Heterogeneity in Competing Auctions*

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Abstract

This paper studies a model of competing auctions in which bidders attach different valuations to the items offered by sellers. We provide a novel characterization of the set of (symmetric) participation rules used by bidders and show that contrary to models with homogeneous goods, heterogeneity rules out randomization when bidders choose trading partners. We also show that changes in some reserve price alter the participation decision of every buyer regardless of her valuation of the item. This implies that such changes not only affect the distribution of valuations of those buyers participating in a given auction but also modify the probability with which every buyer visits the auctions. We illustrate this novel trade–off between screening and traffic effect by showing that it is possible to construct an equilibrium in which both sellers post reserve prices equal to production costs with just two sellers and two bidders.

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1 Introduction

This paper is devoted to the analysis of competition among sellers who wish to sell their items to a pool of bidders using auctions. In simple terms, competing auction models feature several sellers who simultaneously post reserve prices before buyers select the auction where they want to bid[1]. When the number of buyers and sellers is large, the literature (Peters and Severinov, 1997; Virag, 2010) has shown that in equilibrium, sellers post reserve prices close to production costs. This result is in line with the idea that competition should boost equilibrium mechanisms that look much simpler than the type of mechanisms predicted by the theory of monopolistic mechanism design[2]. In this paper we show that a similar conclusion can be obtained with just two sellers and two bidders provided that there is heterogeneity in buyers' preferences.

There are at least two reasons why we may want to consider introducing heterogeneity in models of competing auctions. The first is a practical one. There are many cases in which goods are considered different objects ex ante, i.e., before bidders submit bids, even if these items turn out to be physically identical ex post. For example, it is not uncommon that sellers in online platforms (such as eBay) release information about their objects before the auction begins in order to help buyers have a better assessment of how much the item is worth to them. Since different sellers may provide different amounts of information, it is natural to think of bidders as placing different valuations to the items. The second reason is more theoretical oriented. It is perfectly clear that taking items to be perfect substitutes is a simplifying assumption. What it is not clear is up to what point the current findings in the literature depend on this assumption. Our paper is the first formal attempt to provide an answer to this question. In fact, we could think of the methodology developed in this paper as a robustness test for models of competing auctions studied up to date in the literature.

We consider a standard model of competing auctions with two risk–neutral sellers with unit supply who post reserve prices, and n risk–neutral buyers ($n \geq 2$) with unit demands who value each item differently. Evidently, the fact that bidders have different valuations implies that their types are collections of random variables, which introduce important technical challenges when modeling bidders' participation decisions. One of the contributions of this paper is to provide a complete characterization of the participation rules used by bidders in any symmetric equilibrium in terms of cutoff strategies, similar to those used in (monopolistic) auction models with costly participation (Green and Laffont, 1984; Samuelson, 1985; Vagstad, 2007). Our first result establishes that no matter what participation strategy bidders may use, there always is a best response to this strategy that can be described in terms of a nondecreasing and continuous

[1] Competing auction games belong to the class of competing mechanism games in which the set of available mechanisms is restricted to auctions. Although this restriction is with loss of generality (as identified by McAfee, 1993 and formally proved by Epstein and Peters, 1999), **?** have suggested that this kind of restrictions may be necessary if we want the theory to have some predictive power.

[2] For a complete account of how competition among mechanism designers promotes simple mechanisms, see Peters (2010).

function ρ with the property that a bidder with valuations (v_1, v_2) visits seller 1 if and only if $v_2 \leq \rho(v_1)$, and visits seller 2 with probability one if and only if $v_2 > \rho(v_1)$. This ensures that whenever the game possesses a continuation equilibrium it must also posses an equilibrium in which bidders use pure strategies. This is important because it highlights one of the main differences with models with homogeneous: when items are heterogeneous the coordination failure that arises in models with homogeneous goods disappears because buyers choose trading partners deterministically. Thus, heterogeneity acts as a coordination device by eliminating coordination failures as a source of friction in the market.

Another difference introduced by heterogeneity is the way in which a change in reserve prices affects the demand faced by each seller. With homogeneous goods, a decrease in seller j's reserve price has two consequences on the visiting decisions of bidders: (i) those bidders with valuations just below r_j begin to visit seller j with probability one; and (ii) some bidders who were mixing among sellers find profitable to bid for sure at seller j's auction. The interesting observation is the fact that this change in seller j's reserve price not only affects the types that visit each auction but also the probability with which each bidder visits each seller. In other words, a change in seller j's reserve price affects the distribution of types faced by the seller as well as the probability with which each buyer visits the auctions. This is due to the fact that with heterogeneous goods buyers' participation rules are characterized by a collection of continuous an increasing functions, and a change in some reserve price alters the whole function affecting the participation decisions of bidders with low and high valuations who were indifferent before the change in the reserve price took place. This introduces a novel trade–off between traffic and screening effects not present in models with homogeneous goods.

To illustrate this last point, we show that the existence of a pure strategy equilibrium in which sellers post reserve prices equal to production costs no longer requires the number of bidders and sellers to be large. This is in complete contrast with previous results in the literature where the existence of this kind of equilibrium requires the sellers-to-buyers ratio to be large[3] (Peters and Severinov, 1997; Virag, 2010). For the finite case, Virag (2010) has shown the existence of a mixed strategy equilibrium for every finite version of the game and proved that if the lowest possible valuation is above sellers' production costs then the distribution of equilibrium reserve prices will converge (in distribution) to the sellers' production costs. The first of these findings can be seen as an extension of the analysis of Burguet and Sakovics (1999), who studied a model with two sellers and $n \geq 2$ buyers and showed that the equilibrium probability of posting a zero reserve price in equilibrium is nil independent of how many bidders are in the market. In contrast, we show that in the 2–sellers 2–bidder case there is a pure strategy equilibrium where both sellers post reserve prices equal to production costs. Intuitively, this

[3]The seminal paper of Peters and Severinov (1997) showed the existence of an equilibrium where reserve prices are equal to production costs only when the number of buyers and sellers is large but they did not provide a proof of equilibrium existence when the number of agents is assumed finite. This question was settled by a recent paper by Virag (2010), who demonstrated that a pure strategy equilibrium exists for every finite version of the game whenever the lowest possible valuation is above sellers' production costs.

equilibrium arises because heterogeneous goods makes the participation decisions of low and high valuation types change in response to a unilateral increase in some reserve price, which reduces traffic and due to the screening effect providing sellers with enough incentives to post reserve prices equal to production costs.

The rest of the paper is organized as follows. We outline the model in the next section and the provide a characterization of the equilibrium set of the continuation game in which bidders select trading partners. We then analyze the simultaneous-move game induced by the continuation equilibrium described in the preceding section and analyze in detail the 2–sellers–2–bidders case. The paper ends with some conclusions and final remarks.

2 The Model

Consider an economy in which trade takes place using second-price sealed bid auctions. The economy is populated by two risk-neutral sellers (seller 1 and seller 2) with unit supply, and n risk neutral buyers with unit demands. Sellers are indexed by $j \in \{1, 2\}$, and buyers are indexed by $i = \{1, \cdots, n\}$, $n \geq 2$. Bidders value each item differently. Buyer i's true valuation of item j is V_{ij}. The vector V_i is assumed to be a collection of independently and identically distributed random variables, each following a cumulative distribution function F with continuously differentiable, bounded and positive density $f > 0$, and full support on $[0, 1]$. Sellers have a production cost equal to $c > 0$, which is common knowledge among players[4]. A bidder i with valuation V_{ij} who trades with seller j at price p_j gets a surplus $V_{ij} - p_j$, while seller j gets a surplus $p_j - c$. In case of no trade both the seller and the bidder get an exogenously given payoff normalized to 0. In what follows, if X_l is a set and $l \in \{1, \cdots, L\}$ then $X = \Pi_{l=1}^{L} X_l$ and $X_{-l} = \Pi_{k \neq l} X_k$; thus $X = X_l \times X_{-l}$. Furthermore, $x \in X$ then $x = (x_i, x_{-i})$ with $x_{-i} \in X_{-i}$, $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_L)$.

The game we study is similar in almost all respects to the standard competing auction model with homogeneous goods (e.g. Peters and Severinov (1997); Burguet and Sakovics (1999); Virag (2010)) with the exception that bidders have heterogeneous preferences and hence, their valuations may differ across items. The game begins when Nature draws a pair of independent realizations (v_{i1}, v_{i2}) of the vector (V_{i1}, V_{i2}) using the common prior distribution F, and privately communicates it to bidder i, $i = 1, \cdots, n$. After Nature has moved, sellers simultaneously announce reserve prices which become common knowledge right after announced. We assume that these reserve prices belongs to the closed interval $[c, 1]$, i.e., that no seller can announce a reserve price below the production cost. After observing the reserve prices announced by the sellers, bidders simultaneously choose trading partners. Bidders are assumed to participate in one and only one auction, that is, to choose only one seller as her trading partner. After bidders have selected their trading partners, the bidding process takes place. Thus, all participants in

[4]As we will later show, the existence of a positive production cost is used to ensure the existence of a continuation equilibrium in the bidders subgame.

auction j simultaneously submit their bids to this seller, who then awards the item to the highest bidder (in case of a tie, the item is randomly assigned among the highest bidders) who pays a price equal to the second-highest bid. Finally, the game ends and all payoffs are realized.

3 Analysis

A strategy for seller j is his choice of reserve price $r_j \in [c, 1]$. A strategy for a bidder is a rule that specifies a participation and a bidding decision as a function of the bidder's information in stage two of the game. As it is customary in the literature of competing auctions (e.g., Peters and Severinov (1997); Virag (2010)), we will assume that conditional on participating every bidder bids her estimate truthfully (v_{i1} or v_{i2} depending on which auction bidder i has chosen to bid) since truthful bidding is a Bayesian equilibrium of the bidding stage game. The main advantage of this assumption is the reduction of bidder's strategies to rules that specify the probabilities with which each bidder visits each seller. Thus, a strategy for bidder i is a mapping $\pi_i : [0,1]^2 \times [c,1]^2 \rightarrow [0,1]^2$ with $\pi_i(v_i, r) = \{\pi_{i1}(v_i, r), \pi_{i2}(v_i, r)\}$, $\pi_{ij} \geq 0$, $v_i = (v_{i1}, v_{i2})$, and $\pi_{i1} + \pi_{i2} \leq 1$, such that $\pi_{ij}(v_i, r)$ delivers the probability with which bidder i bids in auction j as a function of her vector of valuations v_i and the vector of reserve prices $r = (r_1, r_2)$. Our equilibrium concept is Perfect Bayesian Equilibrium. We restrict attention to equilibria in which every bidder uses *symmetric participation rules*. A participation rule is symmetric if for a given vector of reserve prices, two bidders with the same vector of valuations visit seller j with the same probability, $\pi_{ij}(\cdot) = \pi_{kj}(\cdot) \equiv \pi_j(\cdot)$, $i \neq k$. We adhere to the convention to treat the decision not to bid in any auction as equivalent to the decision to submit a non serious bid in auction 1 and thus, if $\pi(v, r)$ stands for the probability that a bidder with valuations $v = (v_1, v_2)$ visits auction 1 then $\pi_2(v, r) = 1 - \pi(v, r)$ is the corresponding probability that this bidder visits seller 2. Finally, we let S be the strategy space for bidder i, i.e., the set of all (measurable) mappings π.

3.1 Bidder's Participation Game

Consider the stage game in which bidders must choose trading partners. Suppose that bidder 1 choose seller 1 as trading partner. Take bidder 1 with valuations (v_1, v_2) and suppose that this bidder selects seller 1 as trading partner. From McAfee (1993), the probability that this type of bidder 1 wins the item in auction 1 must be equal to the probability of being the unique bidder in the auction plus the probability that every other buyer who comes to auction 1 does so with a valuation of item 1 below v_1,

$$Q_1(v_1; \pi, r) = 1 - \int_{v_1}^{1} \int_{0}^{1} \pi((t_1, t_2), r) dF(t_1) dF(t_2)$$

if $v_1 \geq r_1$, and $Q_1(v_1; \pi, r) = Q_1(r_1; \pi, r)$ for all $v_1 < r_1$. Then, the (reduced-form) payoff that this bidder expects in auction 1, $\mathcal{U}_1(v_1; \pi, r)$, can be written as the difference of two terms: her probability of trading with seller 1 times his valuation of this item, minus the expected price she pays,

$$\mathcal{U}_1(v_1;\pi,r) = v_1 Q_1(v_1;\pi,r) - P_1(v_1;\pi,r)$$

and similarly for $\mathcal{U}_2(v_2;\pi,r_2)$. It is fairly clear that $\mathcal{U}_j(v_j;\pi,r) = 0$ whenever $v_j \le r_j$. For all types whose valuation v_1 (resp. v_2) is above r_1 (resp. r_2), her payoff should be positive because there always is a positive chance to trade with seller 1 (resp. seller 2) because the probability of having everybody else's valuations below v_1 (resp. v_2) is strictly positive even if $v_1 = r_1$ (since $r_1 \ge c > 0$). Notice that this event is independent of the participation decisions of other bidders and hence, $Q_1(v_1;\pi,r_1) > 0$ (resp. $Q_2(v_2;\pi,r_2) > 0$) for every type such that $v_1 \ge r_1$ (resp. $v_2 \ge r_2$).

Lemma 3.1. *$\mathcal{U}_j(\cdot;\pi,r)$ is nondecreasing and continuous with respect to $v_j \in [0,1]$. Moreover,*

$$\mathcal{U}_j(v_j;\pi,r) = \max\left\{0; \int_{r_j}^{v_j} Q_j(\xi;\pi,r)d\xi\right\} \tag{1}$$

$j = 1,2$.

