

STRATEGY-PROOF AND EFFICIENT SCHEDULING

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ABSTRACT. I construct a class of strategy-proof and Pareto efficient fair division mechanisms among players with dichotomous preferences. Such mechanisms offer a flexible platform to allocate a potentially unbounded continuum of heterogeneous goods according to various distributional objectives such as envy-freeness or arbitrary guaranteed shares of reported demand. The characterization of the strategy-proofness of the mechanisms also serves as a sufficient condition for the non-inferiority of consumer's demand subject to arbitrary submodular ceilings on quantities supplied under fixed prices.

Keywords: Fair division, strategy-proof, lattice programming, monotone comparative statics;

JEL Codes: D47, D63, D82.

1. INTRODUCTION

In this paper, I study the fair division of heterogeneous goods without monetary transfers among players with dichotomous preferences. Real-life applications of such division problems include the assignment of houses, courses, and schools seats to students, and arrival slots at airports to airlines, just to name a few. When players only see the goods as either acceptable or unacceptable, more closely-related examples include scheduling workers to a workstation¹ and circulating the books in a public library among potential readers². From now on, I shall refer to this problem as the library problem: the players will be referred to as readers and the goods to be allocated are reading times.

Common circulation policies such as first- or last-come-first-serve do not provide the readers with sufficient incentives to truthfully demonstrate their need for reading: for example, in the first-come-first-serve mechanism, if a second reader's demand immediately follows the first reader's, the second reader has a strong

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¹Specifically, consider a firm using only labor as its input. The firm owns a single workstation. Workers have to work on the workstation to produce output and only one worker can work on the workstation at any time. Different workers are available for work at different times. Workers all exhibit (potentially different) positive but decreasing marginal product and the wage rate is fixed. The firm aims to maximize the sum of all workers' output and the workers want to maximize their own wage incomes. All workers have report the time they are available for work and the firm has to decide which worker gets to work on the workstation. I thank Balázs Szentes for suggesting this example.

²Among often-cited reasons for excluding monetary transfers in these settings are morality and entitlement of access to public services. For example, extra room and board or library fees will be considered impairing students' rights since students have already paid their tuitions, which grants them access to housing facilities or library books.

incentive to check out the book just before the first reader does. To eliminate such incentives, I set out to construct a class of strategy-proof and Pareto efficient deterministic offline³ interval scheduling mechanisms.

I first formulate the library as a benevolent social planner aiming to maximize the Nash collective utility⁴ of the readers. The constraints faced by the library describe the maximum total reading time any group of readers can be assigned to. The library first selects the reading times to be allocated to each reader by solving this constrained maximization problem, then chooses a schedule according to the selected reading times. Pareto-efficiency is a direct result of maximization. Strategy-proofness is established by a key non-inferiority condition which states that, if a group of readers are allocated all the time they demand as a group, then a group member’s increased demand never decreases the reading time allocated to any group member. This class of mechanisms is flexible enough to accommodate rigid conditions such as proportionality and envy-freeness where readers are to be treated symmetrically and anonymously or more general criteria such as arbitrary guaranteed shares of demonstrated demand when anonymity is less suitable.

1.1. Related literature. By focusing on offline interval scheduling mechanisms, I reside this paper in the literature of the cake cutting problem. Cake cutting is an often used analogy for the division of a continuum of heterogeneous objects among a set of players with different preferences. Fairness is a natural objective in such problems. Fair division problems are first introduced by [Steinhaus \(1948\)](#). Fairness, an initially normative notion, is later formalized to include Pareto efficiency and envy-freeness—both positive concepts—simultaneously. Envy-freeness simply means that no player should prefer any other player’s allocation to her own. Early explorations in economics of fair division include [Foley \(1967\)](#) (who first defined the notion of “envy-freeness”) and [Varian \(1974\)](#) for divisible goods and [Maskin and Feiwel \(1987\)](#) (indivisible goods with same number of goods and players) and [Alkan et al. \(1991\)](#) (indivisible goods with transfers).

This paper also benefits from a more recent literature on random assignments. This literature studies the probabilistic assignment problem of a finite set of goods among players who can demonstrate cardinal or ordinal preferences, a discrete version of the current paper with more general scopes on the players’ preferences. Among the most prominent work with cardinal preferences are [Hylland and Zeckhauser \(1979\)](#) and [Zhou \(1990\)](#). Since [Zhou \(1990\)](#), a fair share of the literature offers negative results on the compatibility of efficiency, fairness, and strategy-proofness. A shift in focus from cardinal to ordinal preferences and mechanisms happened around the turn of the century, with early explorations including [Crès and Moulin \(2001\)](#). [Bogomolnaia and Moulin \(2001\)](#) developed a widely popular and extensively studied mechanism called “Probabilistic Serial” (PS), followed by a work with dichotomous preferences in [Bogomolnaia and](#)

³“Offline” means that all readers have to report to the library their intervals of demand before the allocation plan of the book for any given time is determined. That is, the mechanism is static both in terms of the set of players and their preferences.

⁴Recall that a Nash collective utility function is an additive, continuous, increasing, and concave function of the utilities of the players. See [Moulin \(2004\)](#).

Moulin (2004) closely related to this current research. PS fares very well in terms of efficiency and fairness standards but imperfectly on the incentive front. To this end, Kojima and Manea (2010) shows that PS recovers strategy-proofness in sufficiently large markets. For more recent and complete characterizations of PS and related literature, see Hashimoto et al. (2014) and the references therein.

The cake cutting problem has also attracted interests from political scientists, mathematicians, and computer scientists. In addition to the notion of fairness described above, proportionality (in computer science) or fair share (in economics) is another basic component of fairness. Proportionality requires that each reader/cake eater must be assigned at least $1/n$ of the reading time/piece of cake she desires. In the two-person case, envy-freeness is equivalent to proportionality while only the former implies that latter going beyond two people. Classical cake-cutting algorithms that always result in proportional cuts include cut-and-choose, Dubins-Spanier and Even-Paz (the “moving knife” algorithm), and Selfridge-Conway. Brams and Taylor (1996) and Robertson and Webb (1998) are two classics surveying the literature and Procaccia (2013a) contains a great up-to-date review.

A latest development in the computer science literature on cake cutting is the attention to strategy-proof (or truthful, in computer science parlance) mechanisms. Initial efforts by Brams et al. (2006) and Brams et al. (2008) propose a very weak strategy-proofness notion—they only require truth-telling to be not always dominated, in contrast to the strategy-proofness in economics dictating that truth-telling must be at least a weakly dominant strategy. To the end of reconciling the economic notion of strategy-proofness into the cake cutting problem, Chen et al. (2013) is one of the first (another one is Mossel and Tamuz (2010) with a stochastic—instead of deterministic—algorithm) in proposing a strategy-proof and Pareto efficient cake-cutting algorithm. Procaccia (2013b) offers a comprehensive survey of cake cutting algorithms in computer science whose focus is fairness concepts but also includes work with strategic considerations.

1.2. A brief preview. The key insight in the construction of strategy-proofness in cake cutting problems with piecewise uniform value densities⁵ can be conveyed by the simplest example of the two-agent case. Let l_1 and l_2 be the lengths of the intervals of demand for agents 1 and 2, respectively. Let l be the length of the union of the two agents’ intervals of demand—hence, $l_1 + l_2 \geq l \geq l_i, i \in \{1, 2\}$, and $l < l_1 + l_2$ implies overlapping demand between the two agents. Consider the following approach to allocating the cake, which turns out to be strategy-proof:

- If $l_i \leq l/2$, then agent i gets the entire intervals she demanded;
- Otherwise, both agents split the union of their demands equally.

⁵Cake eaters only care about the length of the desired pieces they get but not the composition of the pieces.

Chen et al. (2013), which focuses on fairness and truthfulness, extends this two-agent mechanism into a sequence of “subroutines” (the Mechanism 1 in the original paper) applicable to any fixed and finite number of agents that result in envy-free allocations. *Mechanism 1* roughly works as follows in the current context of scheduling: given any unallocated reading time and the group of readers who have not been assigned any reading time, select the subgroup of readers whose *average* demand—defined as the length of the union of their demands divided by the number of readers within the subgroup ($l/2$ in the two-reader example above)—is the smallest and allocate the union of their demand to this subgroup (and break ties arbitrarily). Start with all the readers before any reading time is allocated and repeat this subroutine described above until all readers are allocated with some reading time.

Alternatively, observe that the strategy-proofness of the two-agent mechanism simply stems from the fact that the reading times of the two readers tend to increase or decrease simultaneously according to any reader’s (reported) length of demand. This intuition is formalized in Proposition 3 and essentially motivates the construction of the class of mechanisms in this paper. In extending this intuition to more than two readers, I establish a non-inferiority condition extending the comparative statics results of constrained optimization problems in Quah (2007). In the next section, I introduce the setup of the library problem. In section 3, I provide results on characterizing feasible schedules, the Pareto efficiency of the mechanism, the non-inferiority condition, and strategy-proofness. Discussions are made in section 4, comparing this paper to Chen et al. (2013) and previous work in monotone comparative statics with lattice programming. Concluding remarks and future research suggestions are included in section 5. The appendix presents omitted proofs.

2. THE MODEL

A public library owns a single copy of a book that a fixed and finite number n of readers, indexed by i , want to read in continuous time starting at time zero. Let N represent the set of all readers. The library determines and enforces the circulation of the book. A **schedule** of circulation is a profile $\mu \equiv (\mu_1, \mu_2, \dots, \mu_n)$ of functions where

$$\mu_i : \mathbb{R}_+ \rightarrow \{0, 1\}, \quad i \in N,$$

where \mathbb{R}_+ is the set of nonnegative real numbers and $\mu_i(\cdot)$ is integrable with respect to the Lebesgue measure for all i . $\mu_i(r) = 1$ if reader i is assigned the book at time r according to schedule μ . Otherwise, $\mu_i(r) = 0$. The book is indivisible and cannot be read by more than one reader at the same time. Hence, a schedule is **feasible** if and only if

$$\sum_{i \in N} \mu_i(r) \leq 1, \quad \forall r \in \mathbb{R}_+.$$

Let \mathcal{M} be the set of all feasible schedules, to which I restrict my attention from now on.

