hat{b}) - \frac{\varepsilon}{2} > \frac{1}{3}P^* + \frac{2}{3}P^*(\hat{b}) > \frac{1}{3}P^* + \frac{2}{3}P^*(\hat{b}) - \frac{\varepsilon}{2} > P^*(\hat{b}) + \varepsilon$. Let $b_j \equiv \inf\{b : P^j(b) \in (P^* - \varepsilon, P^*)\}$ and $s_j \equiv s^\alpha_{b_j}$. Let $K_j \leq \infty$ be the first round of screening, in which seller type s_j makes a price offer below $\frac{2}{3}P^* + \frac{1}{3}P^*(\hat{b})$ and allocates to all buyer types above some β_j. In the proof, we restrict that δ_j is sufficiently close to one so that the conclusions of the following claim obtain.

Claim 21. We have

$$\lim_{j\to\infty} P^j(b_j) = P^* - \varepsilon, \tag{72}$$

and for δ_j sufficiently large,

$$\frac{2}{3}P^* + \frac{1}{3}P^*(\hat{b}) > \hat{P}^j(\beta_j) > \frac{2}{3}P^* + \frac{1}{3}P^*(\hat{b}) - \frac{\varepsilon}{2}, \tag{73}$$

$$b_j < \hat{b} + \frac{\varepsilon}{2\ell} \text{ and } c(s_j) \leq P^*(\hat{b}) + \frac{\varepsilon}{2}. \tag{74}$$

Proof. By Lemma 21 for any ε there exists $J(\varepsilon)$ such that $P^j(b)$ is ε-continuous for $j \geq J(\varepsilon)$ and so, (72) obtains. Inequality (73) follows from the definition of β_j and (69) in Lemma 21. By the pointwise convergence of $P^j(b)$, $\lim_{j\to\infty} b_j = \hat{b}$ and so, $b_j < \hat{b} + \frac{\varepsilon}{2\ell}$ for δ_j sufficiently large . This, in turn, implies $c(s_j) < c(s^\alpha_{\hat{b}} + \frac{\varepsilon}{2\ell}) < c(s^\alpha_{\hat{b}}) + \frac{\varepsilon}{2} = P^*(\hat{b}) + \frac{\varepsilon}{2}$ where the second inequality is by Lipschitz continuity of $c(s)$. *Q.E.D.*

Optimality of strategy of type s_j. In the first K_j rounds of screening, seller type s_j allocates to the mass of buyer types $x_{Kj} \equiv b_j - \beta_j$. Since buyer type b_j prefers to buy at price $P^j(b_j)$ rather than wait until price drops to $\hat{P}^j(\beta)$,

$$\frac{v(b_j) - P^j(b_j)}{v(b_j) - \hat{P}^j(\beta_j)} \geq \delta_j^{2K_j}. \tag{75}$$

By (72) and (73), the upper bound on $\delta_j^{2K_j}$ in (75) converges to at most $\frac{v(\hat{b})-P^*+\varepsilon}{v(\hat{b})-\frac{2}{3}P^*-\frac{1}{3}P^*(\hat{b})} < 1$. Therefore, $\delta_j^{2K_j}$ converges to some limit $\lambda_K < 1$ as $\delta_j \to 1$ and so, $\lim_{j\to\infty}(1-\delta_j^2)K_j = -\ln\lambda_K > 0$.

For any integer M_{Kj}, consider an alternative screening strategy, in which seller type s_j speeds up screening in the first $\lfloor K_j/M_{Kj}\rfloor$ rounds. Let A_k be the price offer that seller type s_j makes in round k. Define $q_k = P^j(b_j) + \frac{kM_{Kj}}{K_j}\left(A_{K_j-1} - P^j(b_j)\right)$, $k = 1, 2, .., \lfloor K_j/M_{Kj}\rfloor$. In the alternative strategy, seller type s_j makes price offer $p_k \equiv \min\{q_k, A_k\}$ in rounds $k \leq \lfloor K_j/M_{Kj}\rfloor$, makes offer A_{K_j} in round $\lfloor K_j/M_{Kj}\rfloor + 1$ and continues following equilibrium strategy from then on. The total loss from using the alternative strategy is at most $M_{Kj}x_{Kj}\left(\frac{1}{3}P^* - \frac{1}{3}P^*(\hat{b})\right)/K_j$. Indeed, in each round the loss of seller type s_j compared to the maximum surplus that could be extracted is at most $\frac{P^j(b_j)-\hat{P}^j(\beta_j)}{K_j/M_{Kj}} \leq M_{Kj}\left(\frac{1}{3}P^* - \frac{1}{3}P^*(\hat{b})\right)/K_j$ where the inequality follows from (72) and (73). Moreover, there is no loss due to discounting, as the allocation to all buyer types happens sooner under the alternative strategy than under the equilibrium strategy.