Proof. Pick any $\pi \in S$. If $r_j = 1$, $\mathcal{U}_j(v_j;\pi,r_j) = 0$ for all $v_j \in [0,1]$. Similarly, if $\max\{v_j,\hat{v}_j\} < r_j$, where v_j and $\hat{v}_j$ are two valuations of item j, then $\mathcal{U}_j(v_j;\pi,r) = \mathcal{U}_j(\hat{v}_j;\pi,r) = 0$. In both cases we obtain a monotonic and continuous payoff function. Hence, suppose that $r_j < 1$ and let v_j and $\hat{v}_j$ satisfy $v_j \le r_j < \hat{v}_j$. Then $\mathcal{U}_j(v_j;\pi,r) = 0 < \mathcal{U}_j(\hat{v}_j;\pi,r)$ because $Q_j(\hat{v}_j;\pi,r) > 0$ since $F(\hat{v}_j) \ge F(c) > 0$. Thus, $\hat{v}_j > v_j$ implies $\mathcal{U}_j(\hat{v}_j;\pi,r) > \mathcal{U}_j(v_j;\pi,r)$, and $\mathcal{U}_j$ is monotonic. Next, let v_j and $\hat{v}_j$ satisfy $\min\{v_j,\hat{v}_j\} > r_j$. Incentive compatibility conditions imply that:

$$\mathcal{U}_j(v_j;\pi,r) - \mathcal{U}_j(\hat{v}_j;\pi,r) \ge Q_j(\hat{v}_j;\pi,r)(v_j - \hat{v}_j) \tag{2}$$

The right-hand side of this expression is strictly positive so long as $v_j > \hat{v}_j$ because $Q_j(\hat{v}_j;\pi,r) > 0$. Therefore, $\mathcal{U}_j$ is strictly increasing whenever $v_j > r_j$. Furthermore, incentive compatibility also implies that $\frac{d\mathcal{U}_1(v_1;\pi,r)}{dv_1} = Q_1(v_1;\pi,r)$ (see Myerson (1981)). Since $Q_j(\cdot;\pi,r)$ is monotonic, it is Riemann integrable. Therefore, for $v_j > r_j$,

$$\begin{aligned}\mathcal{U}_j(v_j;\pi,r) &= \mathcal{U}_j(r_j;\pi,r) + \int_{r_j}^{v_j} Q_j(\xi;\pi,r)d\xi \\ &= \int_{r_j}^{v_j} Q_j(\xi;\pi,r)d\xi\end{aligned}$$

because $\mathcal{U}_j(r_j;\pi,r) = 0$. Continuity of this function stems from the fact that its derivative is Lebesgue integrable for all $v_j \in [r_j,1]$, $r_j \in [c,1)$. □

Lemma 3.1 is important because it helps us derive the following useful property of the set of best responses of any bidder to any participation rule π used by the remaining ones.

Proposition 3.1. *Take any bidder and let π' be any best response to the symmetric participation rule π used by other bidders. Then, there exists a $\pi'' \in S$ and a nondecreasing and continuous function $\rho : [0,1] \to \mathbb{R}$ with the property that $\pi''(v,r) = 1$ if and only if $v_2 \leq \rho(v_1)$, and $\pi''(v,r) = 0$ if and only if $v_2 > \rho(v_1)$, and such that π'' is also a best response to π.*

HERE...We outline the proof for the case in which both reserve prices are strictly below one, relegating the other cases (together with the proofs of monotonicity and continuity of ρ) to the appendix. Let $\max\{r_1; r_2\} < 1$ and suppose that every bidder other than bidder 1 uses the participation rule π to choose trading partners. A necessary and sufficient condition for the participation rule ω' to be bidder 1's best response to π is that for every type $(v_1, v_2) \in [0,1]^2$, and every $(r_1, r_2) \in [v_0, 1]^2$,

$$\omega'(v,r) = \begin{cases} 0 & \text{if } \mathcal{U}_1(v_1;\pi,r_1) < \mathcal{U}_2(v_2;\pi,r_2) \\ 1 & \text{if } \mathcal{U}_1(v_1;\pi,r_1) > \mathcal{U}_2(v_2;\pi,r_2) \\ \in [0,1] & \text{if } \mathcal{U}_1(v_1;\pi,r_1) = \mathcal{U}_2(v_2;\pi,r_2) \end{cases} \tag{3}$$

where $\mathcal{U}_1(\cdot;\pi,r_1)$ is bidder 1's payoff when she bids in auction 1, and $\mathcal{U}_2(\cdot;\pi,r_2)$ is her payoff when she bids in auction 2. Since $\mathcal{U}_1$ and $\mathcal{U}_2$ are both continuous functions (lemma 3.1) defined on the closed interval $[0,1]$, we can use the intermediate value theorem to claim the existence of a pair of numbers $(v_1^*, v_2^*) \in [0,1]^2$ such that $u_1 = \mathcal{U}_1(v_1^*;\pi,r_1)$ and $u_2 = \mathcal{U}_1(v_2^*;\pi,r_2)$ for every number u_1 between $\mathcal{U}_1(0;\pi,r_1)$ and $\mathcal{U}_1(1;\pi,r_1)$, and every number u_2 between $\mathcal{U}_2(0;\pi,r_2)$ and $\mathcal{U}_2(1;\pi,r_2)$ respectively. First, suppose that $\mathcal{U}_1(1;\pi,r_1) \leq \mathcal{U}_2(1;\pi,r_2)$ then we can assign to every $v_1 \in [0,1]$ a number $\rho(v_1) \in [0,1]$ such that $\mathcal{U}_1(v_1;\pi,r_1) = \mathcal{U}_2(\rho(v_1);\pi,r_2)$. This mapping ρ has the property that $\omega'(v,r) = 1$ if and only if $v_2 \leq \rho(v_1)$ because lemma 3.1 ensures that $\mathcal{U}_2(v_2;\pi,r_2) = \mathcal{U}_2(\rho(v_1);\pi,r_1) = \mathcal{U}_1(v_1;\pi,r_1) = 0$ whenever $v_1 \leq r_1$ (and hence, we can assign the same number $\rho(v_1)$ to every such v_1), and $\mathcal{U}_2(v_2;\pi,r_2) < \mathcal{U}_2(\rho(v_1);\pi,r_1) = \mathcal{U}_1(v_1;\pi,r_1)$ whenever $v_1 > r_1$. Second, if $\mathcal{U}_1(1;\pi,r_1) > \mathcal{U}_2(1;\pi,r_2)$ then there are values of v_1 for which bidder 1 strictly prefers to visit seller 1. Let $\bar{v}_1$ be implicitly defined by $\mathcal{U}_1(\bar{v}_1;\pi,r_1) = \mathcal{U}_2(1;\pi,r_2)$. Clearly, $\bar{v}_1 > r_1$ because $\mathcal{U}_1$ and $\mathcal{U}_2$ are increasing functions and hence, $\mathcal{U}_1(1;\pi,r_2) > \mathcal{U}_2(r_2;\pi,r_2) = 0$. Using a similar argument to the one employed in the previous case we can assign to every $v_1 \in [0,\bar{v}_1]$ a number $\rho(v_1) \in [0,1]$ such that $\mathcal{U}_1(v_1;\pi,r_1) = \mathcal{U}_2(\rho(v_1);\pi,r_2)$. For values of v_1 outside $[0,\bar{v}_1]$, we let $\rho(v_1)$ take the value of one such that $\mathcal{U}_1(v_1;\pi,r_1) > \mathcal{U}_2(\rho(v_1);\pi,r_2)$ holds for every $v_1 > \bar{v}_1$. Then, $\omega'(v,r) = 1$ if and only if $v_2 \leq \rho(v_1)$. Figure 3.1 provides a graphical interpretation of the ρ function and the best response ω'.

An interesting implication of proposition 3.1 is the fact that no matter what participation rule bidders may use, every best response to it can be characterized by a pure strategy that is defined in terms of the associated function ρ. This suggests that for every continuation equilibria (if one exists at all) we can find another one in which bidders use pure strategies.

Corollary 3.2. *Take any pair of reserve prices $(r_1, r_2) \in [v_0, 1]^2$ and consider the continuation game in which bidders simultaneously select trading partners. If this continuation game*

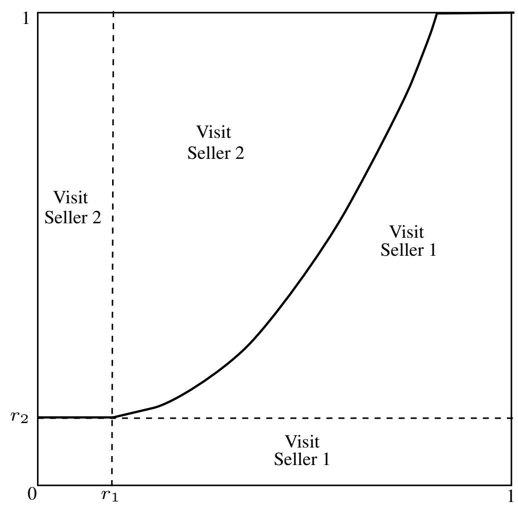

Figure 1: ρ function

possesses an equilibrium then it must also posses an equilibrium in which bidders use pure strategies.

A second interesting implication of proposition 3.1 is the existence of a function ρ that is associated to the best response ω'. The existence of this function allows us to write the payoff of a bidder with valuations (v_1, v_2) directly in terms of ρ:

$$\mathcal{U}_1(v_1; \rho, r) = \max\left\{0; \int_{r_1}^{v_1} \left[1 - \int_{t_1}^{1} F(\rho(\hat{t}_1)) f(\hat{t}_1) d\hat{t}_1\right]^{n-1} dt_1\right\} \tag{4}$$

$$\mathcal{U}_2(v_2; \rho, r) = \max\left\{0; \int_{r_2}^{v_2} \left[F(t_2) F(\rho^{-1}(t_2)) + \int_{\rho^{-1}(t_2)}^{1} F(\rho(\hat{t}_2)) f(\hat{t}_2) d\hat{t}_2\right]^{n-1} dt_2\right\} \tag{5}$$

where $\rho^{-1}(t_2)$ is defined as follows:

$$\rho^{-1}(t_2) = \begin{cases} 0 & \text{if } t_2 < \rho(0) \\ \max\{t_1 \in [0,1] : t_2 \geq \rho(t_1)\} & \text{if } t_2 \geq \rho(0) \end{cases} \tag{6}$$

Notice that in the construction of the payoff functions we have implicitly used the insight of McAfee (1993) regarding the probability with which any given bidder trades with each seller: any bidder with valuations (v_1, v_2) who plans to bid in seller 1's auction wins the item when (i) no other bidder visits this seller; or (ii) any other participant has a valuation lower than this bidder's valuation of the item.

Take any bidder (say bidder 1) and suppose that every other bidder is using a function ρ to select trading partners[5]. Intuitively, for values of v_1 not too high (and reserve prices below one), bidder 1's best response function should deliver a value $\rho'(v_1)$ such that the type $(v_1, \rho'(v_1))$ is indifferent about which seller to visit. This value $\rho(v_1)$ can, in principle, be obtained by equating the expected payoffs given in Eq. (4) and (5). Thus, given v_1 the number $\rho(v_1)$ that satisfies this equality will have the property that bidder 1 wants to visit seller 1 if and only if $v_2 \leq \rho'(v_1)$ else she visits seller 2 with probability one. The only problem with this approach is the possibility that there is no value of v_2 such that payoffs are equal since Eq. (4) and (5) depend on the particular ρ function being used by other bidders. In this case, there will be types of bidder 1 who strictly prefer to visit seller 1 and hence, bidder 1's best response should deliver a value of one for any such type.

Formally, bidder 1's best response is a mapping T taking elements from the set of non-decreasing and continuous functions defined on $[0,1]$, and delivering another function ρ' that represents bidder 1's best response to the function ρ used by other bidders. Let $\mathcal{R}$ be the set of continuous and non decreasing functions mapping elements from $[0,1]$ into $\mathbb{R}$. Bidder 1's best response mapping T on $\mathcal{R}$ can be defined by: (

$$(T\rho)(v_1) = \max\{v_2 \in [0,1] : \mathcal{U}_2(v_2; \rho, r_2) \leq \mathcal{U}_1(v_1; \rho, r_1)\} \tag{7}$$

where $\mathcal{U}_1(v_1; \rho, r_1)$ and $\mathcal{U}_2(v_2; \rho, r_2)$ are given by Eq. (4) and Eq. (5) respectively. Figure 3.1 gives a graphical representation of the procedure used to obtain bidder 1's best response function.

We should point out that any fixed point ρ^* of T can be used to construct a pure strategy ω^* that constitutes a symmetric continuation equilibrium. To see how this works, let ρ^* be a fixed point of T and consider the following symmetric pure strategy: $\omega^*(v,r) = 1$ if and only if $v_2 \leq \rho^*(v_1)$ and $\omega^*(v,r) = 0$ if and only if $v_2 > \rho^*(v_1)$. Suppose that every bidder but bidder 1 conforms to this strategy. We can compute bidder 1's payoffs as done in Eq. (4) and Eq. (5) above. Since ρ^* is a fixed point of T it satisfies $T\rho^* = \rho^*$ and $\mathcal{U}_2(\rho^*(v_1); \rho^*, r_2) \leq \mathcal{U}_1(v_1; \rho^*, r_1)$ for all $v_1 \in [0,1]$. Take any type (v_1, v_2) of bidder 1. First, suppose that $\mathcal{U}_2(\rho^*(v_1); \rho^*, r_2) < \mathcal{U}_1(v_1; \rho^*, r_1)$. Then $\rho^*(v_1)$ must equal one as otherwise there would be some $\tilde{v}_2$ such that $\rho^*(v_1) < \tilde{v}_2 < 1$ and $\mathcal{U}_2(\rho^*(v_1); \rho^*, r_2) < \mathcal{U}_2(\tilde{v}_2; \rho^*, r_2) \leq \mathcal{U}_1(v_1; \rho^*, r_1)$, contradicting the fact that $\rho^*(v_1) = T\rho^*(v_1)$ is the highest such number. It follows that $v_2 \leq \rho^*(v_1)$ and $\mathcal{U}_2(v_2; \rho^*, r_2) \leq \mathcal{U}_2(\rho^*(v_1); \rho^*, r_2)$ (because $\mathcal{U}_2$ is increasing in v_2 from lemma 2) and bidder 1 should visit seller 1 for sure. Second, suppose that $\mathcal{U}_2(\rho^*(v_1); \rho^*, r_2) = \mathcal{U}_1(v_1; \rho^*, r_1)$. If $v_1 \leq r_1$ then $\rho^*(v_1) = r_2$ and it is (weakly) better for bidder 1 to visit seller 1 whenever $v_2 \leq \rho^*(v_1)$ and seller 2 for sure whenever $v_2 > \rho^*(v_1)$. If $v_1 > r_1$ then $\rho^*(v_1) > r_2$ and bidder 1 should visit

[5] Strictly speaking, the function ρ is not a strategy but the function used to describe one. However, once we know the function ρ we can define the strategy ω' associated to it as done in proposition 3.1.