Let Θ be the set of all *finite* unions of bounded open intervals on the nonnegative real line. Each reader i is endowed at time zero with a type $\theta_i \in \Theta$. I may also refer to θ_i as reader i 's intervals of demand henceforth. Let $\theta \in \Theta^n$ represent the profile of the readers' types and $\theta_{-i} \in \Theta^{n-1}$ the profile of all readers' types except for reader i 's. I may also use the notation

$$(\theta_i, \theta_{-i}) = \theta$$

to emphasize reader i 's type in the profile of all readers' types. Since my objective is to construct a strategy-proof mechanism, no specification is required on the distribution of types and belief systems of the players. In other words, the model is prior-free.

All readers are self-interested and rational and only strictly prefer more reading time within their own intervals of demand, without distinguishing between any two assigned intervals of the same measure there within—any assigned reading time within intervals of demand is treated as perfectly substitutable⁶. Without loss of generality, define reader i 's utility from a schedule μ as

$$u_i(\mu; \theta_i) = \int_{\theta_i} \mu_i(r) dr.$$

A schedule μ **Pareto dominates** a schedule μ' if and only if

$$u_i(\mu; \theta_i) \geq u_i(\mu'; \theta_i), \forall i \in N,$$

with the inequality being strict for at least one $i \in N$. A *feasible* schedule μ is **Pareto efficient** if and only if there does not exist another *feasible* schedule μ' that Pareto dominates μ . A (feasible) schedule is **envy-free** if all readers prefer their own schedule to any other reader's, or more formally, for all readers $i, j \in N$ and $j \neq i$,

$$u_i(\mu; \theta_i) = \int_{\theta_i} \mu_i(r) dr \geq \int_{\theta_i} \mu_j(r) dr.$$

A direct circulation mechanism is a function of the readers' types reported at time zero, whose value is the feasible schedule of circulation chosen by the library. Let $t_i \in \Theta$, $t \in \Theta^n$, and $t_{-i} \in \Theta^{n-1}$ be reader i 's reported type, the profile of all readers' reported types, and the profile of all readers' reported types except for reader i 's, respectively. Formally, a (deterministic) **direct circulation mechanism** (henceforth, mechanism) C is defined as

$$C : \Theta^n \rightarrow \mathcal{M}.$$

⁶That is, the value density functions of the readers are *piecewise uniform* in the context of cake cutting, as defined in [Chen et al. \(2013\)](#)

A mechanism C is **strategy-proof** if and only if

$$u_i(C(\theta_i, t_{-i}); \theta_i) \geq u_i(C(t_i, t_{-i}); \theta_i), \quad \forall (t_i, t_{-i}, \theta_i, i) \in (\Theta \times \Theta^{n-1} \times \Theta \times N).$$

A strategy-proof mechanism C is (ex-post) **Pareto efficient** if and only if $C(\theta)$ is a Pareto efficient schedule for all $\theta \in \Theta^n$. Note that I restrict the definition of a Pareto efficient mechanism to the truth-telling dominant strategy equilibrium, first requiring the mechanism to be strategy-proof. I may also sometimes refer to the i -th component in the schedule chosen by C . Thus, let

$$(C_i(t))(\cdot) \equiv \mu_i(\cdot) \text{ where } C(t) \equiv \mu \equiv (\mu_1, \mu_2, \dots, \mu_n).$$

I set out to characterize a class of such mechanisms in the next section.

3. THE RESULTS

3.1. A class of mechanisms. In designing a class of strategy-proof and Pareto efficient circulation mechanisms, I characterize a family of constrained optimization problems whose solutions satisfy a specific non-inferiority condition. I interpret my approach as reverse engineering the mechanisms from a set of conditions the chosen schedules must fulfill. In contrast, [Chen et al. \(2013\)](#) directly formulate one specific truthful and Pareto efficient mechanism in the format of an algorithm. More detailed comparisons between [Chen et al. \(2013\)](#) and my mechanisms can be found in the next section.

Let $g \equiv (g_1(\cdot), g_2(\cdot), \dots, g_n(\cdot))$ be a profile of functions where

$$g_i : \mathbb{R}_+ \rightarrow \mathbb{R}, \quad \forall i \in N.$$

$g_i(\cdot)$ is **continuous, strictly increasing**, and **strictly concave** for all $i \in N$. Let

$$(1) \quad G(x) \equiv \sum_{i \in N} g_i(x_i), \quad x \equiv (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n.$$

Moreover, for any $S \subseteq N$, let

$$b(S; t) \equiv \lambda \left(\bigcup_{i \in S} t_i \right) \text{ where } t \equiv (t_1, t_2, \dots, t_n),$$

and $\lambda(\cdot)$ represents the Lebesgue measure of subsets of real numbers. Define

$$(2) \quad B(t) \equiv \left\{ x \in \mathbb{R}_+^n \mid \sum_{i \in S} x_i \leq b(S; t), \quad \forall S \subseteq N \right\}.$$

Either the values $b(S; t)$ or the inequalities $\sum_{i \in S} x_i \leq b(S; t)$ will be referred to as *budgets* from now on whenever the reference is clear from the context. I refer to the situation where

$$\sum_{i \in S} x_i = b(S; t),$$

as *the budget for S binding at x* . $B(t)$ is referred to as the *feasible set*.

Remark 1. From now on, I shall focus on the case that the readers' reported intervals, combined, contain all the time between $\inf \{\cup_{i \in N} t_i\}$ and $\sup \{\cup_{i \in N} t_i\}$. It can be easily seen that such a restriction is without loss of generality. Specifically, if there exists an interval $[\underline{r}, \bar{r}]$ where no reader reports demand for the book, the mechanism simply shifts all the reported intervals after \bar{r} earlier by exactly $(\bar{r} - \underline{r})$ ⁷ and treats the shifted intervals as the initial reports and determines the circulation accordingly. This shifting procedure can be repeated to eliminate all intervals without any reported demand. It is only routine to shift the chosen schedule later by exactly the same amount $(\bar{r} - \underline{r})$ to generate the schedule for the original reported types.

I formulate a mechanism \mathcal{G} with the solution to the constrained maximization problem:

Definition 1 (Mechanism \mathcal{G}). Consider the constrained maximization problem

$$(CM) \quad \max_{x \in B(t)} G(x).$$

Let

$$x^*(t) \equiv \arg \max_{x \in B(t)} G(x).$$

Mechanism \mathcal{G} chooses a schedule $\mathcal{G}(t)$ satisfying the following conditions:

$$(3) \quad r \notin t_i \Rightarrow (\mathcal{G}_i(t))(r) = 0, \quad \forall i \in N,$$

and

$$(4) \quad \int_{t_i} (\mathcal{G}_i(t))(r) dr = x_i^*(t) \text{ where } x^*(t) \equiv (x_1^*(t), x_2^*(t), \dots, x_n^*(t)).$$

$x^*(t)$ will be referred to as the *optimum* or the *maximizer*.

Remark 2. An objective function $G(x)$ gives rise to a mechanism \mathcal{G} . The class of all such mechanisms are generated by the set of objective functions $G(x)$ that are additive in its components with all the components being continuous, strictly increasing, and strictly concave. However, objective functions that are monotonic

⁷Shifting by more than $(\bar{r} - \underline{r})$ may create overlapping demand that did not exist initially while shifting by less does not eliminate the interval of no demand.

transformations of each other are equivalent in the sense that they all generate the same mechanism. Thus, the class of all the mechanisms that can be characterized by constrained optimization problems such as (CM) is, strictly speaking, the set of equivalence classes of the objective functions with the equivalence relation defined as monotonic transformations.

Remark 3. On the other hand, the definition of $B(t)$ is invariant with respect to $G(x)$. $B(t)$ is a nonempty ($x = 0 \in B(t)$), convex, and closed and bounded—therefore compact—subset of a finite-dimensional Euclidean space for all t . The **regularity conditions** on $G(x)$ along with the compactness of $B(t)$ guarantee the existence of a **unique** maximizer $x^*(t)$ ⁸ for all t .

Remark 4. (3) serves as a partially necessary condition for Pareto efficiency of $\mathcal{G}(t)$. If reader i reports her type truthfully and the mechanism allocates her some reading time that she did not demand, other readers demanding such reading time allocated to i , when such other readers exist, can benefit from the mechanism reallocating such time from i . In other words, Pareto efficiency requires a schedule to respect any reader's declared non-demand, which is characterized by (3).

Remark 5. (4) sets an upper bound on the payoff reader i can obtain from $\mathcal{G}(t)$. If reader i is reporting truthfully, then $x_i^*(t)$ will be her eventual payoff. Otherwise, Mechanism \mathcal{G} could allocate her some reading time that falls out of her true type but within her reported demand: reader i 's payoff will be less than $x_i^*(t)$.

Remark 6. The mechanism works as follows: when presented with the reported types t , the mechanism first solves the constrained maximization (CM) and stores the solution $x^*(t)$. Then it selects a schedule that does not assign any reader any unreported demand (condition (3)) but assigns the exact total reading time to the readers' according to their component in $x^*(t)$ (condition (4)). The existence of a schedule satisfying (3) and (4) is intuitive but not trivial⁹. I will prove the existence of such a schedule by slightly generalizing the network-flow proof of existence in Chen et al. (2013).

3.2. Feasibility and Pareto efficiency.

Proposition 1 (Existence of a feasible schedule). *Given any $t \in \Theta^n$, for all $x \in B(t)$ as defined in (2), there exists a schedule μ such that*

$$(5) \quad r \notin t_i \Rightarrow \mu_i(r) = 0, \quad \forall i \in N,$$

⁸See, for example, Chapter 2 in Intriligator (2002) for the existence theorem. The uniqueness of the solution is an immediate result of the concavity of $G(x)$

⁹I thank Alex Frankel for pointing out the necessity of constructing such a schedule from the solution to (CM).

and

$$(6) \quad \int_{t_i} \mu_i(r) dr = x_i \text{ where } x \equiv (x_1, x_2, \dots, x_n) \text{ and } \mu \equiv (\mu_1, \mu_2, \dots, \mu_n).$$

Proof of Proposition 1. For every pair (t, x) with $x \in B(t)$, I construct a directed network¹⁰ $\Gamma(x; t)$ to show the existence of a feasible schedule with the desired properties described above. Recall that t is a profile of finite unions of open intervals on the nonnegative real line. Equivalently, the i -th component in t , t_i , can be expressed as a unique collection $T_i(t_i)$ of disjoint open intervals. Formally, $T_i(t_i)$ is the partition of t_i with the lowest cardinality. For example,

$$t_i = (1.5, 1.7) \cup (2.7, 3.7) \Rightarrow T_i(t_i) = \{(1.5, 1.7), (2.7, 3.7)\} \subset 2^{\mathbb{R}^+},$$

where $2^{\mathbb{R}^+}$ is the power set of nonnegative real numbers. Collect all the end points of all the intervals in $T_i(t_i)$. Formally, let $\mathcal{T}_i(t_i)$ be the set of all supremum and infimum of the elements of $T_i(t_i)$. Again, as in the example above,

$$t_i = (1.5, 1.7) \cup (2.7, 3.7) \Rightarrow \\ \mathcal{T}_i(t_i) = \{(1.5, 1.7), (2.7, 3.7)\} \Rightarrow \mathcal{T}_i(t_i) = \{1.5, 1.7, 2.7, 3.7\} \subset \mathbb{R}_+.$$