At the same time, by speeding up the screening seller type s_j gains at least $\left(\delta_j^{2K_j/M_{Kj}} - \delta_j^{2K_j}\right)V_{Kj}$, where V_{Kj} is the continuation utility of seller type s_j after she makes price offer A_{K_j} and follows the equilibrium strategy further. By the optimality of strategy of seller type s_j,

$$\frac{M_{Kj}}{K_j}x_{Kj}\left(\frac{1}{3}P^* - \frac{1}{3}P^*(\hat{b})\right) \geq \left(\delta_j^{2K_j/M_K} - \delta_j^{2K_j}\right)V_{Kj} \tag{76}$$

Optimality of strategy of type σ_j. Consider seller type $\sigma_j \equiv s^{\alpha}_{\beta_j}$ and let L_j be the first round of screening, in which seller type σ_j makes a price offer below $\frac{1}{3}P^* + \frac{2}{3}P^*(\hat{b})$. By the analogous argument as with K_j and seller type s_j, we have $\delta_j^{2L_j}$ converges to the limit $\lambda_L < 1$ (correspondingly, $(1-\delta_j^2)L_j \to -\ln \lambda_L > 0$), and for the optimality of strategy of seller type σ_j it is necessary that

$$\frac{M_{Lj}}{L_j} x_{Lj} \left(\frac{1}{3}P^* - \frac{1}{3}P^*(\hat{b}) \right) \geq \left(\delta_j^{2L_j/M_{Lj}} - \delta_j^{2L_j} \right) V_{Lj}, \tag{77}$$

for any integer M_{Lj}. In inequality (77), x_{Lj} denotes the mass of buyer types to whom seller type σ_j allocates in the first L_j rounds, and V_{Lj} denotes the continuation utility of seller type σ_j after price offer in round L_j and follows the equilibrium strategy further.

Lower bound on V_{Kj} . Observe that seller type s_j could post price $\frac{1}{3}P^* + \frac{2}{3}P^*(\hat{b}) - \frac{\varepsilon}{2}$ after price offer A_{K_j}. The mass of buyer types that accept such price is x_{Lj}, and the profit from each such buyer is $\frac{1}{3}P^* + \frac{2}{3}P^*(\hat{b}) - \frac{\varepsilon}{2} - c(s_j) \geq \frac{1}{3}P^* - \frac{1}{3}P^*(\hat{b}) - \varepsilon$ by (74). Hence,

$$V_{Kj} \geq x_{Lj} \left(\frac{1}{3}P^* - \frac{1}{3}P^*(\hat{b}) - \varepsilon \right). \tag{78}$$

Lower bound on V_{Lj} . Suppose that the seller allocated in previous rounds to all buyer types with $P^j(b) > \frac{1}{3}P^* + \frac{2}{3}P^*(\hat{b}) - \frac{\varepsilon}{2}$. If the seller posts price $P^*(\hat{b}) + \varepsilon$ after such history, then by Lemma 22, the mass of buyer types who accept such price is at least $\gamma(\delta_j) > 0$. The profit of seller type σ_j from such buyer types is $P^*(\hat{b}) + \varepsilon - c(\sigma_j) \geq P^*(\hat{b}) + \varepsilon - (P^*(\hat{b}) + \frac{\varepsilon}{2}) = \frac{\varepsilon}{2}$ (by (74) and $\sigma_j < s_j$). Hence,