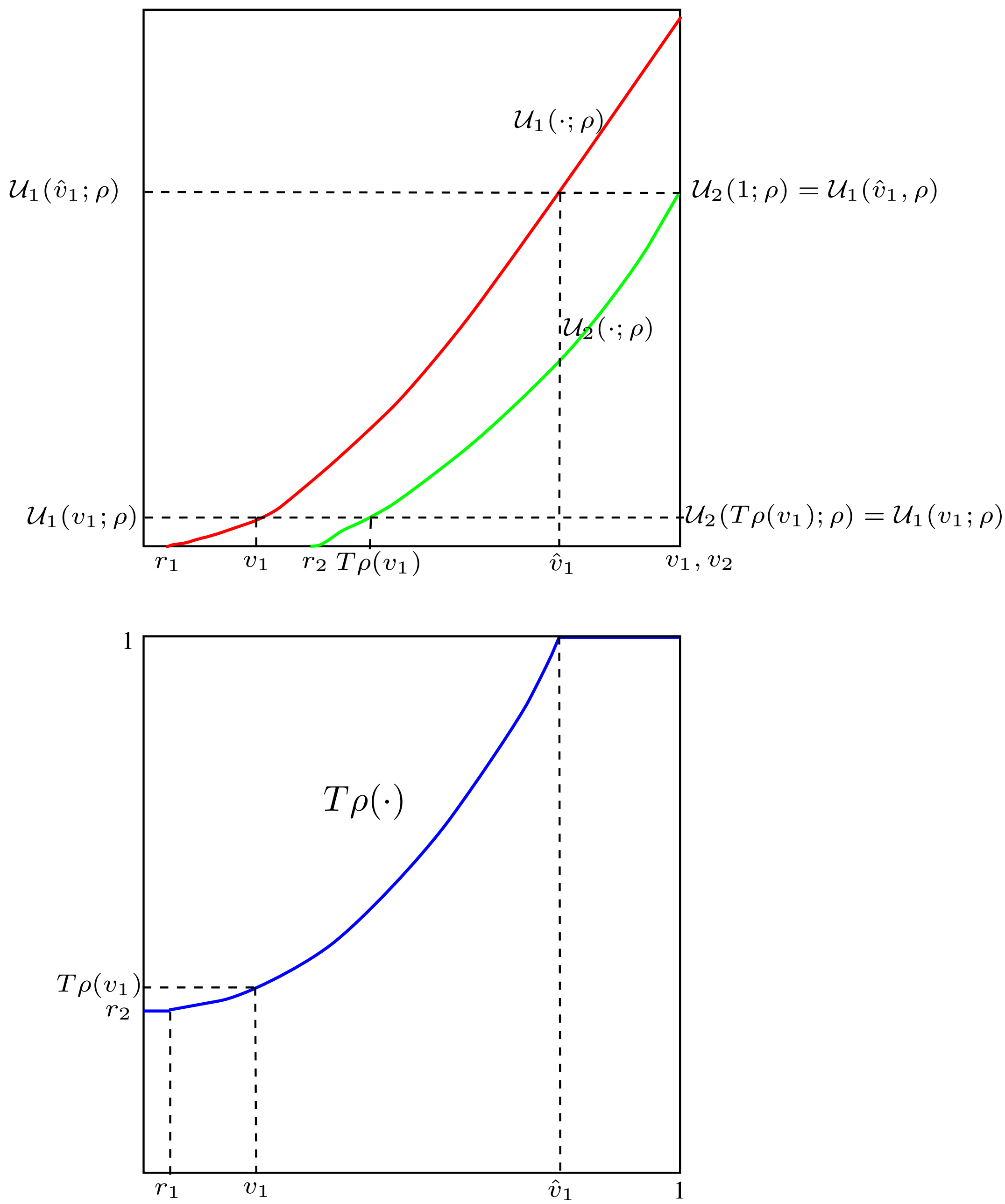

Figure 2: Bidder 1's Best Response Mapping

seller 1 if and only if $v_2 \leq \rho^*(v_1)$ and seller 2 for sure otherwise. Overall, this means that the pure strategy ω^* must be bidder 1's best response to ω^* and thus, ω^* is a symmetric (Bayesian) equilibrium of the bidders' participation game.

The converse of the previous statement is also true. That is, the function associated to any equilibrium strategy must necessarily be a fixed point of T. To see why, suppose that a symmetric continuation equilibrium exists and let π^* be the strategy used by bidders in this equilibrium. From proposition 7, the set of best responses to π^* must contain a strategy ω^* that is characterized by a nondecreasing and continuous function ρ^* such that $\omega^*(v, r) = 1$ if and only if $v_2 \leq \rho^*(v_1)$ and $\omega^*(v, r) = 0$ if and only if $v_2 > \rho^*(v_1)$. Since π^* is a symmetric equilibrium, the strategy ω^* must be a best response to itself. If this were not the case, we could construct a strategy ω' different from ω^* that yields a strictly higher payoff than strategy ω^* for some type of bidder 1. However, ω^* is a best response to π^* and hence, these two strategies must give the same payoff to every type of bidder 1. This means that the strategy ω' must yield a strictly higher payoff to this type of bidder 1 than the payoff associated to strategy π^*, contradicting the fact that π^* is a symmetric continuation equilibrium strategy. Since $\omega^*(v, r) = 1$ if and only if $v_2 \leq \rho^*(v_1)$ and ω^* is a best response to itself, the function ρ^* must necessarily satisfy $T\rho^*(v_1) = \rho^*(v_1)$ for all $v_1 \in [0, 1]$, which implies that ρ^* is a fixed point of T.

The previous discussion allows us to redirect questions about existence and uniqueness of a continuation equilibrium to questions about existence and uniqueness of a fixed point of the best response operator T. The next theorem establishes existence and uniqueness of such fixed point.

Theorem 3.3. *Consider the bidders' participation stage game following any history in which reserve prices (r_1, r_2) belong to the close interval $[c, 1]$, $c > 0$. Then, there exists a unique continuous and nondecreasing function ρ^* such that $T\rho^* = \rho^*$. The function ρ^* is defined by:*

$$\rho^*(v_1) = \begin{cases} \min\{1, r_2\} & \text{if } \max\{r_1, r_2\} = 1 \\ \varphi^*(v_1) & \text{if } \max\{r_1, r_2\} < 1 \end{cases}$$

where,

$$\varphi^*(v_1) = \begin{cases} r_2 & \text{if } v_1 < r_1 \\ \min\{z(v_1); 1\} & \text{if } v_1 \geq r_1 \end{cases}$$

and the function z solves the following equation:

$$\frac{d}{dt} z(t) = \left(\frac{1 - \int_t^1 F(z(\tau)) f(\tau) d\tau}{F(z(t))F(t) + \int_t^1 F(z(\tau)) f(\tau) d\tau} \right)^{n-1} \qquad t \in [r_1, 1]$$

with initial condition $z(r_1) = r_2$.

Proof. In the appendix. □

The proof of the theorem makes extensive use of some properties of the best response operator that must hold true no matter what function other bidders use to select trading partners. For these properties to hold, it is not necessary that v_0 is strictly positive (they also hold if $v_0 = 0$) but we need a strictly positive v_0 in the course of proving existence and uniqueness of the function z for any arbitrary pair of reserve prices[6].

As an example of the properties of T that we exploit, take the case in which $r_1 = 1$. From Eq. (7), bidder 1's best response to any function ρ used by other bidders must be constant and equal to $\min\{1; r_2\}$. To see why, observe that the payoff that bidder 1 expects if she attends to auction 1 is nonpositive no matter what ρ or v_1 is. If $r_2 = 1$ then her expected payoff at auction 2 is also nonpositive and hence, bidder 1 should visit seller 1 with probability one (where she submits a non-serious bid). If $r_2 < 1$ then bidder 1 will select seller 2 with probability one whenever her valuation of item 2 is above r_2 regardless of the function ρ used by bidders other than bidder 1. Thus, $T\rho(v_1) \equiv r_2 = \min\{1; r_2\}$.

Perhaps, the most interesting property arises in cases where both reserve prices are strictly below one. As our previous discussion suggests, we can –at least in principle, find bidder 1's best response to ρ by equating the expected payoffs that bidder 1 would obtain when the other bidders use the function ρ to select trading partners. This idea works fine so long as the value of v_1 given ρ is not too high as to make impossible to find a value of v_2 such that payoffs are equal. Nonetheless, one would suspect that payoff should be equal at least within some subinterval of $[0, 1]$. Part (ii) of the next lemma shows that this is indeed the case and it also summarizes some other useful properties of the best response operator T.

Lemma 3.4. *Suppose that $v_0 \in [0, 1)$ and let $r_1 \in [v_0, 1]$ and $r_2 \in [v_0, 1]$ be any two reserve prices announced by sellers 1 and 2 respectively. Then, for any $\rho \in \mathcal{R}$:*

1. *If $\max\{r_1, r_2\} = 1$, then $T\rho(v_1) = \min\{1; r_2\}$ for all $v_1 \in [0, 1]$;*
2. *If $\max\{r_1, r_2\} < 1$, then:*
   (i) $T\rho(v_1) = r_2$ for all $v_1 \leq r_1$;
   (ii) there exists some $\bar{v}_1$ (that may depend on ρ) satisfying $r_1 < \bar{v}_1 \leq 1$ such that $\mathcal{U}_1(v_1; \rho, r_1) = \mathcal{U}_2(T\rho(v_1); \rho, r_2)$ for all $v_1 \in [r_1, \bar{v}_1]$.
   (iii) If $\bar{v}_1 < 1$ then $T\rho(v_1) = 1$ for all $v_1 \geq \bar{v}_1$.

Proof. In the appendix. □

As already mentioned, part (ii) of lemma 3.4 is perhaps the most useful property of the best response operator that we use to show existence and uniqueness of a fixed point for T. To understand why this is so, recall our discussion about the proof of Proposition 3.1 for the case in which both reserve prices are strictly below one. The idea was to assign to every $v_1 \in [0, 1]$ a number $v_2^* \in [0, 1]$ such that $\mathcal{U}_1(v_1; \pi, r_1) = \mathcal{U}_2(v_2^*; \pi, r_2)$. Using the payoff functions given by Eq. (4) and (5), we can use a similar argument to show existence of a number $T\rho(v_1)$

[6]Proposition 3.5 below shows how to extend Theorem 3.3 to continuation games in which $r_1 = r_2$ and $v_0 = 0$.

such that $\mathcal{U}_1(v_1;\rho,r_1) = \mathcal{U}_2(T\rho(v_1);\rho,r_2)$ regardless of whether $\mathcal{U}_1(1;\rho,r_1) \leq \mathcal{U}_2(1;\rho,r_2)$ or $\mathcal{U}_1(1;\rho,r_1) > \mathcal{U}_2(1;\rho,r_2)$. In the former case, we can assign a number between r_2 and one to every $v_1 \in [r_1, 1]$ because the highest payoff that any bidder expects if bidding in auction 1 is never greater than the highest payoff that she expects in auction 2. In the latter case, we can repeat the above process but this time within some non–empty interval of the form $[r_1, \bar{v}_1]$, $r_1 < \bar{v}_1 \leq 1$. The value $\bar{v}_1$ may depend on the particular function ρ used by other bidders but it is not difficult to show that it must always lie strictly above r_1.

The above discussion suggests to use part (ii) of lemma 3.4 to construct a necessary condition for the best response operator in the form of a integro-differential equation that must hold everywhere with respect to $v_1 \in [r_1, \bar{v}_1]$,

$$\frac{dT\rho(v_1)}{dv_1} = \left(\frac{1 - \int_{v_1}^{1} F(\rho(t))f(t)dt}{F(T\rho(v_1))F(\rho^{-1}(T\rho(v_1)) + \int_{\rho^{-1}(T\rho(v_1))}^{1} F(\rho(t))f(t)dt} \right)^{n-1} \tag{8}$$

plus an initial condition $T\rho(r_1) = r_2$ that follows from part (i) of the lemma. The numerator and denominator of the right–hand–side of this last expression are the probabilities of trading with seller 1 and seller 2 respectively, for a type of bidder whose valuations are $(v_1, T\rho(v_1))$, when every of the remaining $(n-1)$ bidders use the function ρ to select trading partners. Since any fixed point of T must satisfy $T\rho^* = \rho^*$ for all $v_1 \in [0, 1]$, the above equation gives a condition that we can exploit to find a fixed point of T. There are two technical difficulties with this approach. First, the interval within which the above equation holds true is endogenous. Second, Eq. (8) is an integro–differential equation and hence, it is not possible to directly apply any of the standard tools from the theory of differential equations to this problem. To overcome these difficulties we construct an auxiliary problem where we establish existence and uniqueness of a pair of functions that solve a system of two differential equations related to Eq. (8) that holds for all $v_1 \in [r_1, 1]$. In order for this auxiliary problem to possess a unique solution it is sufficient that v_0 be strictly positive. We then use the solution to this auxiliary problem to construct a unique function ρ^* and show that this function must be the unique fixed point of T.

A class of continuation games that will arise in chapters 3 and 4 of this dissertation and that is not covered by theorem 3.3 is the class of continuation games following histories in which $r_1 = r_2 = 0$. As the proof of theorem 3.3 shows in more detail, a $v_0 > 0$ is sufficient to make the denominator of the left-hand side of Eq. (8) well defined under any possible combination of reserve prices that sellers may choose. However, a positive value of v_0 is stronger than needed in cases where both reserve prices are equal. Intuitively, if $r_1 = r_2$ then sellers can be considered ex-ante identical so long as valuations are equal. Thus, we may guess that a bidder with valuations (v_1, v_2) should prefer to bid in auction 1 (resp. auction 2) whenever her valuation of item 1 (resp. item 2) is above her valuation of item 2 (resp. item 1) if this bidder expects everybody else to use this same participation strategy. This gives us $\rho^*(v) = v$, $v \in [0, 1]$, as a candidate for a fixed point of T even if $r_1 = r_2 = 0$. The next result formalizes this intuition.