Now, let

$$\mathcal{T}(t) \equiv \bigcup_{i \in N} \mathcal{T}_i(t_i) \subset \mathbb{R}_+.$$

Order the elements of $\mathcal{T}(t)$ from the minimum to the maximum and let $\varepsilon_j(t)$ be the j -th least of the $m(t)$ elements of $\mathcal{T}(t)$. Observe that under Remark 1, $\cup_{i \in N} t_i$ can be decomposed into

$$[\varepsilon_1(t), \varepsilon_2(t)) \cup [\varepsilon_2(t), \varepsilon_3(t)) \cup \dots \cup [\varepsilon_j(t), \varepsilon_{j+1}(t)) \cup \dots \cup [\varepsilon_{m(t)-1}(t), \varepsilon_{m(t)}(t)].$$

That is, the collection of all the intervals—which I will refer to as **bins**—bounded by adjacent elements of $\mathcal{T}(t)$, when ordered increasingly, forms a partition of the entire interval of demand from all readers. Moreover, for any $i \in N$ and $1 \leq j \leq m(t) - 1$, either

$$(\varepsilon_j(t), \varepsilon_{j+1}(t)) \subseteq t_i$$

or

$$(\varepsilon_j(t), \varepsilon_{j+1}(t)) \cap t_i = \emptyset.$$

This is because, by construction, the set of readers who reported demand for any bin does not change **within** the bin, the bin must be either entirely in a reader's reported demand or not at all.

¹⁰See [Vohra \(2011\)](#) for an introduction to networks in the mechanism design context.

Definition 2. Define the following directed network $\Gamma(x; t)$: create a source Ψ , a node (called a **bin node**) for each bin $(\varepsilon_j(t), 1 \leq j \leq m(t-1))$, a node (called a **reader node**) for each reader (represented in lowercase Roman numerals), and a sink Ω . Create the arcs as follows.

- Connect the source to all bin nodes with arcs pointing to the bin nodes—I will refer to such arcs as **source-bin arcs**;
- Assign capacity $(\varepsilon_{j+1}(t) - \varepsilon_j(t))$ to the arc connecting the source to the node corresponding to the bin with end points $\varepsilon_j(t)$ and $\varepsilon_{j+1}(t)$;
- Connect the reader nodes to the sink with arcs pointing to the sink—I will refer to such arcs as **reader-sink arcs**;
- Assign capacity x_i to the arc connecting reader i 's node to the sink;
- Connect a bin node to a reader node with an arc pointing to the reader node—referred to as a **bin-reader arc**—if and only if the bin is a subset of the reader's reported demand;
- Assign infinite capacities to all bin-reader arcs.

As a quick example, suppose there are three readers whose reports are

$$t_1 = (0, 12) \cup (23, 35); \quad t_2 = (7, 17); \quad t_3 = (13, 25).$$

This implies that the set $B(t)$ is given by

$$\begin{cases} 0 \leq x_1 \leq 24, & 0 \leq x_2 \leq 10, & 0 \leq x_3 \leq 12; \\ x_1 + x_2 \leq 29, & x_2 + x_3 \leq 18, & x_1 + x_3 \leq 34; \\ x_1 + x_2 + x_3 \leq 35. \end{cases}$$

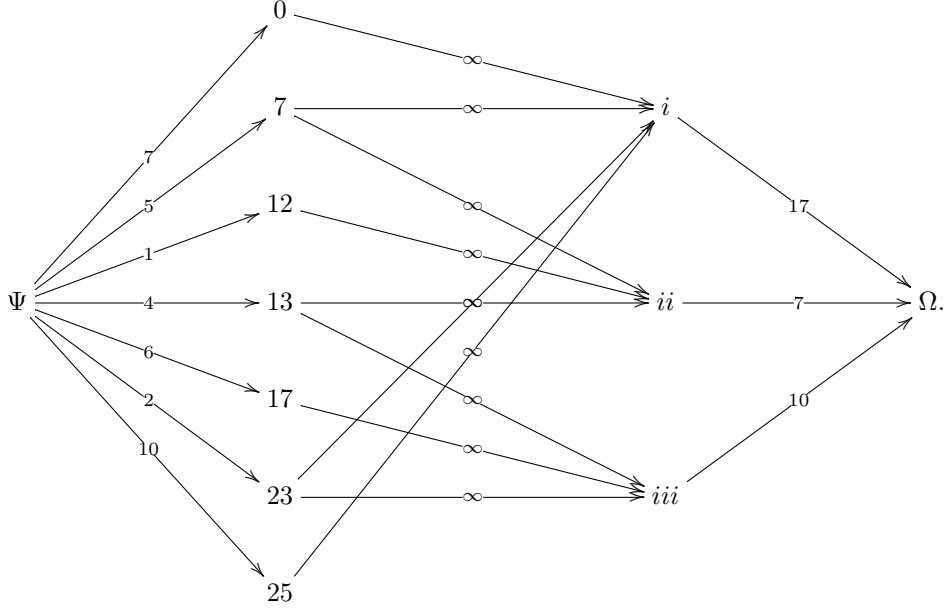
And t gives rise to $\mathcal{T}(t)$ as

$$\{0, 7, 12, 13, 17, 23, 25, 35\}.$$

Take, say, $(17, 7, 10) \in B(t)$. Therefore, $\Gamma(x; t)$ in this case is represented by Figure 1.

In $\Gamma(x; t)$, the maximum flow size must be less than or equal to $\sum_{i \in N} x_i$. On the other hand, if the maximum flow is of size $\sum_{i \in N} x_i$, then there exists a schedule that satisfies conditions (5) and (6). The construction of a schedule from the maximum flow is natural: the flow size assigned to a certain bin-reader arc will be the amount of reading time in that bin assigned to that reader. (5) is satisfied by the schedules constructed from this maximum flow since when a certain bin is not a subset of the reported demand of a certain reader i , that bin node is not connected to the reader i node. (6) is satisfied because when the

FIGURE 1. Example: Network Flow Representation



maximum flow is of size $\sum_{i \in N} x_i$, the flow size on each reader-sink arc must be of full capacity. Therefore, to prove Proposition 1, the following lemma would suffice:

Lemma 1. *For all (t, x) with $x \in B(t)$, $\Gamma(x; t)$ as defined in Definition 2 has a maximum flow of*

$$\sum_{i \in N} x_i.$$

Proof of Lemma 1. By the Max Flow Min Cut Theorem¹¹, the maximum flow in $\Gamma(x; t)$ is equal to the minimum cut (the minimum sum of the capacities of the arcs that are cut to disconnect the sink from the source). A cut is a set of arcs in $\Gamma(x; t)$ such that removing this set of arcs eliminates any path from the source to the sink. Notice that cutting all reader-sink arcs will disconnect the sink Ψ from the source Ω —it is only left to show that this yields the **minimum** cut. I will refer to this cut of all the reader-sink arcs as the cut K . The total capacity of the arcs in K is simply $\sum_{i \in N} x_i$.

Suppose, instead, not all the reader-sink arcs are cut in the minimum cut. Then let $K' \neq K$ be the minimum cut. Let S' represent the set of readers whose reader-sink arcs are **not** cut in K' . Let $S = N \setminus S'$. Note that, to disconnect the source from the sink, any source-bin arc with the corresponding bin being a subset of the reported demand of any reader in S' must be cut¹². Observe that all such bins whose nodes

¹¹See, for example, Chapter 6 in Papadimitriou and Steiglitz (1998)

¹²Otherwise, some bin-reader arc must be cut, which induces a cut of total capacity of infinity—definitely not the minimum cut since cutting all reader-sink arcs results in a lower capacity.

are disconnected from the source form a partition of $\cup_{i' \in S'} t_{i'}$:

$$\bigcup \{(\varepsilon_j(t), \varepsilon_{j+1}(t)) \mid \exists i' \in S' \text{ such that } (\varepsilon_j(t), \varepsilon_{j+1}(t)) \in t_{i'}, 1 \leq j \leq m(t) - 1\} = \bigcup_{i' \in S'} t_{i'}.$$

Thus, the total capacity of the cut source-bin arcs is

$$\lambda \left(\bigcup_{i' \in S'} t_{i'} \right),$$

which, in turn, implies the total capacity of the arcs in the cut K' is at least

$$\lambda \left(\bigcup_{i' \in S'} t_{i'} \right) + \sum_{i \in S} x_i.$$

Recall that $x \in B(t)$, which implies that

$$\sum_{i' \in S'} x_{i'} \leq \lambda \left(\bigcup_{i' \in S'} t_{i'} \right).$$

Therefore, the total capacity of the arcs in K' must be greater than or equal to that of K , which means that the maximum flow in $\Gamma(x, t)$ must be precisely $\sum_{i \in N} x_i$. This proves Lemma 1. \square

The proof of Lemma 1 concludes the proof of Proposition 1. \square

The following corollary is an immediate result following Proposition 1 and Lemma 1.

Corollary 1. *Given any $t \in \Theta^n$, for all $x \in B(t)$, there exists a schedule μ such that, in addition to satisfying (5) and (6), $\mu_i(r)$ is equal to 1 over a finite union of open intervals on the nonnegative real line for all $i \in N$.*

Proof. Since $t \in \Theta^n$, $\mathcal{T}(t)$ is a finite set with cardinality $m(t) < +\infty$. Hence, in $\Gamma(x, t)$, there are $m(t) - 1$ bin nodes, which implies that there are at most $n \cdot (m(t) - 1)$ bin-reader arcs. By construction, the number of disjoint intervals over which any reader can potentially be assigned the book is equal to the number of bin-reader arcs she is connected to. The total number of disjoint intervals over which all readers can jointly be assigned the book is thus at most

$$n \cdot (m(t) - 1) < +\infty. \quad \square$$

With the existence of feasible schedules established, I now turn to the Pareto efficiency of \mathcal{G} .