$$V_{Lj} \geq \gamma(\delta_j)\frac{\varepsilon}{2} \tag{79}$$

Lower bound on x_{Kj}. Combining inequalities (76), (77), (78), (79) we get

$$Cx_{Kj} \geq \frac{K_j(1-\delta_j)}{M_{Kj}} \frac{L_j(1-\delta_j)}{M_{Lj}} \left(\delta_j^{2K_j/M_{Kj}} - \delta_j^{2K_j} \right) \left(\delta_j^{2L_j/M_{Lj}} - \delta_j^{2L_j} \right) \frac{\gamma(\delta_j)}{(1-\delta)_j^2} \frac{\varepsilon}{2}. \tag{80}$$

where we collect all the constants into a positive constant C. Since $\frac{K_j(1-\delta_j)}{M_{Kj}} \sim \frac{K_j \ln(\delta_j)}{M_{Kj}}$, $\frac{L_j(1-\delta_j)}{M_{Lj}} \sim \frac{L_j \ln(\delta_j)}{M_{Lj}}$ and $\gamma(\delta_j) \sim (1-\delta_j)^2$, we could find M_{Kj} and M_{Lj} (in general dependent on δ_j) such that right-hand side of inequality (80) converges to a positive number. On the other hand, $x_{Kj} \leq b_j - \hat{b} \leq \frac{\varepsilon}{2\ell}$ by (74) . This contradicts the fact that ε was chosen arbitrary. $\square$

Proof of Corollary 3. Suppose to contradiction that there exists some $\tilde{b} \geq b^{\omega}_{s+}$ such that $P^*(\tilde{b}) > c(s^{\alpha}_{\tilde{b}})$. For $b < b^{\omega}_{s+}$, $c(s^{\alpha}_{b}) > P*(b)$ and combined with Lemma 7, this implies that there is $\hat{b} \geq b^{\omega}_{s+}$ such that $P^*(b)$ is discontinuous at $\hat{b}$ which contradicts Lemma 8. □

Lemma 23. *Sequence $P^j(b)$ converges uniformly to $P^*(b)$ on* $[0,1]$.

Proof. We show that the function $f^j(b) = \hat{P}^j(b) - P^*(b)$ converges uniformly to zero on $[0,1]$, which would imply the desired uniform convergence of $P(b)$ by the following claim.

Claim 22. For any $b \in [0,1]$, $0 < P^j(b) - P^*(b) \leq f^j(b)$.

Proof. First, for all $b \in [0,1]$, $P^j(b) \leq \hat{P}^j(b)$ by the definition and so, $P^j(b) - P^*(b) \leq f^j(b)$. Second, by Lemma 3, $P^j(b) \geq \frac{v(0)+\delta c(0)}{1+\delta} > \frac{v(0)+c(0)}{2}$ for all $b \in [0,1]$. Moreover, by Lemma 16, $P^j(b) > c(s^{\alpha}_{b})$ for all $b \in [0,1]$. Therefore, $0 < P^j(b) - P^*(b)$ for all $b \in [0,1]$. *Q.E.D.*

Claim 23. Function $f^j(b)$ is upper-semicontinuous, and for any $\varepsilon > 0$, $f^j(b+\frac{\varepsilon}{\ell}) \geq f^j\,(b)-\varepsilon$.

Proof. To show that $f^j(b)$ is upper-semicontinuous, consider a sequence $\{b_i\}_{i=1}^{\infty}$ converging to some $b \in [0,1]$. Then by continuity of $P^*(b)$ and right-continuity of $\hat{P}^j(b)$, $\limsup_{i\to\infty}(\hat{P}(b_i) - P^*(b_i)) = \limsup_{i\to\infty} \hat{P}(b_i) - P^*(b) \leq \hat{P}(b) - P^*(b)$.

Next, choose any $\varepsilon > 0$. Since $\hat{P}^j(b)$ is increasing, $\hat{P}^j\left(b + \frac{\varepsilon}{\ell}\right) \geq \hat{P}^j(b)$. Moreover, $P^*(b) = \max\left\{\frac{v(0)+c(0)}{2}, c(s^{\alpha}_{b})\right\}$ and the derivative of $c(s)$ is bounded above by ℓ and so, $-P^*\left(b + \frac{\varepsilon}{\ell}\right) \geq -P^*(b) - \varepsilon$. Therefore, $f^j(b + \frac{\varepsilon}{\ell}) \geq f^j\,(b) - \varepsilon$. *Q.E.D.*

Claim 24. Function $f^j(b)$ converges uniformly to 0 on $[0,1]$.