Proposition 3.5. *Consider any continuation game following a history in which* $r_1 = r_2$*, with* $r_1 \in [0,1]$ *and* $r_2 \in [0,1]$*. Then, the participation strategy:*

$$\pi(s,r) \quad = \quad \begin{cases} 1 & \text{if } v_1 \geq \rho^*(v_1) \\ 0 & \text{if } v_1 < \rho^*(v_1) \end{cases}$$

constitutes the unique symmetric equilibrium of this continuation game. The function $\rho^* : [0,1] \to [0,1]$ *is defined by:*

$$\rho^*(v_1) = \begin{cases} \min\{1; r_2\} & \text{if } \max\{r_1; r_2\} = 1 \\ \varphi^*(v_1) & \text{if } \max\{r_1; r_2\} < 1 \end{cases}$$

where:

$$\varphi^*(v_1) = \begin{cases} r_2 & \text{if } v_1 < r_1 \\ v_1 & \text{if } v_1 \geq r_1 \end{cases}$$

We outline the proof of the function $\rho^*(v_1) = v_1$ being a fixed point of T relegating the proof of uniqueness to the appendix. From lemma 3.4, the best response operator $T\rho^*$ must satisfies $T\rho^*(0) = 0$ and $\mathcal{U}_2(T\rho^*(v_1); \rho^*, r) = \mathcal{U}_1(v_1; \rho^*, r)$ for $v_1 \in [0, \bar{v}]$ where $\bar{v}$ is implicitly defined by $\mathcal{U}_1(\bar{v}; \rho^*, r) = \mathcal{U}_2(1; \rho^*, r)$. As previously mentioned, these two properties of the best response operator hold true even if v_0 is equal to zero. Using Eq. (4) and (5) we obtain the following payoff functions when bidders other than bidder 1 use this function ρ^* to select trading partners:

$$\mathcal{U}_1(v_1; \rho^*, r) = \int_0^{v_1} \left(\frac{1}{2} + \frac{F^2(t)}{2} \right)^{n-1} dt$$

and,

$$\mathcal{U}_1(v_2; \rho^*, r) = \int_0^{v_2} \left(\frac{1}{2} + \frac{F^2(t)}{2} \right)^{n-1} dt$$

Hence, $\bar{v} = 1$ and $\mathcal{U}_1(v_1; \rho^*, r) = \mathcal{U}_2(v_2; \rho^*, r)$ if and only if $v_1 = v_2$. It follows that:

$$\begin{aligned} T\rho^*(v_1) &= \max\{v_2 : \mathcal{U}_2(v_2; \rho^*, r) \leq \mathcal{U}_1(v_1; \rho^*, r)\} \\ &= v_1 \\ &= \rho^*(v_1) \end{aligned}$$

for every $v_1 \in [0,1]$, which shows that $\rho^*(v_1) = v_1$, $v_1 \in [0,1]$, is a fixed point of T when $r_1 = r_2$ and $v_0 \geq 0$.

3.2 Heterogeneity as a Coordination Device

We end this section with some discussion about the role played by heterogeneity in the selection of trading partners, and how this selection rule is affected by changes in reserve prices. Observe that apart from providing a complete and novel characterization of the set of (symmetric) participation rules, Theorem 3.3 also rules out randomization in the selection of trading partners. Indeed, theorem 3.3 shows that the set of types who wish to randomize between sellers (those lying on the cutoff function) must have zero measure. Perhaps more importantly, this lack of randomization is a property that must hold true in every symmetric continuation equilibrium of the game. This differentiates our model from current models in the literature where bidders always randomize in their choices of trading partners[7], which in turn introduces frictions in the market due to a coordination failure. Contrary to this, the introduction of heterogeneity eliminates this market friction by coordinating the visiting decisions of bidders. In this sense, heterogeneity acts as a device that rules out market frictions due to coordination problems in the selection of trading partners.

A second issue closely related to the previous one is the way in which changes in some reserve price affect bidders' visiting decisions. The literature on competing auctions (Peters and Severinov, 1997; Burguet and Sakovics, 1999; Virag, 2010) has shown that in the case of homogeneous goods, a change in some seller's reserve price changes the set of types that visits an auction but it does not change the probability with which each bidder participates. Thus, a higher reserve price has the effect of shutting down the participation of those bidders whose valuations are close to the reserve price, making less likely that bidders with high valuations face an opponent. As Peters (2010) points out, this means that sellers who compete in auctions do not compete directly for the high valuation bidders since only low valuation types alter their behavior in response to changes in reserve prices. This is no longer true when items are assumed to be heterogeneous. As shown by theorems 3.2 and 3.3, bidders use functions to select trading partners and hence, changes in some reserve price will have an effect on the participation decisions of the whole set of types. In particular, some bidders with high valuations will also respond by shifting from the high-reserve to the low–reserve price auction. This additional shift in the number of bidders who now find profitable to attend to the low–reserve price auction will reduce expected traffic, adding a new channel through which reserve prices affect sellers profits.

Proposition 3.6. *Take any two distinct pair of reserve prices* (r_1, r_2) *and* $(\hat{r}_1, r_2)$*, with* $r_1 < \hat{r}_1$*. Let* $\rho(\cdot;(r_1, r_2))$ *and* $\rho(\cdot;(\hat{r}_1, r_2))$ *be the equilibrium functions used by bidders to select trading partners when reserve prices are* (r_1, r_2) *and* $(\hat{r}_1, r_2)$ *respectively. Then,* $\rho(v_1;(r_1, r_2)) \geq \rho(v_1;(\hat{r}_1, r_2))$ *for every* $v_1 \in [0, 1]$*. Furthermore, if* $r_2 < 1$ *then there exists a nonempty interval* $\Omega \subseteq [0, 1]$ *such that* $\rho(v_1;(r_1, r_2)) > \rho(v_1;(\hat{r}_1, r_2))$ *for all* $v_1 \in \Omega$*.*

Proof. In the appendix. □

[7] These models also admit continuation equilibria in which bidders choose trading partners using pure strategies. However, such continuation equilibria require some sort of sunspot that allows bidders to coordinate on their visiting decisions.

3.3 Sellers' Game

The existence of a function ρ^* that can be associated to a (pure) strategy equilibrium in the bidders' continuation game makes it possible to describe sellers' reduced form payoffs using the function ρ^*. Suppose that sellers announce some pair of reserve prices $r_1 \in [c, 1]$ and $r_2 \in [c, 1]$ respectively, and that bidders' choice of trading partners is described by the continuation equilibrium strategy ω^* characterized by the function ρ^* . Take seller 1. It is not difficult to check that if $r_1 = 1$ then seller 1's expected profit must be equal to zero[8]. In all other cases, seller 1's (resp. seller 2's) expected revenue must be equal to the sum of the revenue that the seller expects when a single bidder visits his auction plus the revenue that he expects when two or more bidders visits him. Therefore, all that we need to know to compute seller's payoff is the probability with which any given bidder visits his auction, and the expected value of the second highest type of those bidders who chooses to visit the auction, whenever two or more bidders visit. From proposition 3.1 any bidder with valuations (v_1, v_2) visits seller 1 with probability one if and only if $v_2 < \rho^*(v_1)$; otherwise she visits seller 2 with probability one. Therefore, the probability that a bidder visits seller 1, $q^* := q(\rho^*)$, is:

$$\begin{aligned} q^* &= \int_0^1 \left\{ \int_0^{\rho^*(v_1)} dF(v_2) \right\} dF(v_1) \\ &= \int_0^1 F(\rho^*(v_1))dF(v_1) \end{aligned}$$

which yields an expected revenue for seller 1 in case he receives a single visitor equal to:

$$n(r_1 - c)q^* \left[1 - q^*\right]^{n-1}$$

Let $G_1^*(v_1) := G_1(v_1; \rho^*)$ be the probability that a bidder with valuation v_1 trades with seller 1 when there are two or more bidders bidding in auction 1. From proposition 3.1, this probability is given by:

$$\begin{aligned} G_1^*(v_1) &= \left[1 - q^* + \int_0^{v_1} F(\rho^*(\xi))dF(\xi)\right] \\ &= \left[1 - \int_{v_1}^1 F(\rho^*(\xi))dF(\xi)\right] \end{aligned}$$

which gives us the expected revenue when two or more bidders bids in auction 1:

$$n(n-1) \int_{r_1}^1 (t_1 - c) \left[1 - G_1^*(t_1)\right] \left[G_1^*(t_1)\right]^{n-2} dG_1^*(t_1)$$

Let $V^+(r_1, r_2; \rho^*)$ be defined by:

[8]If $r_1 = r_2 = 1$ then every bidder visits seller 1 for sure. However, as $v_1 \leq r_1$ for all $v_1 \in [0, 1]$, bidders submit non-serious bid equal to c. If $r_2 < 1$ then $\rho^*(v_1) \equiv r_2$ and again, any bidder who visits seller 1 submits a non–serious bid equal to c. Hence, seller 1's profit is equal to zero whenever $r_1 = 1$, $r_2 \in [c, 1]$.

$$V_1^+(r_1, r_2; \rho^*) =$$
$$n(r_1 - c)q^* \left[1 - q^*\right]^{n-1} + n(n-1) \int_{r_1}^{1} (t_1 - c) \left[1 - G_1^*(t_1)\right] \left[G_1^*(t_1)\right]^{n-2} dG_1^*(t_1) - c(1 - q^*)^n$$

Therefore, seller 1's payoff when $r_1 \in [c, 1)$ (seller 2's payoff can be derived likewise) is:

$$V_1(r_1, r_2; \rho^*) = \begin{cases} 0 & \text{if } r_1 = 1 \\ V_1^+(r_1, r_2; \rho^*) & \text{if } c \leq r_1 < 1 \end{cases}$$

Although the payoff functions may have discontinuities, such discontinuities will occur at $r_1 = 1$ (resp. $r_2 = 1$). As we show in the appendix, this kind of discontinuities do not preclude the existence of an equilibrium (in mixed strategies) for our game.

Proposition 3.7. *The competing auction game with heterogeneous goods admits a Perfect Bayesian equilibrium in which bidders follow symmetric strategies.*

Proof. In the appendix. □

The previous proposition settles the question of equilibrium existence. However, the theorem does not ensure that there exists an equilibrium in which sellers use pure strategies. In general, showing existence of pure-strategy equilibria in competing auction games is a complex task because it usually requires some form of concavity of payoff functions, which is an endogenous component in this class of games. Furthermore, the lack of a close–form solution for the continuation equilibrium function ρ^* adds another layer of complexity to the analysis of pure strategy equilibria.

3.4 The 2×2 case

Apart from Virag (2010), the only other paper that addresses the question of existence of pure–strategy equilibria is Burguet and Sakovics (1999). In their model, two sellers with unit supply compete by positing reserve prices. These authors have shown that the equilibrium probability of posting a reserve price equal to zero is nil, and that this probability remains nil even if the number of bidders grows very large. Intuitively, when the number of sellers is restricted to two, a unilateral increase of a reserve price only affects the pool of types that visits each seller but it does not affect the probability with which each seller visits. This means that a higher reserve price increases the probability of selling to bidders with higher valuations, which is achieved by eliminating bidders with relatively low valuations. Peters and Severinov (1997), Burguet and Sakovics (1999), and Virag (2010) have all shown that rising a reserve price has an effect only on the participation decision of bidders whose valuations are close to the reserve prices leaving unchanged this probability for high valuation bidders. Thus, when reserve prices are close to production costs, unilaterally increasing a reserve price has almost no cost to the seller because only bidders with very low valuations are banned from participating in the auction. This effect remains positive so long as sellers have the ability to affect the utility levels of bidders (by changing the composition of the pool of visitors), which does not depend on the number of buyers present in the market. With heterogeneous goods, bidders use functions to select trading partners and changes in some reserve price have the potential to affect the participation of the whole set of types. In particular, changes in reserve prices affect the participation decisions of bidders with high valuations who were just indifferent before the change in the reserve price took place, lowering the expected traffic. This traffic effect tends to offset the positive effect (on profits) of the screening effect and whether this last effect is strong enough to countervail the first depends on the number of potential customers.

Proposition 3.8. *Take any two distinct pair of reserve prices* (r_1, r_2) *and* $(\hat{r}_1, r_2)$*, with* $r_1 < \hat{r}_1$*. Let* ρ^* *and* $\hat{\rho}$ *be the functions used by bidders to select trading partners when reserve prices are* (r_1, r_2) *and* $(\hat{r}_1, r_2)$ *respectively. Then,* $\rho^*(v_1) \geq \hat{\rho}(v_1)$ *for every* $v_1 \in [0, 1]$ *with strict inequality for some nonempty subset* $\Omega \subseteq [0, 1]$*.*

Proof. In the appendix. □

We are now ready to state the main result of this subsection. Suppose that seller 1 unilateral increases reserve price r_1. Then, Proposition 3.8 ensures that such increase will induce bidders with low and high valuations to stop visiting seller 1. This negative traffic effect becomes stronger the larger the number of bidders in the market. Thus, for a number of bidders high enough, we should expect sellers' profits to be decreasing in their own reserve prices leading to the existence of a pure strategy equilibrium in which both sellers post reserve prices equal to production costs.

Proposition 3.9. *Suppose that there are two sellers and two bidders participating in the market. Then there is an equilibrium in which both sellers post reserve prices equal to production costs.*

Proof. In the appendix. □

4 Concluding Remarks

The purpose of this paper is to show the consequences of the introduction of heterogeneity in bidders' tastes in models where seller compete for the attention of bidders through reserve price offers. In our model, two sellers running second-price auctions post nonnegative reserve prices which direct the attention of several bidders who attach different valuations to the items offered. We provide a complete and novel characterization of the set of continuation equilibria in which bidders use symmetric participation rules. This characterization allows us to show that heterogeneity acts as a coordinating device by reducing the probability that any two given bidders meet at the same auction. Indeed, we show that the set of types willing to choose trading partners at random has measure zero. Intuitively, this is true because once a bidder has decided to attend to auction j, her payoff in this auction depends on his valuation of this particular item alone. This creates a difference with respect to models in which items are assumed to be homogeneous. In these models, the only asymmetry among sellers is given by the reserve price set by each seller. Thus, after adjusting bidders' payoffs to incorporate these asymmetries, items become essentially the same thing to the eyes of bidders and it results natural that bidders select sellers at random. However, when items are heterogeneous differences among sellers remain even after we have accounted for the asymmetries produced by reserve prices. As bidders attach different values to different items, the decision about attending to auction j or some other auction reduces to the comparison of expected rents. Since the expected rent at seller j's depends on v_j alone (for any given vector of reserve prices) and this rent is increasing in the valuation of item j, the event of having a bidder indifferent among all auctions should be a zero-measure event.

A consequence of the way bidders select trading partners is the effect caused by changes in reserve prices. When items are assumed homogeneous, a unilateral decrease in seller j's reserve price affects the participation decisions of two types of bidders: (i) those who were not visiting but now find profitable to do so; (ii) some bidders who were mixing among some subset of sellers. However, bidders with high valuations do not change the probability with which they visits each seller. This is no longer true when goods are heterogeneous: unilateral changes in reserve prices affect not only the composition of the pool of types who visit but also the probability with which every bidder visits the auctions, including those with high valuations. This introduces a novel trade–off between traffic and screening effects is not present in models with homogeneous goods.