Remark 7. If Mechanism \mathcal{G} is strategy-proof, the Pareto efficiency of \mathcal{G} is obtained by the following conditions combined, when all readers are reporting their types truthfully:

¹³Recall that there are only a finite number of readers.

- (1) No reader is assigned any reading time outside her reported (hence truthful) demand;
- (2) Any reading time that is demanded by any reader must be assigned to one of these readers.

The first condition corresponds to (3) while the second condition will be established jointly by the fact that all the components of $G(x)$ is increasing and (4).

Proposition 2 (Pareto efficiency of \mathcal{G}). *If \mathcal{G} is strategy-proof, then $\mathcal{G}(t)$ is a Pareto efficient schedule when $t = \theta$ for all $\theta \in \Theta^n$. Hence, Mechanism \mathcal{G} is a Pareto efficient mechanism.*

I omit the obvious proof here since Pareto efficiency is a direct result of maximizing $G(\cdot)$ —please see Appendix A for the details.

Remark 8. The approach to obtaining the second condition in Remark 7 demonstrates the fundamental difference between formulating mechanisms as constrained optimization problems and the approach Chen et al. (2013) took. Chen et al. (2013) relies on the exactness of the total reading time assigned to specific groups of readers while my approach exploits the design of (CM) simply being a maximization problem with all components of the objective function being strictly increasing.

3.3. Strategy-proofness. The major difficulty in constructing a strategy-proof and Pareto efficient mechanism lies in the tension between these two concepts that can only be reconciled by some sense of uniform monotonicity. More concretely, readers must be assigned more reading time when they report larger intervals (or more precisely, supersets) of demand, which naturally gives the readers the incentive to exaggerate how much they need the book. To tackle this issue, a strategy-proof mechanism, without identifying which part of the reported demand is the true demand for any reader, must decrease the amount of reading time assigned within the true intervals of demand while increasing the total assigned reading time over the reported intervals when readers are reporting larger intervals of demand. Simultaneously achieving these two objectives directly motivates my construction of the strategy-proof mechanisms.

Due to the richness of the type space (all finite unions of open intervals on the nonnegative real line), tractably categorizing readers' reports is much needed for comparisons between truthful reports and potentially strategic deviations. I provide the following lemma as a sensible start.

Lemma 2. *Mechanism \mathcal{G} is strategy-proof if and only if the following two conditions hold simultaneously for all $(i, t_i, t_{-i}; \theta_i) \in N \times \Theta \times \Theta^{n-1} \times \Theta$:*

$$(7) \quad t_i \subset \theta_i \Rightarrow u_i(\mathcal{G}(t); \theta_i) \leq u_i(\mathcal{G}(\theta_i, t_{-i}); \theta_i);$$

$$(8) \quad t_i \supset \theta_i \Rightarrow u_i(\mathcal{G}(t); \theta_i) \leq u_i(\mathcal{G}(\theta_i, t_{-i}); \theta_i).$$

Proof of Lemma 2. Take any given t_{-i} and let $t_i \neq \theta_i$.

If $t_i \subset \theta_i$ or $t_i \supset \theta_i$, conditions (7) and (8) imply that reader i cannot profitably deviate with such reports. If $t_i \cap \theta_i = \emptyset$, (3) implies that reader i 's payoff will be zero, which cannot be greater than her payoff when reporting truthfully since $u_i(\mathcal{G}(\theta_i, t_{-i}); \theta_i) = x_i^*(\theta_i, t_{-i}) \geq 0$. So reader i will not deviate with such reports, either.

Suppose $\theta'_i \equiv t_i \cap \theta_i \neq \emptyset$. Also, $t_i \not\subset \theta_i$ and $t_i \not\supset \theta_i$. By the definition of $u_i(\cdot)$ and (3),

$$(9) \quad u_i(\mathcal{G}(t); \theta_i) = \int_{\theta_i} (\mathcal{G}_i(t))(r) dr = \int_{\theta'_i} (\mathcal{G}_i(t))(r) dr.$$

Notice that $\theta'_i \subset t_i$ and $\theta'_i \subset \theta_i$ and consider the case when reader i 's true type is θ'_i . (8) implies

$$\int_{\theta'_i} (\mathcal{G}_i(t))(r) dr = u_i(\mathcal{G}(t); \theta'_i) \leq u_i(\mathcal{G}(\theta'_i, t_{-i}); \theta'_i) = \int_{\theta'_i} (\mathcal{G}_i(\theta'_i, t_{-i}))(r) dr.$$

But when reader i 's true type is θ_i , (7) implies that

$$(10) \quad u_i(\mathcal{G}(\theta'_i, t_{-i}); \theta_i) \leq u_i(\mathcal{G}(\theta_i, t_{-i}); \theta_i).$$

By (3),

$$(11) \quad u_i(\mathcal{G}(\theta'_i, t_{-i}); \theta_i) = \int_{\theta_i} (\mathcal{G}_i(\theta'_i, t_{-i}))(r) dr = \int_{\theta'_i} (\mathcal{G}_i(\theta'_i, t_{-i}))(r) dr = u_i(\mathcal{G}(\theta'_i, t_{-i}); \theta'_i).$$

Combining (9) through (11) yields

$$u_i(\mathcal{G}(t); \theta_i) = u_i(\mathcal{G}(t); \theta'_i) \leq u_i(\mathcal{G}(\theta'_i, t_{-i}); \theta'_i) = u_i(\mathcal{G}(\theta'_i, t_{-i}); \theta_i) \leq u_i(\mathcal{G}(\theta_i, t_{-i}); \theta_i),$$

which completes the proof. \square

Remark 9. Lemma 2 asserts that, to ensure strategy-proofness, it suffices to check only the deviations that are proper subsets ((7)) or supersets ((8)) of the true types. This restriction is particularly convenient in the context of Proposition 3 since any potentially profitable deviations from the truth will only (weakly) affect all the budgets in the same direction.

Consider a reader i who is comparing the options of reporting her true type and deviating from the truth. Her report can only change the allocated reading times to her (and potentially to other readers as well) only if her report changes the maximizer x^* of $G(x)$. The function $G(x)$ does not depend on any reader's report

hence the only way to affect the final allocation is through changing the feasible set $B(t)$. Specifically, reader i can only influence the values $b(S; t)$ of the budgets of the subsets S which she is a member of ($i \in S$). Moreover, only changing the $b(S; t)$ where the budget was initially binding for the subgroup S potentially changes x^* . On the other hand, what complicates the matter is that any change in i 's report can change up to $(2^{n-1} - 1)$ (binding or slacking) budgets. To characterize the effects of a change in a reader's report on the allocated reading times, I now take a quick detour to the comparative statics of constrained optimization problems. I document the desired monotonicity results as a *non-inferiority condition* below in Proposition 3 after introducing some new definitions and relevant results that will facilitate establishing Proposition 3.

Definition 3. Let $N = \{1, 2, 3, \dots, n\}$ and 2^N represent the power set of N . A budget function $b : 2^N \rightarrow \mathbb{R}_{++}$ is **submodular** if and only if

$$b(S_1) + b(S_2) \geq b(S_1 \cap S_2) + b(S_1 \cup S_2), \forall S_1, S_2 \subseteq N.$$

A budget function $\hat{b}(\cdot)$ is **marginally relaxed from $b(\cdot)$ by i** if and only if

$$\begin{cases} \hat{b}(S) = b(S), & \forall S \subseteq N \text{ such that } i \notin S; \\ \hat{b}(S') \geq b(S'), & \forall S' \subseteq N \text{ such that } i \in S'. \end{cases}$$

Moreover, for any $x \in \mathbb{R}^n$ and any submodular budget function $b(\cdot)$, x **respects $b(\cdot)$** if and only if

$$(12) \quad \sum_{i \in S} x_i \leq b(S), \forall S \subseteq N.$$

Otherwise, I say that x **violates $b(\cdot)$** . When the inequality in (12) is an equality, I say that the budget for S is **binding** at x and when the inequality is strict, the budget **slacking**, both with respect to $b(\cdot)$.

Submodularity implies potential relevance of budgets for the larger sets. See Che et al. (2013) for more general conditions regarding ceilings of quantity supplied and floors of quantity demanded in the reduced-form auctions context. Quick observations yield the following result about submodular budget functions.

Lemma 3. Let $b(\cdot)$ be a submodular budget function. Take $x \in \mathbb{R}^n$ such that

$$\sum_{i \in S} x_i \leq b(S), \forall S \subseteq N.$$

Then, for any $S_1, S_2 \subseteq N$,

$$\left\{ \sum_{i \in S_1} x_i = b(S_1) \text{ and } \sum_{i \in S_2} x_i = b(S_2) \right\}$$

$$\Rightarrow \left\{ \sum_{i \in S_1 \cup S_2} x_i = b(S_1 \cup S_2) \text{ and } \sum_{i \in S_1 \cap S_2} x_i = b(S_1 \cap S_2) \right\}.$$

That is, with submodular budget functions, binding budgets for two sets of variables imply binding budgets for their intersection and union.

Proof. Observe that

$$b(S_1 \cup S_2) + b(S_1 \cap S_2) \geq \sum_{i \in S_1 \cup S_2} x_i + \sum_{i \in S_1 \cap S_2} x_i = b(S_1) + b(S_2) \geq b(S_1 \cup S_2) + b(S_1 \cap S_2).$$

The first inequality and equality above are by assumptions of respecting the budgets of $S_1 \cup S_2$ and $S_1 \cap S_2$ and the budgets being binding for S_1 and S_2 . The second inequality is given by submodularity. Therefore, both inequalities above must be equalities and since, by assumption,

$$\sum_{i \in S_1 \cup S_2} x_i \leq b(S_1 \cup S_2) \text{ and } \sum_{i \in S_1 \cap S_2} x_i \leq b(S_1 \cap S_2),$$

both budgets of $S_1 \cup S_2$ and $S_1 \cap S_2$ must be binding. \square

Now I introduce a device essential in the proof of the non-inferiority condition and naturally motivated by submodular budget functions. Take any vector x and a submodular budget function $b(\cdot)$. Let $\mathcal{S}_j(x; b(\cdot))$ be the collection of all the binding budgets for j at x according to $b(\cdot)$. That is

$$(13) \quad \mathcal{S}_j(x; b(\cdot)) \equiv \left\{ S \subseteq N \mid j \in S \text{ and } \sum_{k \in S} x_k = b(S) \right\}.$$

Taking the intersection of all the sets in $\mathcal{S}_j(x; b(\cdot))$ yields a subset of N to which I refer as the *core*¹⁴ of j at x according to $b(\cdot)$. Formally,

Definition 4. Let $x \in \mathbb{R}^n$ and $b(\cdot)$ be a submodular budget function. For any $j \in N$, let $\mathcal{S}_j(x; b(\cdot))$ be defined as in (13). The **core of j at x according to $b(\cdot)$** is defined as

$$M_j(x; b(\cdot)) \equiv \bigcap_{S \in \mathcal{S}_j(x; b(\cdot))} S.$$

Remark 10. Lemma 3 immediately implies that $M_j(x; b(\cdot)) \in \mathcal{S}_j(x; b(\cdot))$. Thus, the core of j at x according to $b(\cdot)$ is the unique minimal element of $\mathcal{S}_j(x; b(\cdot))$ with sets being partially ordered by inclusion. I document this result in the following lemma and omit the obvious proof.