Proof. Function $f^j(b)$ converges poinwise to 0 on $[0,1]$. Since $f^j(b)$ is upper-semicontinuous function on a compact set by Claim 23, $f^j(b)$ achieves its maximum at some $b_j \in [0,1]$.

We next show that $f^j(b_j)$ converges to 0 as $j \to \infty$. Suppose to contradiction that for all $j \in \mathbb{N}$ there exists $\varepsilon > 0$ so that $f^j(b_j) > \varepsilon$. By Claim 23, $f^j(b) > \frac{\varepsilon}{2}$ for all $b \in \left[b_j, b_j + \frac{\varepsilon}{2\ell}\right]$. By compactness of $[0,1]$, sequence b_j converges (over subsequence) to some $b^* \in [0,1]$ as $j \to \infty$ and so, there exists J such that for all $j \geq J$, $b_j \in \left[b^* - \frac{\varepsilon}{8}, b^* + \frac{\varepsilon}{8}\right]$. Hence, for $b \in \left[b^* + \frac{\varepsilon}{8}, b^* + \frac{3\varepsilon}{8}\right]$, $f^j(b_j) > \frac{\varepsilon}{2}$ for all $j \geq J$. This contradicts the pointwise convergence of $f^j(b)$ to 0. *Q.E.D.* □

Proof of Corollary 1. Observe that continuation utility of seller type s in the seller punishing equilibrium is bounded above by $P^j(b^{\omega}_{s}) - c(s)$. By Lemma 23, $\sup_{s\in[0,1]} |P^j(b^{\omega}_{s}) - c(s) - \max\{\frac{v(0)+c(0)}{2} - c(s), 0\}|$ converges to zero as $\delta_j \to 0$ which gives the desired conclusion. □

Numerical Simulations

Lemma 24. *Suppose $v(b) = b$ and $c(s) = s - 1$. For all $\eta \in (0, \frac{1}{2})$ there exists a (seller) punishing equilibrium, in which buyer types in $[0, \eta]$ pool on offer $-\frac{1}{1+\delta}$ and seller type 0 accepts buyer's price offer $-\frac{1}{1+\delta}$. Moreover, all seller types $s > 0$ reject $-\frac{1}{1+\delta}$ and make a counter-offer.*

Proof. Suppose that strategies of seller type 0 and buyer types in $[0, \eta]$ are as described in the statement of the theorem. To show that they constitute equilibrium, we prove optimality of such strategies. For $b \in [0, \eta]$, $P(b) = (1-\delta)b - \frac{\delta}{1+\delta}$, i.e. buyer type b is indifferent between accepting $P(b)$ in the current round, and having seller accept $\frac{1}{1+\delta}$ in the next round. The profit of seller type 0 is at most $M(b) = (\eta - b)(P(b)+1) + \delta \int_0^b (P(t)+1)dt$. Then $M'(b) = -(1-\delta)(P(b)+1) + (\eta - b)P'(b) = (1-\delta)(\eta - (2-\delta)b - \frac{1}{1+\delta}) < 0$, as $\eta < \frac{1}{2} < \frac{1}{1+\delta}$ and $b \geq 0$. Therefore, it is optimal for seller type 0 to make price offer $-\frac{\delta}{1+\delta}$. Moreover, seller type 0 is indifferent between accepting $-\frac{1}{1+\delta}$ in the current round and making offer $-\frac{\delta}{1+\delta}$ in the next round.

Equilibrium after buyer deviation is described in Lemma 5. In particular, seller type 0 puts probability one on buyer η and, hence, would accept only price offers above or equal to $\frac{\eta-\delta}{1+\delta}$. For deviation from the punishing equilibrium to be not profitable it is sufficient that for all $b \in [0, \eta]$, $\delta\left(b + \frac{1}{1+\delta}\right) \geq \delta\left(b - \frac{\eta-\delta}{1+\delta}\right)$, which holds for $\eta > 0$.

We show that any seller type $s > 0$ prefers to reject price offer $-\frac{1}{1+\delta}$ and make counter-offer $-\frac{\delta}{1+\delta}$. For this to be the case it is sufficient $\delta\left(-\frac{\delta}{1+\delta} - s + 1\right) > -\frac{1}{1+\delta} - s + 1 \iff s > 0$. □

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