Finally, we show that in the 2–seller 2–bidders case there exists a pure-strategy equilibrium in which sellers post reserve prices equal to production costs. As unilaterally increases in some reserve price eliminates not only low-valuation but also eliminates high-valuation bidders who were just indifferent before the change in the reserve price, when there are just two bidders the fierce competition to capture these bidders forces sellers to post reserve prices equal to production costs.

Appendix

Proof of Theorem 3.3

The following lemma will prove useful in the main proof of the theorem.

Lemma 4.1. *Let $r_1 \in [c,1]$ and $r_2 \in [c,1]$ be any two reserve prices announced by sellers 1 and 2 respectively. Then, for any $\rho \in \mathcal{R}$:*

1. *If $\max\{r_1, r_2\} = 1$, then $T\rho(v_1) = \min\{r_1; r_2\}$ for all $v_1 \in [0,1]$;*
2. *If $\max\{r_1, r_2\} < 1$, then:*
   - *(i) $T\rho(v_1) = r_2$ for all $v_1 \le r_1$;*
   - *(ii) there exists some $\bar{v}_1$ (that may depend on ρ) satisfying $r_1 < \bar{v}_1 \le 1$ such that $\mathcal{U}_1(v_1; \rho, r_1) = \mathcal{U}_2(T\rho(v_1); \rho, r_2)$ for all $v_1 \in [r_1, \bar{v}_1]$.*
   - *(iii) If $\bar{v}_1 < 1$ then $T\rho(v_1) = 1$ for all $v_1 \ge \bar{v}_1$.*

Proof. To prove part (1), consider the case in which $r_1 = 1$ and $r_2 < 1$ (the other cases are handled likewise). Then, $\mathcal{U}_1(\cdot; \rho, r_1) = 0$ no matter what v_1 or ρ is, and $\mathcal{U}_2(v_2; \rho, r_2) > 0$ for $v_2 \in [r_2, 1]$. Therefore, bidder 1's optimal response to any ρ must be to visit seller 1 (resp. seller 2) with probability zero (resp. probability one) whenever her valuation of item 2 is above r_2. This means that $T\rho(v_1) = r_2 = \min\{r_1, r_2\}$ for all $v_1 \in [0,1]$ and all $\rho \in \mathcal{R}$.

Next, we prove (i) of part (2). Take any type (v_1, v_2) such that $v_1 \le r_1$. Then, $\mathcal{U}(v_1; \rho, r_1) = 0$ for all $\rho \in \mathcal{R}$ and hence,

$$\begin{aligned} T\rho(v_1) &= \max\{v_2 \in [0,1] : \mathcal{U}_2(v_2; \rho, r_2) \le \mathcal{U}_1(v_1; \rho, r_1)\} \\ &= r_2 \end{aligned}$$

for all $v_1 \le r_1$ and all $\rho \in \mathcal{R}$. To prove (ii) and (iii), let $I_1 = [0, \bar{u}_1]$ and $I_2 = [0, \bar{u}_2]$ be the (compact) image of $\mathcal{U}_1(\cdot; \rho, r_1)$ and $\mathcal{U}_2(\cdot; \rho, r_2)$ on $[r_1, 1]$ and $[0,1]$ respectively. It is almost immediate that $I_1 \cap I_2 \neq \emptyset$ because $\mathcal{U}_1(r_1; \rho, r_1) = 0 = \mathcal{U}_2(r_2; \rho, r_2)$, regardless of ρ. Consider the following two cases.

(i) $I_1 \subseteq I_2$. Then, $\mathcal{U}_1(\cdot; \rho, r_1) \in I_2$ for every $v_1 \in [r_1, 1]$. From the intermediate value theorem we can assign to every $v_1 \in [r_1, 1]$ a number $v_2^* \in [0,1]$ such that $\mathcal{U}_1(v_1; \rho, r_1) = \mathcal{U}_2(v_2^*; \rho, r_2)$. Moreover, the fact that $\mathcal{U}_2$ is increasing in $v_2 > r_2$ and $\mathcal{U}_2(v_2; \rho, r_2) = 0$ for all $v_2 \in [0, r_2]$, $\rho \in \mathcal{R}$, implies that this number must be unique. Since $T\rho(v_1)$ delivers the maximum number such that $\mathcal{U}_2(v_2; \rho) \le \mathcal{U}_1(v_1; \rho)$ holds, $T\rho(v_1) = v_2^*$, and $\mathcal{U}_1(v_1; \rho, r_1) = \mathcal{U}_2(T\rho(v_1); \rho, r_1)$ for all $v_1 \in [r_1, 1]$.

(ii) $I_2 \subset I_1$. Then, there are values of v_1 such that $\mathcal{U}_1(\cdot; \rho, r_1)$ falls outside the range of $\mathcal{U}_2(\cdot; \rho, r_2)$. From Eq. (5), $\mathcal{U}_2(1; \rho, r_2) \ge [F(c)]^{n-1}(1 - r_2) > 0$ because $r_2 < 1$ and $c > 0$.

Moreover, lemma 2 ensures that $\mathcal{U}_1$ is a continuous function of v_1 for any $\rho \in \mathcal{R}$. Therefore, there must be a number $\bar{v} \leq 1$ such that $\bar{v} = \max\{v_1 \in [0,1] : \mathcal{U}_1(v_1; \rho, r_1) \leq \mathcal{U}_2(1, \rho, r_2)\}$. Furthermore, the number $\bar{v}$ must be strictly greater than r_1 because $\bar{v} = r_1$ would imply the existence of some other number $\tilde{v} > \bar{v}$ such that $\mathcal{U}_1(\tilde{v}; \rho, r_1) = \mathcal{U}_2(1, \rho, r_2)$, contradicting the fact that $\bar{v}$ is the maximum such number. Using a similar argument to the one employed in case (i) above allows to assign to every $v_1 \in [0, \bar{v}_1]$ a number $v_2^* \in [0,1]$ such that $\mathcal{U}_1(v_1; \rho, r_1) = \mathcal{U}_2(v_2^*; \rho, r_2)$, and since $T\rho(v_1)$ delivers the maximum number such that $\mathcal{U}_2(v_2; \rho) \leq \mathcal{U}_1(v_1; \rho)$, $\mathcal{U}_1(v_1; \rho, r_1) = \mathcal{U}_2(T\rho(v_1); \rho, r_1)$ for all $v_1 \in [r_1, \bar{v}]$. For values of v_1 greater than $\bar{v}$, $\mathcal{U}_1(v_1; \rho, r_1) > \mathcal{U}_2(1; \rho, r_2)$ and hence, $T\rho(v_1) = 1$ for all $v_1 \geq \bar{v}$. This completes the proof of (ii) and (iii) of the lemma. □

The proof of the theorem is organized in two cases. The first case covers continuation games following histories in which at least one reserve price is equal to one whereas the second one covers continuation games in which both reserve prices are strictly below one.

Case 1. Suppose that $\max\{r_1, r_2\} = 1$. Then, part (1) in lemma 4.1 ensures that $T\rho(v_1) = \min\{r_1, r_2\}$ holds for all $v_1 \in [0,1]$ and all $\rho \in \mathcal{R}$. In particular, this must hold for $\rho^*(v_1) \equiv \min\{r_1, r_2\}$ and therefore, $T\rho^*(v_1) = \min\{1, r_2\} \equiv \rho^*(v_1)$, which implies that ρ^* is the unique fixed point of T.

Case 2. Suppose that $\max\{r_1, r_2\} < 1$. Then, part (2) of lemma 4.1 ensures the existence of some nonempty interval $[r_1, \bar{v}_1]$ such that $\mathcal{U}_1(v_1; \rho, r_1) = \mathcal{U}_2(T\rho(v_1); \rho, r_2)$, $v_1 \in [r_1, \bar{v}_1]$. Since this equation holds for every $v_1 \in [r_1, \bar{v}_1]$, $\mathcal{U}_1$ is strictly increasing in v_1, $T\rho(r_1) = r_2$ from part (i) of lemma 4.1, and $\mathcal{U}_2$ is increasing in $v_2 > r_2$, the function $T\rho$ must be strictly increasing with respect to v_1 within the interval $[r_1, \bar{v}_1]$ and hence, differentiable everywhere with respect to v_1 in $(r_1, \bar{v}_1)$:

$$\frac{dT\rho(v_1)}{dv_1} = \left(\frac{1 - \int_{v_1}^1 F(\rho(t)) f(t) dt}{F(T\rho(v_1)) F(\rho^{-1}(T\rho(v_1)) + \int_{\rho^{-1}(T\rho(v_1))}^1 F(\rho(t)) f(t) dt} \right)^{n-1}$$

where the numerator (resp. denominator) is the slope of $\mathcal{U}_1$ (resp. $\mathcal{U}_2$), i.e., the probability of trading with seller 1 (resp. seller 2) when bidder 1's valuations are $(v_1, T\rho(v_1))$ and the remaining $(n-1)$ bidders use the function ρ. Moreover, if ρ^* is a fixed point of T, then $T\rho^* = \rho^*$ and the above equation becomes:

$$\frac{d\rho^*(v_1)}{dv_1} = \left(\frac{1 - \int_{v_1}^1 F(\rho^*(t)) f(t) dt}{F(\rho^*(v_1)) F(\rho^{*-1}(\rho^*(v_1)) + \int_{\rho^{*-1}(\rho^*(v_1))}^1 F(\rho^*(t)) f(t) dt} \right)^{n-1}$$

where $\rho^{*-1}(\rho^{-1}(v_1))$ may differ from v_1 if $\bar{v}_1 < 1$ (if $\bar{v}_1 = 1$ then ρ^* is increasing in $[r_1, 1]$ and hence, $\rho^{*-1}(\rho^*(v_1)) = v_1$). Although this last equation holds for v_1 less or equal to $\bar{v}_1$ (and $\bar{v}_1$ depends on the particular function ρ), we can use it to construct a fixed point under the assumption that it holds for all $[r_1, 1]$, as the next lemma shows.

Lemma 4.2. *Suppose that there exists a continuous and increasing function $z : [r_1, 1] \to \mathbb{R}$ that satisfies:*

$$\begin{aligned}\frac{dz(v_1)}{dt} &= \left(\frac{1-\int_{v_1}^{1} F(z(t_1))f(t_1)dt_1}{F(z(v_1))F(v_1)+\int_{v_1}^{1} F(z(\tau))f(\tau)d\tau}\right)^{n-1}\\ z(r_1) &= r_2\end{aligned}$$

where F is an absolutely continuous distribution function with strictly positive and bounded density f, and support $[0,1]$ (and hence, $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$), and $r_1 \in (c, 1)$, $r_2 \in (c, 1)$, $c > 0$. Define ρ^* as follows:

$$\rho^*(v_1) = \begin{cases} r_2 & \text{if } v_1 < r_1 \\ \min\{z(v_1), 1\} & \text{if } v_1 \geq r_1 \end{cases}$$

Then, ρ^* is a fixed point of T.

Proof. From part (i) in lemma 4.1, $T\rho^*(v_1) = r_2 = \rho^*(v_1)$ whenever $v_1 < r_1$. Hence, let $v_1 \geq r_1$. Suppose that bidders other than bidder 1 uses the function ρ^* defined in the lemma. From McAfee (1993), the probability that bidder 1 trades with seller j when her valuation is v_j is equal to the probability that no other bidder visits seller j plus the probability that any other participant has a valuation lower than v_j. Then,

$$H_1(t_1; \rho^*, r_1) = \left(1 - \int_{t_1}^{1} F(\min\{z(\hat{t}_1), 1\})f(\hat{t}_1)d\hat{t}_1\right)^{n-1}$$

for all $t_1 \geq r_1$. Since $F(x) = 1$ for all $x \geq 1$, $F(\min\{z(v_1), 1\}) = F(z(v_1))$ for all $v_1 \in [r_1, 1]$ and hence,

$$\begin{aligned}H_1(t_1; \rho^*, r_1) &= \left(1 - \int_{t_1}^{1} F(\min\{z(\hat{t}_1), 1\})f(\hat{t}_1)d\hat{t}_1\right)^{n-1}\\ &= \left(1 - \int_{t_1}^{1} F(z(\hat{t}_1))f(\hat{t}_1)d\hat{t}_1\right)^{n-1}\\ &= H_1(t_1; z, r_1)\end{aligned}$$

for all $t_1 \in [r_1, 1]$. Therefore,

$$\begin{aligned}\mathcal{U}_1(v_1; \rho^*, r_1) &= \int_{r_1}^{v_1} \left(1 - \int_{t_1}^{1} F(z(\hat{t}_1))f(\hat{t}_1)d\hat{t}_1\right)^{n-1} dt_1\\ &= \mathcal{U}_1(v_1; z, r_1)\end{aligned}$$

Let $u(t_1) = \int_{r_2}^{z(t_1)} H_2(t_2, z, r_2) dt_2$ where,

$$H_2(t_2; z, r_2) \;=\; \left(F(t_2)F(z^{-1}(t_2)) + \int_{z^{-1}(t_2)}^{1} F(z(\hat{t}_1)) f(\hat{t}_1) d\hat{t}_1 \right)^{n-1}$$