Lemma 4. Let $x \in \mathbb{R}^n$ and $b(\cdot)$ be a submodular budget function. For all $j \in N$,

$$M_j(x; b(\cdot)) \in \mathcal{S}_j(x; b(\cdot)) \text{ and } M_j(x; b(\cdot)) \subseteq S, \forall S \in \mathcal{S}_j(x; b(\cdot)).$$

¹⁴Open to suggestions for a better name.

With the notion of core in hand, I now introduce the non-inferiority condition, the proof of which relies heavily on the minimum property of the core provided in Lemma 4.

Proposition 3 (Non-inferiority). *Let $G : \mathbb{R}_+^n \rightarrow \mathbb{R}$ satisfy all the *regularity conditions*. Let $\hat{b}(\cdot) \neq b(\cdot)$ be two submodular budget functions such that $\hat{b}(\cdot)$ is marginally relaxed from $b(\cdot)$ by i . Let*

$$\begin{cases} x^* \equiv \arg \max G(x) \text{ such that } \sum_{i \in S} x_i \leq b(S), \forall S \subseteq N \equiv \{1, 2, 3 \dots, n\}; \\ \hat{x}^* \equiv \arg \max G(x) \text{ such that } \sum_{i \in S} \hat{x}_i \leq \hat{b}(S), \forall S \subseteq N. \end{cases}$$

Then, there exists an $S^ \subseteq N$ such that $i \in S^*$ and*

$$\sum_{j \in S^*} x_j^* = b(S^*) \Rightarrow x_j^* \leq \hat{x}_j^*, \forall j \in S^*.$$

Remark 11. The non-inferiority condition states that, when an (old) submodular budget function is marginally relaxed to another (new) in some dimension i , there must exist a subset of dimensions containing i in which the new maximizer must be no less than the old, coordinate-wise, within this subset. In the context of library problem, this translates into the joint monotonicity in the allocated reading times of some subgroup of readers if one of the readers changes her report from a subset to a superset. If some reader i can increase her reading time by unilaterally reporting a superset of her true type, she must be increasing the reading times of other group members of some $S^* \subseteq N$ as well. Since the maximal sum of the payoffs of the group members of S^* is fixed when i was reporting the truth, misreports to supersets must decrease i 's payoff, suggesting strategy-proofness.

Remark 12. The negation of Proposition 3 says that for any binding budget of S including i at x^* , $\hat{x}_j^* < x_j^*$ for some $j \in S$. The proof of the Proposition then boils down to searching for a pair $y^*, \hat{y}^* \in \mathbb{R}^n$ between x^* and \hat{x}^* , coordinate-wise, such that y^* respects the budget function $b(\cdot)$ while \hat{y}^* respects $\hat{b}(\cdot)$ and

$$x^* + \hat{x}^* = y^* + \hat{y}^*.$$

Then additivity and concavity of $G(\cdot)$ implies that at least one of x^* and \hat{x}^* is not optimal—see Remark 18 in Appendix B for a detailed explanation. Additivity implies supermodularity, although $G(\cdot)$ being supermodular and concave alone would not suffice for the proof. The search for y^* and \hat{y}^* is detailed in Appendix B by an Algorithm \mathcal{O} .

Remark 13. It should be clearly acknowledged that the proof, and the class of mechanisms itself in some sense, is largely inspired by the monotonic comparative statics results of constrained optimization problems

presented in [Quah \(2007\)](#). Moreover, the search for the pair of vectors described in the previous remark is mostly inspired by the linear algebra research in [Goberna et al. \(2003\)](#) and [Goberna and Rodriguez \(2006\)](#). Although, none of the aforementioned work can be directly applied, all offer fundamental insights.

Proposition 3 conveniently implies the strategy-proofness in Mechanism \mathcal{G} as long as the budgets in the library problem exhibit submodularity and reports to supersets by i result in marginal relaxation. The latter condition is apparent given the definition of the budgets $b(S; t)$. I document the former condition and its short proof in the following lemma.

Lemma 5. $b(\cdot; t)$ is a submodular budget function for all $t \in \Theta^n$.

Proof. Take $S_1, S_2 \subseteq N$ and any $t \in \Theta^n$.

$$b(S_1 \cup S_2; t) = \lambda \left(\bigcup_{j \in (S_1 \cup S_2)} t_j \right) = \lambda \left(\bigcup_{j \in S_1} t_j \right) + \lambda \left(\bigcup_{j \in S_2} t_j \right) - \lambda \left(\left(\bigcup_{j \in S_1} t_j \right) \cap \left(\bigcup_{j \in S_2} t_j \right) \right).$$

Notice that

$$\left(\bigcup_{j \in S_1} t_j \right) \cap \left(\bigcup_{j \in S_2} t_j \right) \supseteq \bigcup_{j \in S_1 \cap S_2} t_j \Rightarrow \lambda \left(\left(\bigcup_{j \in S_1} t_j \right) \cap \left(\bigcup_{j \in S_2} t_j \right) \right) \leq b(S_1 \cap S_2; t).$$

To see this, take, for example, $S_1 = \{1, 2\}, S_2 = \{2, 3\}$. $\forall r \in t_2, r \in (t_1 \cup t_2)$ and $r \in (t_2 \cup t_3)$ hence $r \in (t_1 \cup t_2) \cap (t_2 \cup t_3)$, which can be easily generalized to any subsets S_1 and S_2 . Therefore,

$$b(S_1 \cup S_2; t) = b(S_1; t) + b(S_2; t) - \lambda \left(\left(\bigcup_{j \in S_1} t_j \right) \cap \left(\bigcup_{j \in S_2} t_j \right) \right) \leq b(S_1; t) + b(S_2; t) - b(S_1 \cap S_2; t). \quad \square$$

With Lemma 5 providing the applicability of Proposition 3, I can now establish strategy-proofness.

Proposition 4 (Strategy-proofness). *Mechanism \mathcal{G} is strategy-proof.*

Proof. Let i be the reader contemplating on deviating from her true report θ_i . Other readers' reports are fixed at t_{-i} . According to Lemma 2, I only have to consider reports $\underline{t}_i \subset \theta_i$ and $\bar{t}_i \supset \theta_i$. Let \underline{x}^* be the maximizer of $G(\cdot)$ subject to the corresponding constraints when i reports \underline{t}_i and \bar{x}^* when i reports \bar{t}_i , and x^* when i reports truthfully. By Lemma 5 and Proposition 3,

$$\underline{x}_i^* \leq x_i^* (= u_i(\mathcal{G}(\theta_i, t_{-i}); \theta_i)) \leq \bar{x}_i^*.$$

I first compare i 's payoffs when reporting \underline{t}_i and θ_i :

$$u_i(\mathcal{G}(\underline{t}_i, t_{-i}); \theta_i) = \int_{\theta_i} \mathcal{G}_i(\underline{t}_i, t_{-i})(r) dr = \int_{\underline{t}_i} \mathcal{G}_i(\underline{t}_i, t_{-i})(r) dr = \underline{x}_i^* \leq x_i^*.$$

The second equality above follows from the construction that $\mathcal{G}_i(\underline{t}_i, t_{-i})(\cdot)$ vanishes beyond \underline{t}_{-i} . Thus, deviations to subsets of true types is never profitable. For \bar{t}_i , if no budget containing i was binding at report (θ_i, t_{-i}) , $x_i^* = \bar{x}_i^*$ and i cannot benefit from misreporting. Suppose otherwise. Then Proposition 3 implies the existence of a subset $S^* \subseteq N$ such that $i \in S^*$ and

$$(14) \quad \sum_{j \in S^*} x_j^* = b(S^*; (\theta_i, t_{-i})) \text{ and } x_j^* \leq \bar{x}_j^*, \forall j \in S^*.$$

If $S^* = \{i\}$, $x_i^* = \lambda(\theta_i) \geq u_i(\mathcal{G}(\bar{t}_i, t_{-i}); \theta_i)$, since $\lambda(\theta_i)$ is the upper bound on i 's payoff when her true type is θ_i . Suppose $S^* \neq \{i\}$. Feasibility implies that

$$\begin{aligned} b(S^*; (\theta_i, t_{-i})) &\geq \int_{(\bigcup_{j \in S^* \setminus \{i\}} t_j) \cup \theta_i} \sum_{j' \in S^*} \mathcal{G}_{j'}(\bar{t}_i, t_{-i})(r) dr \\ &= \int_{(\bigcup_{j \in S^* \setminus \{i\}} t_j) \cup \theta_i} \mathcal{G}_i(\bar{t}_i, t_{-i})(r) dr + \sum_{j \in S^* \setminus \{i\}} \bar{x}_j^* \geq \int_{\theta_i} \mathcal{G}_i(\bar{t}_i, t_{-i})(r) dr + \sum_{j \in S^* \setminus \{i\}} \bar{x}_j^* \\ &= u_i(\mathcal{G}(\bar{t}_i, t_{-i}); \theta_i) + \sum_{j \in S^* \setminus \{i\}} \bar{x}_j^*. \end{aligned}$$

The equality in the second line above is due to the fact that, for $j \in S^* \setminus \{i\}$, $\mathcal{G}_j(\bar{t}_i, t_{-i})(\cdot)$ vanishes beyond $(\bigcup_{j \in S^* \setminus \{i\}} t_j) \cup \theta_i$ since t_{-i} is held constant. The inequality in the second line above is due to the shrink in the domain of integration. Rewriting the inequality above, I obtain

$$u_i(\mathcal{G}(\bar{t}_i, t_{-i}); \theta_i) \leq b(S^*; (\theta_i, t_{-i})) = x_i^* + \sum_{j \in S^* \setminus \{i\}} (x_j^* - \bar{x}_j^*) \leq x_i^*$$

by (14). Therefore, \bar{t}_i is not a profitable misreport either and strategy-proofness is established. \square

4. DISCUSSION

4.1. As a generalization of Chen et al. (2013). The mechanisms developed in this paper generalize the Mechanism 1 constructed in Chen et al. (2013), treating mechanisms in general with a very different perspective. Formulating mechanisms with constrained optimizations expands the horizon where mechanism designers are able to select their mechanisms. Granted, such flexibility in the choice of the mechanism is only possible when desirable properties such as proportionality and envy-freeness are set aside in the first place. It is then natural to ask whether the same objectives described in Chen et al. (2013) can still be achieved by a subclass of mechanisms within the class constructed in this paper. Fortunately, the answer is an affirmative “yes”. I present this relationship as:

Corollary 2. *Let*

$$\hat{G}(x) \equiv \sum_{i \in N} g(x_i),$$

where $g(\cdot)$ satisfies all the *regularity conditions* and is continuously differentiable. Then the mechanism $\hat{\mathcal{G}}$ generated by $\hat{G}(x)$ replicates the *Mechanism 1* in [Chen et al. \(2013\)](#), so does any mechanism $\tilde{\mathcal{G}}$ generated by any monotonic transformation $\tilde{G}(x)$ of $\hat{G}(x)$.