Then, $du = H_2(z(t), z, r_2)\dot{z}(t)dt$ and,

$$\begin{aligned}
\int_{r_1}^{v_1} du &= u(v_1) - u(r_1) \\
&= \int_{r_2}^{z(v_1)} H_2(t_2; z, r_2) dt_2 \\
&= \int_{r_1}^{v_1} H_1(t_1; z, r_1) dt_1
\end{aligned}$$

where:

$$H_1(v_1; z, r_1) = \left(1 - \int_{t_1}^{1} F(z(\hat{t}_1)) f(\hat{t}_1) d\hat{t}_1 \right)^{n-1}$$

because z solves problem P. From part (ii) of lemma 4.1, bidder 1's best response $T\rho^*(v_1)$ must satisfy

$$\int_{r_1}^{v_1} \left(1 - \int_{t_1}^{1} F(\rho^*(\hat{t}_1)) f(\hat{t}_1) d\hat{t}_1 \right)^{n-1} dt_1 =$$
$$\int_{r_2}^{T\rho^*(v_1)} \left(F(t_2)F(\rho^{*-1}(t_2)) + \int_{\rho^{*-1}(t_2)}^{1} F(\rho^*(t_1)) f(t_1) dt_1 \right) dt_2$$

for some nonempty interval $[r_1, \bar{v}_1]$. Since $\mathcal{U}_1(v_1; \rho^*, r_1) = \mathcal{U}_1(v_1; z, r_1)$ for all $v_1 \in [r_1, 1]$,

$$\begin{aligned}
\mathcal{U}_2(T\rho^*(v_1); \rho^*, r_1) &= \mathcal{U}_2(z(v_1); z, r_1) \\
&= \int_{r_2}^{z(v_1)} \left(F(t_2)F(z^{-1}(t_2)) + \int_{z^{-1}(t_2)}^{1} F(z(t_1)) f(t_1) dt_1 \right) dt_2
\end{aligned}$$

for all $v_1 \in [r_1, \bar{v}_1]$ because $\mathcal{U}_1(v_1; z, r_1) = \mathcal{U}_2(z(v_1); z, r_2)$ since z solves problem P. Since $v_1 \geq r_1$, and $H_2(v_2; \rho^*, r_2) > 0$, $\mathcal{U}_2$ is strictly increasing in $v_2 \in [r_2, 1]$. Therefore, the value $v_2^* \in [0, 1]$ satisfying $\mathcal{U}_1(v_1; \rho^*, r_1) = \mathcal{U}_2(v_2^*; \rho^*, r_2)$ must be unique and thus, $T\rho^*(v_1) = z(v_1) \leq 1$ for all $v_1 \in [r_1, \bar{v}_1]$. If $\bar{v}_1 = 1$ then, $T\rho^*(v_1) = z(v_1)$ for all $v_1 \in [r_1, 1]$ and $T\rho^*(v_1) = \min\{z(v_1), 1\} = \rho^*(v_1)$. If $\bar{v}_1 < 1$, then $T\rho^*(\bar{v}_1) = z(\bar{v}_1) = 1$ because $T\rho^*(v_1) = 1$ for all $v_1 \in [\bar{v}_1, 1]$ from part (ii) in lemma 4.1. Since z solves problem P, it is a strictly function of $v_1 \in [r_1, 1]$. Therefore, $z(v_1) > z(\bar{v}_1) = 1$ for all $v_1 \in [\bar{v}_1, 1]$ and thus, $\min\{z(v_1); 1\} = 1$ whenever $v_1 \in [\bar{v}_1, 1]$ and $T\rho^*(v_1) = 1 = \min\{z(v_1); 1\} = \rho^*(v_1)$ if $\bar{v}_1 < 1$. Therefore, $T\rho^*(v_1) = \rho^*(v_1)$ for all $v_1 \in [0, 1]$ and ρ^* as defined in the lemma must be a fixed point of T. □

The rest of the proof is intended to show existence of a function z.

Proposition 4.3. *Let F be an absolutely continuous distribution function with support $[0,1]$ (hence, it satisfies $F(s)=0$ for all $s<0$, $F(s)=1$ for all $s>1$), and strictly positive and bounded density f, and let r_1 and r_2 be scalars satisfying $r_1 \in (c,1)$, $r_2 \in (c,1)$, with $c \in (0,1)$. Then, there exists a unique continuous and increasing function $z^* : [r_1, 1] \to \mathbb{R}$ that solves the following (Problem P):*

$$\frac{dz^*(v_1)}{dv_1} = \left(\frac{1-\int_{v_1}^1 F(z^*(t))f(t)dt}{F(z^*(v_1))F(v_1)+\int_{v_1}^1 F(z^*(t))f(t)dt}\right)^{n-1} \tag{9}$$

$$z^*(r_1) = r_2 \tag{10}$$

Proof. A solution to the above Problem P is a continuous and increasing function defined on the closed and compact interval $[r_1, 1]$ that satisfies the integro–differential equation (9), and the initial condition (10). Our plan to demonstrate that this problem admits a unique solution is the following. First, we will use standard tools from the theory of differential equations to show existence and uniqueness of a pair of continuous functions that solves the following initial value problem:

$$\begin{aligned}
\frac{d\phi(t)}{dt} &= \left(\frac{1-\phi_2(t)}{F(\phi_1(t))F(t)+\phi_2(t)}\right)^{n-1} \\
\frac{d\phi_2(t)}{dt} &= -F(\phi_1(t))f(t) \\
\phi_1(r_1) &= r_2 \\
\phi_2(r_1) &= \theta
\end{aligned}$$

where $\theta \in (0,1)$. Second, we will use this solution –call it $(\phi_1^\theta; \phi_2^\theta)$, to show existence of a unique root θ^* to the equation:

$$\phi_2^\theta(1) = 0 \tag{11}$$

that will allow us to uniquely express $\phi_2^{\theta^*}$in terms of $\phi_1^{\theta^*}$:

$$\begin{aligned}
\phi_2^{\theta^*}(t) &= \phi_2^{\theta^*}(1) + \int_t^1 F(\phi_1^{\theta^*}(t))f(t)dt \\
&= \int_t^1 F(\phi_1^{\theta^*}(t))f(t)dt
\end{aligned}$$

Third, we will use $\phi_2^{\theta^*}$ into the above initial value problem to obtain a unique continuous and increasing function $\phi_1^{\theta^*}$ that satisfies:

$$\begin{aligned}
\frac{d\phi_1^{\theta^*}(t)}{dt} &= \left(\frac{1-\int_t^1 F(\phi_1^{\theta^*}(t))f(t)dt}{F(\phi_1^{\theta^*}(t))F(t)+\int_t^1 F(\phi_1^{\theta^*}(t))f(t)dt}\right)^{n-1} \\
\phi_1^{\theta^*}(r_1) &= r_2
\end{aligned}$$

showing that Problem P has indeed a unique solution with the desired properties.

Lemma 4.4. *There exists a unique pair of continuous functions defined for all $t \in [r_1, 1]$ that solves the following initial value problem:*

$$
\begin{aligned}
\frac{d\phi(t)}{dt} &= \left(\frac{1-\phi_2(t)}{F(\phi_1(t))F(t)+\phi_2(t)}\right)^{n-1} \\
\frac{d\phi_2(t)}{dt} &= -F(\phi_1(t))f(t) \\
\phi_1(r_1) &= r_2 \\
\phi_2(r_1) &= \theta
\end{aligned}
$$

with $\theta \in (0, 1)$.

Proof. Consider the domain:

$$D = \left\{(t, y_1, y_2) \in \mathbb{R}^3 : c \le t \le 1; c \le y_1 < \infty; 0 \le y_2 < \infty\right\}$$

and the mapping $h : D \to \mathbb{R}^2$:

$$
\begin{aligned}
h(t, \mathbf{y}) &= [h_1(t, \mathbf{y}); h_2(t, \mathbf{y})] \\
h_1(t, \mathbf{y}(t)) &= \left(\frac{1-y_2}{F(y_1)F(t)+y_2}\right)^{n-1} \\
h_2(t, \mathbf{y}(t)) &= -F(y_1)f(t)
\end{aligned}
$$

where $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. Notice that the denominator of h_1 is positive on D because y_1 and t both bounded away from zero and F increasing ensure that $F(y_1)F(t) + y_2 \ge F^2(c) > 0$ for all $(t, \mathbf{y}) \in D$.

Claim 4.5. The mapping $h(t, \mathbf{y})$ is uniformly continuous with respect to t, bounded, and Lipschitz continuous in $\mathbf{y}$ on D.

Proof. As F is a continuous function, $F(s) > 0$ for $s > 0$, $y_1 \ge c > 0$, and $t \ge r_1 > 0$, $h_1(t, \mathbf{y})$ and $h_2(t, \mathbf{y})$ are continuous functions with respect to t. Moreover, t belongs to the compact interval $[r_1, 1]$ and by the Heine–Cantor theorem, both $h_1(t, \mathbf{y})$ and $h_2(t, \mathbf{y})$ must be uniformly continuous functions of t.

In what follows, if $x \in \mathbb{R}$ then $|x|$ denotes Euclidean norm in $\mathbb{R}$ whereas if $x \in \mathbb{R}^n$ $|x|$ denotes the 1-norm, i.e., $|x| := |x|_1 = \sum_{i=1}^{2} |x_1|$. We now show that there exists a constant $B > 0$ such that $|h(t, \mathbf{y})| \le B$ for all $(t, \mathbf{y}) \in D$. First, as $\bar{f}$ is a bound for f and $F(s) = 1$ for all $s \ge 1$, $|-F(y_1)f(t)| \le \bar{f}$ for all $(t, \mathbf{y}) \in D$. Second, for every $(t, \mathbf{y}) \in D$:

$$
\begin{aligned}
\frac{1-y_2}{F(y_1)F(t)+y_2} &\le \frac{1-y_2}{F^2(c)} \\
&\le \frac{1}{F^2(c)}
\end{aligned}
$$

because $F(y_1)F(t) + y_2 \geq F^2(c)$ and $y_2 \geq 0$. Hence, $({}^1/_{F^2(c)})^{n-1}$ is an upper bound for h_1. Similarly,

$$\frac{1-y_2}{F(y_1)F(t)+y_2} \quad \geq \quad \frac{1-y_2}{1+y_2}$$

since $F(s) \leq 1$ for all $s \in \mathbb{R}$. Let y_2 tend to ∞. It is not difficult to check that the right–hand side of this expression tends to -1. Therefore,

$$\begin{aligned} \frac{1-y_2}{F(y_1)F(t)+y_2} &\geq \frac{1-y_2}{1+y_2} \\ &> -1 \end{aligned}$$

and $h_1(t, \mathbf{y}) \leq (-1)^{n-1}$ for all $(t, \mathbf{y}) \in D$. Since $F(c) < 1$, $-\left(\frac{1}{F^2(c)}\right)^{n-1} < (-1)^{n-1}$ for any $n \geq 2$ finite. It follows that $|h_1(t, \mathbf{y})| \leq \left(\frac{1}{F^2(c)}\right)^{n-1}$ and $B = \max\left\{\left(\frac{1}{F^2(c)}\right)^{n-1}; \bar{f}\right\}$ is a bound for $h(t, \mathbf{y})$, $(t, \mathbf{y}) \in D$.

Finally, we show that $h(t, \mathbf{y})$ is (globally) Lipschitz continuous with respect to $\mathbf{y}$. To demonstrate this, we need to produce a positive constant $M > 0$, independent of $(t, \mathbf{y}) \in D$, satisfying:

$$|H(t; \mathbf{y}_1) - H(t; \mathbf{y}_2)| \leq M\, |\mathbf{y}_1 - \mathbf{y}_2|$$

for every $(t, \mathbf{y}_1) \in D$ and $(t, \mathbf{y}_2) \in D$. Simple computations yield:

$$\frac{\partial h_1(t,\mathbf{y})}{\partial y_1} = -(n-1)\left(\frac{1-y_2}{F(y_1)F(t)+y_2}\right)^{n-1}\left(\frac{(1-y_2)f(y_1)F(t)}{(F(y_1)F(t)+y_2)^2}\right) \tag{12}$$

$$\frac{\partial h_1(t,\mathbf{y})}{\partial y_2} = -(n-1)\left(\frac{1-y_2}{F(y_1)F(t)+y_2}\right)^{n-1}\left(\frac{F(y_1)F(t)+1}{(F(y_1)F(t)+y_2)^2}\right) \tag{13}$$

$$\frac{\partial h_2(t,\mathbf{y})}{\partial y_1} = -f(y_1)f(t) \tag{14}$$

It is almost immediate that $|-f(y_1)f(t)| \leq \bar{f}^2$ and hence, $\frac{\partial h_2(t,\mathbf{y})}{\partial y_1}$ is bounded by $\bar{f}^2$. Next,

$$\left|\frac{1-y_2}{F(y_1)F(t)+y_2}\right| \leq \frac{1}{F^2(c)}$$

and,

$$\begin{aligned} \left|\frac{(1-y_2)f(y_1)F(t)}{(F(y_1)F(t)+y_2)^2}\right| &= \left|\frac{(1-y_2)}{F(y_1)F(t)+y_2}\right|\left|\frac{f(y_1)F(t)}{F(y_1)F(t)+y_2}\right| \\ &\leq \left(\frac{1}{F^2(c)}\right)\left(\frac{\bar{f}}{F^2(c)}\right) \end{aligned}$$

and,

$$\left|\frac{F(y_1)F(t)+1}{(F(y_1)F(t)+y_2)^2}\right| \leq \frac{2}{F^2(c)}$$

for every $(t, \mathbf{y}) \in D$. Therefore,

$$\begin{aligned}
\left|\frac{\partial h_1(t,\mathbf{y})}{\partial y_1}\right| &\leq (n-1)\left(\frac{1}{F^2(c)}\right)^{n-1}\left(\frac{\bar{f}}{F^2(c)}\right) \\
&= m_1 \\
\left|\frac{\partial h_1(t,\mathbf{y})}{\partial y_2}\right| &\leq (n-1)\left(\frac{1}{F^2(c)}\right)^{n-1}\left(\frac{2}{F^2(c)}\right) \\
&= m_2 \\
\left|\frac{\partial h_2(t,\mathbf{y})}{\partial y_1}\right| &\leq \bar{f}^2 \\
&= m_3
\end{aligned}$$

and all these three derivatives are continuous and bounded functions in D, with bounds independent of $(t, \mathbf{y}) \in D$. Set $M = \max\{m_1, m_2, m_3\} > 0$. Then, standard arguments imply that:

$$|H(t, \mathbf{y}_1) - H(t, \mathbf{y}_2)| \leq M \, |\mathbf{y}_1(t) - \mathbf{y}_2(t)|$$

holds true for every $(t, \mathbf{y}_1) \in D$ and $(t, \mathbf{y}_2) \in D$. □

Consider the space $\mathcal{C}$ of continuous vector–valued functions $\phi = (\phi_1, \phi_2)$, $\phi_i : [r_1, 1] \to \mathbb{R}$, $i = 1, 2$, equipped with the sup norm, $\|\phi\| = \sup\{|\phi(t)|\,; t \in [r_1, 1]\}$. Let $\mathcal{D} = \{\phi \in \mathcal{C} : \|\phi - \phi_0\| \leq B; \phi_0 = (r_2, \theta)\} \subset \mathcal{C}$, where $B = \max\left\{\left(\frac{1}{F^2(c)}\right)^{n-1}; \bar{f}\right\}$ be the subset of continuous and increasing functions whose graph belong to D . Then, $\mathcal{D}$ is a complete metric space because $\mathcal{D}$ is a closed subset of a complete metric space. Define the operator K by:

$$\begin{aligned}
K\phi(t) &= \phi_0 + \int_{r_1}^{t} h(\tau, \phi(\tau))d\tau \\
\phi_0 &= (r_2, \theta)
\end{aligned}$$

Claim 4.6. K maps $\mathcal{D}$ into itself.

Proof. First, from claim 2 the mapping $h(t, \phi(t))$ is continuous in t on D. Since the integral sign preserves continuity, $K\phi(t)$ must also be continuous in $t \in [r_1, 1]$ when $\phi \in \mathcal{D}$. Second,

$$\begin{aligned}
|K\phi(t) - \phi_0(t)| &\leq \int_{r_1}^{t} |h(\tau, \phi(\tau))|\, d\tau \\
&\leq (t - r_1)B \\
&< B
\end{aligned}$$

and B is an upper bound of $|K\phi(t) - \phi_0(t)|$, $t \in [r_1, 1]$. Hence,

$$\begin{aligned}
\|K\phi - \phi_0\| &= \sup\{|K\phi(t) - \phi_0(t)|\,; t \in [r_1, 1]\} \\
&\leq B
\end{aligned}$$

and $K\mathbf{Y} \in \mathcal{D}$. Therefore, for any given $\phi \in \mathcal{D}$ the operator K delivers a continuous function that satisfies $\|K\phi - \phi_0\| \leq B$ and hence, $K\phi \in \mathcal{D}$. $\square$

Claim 4.7. Let $\mathcal{L}(t) = M \int_{r_1}^{t} d\tau$. For every $t \in [r_1, 1]$ the operator K satisfies:

$$|K^m\phi(t) - K^m\rho(t)| \leq \frac{\mathcal{L}(t)^m}{m!} \sup_{t\in[r_1,1]} |\phi(t) - \rho(t)| \tag{15}$$

where $m \in \mathbb{N}_0$, $K^m\phi(t) = K[K^{m-1}\phi](t)$, $K^0\phi(t) = \phi$, and $\phi \in \mathcal{D}$, $\rho \in \mathcal{D}$.

Proof. Set $m = 1$. Then,

$$\begin{aligned} |K\phi(t) - K\rho(t)| &\leq \int_{r_1}^{t} |h(\tau, \phi(\tau)) - h(\tau, \phi(\tau))| \, d\tau \\ &\leq \int_{r_1}^{t} M \, |\phi(\tau) - \rho(\tau)| \, d\tau \\ &\leq \mathcal{L}(t) \sup_{t\in[r_1,1]} |\phi(t) - \rho(t)| \end{aligned}$$

where the second inequality follows from claim 2. Next, suppose that inequality (15) holds for some $m > 1$. Then,

$$\begin{aligned} \left|K^{m+1}\phi(t) - K^{m+1}\rho(t)\right| &= |K[K^m\phi](t) - K[K^m\rho](t)| \\ &\leq \int_{r_1}^{t} |h(\tau, K^m\phi(\tau)) - h(\tau, K^m\rho)| \, d\tau \\ &\leq \int_{r_1}^{t} M \, |K^m\phi(\tau) - K^m\rho(\tau)| \, d\tau \\ &\leq \int_{r_1}^{t} M \frac{\mathcal{L}(\tau)^m}{m!} \sup_{s\in[r_1,\tau]} |\phi(s) - \rho(s)| \, d\tau \\ &= \int_{r_1}^{t} \mathcal{L}'(\tau) \frac{\mathcal{L}(\tau)^m}{m!} \sup_{s\in[r_1,\tau]} |\phi(s) - \rho(s)| \, d\tau \\ &= \frac{\mathcal{L}(t)^{m+1}}{(m+1)!} \sup_{t\in[r_1,1]} |\phi(t) - \rho(t)| \end{aligned}$$

where the third line follows from claim 2, the fourth line follows from the induction hypothesis, and the sixth line follows from integration by substitution. This shows that (15) also holds for $m + 1$ and hence, it must hold for any $m \in \mathbb{N}_0$. $\square$

Let $\theta_m = \frac{\mathcal{L}(1)^m}{m!}$. Observe that $\sum_{m=1}^{\infty} \theta_m < \infty$ and hence, this sum converges. Moreover,

$$\begin{aligned} |K^m\phi(t) - K^m\rho(t)| &\leq \frac{\mathcal{L}(t)^m}{m!} \sup_{t\in[r_1,1]} |\phi(t) - \rho(t)| \\ &\leq \theta_m \sup_{t\in[r_1,1]} |\phi(t) - \rho(t)| \end{aligned}$$

and $\theta_m \sup_{t\in[r_1,1]} |\phi(t) - \rho(t)|$ is a bound for $|K^m\phi(t) - K^m\rho(t)|$. Therefore,

$$\|K^m\phi(t) - K^m\rho(t)\| \leq \theta_m \|\phi(t) - \rho(t)\|$$

and since $\theta_m \to 0$ as $m \to \infty$, there is some m^* such that $K^{m^*}\phi$ is a contraction. Therefore, Theorem 9-9 in (Kreider, Kuller, and Ostberg, 1968) ensures the existence of a unique fixed ϕ^θ of K. Since ϕ^θ is a fixed point of K its graph must belong to D, which implies that ϕ^θ is defined for all $t \in [r_1, 1]$. □

From lemma 2 there exists a unique vector–valued function $\phi^\theta = (\phi_1^\theta, \phi_2^\theta)$ that satisfies:

$$\frac{d\phi_1^\theta(t)}{dt} = \left(\frac{1-\phi_2^\theta(t)}{F(\phi_1^\theta(t))F(t)+\phi_2^\theta(t)}\right)^{n-1} \tag{16}$$

$$\frac{d\phi_2^\theta(t)}{dt} = -F(\phi_1^\theta(t))f(t) \tag{17}$$

$$\phi_1^\theta(r_1) = r_2 \tag{18}$$

$$\phi_2^\theta(r_1) = \theta \tag{19}$$

Consider the relation:

$$\phi_2^\theta(1) = 0$$

We want to show that there exists a unique $\theta^* \in (0,1)$ that makes the above relation hold true. We begin by showing existence of such root. Integrate Eq. (17) between r_1 and t to obtain:

$$\phi_2^\theta(t) = \theta - \int_{r_1}^{t} F(\phi_1^\theta(\tau))f(\tau)d\tau$$

Since $\phi_2^\theta(r_1) = \theta < 1$, and $\phi^\theta(t) < \theta_2^\theta(r_1)$ because of Eq. (17), $\phi^\theta(t) < 1$ for all $t \in [r_1, 1]$. Hence, $1 - \phi_2^\theta(t) > 0$ and ϕ_1^θ is increasing with respect to $t \in [r_1, 1]$. Moreover, F is increasing and $\phi_1^\theta(t) \geq c$; then $F(\phi_1^\theta(t)) > F(c)$, $t \in (r_1, 1]$. Let $\tilde{\theta}$ be any value of θ living in the open interval $(0, F(c)(1 - F(r_1))$. Then,

$$\begin{aligned}
\phi_2^{\tilde{\theta}}(1) &= \tilde{\theta} - \int_{r_1}^{1} F(\phi_1^{\tilde{\theta}}(\tau))f(\tau)d\tau \\
&< \tilde{\theta} - \int_{r_1}^{1} F(c)f(\tau)d\tau \\
&= \tilde{\theta} - F(c)(1 - F(r_1)) \\
&< F(c)(1 - F(r_1) - 1 + F(r_1)) \\
&= 0
\end{aligned}$$

and $\phi_2^\theta(1)$ must be negative for values of θ close to zero. Likewise, let $\hat{\theta}$ live in the open interval $(1 - F(r_1), 1)$. Then,

$$
\begin{aligned}
\phi_2^{\hat\theta}(1) &= \hat\theta - \int_{r_1}^{1} F(\phi_1^{\hat\theta}(\tau))f(\tau)d\tau \\
&> \hat\theta - \int_{r_1}^{1} f(\tau)d\tau \\
&= \hat\theta - (1-F(r_1)) \\
&> (1-F(r_1)) - (1-F(r_1)) \\
&= 0
\end{aligned}
$$

since $F(s) = 1$ for all $s \geq 1$ and hence, $F(\phi_1^\theta(t)) \leq 1$ for all $t \in [0,1]$ and $\theta \in (0,1)$. Therefore, $\phi_2^\theta(1)$ must be negative for values of θ close to zero and positive for values of θ close to one, implying the existence of some $\theta^* \in (0,1)$ such that $\phi_2^{\theta^*}(1) = 0$. Uniqueness follows from the next claim.

Claim 4.8. ϕ_1^θ is decreasing and ϕ_2^θ is increasing in $\theta \in (0,1)$ for every $t \in [r_1, 1]$.

Proof. The proof of the claim is by contradiction. Standard considerations in the theory of differential equations (e.g. Theorem 9-12 in (Kreider, Kuller, and Ostberg, 1968)) ensures that $\phi^\theta = (\phi_1^\theta, \phi_2^\theta)$ is continuously differentiable with respect to $\theta \in (0,1)$. Furthermore, $\delta^\theta = (\delta_1^\theta, \delta_2^\theta)$, $\delta_i^\theta = \frac{d\phi_i^\theta(t)}{d\theta}$, must solve the following initial value problem:

$$
\begin{aligned}
\frac{d\delta_1^\theta(t)}{dt} &= -(n-1)\left(\frac{1-\phi_2^\theta(t)}{F(\phi_1^\theta(t))F(t)+\phi_2^\theta(t)}\right)^{n-2} \times \\
&\quad \left(\frac{\delta_1^\theta(t)(1-\phi_2^\theta(t))f(\phi_1^\theta(t))F(t)+\delta_2^\theta(t)(F(\phi_1^\theta(t))F(t)+1)}{(F(\phi_1^\theta(t))F(t)+\phi_2^\theta(t))^2}\right) && (20) \\
\frac{d\delta_2^\theta(t)}{dt} &= -f(\phi_1^\theta(t))f(t)\delta_1^\theta(t) && (21) \\
\delta_1^\theta(r_1) &= 0 && (22) \\
\delta_2^\theta(r_1) &= 1 && (23)
\end{aligned}
$$

Suppose that there exists some $t \in [r_1, 1]$ such that $\delta_1^\theta(t) > 0$. Since $\delta_1^\theta(r_1) = 0$ and $\frac{d\delta_1^\theta(r_1)}{dt} < 0$, there is some $\epsilon > 0$ such that $\delta_1^\theta(r_1) = 0$ and $\delta_1^\theta(t) < 0$ for all $t \in [r_1, \epsilon]$. Thus, if $\delta_1^\theta(t) > 0$ at some t^*, there must be the case that $t^* > r_1$, $\delta_1^\theta(t) < 0$ for $t \in (r_1, t^*)$, and $\delta_1^\theta(t^*) = 0$. This requires the slope of δ_1^θ at t^* to be positive because $\delta_1^\theta(r_1) = 0$ and δ_1^θ must cross the x-axis at t^* from below. Evaluating $\frac{d\delta_1^\theta(t)}{dt}$ at $t = t^*$ yields:

$$
\frac{d\delta_1^\theta(t^*)}{dt} =
$$

$$
-(n-1)\left(\frac{1-y_2^\theta(t^*)}{F(y_1^\theta(t^*))F(t^*)+y_2^\theta(t^*)}\right)^{n-2}\left(\frac{\delta_2^\theta(t^*)(F(y_1^\theta(t^*))F(t^*)+1)}{(F(y_1^\theta(t^*))F(t^*)+y_2^\theta(t^*))^2}\right) \tag{24}
$$

I claim that this expression is strictly negative. Integration of Eq. 17 between r_1 and t yields:

$$\phi_2^\theta(t) = \theta - \int_{r_1}^{t} F(\phi_1^\theta(\tau))f(\tau)d\tau$$

and hence,

$$\begin{aligned}\left(\frac{1-\phi_2^\theta(t^*)}{F(\phi_1^\theta(t^*))F(t^*)+\phi_2^\theta(t^*)}\right)^{n-2} &= \left(\frac{1-\theta+\int_{r_1}^{t^*} F(\phi_1^\theta(\tau))f(\tau)d\tau}{F(\phi_1^\theta(t^*))F(t^*)+\theta+\int_{r_1}^{t^*} F(\phi_1^\theta(\tau))f(\tau)d\tau}\right)^{n-2} \\ &> 0\end{aligned}$$

because F and $\phi_1^\theta(t)$ increasing in $t \in [r_1, 1]$, and $\phi_1^\theta(t) \geq c$, imply $\int_{r_1}^{1} F(\phi_1^\theta(\tau))f(\tau)d\tau > F(c)(1-F(r_1)) > 0$. Second, integration of Eq. (21) between r_1 and t gives:

$$\delta_2^\theta(t) = 1 - \int_{r_1}^{t} f(\phi_1^\theta(\tau))\delta_1^\theta(\tau)f(\tau)d\tau$$

since $\delta_2^\theta(r_1) = 1$. Since $\delta_1^\theta(t) < 0$ for all $t \in (r_1, t^*)$ and $f > 0$, $\delta_2^\theta(t^*) > 0$. Therefore, both terms within brackets in Eq. (24) are positive, from where it follows that $\frac{d\delta_1^\theta(t^*)}{dt} < 0$. This creates the contradiction needed to complete the proof of the claim. □