The proof of Corollary 2 is based on the Karush-Kuhn-Tucker (KKT) conditions of (CM). The concavity of $g(\cdot)$ implies that the KKT conditions are *sufficient* to determine the solutions to the constrained maximization problem¹⁵. Thus, it suffices to show that the allocation chosen by *Mechanism 1* satisfies the KKT conditions for some Lagrange multipliers of the constraints. This turns out to be especially convenient given the way *Mechanism 1* was constructed and the complementary slackness between the Lagrange multipliers and the constraints implied by the KKT conditions. The key relationship between $\hat{\mathcal{G}}$ and *Mechanism 1* is the equivalence between binding constraints and exact allocations within certain subgroups of readers.

Proof. Recall that *Mechanism 1* breaks the set of all readers N into a partition $\{S_1, S_2, \dots, S_{\bar{k}}\}$ according to the order they are called into the subroutines. Let $\eta(S)$ be the value of the Lagrange multipliers of the constraint of readers in group S when imposing the allocation chosen by *Mechanism 1*. Then, for all $i \in S_k$ and $1 \leq k \leq \bar{k}$, the following characterization of the values of the Lagrange multipliers satisfy the first-order conditions with respect to x_i and complementary slackness

$$(15) \quad \frac{dg}{dx_i}(x_i^*) = \sum_{\Delta=0}^{\bar{k}-k} \{\eta(S_1 \cup S_2 \cup \dots \cup S_k \cup \dots \cup S_{k+\Delta})\}$$

where x_i^* is chosen by *Mechanism 1*. Since $dg/dx_i(\cdot)$ is a *decreasing* function, notice that, for some $j \in S_{k'}$ for some $\bar{k} \geq k' > k$, (15) implies that $x_i^* \leq x_j^*$, which coincides with Lemma 3.4 in [Chen et al. \(2013\)](#).

Specifically, to solve for the values of the Lagrange multipliers with the allocation given by *Mechanism 1*, start with the readers in the last group $S_{\bar{k}}$. Notice that for these readers, the budget for the entire set of readers N is the only binding budget they are members of. Since $g(\cdot)$ is a strictly increasing function, the Lagrange multiplier $\eta(N)$ is positive and given by

$$\eta(N) = g'(x_i^*), \quad \forall i \in S_{\bar{k}}.$$

For $S_{\bar{k}-1}$,

$$\eta(N) + \eta(N \setminus S_{\bar{k}}) = g'(x_i^*), \quad \forall i \in S_{\bar{k}-1}.$$

Repeat this process until all Lagrange multipliers are solved for.

It should be noted here, however, the solution proposed in (15) is only one of the possibly infinitely many solutions for the Lagrange multipliers, since the exact allocations of *Mechanism 1* can result in linearly

¹⁵See, for example, Chapter 4 in [Intriligator \(2002\)](#)

dependent binding constraints. Nonetheless, this does not pose a problem since the exact allocations still satisfy the KKT conditions, which only require the existence of some Lagrangian multipliers satisfying the first-order conditions and complementary slackness. \square

Remark 14. Corollary 2 demonstrates that the essence of fairness concepts such as proportionality and envy-freeness is the symmetry among the readers—instead of the functional form of $G(x)$, see Corollary 3 below—from the perspective of the mechanism. This should come as no surprise since a necessary condition for envy-freeness is equal treatment of equals, which naturally requires the symmetry among the readers, especially in terms of their components in the Nash collective utility of the benevolent social planner.

Going beyond symmetry, the class of mechanisms developed here can also cater specific distributional objectives. For example, readers taking a certain class could be partially favored¹⁶ if the book in discussion is a popular (and presumably expensive) textbook in a university library. Alternatively, in the context of cake cutting, the sugar-free portion of the cake is to be allocated to patient with diabetes in larger pieces than to people without diabetes who just happen to be on a diet and resisting sugar. In general, reader i can be guaranteed an α_i fraction of her reported demand with the following mechanism:

Corollary 3. *Let*

$$G(x) = \sum_{i \in N} \alpha_i \ln(x_i) \text{ where } \alpha_i > 0, \forall i \text{ and } \sum_{i \in N} \alpha_i = 1.$$

Then

$$x_i^* \geq \alpha_i \cdot \lambda(t_i).$$

Remark 15. The proof of Corollary 3 is almost identical to that of Corollary 2—the only caveat is that now Mechanism 1 must be modified to choose the group of readers with the least weighted average of demand with the weights being the α_i 's of the readers in the group. The rest of the proof proceeds precisely like that of Corollary 2 and is hence trivial. Notice that since the desired property is phrased as guaranteed *shares* of reported demand, the objective function as a sum of weighted *natural-log* functions is vital in implementing such allocations.

4.2. Relationship to the monotone comparative statics literature. The proof of Proposition 3 in Appendix B builds upon lattice programming¹⁷ hence does not rely on local properties of the maximizer of

¹⁶In-class readers should be only partially favored over outside-class ones. Completely discriminatory mechanisms such as serial dictatorship with in-class readers being the first group of dictators introduce significant incentives for readers to change identities by, say, registering for the class that was unnecessary for the initially out-class readers. Assuming registering for the class poses some potent cost, a partially favoring mechanism could deter such changes in identities.

¹⁷Topkis (1978) first introduced lattice programming into comparative statics problems. See Topkis (2001) for the classical introduction of lattice programming in economics. See other applications in Athey (2002) (uncertainty), Quah and Strulovici (2009) (orders of functions), Vives (1990) (strategic complementarities), and Vives (2001) (industrial organizations).

$G(x)$ such as differentiability to generate comparative statics results. In fact, even in the simplest two-reader case, any reader's reading time is not differentiable everywhere as a function of the length of her reported demand. Moreover, because of the sheer size of the constrained maximization problem, typical monotone comparative statics results such as the single-crossing property in [Milgrom and Shannon \(1994\)](#) cannot be applied or may be hard to verify. More importantly, in the current constrained optimization problem, the feasible set is simply not a sublattice of \mathbb{R}^n . Trying to establish sufficient conditions for the normality of consumer's demand in constrained utility maximization problems, [Antoniadou \(2004a\)](#) and [Antoniadou \(2004b\)](#) propose an alternative *direct value order* of vectors in \mathbb{R}^n . Essentially, with the n goods arranged in a predetermined order, say in the increasing order of index from good 1 to good n , consumption bundles are then ordered according to the expenditure of the first k goods in an lexicographic manner. The main result of the papers links the supermodularity of the utility function with respect to the direct value order to the normality of the first k goods in the lexicographic order.

The major difficulty of applying the results in [Antoniadou \(2004a\)](#) and [Antoniadou \(2004b\)](#) lies in two folds: the utility maximization problem in these two papers only contains one single constraint; secondly, no representation result is provided as for what constitutes a supermodular utility function with respect to the direct value order. In simplifying these two issues, [Quah \(2007\)](#) takes a more abstract stance on the feasible set of a constrained optimization problem but a more explicit approach to the objective function. [Quah \(2007\)](#) first defines a set of orders of n -dimensional real vectors indexed by a single parameter $\gamma \in [0, 1]$. Take $x, y \in \mathbb{R}^n$ and let $x \vee y$ and $x \wedge y$ be the join and meet of x and y with respect to the Euclidian order. Fitting for the current problem, I only consider the case where x and y are unordered. Consider maximizing some $G(z)$ first subject to $z \in B_1$ then $z \in B_2$. The main result in [Quah \(2007\)](#) states that if $G(\cdot)$ is supermodular and concave in all directions, then for any unordered x and y such that $x \in B_1$ and $y \in B_2$, if there exists a number $\gamma \in [0, 1]$ such that

$$\gamma x + (1 - \gamma)(x \vee y) \in B_2 \text{ and } \gamma y + (1 - \gamma)(x \wedge y) \in B_1,$$

then the maximizer y^* of $G(\cdot)$ subject to B_2 should be no less than x^* subject to B_1 in all elements. Unfortunately, this condition does not easily apply in the current problem. Simply consider

$$x = (1, 1, 5, 7) \text{ and } y = (5, 5, 4, 4) \Rightarrow x \vee y = (5, 5, 5, 7),$$

and there are only two constraints:

$$\begin{cases} B_1 : z_1 + z_2 + z_3 + z_4 \leq 14, & z_1 + z_4 \leq 9; \\ B_2 : z_1 + z_2 + z_3 + z_4 \leq 18, & z_1 + z_4 \leq 9. \end{cases}$$

Then γ must be 0.5 and $0.5x + 0.5(x \vee y)$ would violate the second constraint in B_2 . To bypass this issue, in the proof of Proposition 3 in Appendix B, instead of a single γ for all dimensions, I allow for a vector $\gamma \in [0, 1]^n$ with potentially different weights on x and the join $(x \vee y)$ in different dimensions. Observe that if the scalar γ exists as in Quah (2007), the vector γ certainly exists—hence, the latter is a relaxation from the former. I prove the existence of such a vector in Appendix B by constructing an explicit algorithm of finding at least one γ (See Algorithm O in Appendix B). With potential applications in consumer theory in mind, I provide the following result.