Hence, there must exist a unique θ^* such that $\phi^{\theta^*}(1) = 0$. Let $\phi^{\theta^*} = (\phi_1^{\theta^*}, \phi_2^{\theta^*})$ be the unique solution to our initial value problem when θ takes the value θ^*. Integrating Eq. (17) between t and one yields:

$$\begin{aligned}\phi_2^{\theta^*}(t) &= \phi_2^{\theta^*}(1) + \int_t^1 F(\phi_1^{\theta^*}(\tau)f(\tau)d\tau \\ &= \int_t^1 F(\phi_1^{\theta^*}(\tau)f(\tau)d\tau\end{aligned}$$

because $\phi^{\theta^*}(1) = 0$. Therefore, if we let $\phi^{\theta^*}(t) := z^*(t)$, the function z^* must be the unique increasing and continuous function that satisfies:

$$\begin{aligned}\frac{dz^*(t)}{dt} &= \left(\frac{1-\int_t^1 F(z^*(t))f(t)dt}{F(z^*(t))F(t)+\int_t^1 F(z^*(t))f(t)dt}\right)^{n-1} \\ z^*(r_1) &= r_2\end{aligned}$$

and z^* is the unique solution to Problem P. □

Proof of Proposition 3.8

Since ρ^* and $\hat{\rho}$ are fixed points of the operator T, they belong to $\mathcal{R}$ and therefore, they are continuous and nondecreasing functions of $v_1 \in [0,1]$. Moreover, $r_1 < \hat{r}_1$, and $[0, r_1]$ is a proper subset of $[0, \hat{r}_1]$. Thus, from part (i) of lemma 4.1, $\rho^*(v_1) = \hat{\rho}(v_1) = r_2$ for $v_1 \in [0, r_1]$, and $\rho^*(v_1) \geq \hat{\rho}(v_1) = r_2$ for $v_1 \in [r_1, \hat{r}_1]$ because ρ^* is a nondecreasing function of v_1. Suppose that there exists some $\tilde{v}_1 \in (r_1, \hat{r}_1)$ such that $\rho^*(\tilde{v}_1) = \hat{\rho}(\tilde{v}_1)$. As $\tilde{v}_1 < \hat{r}_1$, $\hat{\rho}(\tilde{v}_1) = r_2$ implying that $\rho^*(\tilde{v}_1) = r_2$ and $\mathcal{U}_2(\rho^*(\tilde{v}_1), \rho^*) = \mathcal{U}_1(\tilde{v}_1; \rho^*) = 0$ from part (ii) of lemma 4.1. However, $\tilde{v}_1 > r_1$ and $\mathcal{U}_1$ is increasing in v_1 for $v_1 > r_1$ and hence, $\mathcal{U}_1(\tilde{v}_1; \rho^*) > \mathcal{U}_1(r_1; \rho^*) = 0$, a contradiction. This implies that $\rho^*(v_1) > \hat{\rho}(v_1)$ for all $v_1 \in (r_1, \hat{r}_1]$. Next, from part (ii) of lemma 4.1, ρ^* must satisfy $\mathcal{U}_2(\rho^*(v_1); \rho^*, r_2) = \mathcal{U}_1(v_1; \rho^*, r_1)$ at every $v_1 \in (r_1, \bar{v}_1)$. Obviously, if $\bar{v}_1 \leq \hat{r}_1$ then $\rho^*(v_1) \geq \hat{\rho}(v_1)$ for all $v_1 \in [0,1]$, $\rho^*(v_1) > \hat{\rho}(v_1)$ for $v_1 \in (r_1, \hat{r}_1)$, and the proof of the proposition is finished. Hence, suppose that $\bar{v}_1 > \hat{r}_1$.To prove the proposition is sufficient to show that ρ^* and $\hat{\rho}$ do not intersect in the interval $(\hat{r}_1, \bar{v}_1)$. Suppose that this is not true, i.e., suppose that there exists some $\tilde{v}_1 \in (\hat{r}_1, \bar{v}_1)$ such that $\rho^*(\tilde{v}_1) = \hat{\rho}(\tilde{v}_1)$. Let $I = \{v_1 \in (\hat{r}_1, \bar{v}_1) : \rho^*(v_1) = \hat{\rho}(v_1)\}$. If I is empty, then $\rho^*(v_1) = \hat{\rho}(v_1)$ happens only at points in $[0, r_1]$ or points where ρ^* and $\hat{\rho}$ are both equal to one. Hence, suppose that I is not empty. Let $\underline{v}_1 = \inf\{v_1 \in I\}$, i.e., the lowest value of v_1 at which ρ^* and $\hat{\rho}$ intersect. There are two cases of interest.

Case 1. $\rho^*(v_1) < \hat{\rho}(v_1)$ for all $v_1 \in (\underline{v}_1, \bar{v}_1)$. In this case, we must have:

$$\left[1 - \int_{\underline{v}_1}^{1} F(\rho^*(\xi))dF(\xi)\right] > \left[1 - \int_{\underline{v}_1}^{1} F(\hat{\rho}(\xi))dF(\xi)\right] \tag{25}$$

and,

$$\left[F(\rho^*(\underline{v}_1))F(\underline{v}_1) + \int_{\underline{v}_1}^{1} F(\rho^*(\xi))dF(\xi)\right] < \left[F(\hat{\rho}(\underline{v}_1))F(\underline{v}_1) + \int_{\underline{v}_1}^{1} F(\hat{\rho}(\xi))dF(\xi)\right] \tag{26}$$

because $\rho^*(v_1) < \hat{\rho}(v_1)$ holds for for all $v_1 \in (\underline{v}_1, \bar{v}_1)$. Since $\rho^*(v_1) > \hat{\rho}(v_1)$ for all $v_1 \in [r_1, \underline{v}_1)$, $\rho^*(\underline{v}_1) = \hat{\rho}(\underline{v}_1)$, and $\rho^*(v_1) < \hat{\rho}(v_1)$ for all $v_1 \in (\underline{v}_1, \bar{v}_1)$, it must be the case that $\hat{\rho}$ cuts ρ^* from below at $\underline{v}_1$ because both fixed points are continuous functions of v_1. This requires the slope of ρ^* to be lower than the slope of $\hat{\rho}$ at $\underline{v}_1$. By part (ii) of lemma 4.1, the slope of ρ^* at $\underline{v}_1$ can be estimated as the ratio of the probabilities of trading with seller 1 and seller 2 respectively when the bidder's type is equal to $(v_1, \rho^*(v_1))$. However, inequalities 25 and 26 imply that the slope of ρ^* is strictly greater than the slope of $\hat{\rho}$ at $\underline{v}_1$, a contradiction.

Case 2. $\rho^*(v_1) > \hat{\rho}^*(v_1)$ for all $v_1 \in (\tilde{v}_1, \bar{v}_1)$. Similar to the case above, $\rho^*(\tilde{v}_1) = \hat{\rho}^*(\tilde{v}_1)$ and $\rho^*(v_1) > \hat{\rho}^*(v_1)$ for all $v_1 \in (\tilde{v}_1, \bar{v}_1)$ requires the slope of ρ^* to be greater than the slope of $\hat{\rho}^*$at $\tilde{v}_1$. However, with $\rho^*(v_1) > \hat{\rho}^*(v_1)$ for all $v_1 \in (\tilde{v}_1, \bar{v}_1)$, inequalities 25 and 26 are reversed and hence, $\rho'^*(\tilde{v}_1) < \hat{\rho}'^*(\tilde{v}_1)$, a contradiction.

Proof of Proposition 3.9

Let $Z_1(t_1;\rho^*) = [G_1^*(t_1)]^n + n\,[G_1^*(t_1)]^{n-1}\,[1 - G_1^*(t_1)]$. Then,

$$dZ_1 = n(n-1)\,[G_1^*(t_1)]^{n-2}\,[1 - G_1^*(t_1)]$$

and,

$$V_1^+(r_1, r_2; \rho^*) = nr_1 q^*\,[1-q^*]^{n-1} + \int_{r_1}^{1} t_1 dZ_1(t_1;\rho^*) - c$$

Use integration by parts:

$$\begin{aligned}\int_{r_1}^{1} t_1 dZ_1 &= t_1 Z_1(t_1)|_{r_1}^{1} - \int_{r_1}^{1} Z_1(t_1)dt_1 \\ &= 1 - r_1 Z_1(r_1) - \int_{r_1}^{1} Z_1(t_1)dt_1\end{aligned}$$

Corollary 4.9. *Consider the class of continuation games in which $c \leq r_1 = r_2 < 1$. Then,*

$$\rho^*(v) = \begin{cases} r_1 & \text{if } v_1 < r_1 \\ v & \text{if } v_1 \geq r_1 \end{cases}$$

is the unique fixed point of T.

Proof. If $v_1 < r_1$ then $\mathcal{U}_1(v_1;\rho^*, r_1) = 0$ and hence $T\rho^*(v_1) = r_2 = r_1 = \rho^*(v_1)$. Hence, consider the case in which $v_1 > r_1$. Let $\varphi(v) = v$ for all $v \in [0,1]$. Then,

$$\begin{aligned}&\left(F(\varphi(t))F(t) + \int_t^1 F(\varphi(\hat{t}))f(\hat{t})d\hat{t}\right)^{n-1} \\ &= \left(F^2(t) + \int_t^1 F(\hat{t})f(\hat{t})d\hat{t}\right)^{n-1} \\ &= \left(F^2(t) - \frac{F^2(t)}{2} + \frac{1}{2}\right)^{n-1} \\ &= \left(\frac{1}{2} + \frac{F^2(t)}{2}\right)^{n-1} \\ &= \left(1 - \int_t^1 F(\varphi(t))f(t)dt\right)^{n-1}\end{aligned}$$

whenever $t \geq r_1 = r_2$. Therefore, the function φ satisfies:

$$\begin{aligned}\left(F(\varphi(t))F(t) + \int_t^1 F(\varphi(\hat{t}))f(\hat{t})d\hat{t}\right)^{n-1} \dot{\varphi}(t) &= \left(1 - \int_t^1 F(\varphi(\hat{t}))f(\hat{t})d\hat{t}\right)^{n-1} \\ \varphi(r_1) &= r_2\end{aligned}$$

and φ is a solution to problem P in lemma 4.3. Therefore, ρ^* must be the unique fixed point of T and thus, the unique symmetric equilibrium strategy for this class of continuation games. □

From the previous corollary, if reserve prices are both equal to production cost, then $\rho^*(v) = v$ is the unique continuation equilibrium function. Then, payoff becomes:

$$V_1^+(c, c; \rho^*) = ncq^c [1 - q^c]^{n-1} + \int_c^1 t_1 dZ_1^c(t_1) - c$$

Consider any possible admissible reserve price within $[0, 1]$. Let this price be $r_1 > c$. Then,

$$V_1^+(\hat{r}_1, c; \hat{\rho}) = nr_1 q [1 - q]^{n-1} + \int_{r_1}^1 t_1 dZ_1(t_1) - c$$

where it is assumed that bidders select trading partners using the function $\hat{\rho}$ that is a fixed point of T when reserve prices are (r_1, c). From proposition 3.8, $\rho(v_1) \le v_2$ with strict inequality actually. Hence, $Z_1 > \hat{Z}_1$ for all $v_1 \in [0, 1]$. Moreover,

$$V_1^+(c, c; \rho^*) = ncq^c(1 - q^c)^{n-1} + \int_c^{r_1} t_1 dZ_1^c(t_1) + \int_{r_1}^1 t_1 dZ_1^c(t_1) - c$$

expected value in the first integral should be greater than the same with just Z. Since $Z(r_1) = (1 - q)^n + nq[1 - q]^{n-1}$ and $Z^c(c) = (1 - q^c)^n + nq^c[1 - q^c]^{n-1}$, then $nq[1 - q]^{n-1} = Z(r_1) - (1 - q)^n$ and $nq^c[1 - q^c]^{n-1} = Z^c(c) - (1 - q^c)^n$. Integration by parts of $\int_c^{r_1} t_1 dZ_1^c(t_1)$ yields:

$$\int_c^{r_1} t_1 dZ_1^c(t_1) = r_1 Z^c(r_1) - cZ^c(c) - \int_c^{r_1} Z_1^c(t_1) dt_1$$

Therefore,

$$ncq^c(1 - q^c)^{n-1} - nr_1 q [1 - q]^{n-1} + \int_c^{r_1} t_1 dZ_1^c(t_1) =$$

$$= cZ^c(c) - c(1 - q^c)^n - r_1 Z(r_1) + r_1(1 - q)^n + \left[r_1 Z^c(r_1) - cZ^c(c) - \int_c^{r_1} Z_1^c(t_1) dt_1 \right]$$

$$= -c(1 - q^c)^n - r_1 [Z(r_1) - Z^c(r_1)] + r_1(1 - q)^n - \int_c^{r_1} Z_1^c(t_1) dt_1$$

$$> c [(1 - q)^n - (1 - q^c)^n] - r_1 [Z(r_1) - Z^c(r_1)] - \int_c^{r_1} Z_1^c(t_1) dt_1 \text{ because } r_1 > c$$

Let $n = 2$. Then,

$$V_1^+(c, c; \rho^*) - V_1^+(r_1, c; \hat{\rho}) =$$

$$2cq^c(1-q^c) + \int_c^{r_1} t_1 dZ_1^c(t_1) + \int_{r_1}^1 t_1 dZ_1^c(t_1) - 2r_1 q\,[1-q] - \int_{r_1}^1 t_1 dZ_1(t_1)$$

$$= 2cq^c(1-q^c) - 2r_1 q\,[1-q] + \int_c^{r_1} t_1 dZ_1^c(t_1) + \int_{r_1}^1 \{Z - Z^c\} dt_1 + r_1\{Z(r_1) - Z^c(r_1)\}$$

after intregration by parts;

$$> 2cq^c(1-q^c) - 2r_1 q^c\,[1-q^c] + \int_c^{r_1} t_1 dZ_1^c(t_1) + r_1\{Z(r_1) - Z^c(r_1)\}$$

because $\int_{r_1}^1 \{Z - Z^c\} dt_1 > 0$and

$n = 2$implies that $q^c(1-q^c) > q(1-q)$

$$= 2(c - r_1)q^c(1-q^c) + r_1 Z^c(r_1) - cZ^c(c) - \int_c^{r_1} Z_1^c(t_1) dt_1 + r_1\{Z(r_1) - Z^c(r_1)\}$$

after integration by parts;

$$> 2(c - r_1)q^c(1-q^c) + (r_1 - c)Z^c(c) - \int_c^{r_1} Z_1^c(t_1) dt_1 + r_1\{Z(r_1) - Z^c(r_1)\}$$

because $Z^c(r_1) > Z^c(c)$

$$\geq (1-q^c)^2 - \int_c^{r_1} Z_1^c(t_1) dt_1 + r_1\{Z(r_1) - Z^c(r_1)\}$$

because $Z^c(c) \geq Z^c(0) = G^2(0) + 2G(0)(1 - G(0)) = (1-q^c)^2 + 2q^c(1-q^c)$

$$> (1-q^c)^2 - \int_c^{r_1} Z_1^c(t_1) dt_1 + cZ^c(c) - r_1 Z^c(r_1)$$

because $r_1 Z(r_1) > r_1 Z^c(r_1) > cZ^c(c)$

$$= (1-q^c)^2 + t_1 Z^c(t_1)|_c^{r_1} - \int_c^{r_1} Z_1^c(t_1) dt_1$$

$$= (1-q^c)^2 + \int_c^{r_1} t_1 dZ^c(t_1)$$

after integrating by parts;

$$> 0$$

Therefore, $V^c(c,c) \geq V_1(r_1, c)$ for all $r_1 \in [c, 1]$ when $n = 2$. A similar argument should apply to seller 2. Hence, there is no profitable deviation and (c, c) is the unique equilibrium of the game.

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