Corollary 4. *Let $G(\cdot)$ satisfy all the **regularity conditions** and let $b(\cdot)$ and $\hat{b}(\cdot)$ be two submodular budget functions with $\hat{b}(\cdot)$ being marginally relaxed from $b(\cdot)$ by i . Let $p \gg 0 \in \mathbb{R}^n$. Let x^* (\hat{x}^*) be the maximizer of $G(\cdot)$ among x (\hat{x}) such that $p \cdot x$ ($p \cdot \hat{x}$) respects $b(\cdot)$ ($\hat{b}(\cdot)$). Then,*

$$x_i^* \leq \hat{x}_i^*.$$

Remark 16. Replacing the x_i^* 's and \hat{x}_i^* 's in the proof of Proposition 3 with $p_i x_i^*$ and $p_i \hat{x}_i^*$ will establish Corollary 4. Hence, I omit this easy proof.

I interpret Corollary 4 as the utility maximization problem of a consumer subject to rationing or ceilings on the quantities supplied in any arbitrary set of goods with fixed prices across rations¹⁸. This Corollary hence offers a sufficient condition for good i to be non-inferior when only the rations involving i are relaxed.

5. CONCLUDING REMARKS

In this paper, I developed a class of offline strategy-proof and Pareto efficient deterministic circulation mechanisms for finitely many readers (with types given by finite unions of continuous time intervals) of a single copy of a book in a public library. This library problem coincides with the cake cutting problem with piecewise uniform value densities and the class of mechanisms constructed in this paper accommodates distributional concepts ranging from proportionality and envy-freeness to arbitrary guaranteed shares of demonstrated demands.

In addition to the valuable extensions to piecewise constant and piecewise linear value densities in the cake cutting problem suggested by Chen et al. (2013), two directions are particularly interesting for future

¹⁸See Friedman (2009) for an example of a two-good rationing with ration-variant prices.

research, one especially so in the realm of scheduling. The first is the online version of the offline mechanisms created here, where the allocation of the book at any given time does not depend on the readers who have yet to show up. Previous work, spanning from [Lipton and Tomkins \(1994\)](#) to the more recent [Robu et al. \(2012\)](#)¹⁹, largely suggests a ceiling of being 2-competitive in terms of efficiency. Moreover, a rapidly growing literature on dynamic mechanism design (see, for example, [Gallien \(2006\)](#), [Athey and Segal \(2013\)](#), [Bergemann and Välimäki \(2010\)](#), [Pavan et al. \(2013\)](#), [Pai and Vohra \(2013\)](#) and [Gershkov and Moldovanu \(2009\)](#)) offer valuable insights in this aspect, despite the apparent differences in the mechanism designers' objectives and equilibrium concepts of this literature and the current paper. The constrained optimization perspective offered here suggests that an important issue (and perhaps the biggest obstacle) in achieving efficiency with online mechanisms lies in the flows allocated on the arcs corresponding to earlier bins in converting the solutions of the optimization problem into schedules using networks such as [Figure 1](#).

The second meaningful direction is the generalization of the current mechanisms to multiple copies of the book, or even more generally, with different capacities or quotas for different time intervals. Important applications of this problem include random matching mechanisms with dichotomous preferences as in [Bogomolnaia and Moulin \(2004\)](#) that can be extended to approach problems such as college-student matching with college-specific quotas. Combined with future research featuring piecewise constant value densities, this extension into the general capacity and quota environment can offer exciting insights into the connection between the random assignment problems in [Bogomolnaia and Moulin \(2001\)](#) (or in a more general setting, [Budish et al. \(2013\)](#)) and the cake cutting problem.

¹⁹I thank Varun Gupta for suggesting this paper.

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APPENDIX A. PROOF OF PROPOSITION 2

Proposition 5 (Pareto efficiency of \mathcal{G}). *If \mathcal{G} is strategy-proof, then $\mathcal{G}(t)$ is a Pareto efficient schedule when $t = \theta$ for all $\theta \in \Theta^n$. Hence, Mechanism \mathcal{G} is a Pareto efficient mechanism.*

Proof of Proposition 2. Suppose there exists some feasible $\mu' \neq \mathcal{G}(t)$ Pareto dominating $\mathcal{G}(t)$ when $t = \theta$. I maintain the condition $t = \theta$ throughout this proof. Also, it may help to keep in mind that $\mathcal{G}(t)$ satisfies (3). Then

$$\sum_{i \in N} u_i(\mu'; \theta_i) > \sum_{i \in N} u_i(\mathcal{G}(t); \theta_i).$$

On the other hand, for all feasible μ ,

$$\sum_{i \in N} u_i(\mu; \theta_i) = \sum_{i \in N} \int_{\theta_i} \mu_i(r) dr \leq \int_{\bigcup_{i \in N} t_i} \sum_{i \in N} \mu_i(r) dr \leq \int_{\bigcup_{i \in N} t_i} 1 dr \leq \lambda \left(\bigcup_{i \in N} t_i \right).$$

Thus, to prove $\mathcal{G}(t)$ is Pareto efficient, it suffices to show that

$$(16) \quad \sum_{i \in N} u_i(\mathcal{G}(t); \theta_i) = \lambda \left(\bigcup_{i \in N} t_i \right).$$

Remark 17. Note that, in general, (16) does not imply Pareto efficiency if the readers' utility functions are not defined to be the amount of reading time within intervals of demand—simply multiply any one reader's reading time by 2 to obtain her utility function, some schedule satisfying (16) may not be Pareto efficient anymore. However, a Pareto efficient schedule under the current definition of utility will remain so, regardless of different definitions of the utility functions as long as they represent the same preferences of the readers who only strictly prefer more reading time. I have only chosen a utility representation that facilitates the characterizations of Pareto efficient schedules.

I first prove the second condition in Remark 7.

Suppose, instead, $\mathcal{G}(t)$ is such that there exists an interval (r_1, r_2) such that

- (1) $(\mathcal{G}_i(t))(r) = 0$ for all $(i, r) \in N \times (r_1, r_2)$;
- (2) And there exists a reader i' such that $(r_1, r_2) \subseteq t_{i'}$.

I will show that changing $\mathcal{G}_{i'}(r)$ from 0 to 1 for all $r \in (r_1, r_2)$ is still feasible and will increase the value of the objective function $G(x)$. Hence, the amount of reading times in the schedule $\mathcal{G}(t)$ do not maximize the objective function, leading to a contradiction. First notice that changing $\mathcal{G}_{i'}$ over (r_1, r_2) will still yield a feasible schedule since the book is still not shared by more than one reader. It suffices to show that increasing the reading time for reader i' is in the feasible set as defined in (2).

Take any $S \subseteq N$. If $i' \notin S$, the total reading time of readers in S is not changed by the reassignment of (r_1, r_2) , so any constraint corresponding to such S with $i' \notin S$ remains satisfied. On the other hand,

$$i' \in S \Rightarrow (r_1, r_2) \subseteq t_{i'} \subseteq \bigcup_{i \in S} t_i,$$

which implies

$$(17) \quad \sum_{i \in S} \int_{t_i} (\mathcal{G}_i(t))(r) dr = \int_{(\bigcup_{i \in S} t_i) \setminus (r_1, r_2)} \sum_{i \in S} (\mathcal{G}_i(t))(r) dr \leq \lambda \left(\bigcup_{i \in S} t_i \right) - \lambda((r_1, r_2)).$$

Assigning (r_1, r_2) to reader i' will increase the first and last terms of (17) by $\lambda((r_1, r_2))$, so such an adjustment is still within $B(t)$ as defined in (2). I have then established that

$$(18) \quad \sum_{i \in N} (\mathcal{G}_i(t))(r) = 1, \forall r \in \bigcup_{i \in N} t_i.$$

Recall that $\mathcal{G}(t)$ satisfies (3), which implies that, for all $i \in N$,

$$(19) \quad u_i(\mathcal{G}(t); \theta_i) = \int_{\theta_i} (\mathcal{G}_i(t))(r) dr = \int_{t_i} (\mathcal{G}_i(t))(r) dr = \int_{\bigcup_{i \in N} t_i} (\mathcal{G}_i(t))(r) dr.$$

Summing (19) over all readers and using (18) yields (16), which completes the proof. \square

APPENDIX B. PROOF OF PROPOSITION 3

Proposition (Non-inferiority). *Let $G : \mathbb{R}_+^n \rightarrow \mathbb{R}$ satisfy all the **regularity conditions**. Let $\hat{b}(\cdot) \neq b(\cdot)$ be two submodular budget functions such that $\hat{b}(\cdot)$ is marginally relaxed from $b(\cdot)$ by i . Let*

$$\begin{cases} x^* \equiv \arg \max G(x) \text{ such that } \sum_{i \in S} x_i \leq b(S), \forall S \subseteq N \equiv \{1, 2, 3 \dots, n\}; \\ \hat{x}^* \equiv \arg \max G(x) \text{ such that } \sum_{i \in S} \hat{x}_i \leq \hat{b}(S), \forall S \subseteq N. \end{cases}$$

Then, there exists an $S^* \subseteq N$ such that $i \in S^*$ and

$$\sum_{j \in S^*} x_j^* = b(S^*) \Rightarrow x_j^* \leq \hat{x}_j^*, \forall j \in S^*.$$

Remark 18. The proof is motivated by a supermodularity-like property built in the objective function $G(x)$.

Take any two vectors $x, y \in \mathbb{R}_+^n$ and let $w, z \in \mathbb{R}_+^n$ be such that

$$(20) \quad w + z = x + y \text{ and } \min \{x_i, y_i\} \leq w_i, z_i \leq \max \{x_i, y_i\}, \forall i \in N.$$

Take any $i \in N$. Since $g_i(\cdot)$ is strictly concave, (20) implies that

$$(21) \quad g_i(x_i) + g_i(y_i) \leq g_i(w_i) + g_i(z_i)$$

with the inequality being strict if the inequalities in (20) are strict—note that the inequalities in (20) must be strict simultaneously since $w_i + z_i = x_i + y_i$. To see (21), let

$$\gamma_i \equiv \frac{w_i - \min \{x_i, y_i\}}{\max \{x_i, y_i\} - \min \{x_i, y_i\}} \in [0, 1] \Rightarrow 1 - \gamma_i = \frac{z_i - \min \{x_i, y_i\}}{\max \{x_i, y_i\} - \min \{x_i, y_i\}} \in [0, 1],$$

then

$$w_i = \gamma_i \max \{x_i, y_i\} + (1 - \gamma_i) \min \{x_i, y_i\} \text{ and } z_i = \gamma_i \min \{x_i, y_i\} + (1 - \gamma_i) \max \{x_i, y_i\}.^{20}$$

Without loss of generality, assume $x_i \leq y_i$. By the concavity of $g_i(\cdot)$

$$\begin{cases} g_i(w_i) = g_i(\gamma_i y_i + (1 - \gamma_i) x_i) \geq \gamma_i g_i(y_i) + (1 - \gamma_i) g_i(x_i); \\ g_i(z_i) = g_i(\gamma_i x_i + (1 - \gamma_i) y_i) \geq \gamma_i g_i(x_i) + (1 - \gamma_i) g_i(y_i); \end{cases} \Rightarrow g_i(w_i) + g_i(z_i) \geq g_i(x_i) + g_i(y_i).$$

Thus, it must be the case that

$$G(x) + G(y) \leq G(w) + G(z)$$

²⁰ $w_i + z_i$ must be equal to $x_i + y_i$ for γ_i to be well-defined.

with the inequality being strict if the inequalities in (20) is strict for at least one $i \in N$. Indeed, setting one of w_i and z_i to be the minimum between x_i and y_i and the other to be the maximum, this property of $G(x)$ directly translates into supermodularity—the key condition is the additivity. The major problem with such an exercise of directly picking the minimum and maximum hence generating the meet and join of x and y is that, as pointed out by various authors (see, for example, Quah (2007), Antoniadou (2007), and Mirman and Ruble (2008)), the join of x and y may not be in the feasible region, breaking the critical link in the proof for monotonic comparative statics of constrained optimization problems with a revealed-preference argument.

Proof of Proposition 3. Notation-wise, let $S_h(x^*, \hat{x}^*)$ and $S_l(x^*, \hat{x}^*)$ be the subsets of N where the optimizer increase and decrease, respectively, i.e.

$$\begin{cases} S_h(x^*, \hat{x}^*) \equiv \{j \in N | x_j^* < \hat{x}_j^*\}; \\ S_l(x^*, \hat{x}^*) \equiv \{j \in N | x_j^* > \hat{x}_j^*\}. \end{cases}$$

Since there will be no ambiguity when the two optima are given, I drop the arguments of $S_h(\cdot, \cdot)$ and $S_l(\cdot, \cdot)$ henceforth and simply use S_h and S_l to stand for the two subsets *only within this proof*. The union of S_h and S_l may be a proper subset of N . However, this does not hinder any of the following arguments in this proof. Also, it may help to keep in mind that i is the dimension in which the budget function is marginally relaxed while j indexes a generic element of N . First notice that the Proposition is trivially true if no budget including i is binding at x^* or when $i \in S_h$ and $x_i^* = b(\{i\})$. Suppose otherwise and assume the statement of the Proposition is false.

Both S_h and S_l are nonempty: S_h is nonempty due to the optimality of x^* while S_l is nonempty by assuming the negation of the Proposition. Moreover, optimality's also imply that for any $j \in N$, $M_j(x^*; b(\cdot)) \neq \emptyset$ and $M_j(\hat{x}^*; \hat{b}(\cdot)) \neq \emptyset$ —otherwise, increase such an x_j^* by a small enough $\epsilon > 0$, for example, will not violate any budget but increase the value of the objective function $G(\cdot)$. On the other hand, negating the Proposition also implies that

$$(22) \quad \left\{ i \in S \subseteq N \text{ and } \sum_{j \in S} x_j^* = b(S) \right\} \Rightarrow S \cap S_l \neq \emptyset.$$

Consider the following claims, followed immediately by their very short proofs:

$$(1) \quad \forall j \in S_h, M_j(x^*; b(\cdot)) \cap S_l \neq \emptyset;$$

Proof. Take any $j \in S_h$,

$$i \in M_j(x^*; b(\cdot)) \Rightarrow M_j(x^*; b(\cdot)) \cap S_l \neq \emptyset \text{ by (22).}$$

When $i \notin M_j(x^*; b(\cdot))$,

$$b(M_j(x^*; b(\cdot))) = \hat{b}(M_j(x^*; b(\cdot))),$$

since the budget function was only marginally relaxed by i . $M_j(x^*; b(\cdot))$ not including any element of S_l implies a violation of the budget for $M_j(x^*; b(\cdot))$. \square

(2) $\forall j' \in S_l, M_{j'}(\hat{x}^*; \hat{b}(\cdot)) \cap S_h \neq \emptyset$.

Proof. Otherwise, the budget for $M_{j'}(\hat{x}^*; \hat{b}(\cdot))$ is violated at x^* with respect to $b(\cdot)$. \square

Now, consider the following algorithm based on the claims above.

Algorithm \mathcal{O} :

- | | |
|--|----------------------|
| SI Choose any $j_0 \in S_h$ choose any j_1 from $M_{j_0}(x^*; b(\cdot)) \cap S_l$; Let $\kappa \geq 2$; | (Initiation) |
| SE Choose any j_κ from $M_{j_{\kappa-1}}(\hat{x}^*; \hat{b}(\cdot)) \cap S_h$ in even-numbered step $\kappa \geq 2$; | (Even-numbered step) |
| SO Choose any j_κ from $M_{j_{\kappa-1}}(x^*; b(\cdot)) \cap S_l$ in odd-numbered step $\kappa \geq 3$; | (Odd-numbered step) |
| ST Terminate when there is a candidate j_κ in the set $\{j_1, j_2, \dots, j_{\kappa-1}\}$; | (Termination) |
| SC Store the sequence $(j_\phi, j_{\phi+1}, \dots, j_{\kappa-1})$ after termination, | |
| where j_ϕ is a candidate j_κ in the set $\{j_1, j_2, \dots, j_{\kappa-1}\}$. | (Collection) |
-

Remark 19. Algorithm \mathcal{O} always terminates in finitely many steps since n is finite. It is also well-defined in initiation and procedure because of claims 1 and 2 above. Observe that the stored sequence after termination always contains an even number of elements of N , with the sequence alternating between elements of S_h and S_l . Moreover, for a $j_\phi \in S_h$ (or S_l),

$$j_{\phi+1} \in M_{j_\phi}(x^*; b(\cdot)) \text{ (or } M_{j_\phi}(\hat{x}^*; \hat{b}(\cdot))).$$

That is, if the proceeding coordinate is in S_h (S_l), the immediately succeeding coordinate in the stored sequence must be in the core of the proceeding one at x^* (\hat{x}^*) with respect to $b(\cdot)$ ($\hat{b}(\cdot)$). The stored sequence will be the coordinates to be perturbed in order to generate two vectors between x^* and \hat{x}^* to imply the non-optimality of at least one of them, resulting in a contradiction.

I now switch to the slacking budgets. Let $\varepsilon \geq 0$ and $\hat{\varepsilon} \geq 0$ be the smallest surpluses of the slacking budgets $b(\cdot)$ and $\hat{b}(\cdot)$ at x^* and \hat{x}^* , respectively:

$$\begin{cases} \varepsilon \equiv \min \left\{ b(S) - \sum_{j \in S} x_j^* \mid S \subseteq N \text{ and } b(S) > \sum_{j \in S} x_j^* \right\}; \\ \hat{\varepsilon} \equiv \min \left\{ \hat{b}(S) - \sum_{j \in S} \hat{x}_j^* \mid S \subseteq N \text{ and } \hat{b}(S) > \sum_{j \in S} \hat{x}_j^* \right\}. \end{cases}$$

Also, $\sigma^* \equiv \min_{j \in S_h \cup S_l} \{|x_j^* - \hat{x}_j^*|\} > 0$. If neither ε nor $\hat{\varepsilon}$ is well-defined—that is, no budget is slacking at either x^* or \hat{x}^* with respect to $b(\cdot)$ or $\hat{b}(\cdot)$, set $\varepsilon^* = \sigma^*$. When only one of ε and $\hat{\varepsilon}$ is well-defined, set ε^* to be the minimum between σ^* whichever one of ε and $\hat{\varepsilon}$ is well-defined. When both ε and $\hat{\varepsilon}$ are well-defined, set $\varepsilon^* = \min\{\varepsilon, \hat{\varepsilon}, \sigma^*\}$. Last but not least, set $\beta^* \equiv \varepsilon^*/(2N) > 0$. For any given $S \subseteq N$, let $\mathbf{1}_S$ be the vector with ones in the coordinates in S and zeros everywhere else. Let $S^c \equiv \{\phi, \phi + 1, \dots, \kappa - 1\}$, the set of coordinates in the sequence stored after the termination of Algorithm \mathcal{O} . Let $S_h^* \equiv S_h \cap S^c$ and $S_l^* \equiv S_l \cap S^c$. Notice that $S_h^* \cup S_l^* = S^c$. Consider the following two vectors obtained from x^* , \hat{x}^* , and the stored sequence out of Algorithm \mathcal{O} :

$$\begin{cases} y^* = x^* + \beta^* \cdot (\mathbf{1}_{S_h^*} - \mathbf{1}_{S_l^*}); \\ \hat{y}^* = \hat{x}^* - \beta^* \cdot (\mathbf{1}_{S_h^*} - \mathbf{1}_{S_l^*}). \end{cases}$$

By construction, for $j \in (N \setminus S_h^*) \setminus S_l^*$, $x_j^* = y_j^*$ and $\hat{x}_j^* = \hat{y}_j^*$. Furthermore,

$$\begin{cases} x_j^* < y_j^* < \hat{y}_j^* < \hat{x}_j^*, & \forall j \in S_h^*; \\ \hat{x}_j^* < \hat{y}_j^* < y_j^* < x_j^*, & \forall j \in S_l^*. \end{cases} \Rightarrow (x^* \wedge \hat{x}^*) \leq y^*, \hat{y}^* \leq (x^* \vee \hat{x}^*).$$

Therefore, by the **regularity conditions** of $G(\cdot)$,

$$(23) \quad G(y^*) + G(\hat{y}^*) > G(x^*) + G(\hat{x}^*).$$

as suggested in Remark 18. I now verify that y^* respects $b(\cdot)$ —the verification of \hat{y}^* respecting $\hat{b}(\cdot)$ follows identically and is thus omitted. By the construction of β^* , any slacking budget at x^* is still slacking at y^* with respect to $b(\cdot)$. Moreover, by the constructions of S_h^* and S_l^* , any binding budget at x^* with respect to $b(\cdot)$ containing m elements of S_h^* must contain m elements of S_l^* . Thus, any such binding budget is still binding at y^* with respect to $b(\cdot)$. Therefore, y^* respects $b(\cdot)$. Similarly, \hat{y}^* respects $\hat{b}(\cdot)$. Hence, (23) presents a contradiction to the optimality of x^* and \hat{x}^* . \square

