Fairness in Tiebreak Mechanisms: A Market Design Approach for Penalty Shootouts*

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Abstract

In the current penalty shootout mechanism in major soccer elimination tournaments, where a coin toss decides which team will kick first, each team alternately takes five penalty kicks. A team taking the first kick wins the shot-out with more than 60% chance, however. We define a sequentially fair mechanism such that each of the skill-balanced teams has exactly 50% chance to win a shootout whenever the score is tied at the end of any round. It turns out that there is only one such exogenous mechanism and all other sequentially fair mechanisms we find are endogenous, in which kicking-order patterns take the score at that round into consideration. Given this multitude of sequentially fair mechanisms, we resort to other criteria to refine the set of desirable mechanisms further. We show that there is a unique sequentially fair mechanism with minimum possible switches in the kicking order and with maximum goal efficiency.

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1 Introduction

Market design as a general field seeks to provide practical solutions to various resource allocation problems, in which monetary transfers are often unavailable. This relatively new field has already made concrete contributions and important applications in solving many real-life problems. In particular, it has already enjoyed impressive successes in applying economics tools and insights to improve the methods for allocating resources in organizing markets such as the one between medical interns and residents, in assigning students to public schools or to courses at a given university, in allocating housing to new immigrants and dorms to college students, and in creating paired kidney exchanges between kidney donors with medical incompatibilities and transplant patients, among others.[1] What makes these successes of market design more impressive is the limitation or absence of monetary transfers acting as a very serious constraint in these real-world allocation problems (e.g., public school slots and human kidneys are not allowed to be traded for money), which means that efficiency and fairness need to be achieved through other means. As such, in this paper we will explore a new application of this field by attempting to design a practical solution to the problematic penalty shootouts, which currently constitute the only way to determine the winning team worldwide when a match score is tied in major soccer elimination tournament matches after the regular 90-minute period and the 30-minute extra time.

Soccer is not only the leading sports in the world (in terms of its fan base, its revenues, and the number of players playing soccer in organized leagues), but also has profound - albeit at times negative - impact on ordinary people's daily lives and even on countries:[2] Following a World Cup elimination match between Honduras and El Salvador, soccer has been blamed for

[1] For more on recent accomplishments of market design, see for instance Abdulkadiroğlu and Sönmez (2013), Che (2010), Nobel Prize Organization (2012), Sönmez and Ünver (2011), Sönmez and Ünver (2013).

[2] "... Over 3.2 billion people, or 46.4 percent of the global population, saw at least one minute of in-home television coverage of the event, representing an eight percent rise on figures recorded during the 2006 FIFA World Cup in Germany," FIFA World magazine reported in its August/September 2011 issue (http://www.fifa.com/worldcup/archive/southafrica2010/organisation/media/newsid=1473143/).

The sports industry's annual revenues are estimated to exceed $600 billion as of 2009, which in turn exceeds half of the annual export revenues of the U.S. In particular, soccer accounts for 43% of the measurable part of these revenues, namely of all ticketing, media and marketing revenues (see Zygband and Collignon, 2011). In addition, prominent soccer teams easily compare to major conglomerates. Forbes reports that Real Madrid posted a revenue of $650 million during the 2011-12 season and is worth $3.3 billion. Manchester United, which is ranked second on Forbes list with a $3.17 valuation, has its shares trading publicly. "Those who bought in, including legendary investor George Soros, have been richly rewarded. Most recently trading at $17, shares of the soccer team have outperformed the S&P 500 by better than two-to-one since the IPO" (see Ozanian, 2013).

FIFA reported that 265 million people played soccer in organized leagues in 2007 worldwide (http://www.fifa.com/worldfootball/bigcount/).

instigating a 100-hour war that took place in 1969 between these two neighboring countries with devastating consequences.[3] In addition, Edmans et al (2007) report significant market declines ensuing soccer losses; these declines are stronger after important losses such as in the World Cup elimination stage matches. Given these, it is not surprising that a special attention is paid to soccer as a major social and economic phenomenon throughout the world, in particular to major national- or club-team level soccer elimination tournaments (e.g., the World Cup, the European Cup, the Champions League, etc.) - with an elevated interest in its match-deciding penalty shootouts.

In the current shootout mechanism used since 1970[4] - with a minor tweak in 2003 -,[5] each team takes five penalty kicks from the penalty mark in an alternating order, and the order of the kicks is decided by the referee's initial coin toss such that the team that wins the coin toss gets to kick first in each round. An important feature of the current mechanism has been intact since its inception, however: if the shootout score too is tied after each team takes five penalty kicks, sudden-death rounds are reached, which go on until the tie is broken such that the kicking order is preserved in these extra rounds as well.

Currently penalty shootouts in soccer are deeply problematic, since, as recently Apesteguia and Palacios-Huerta (2010) found out, teams, which get to take the first penalty kick, win the penalty shootout with more than 60 percent of the time.[6] Clearly, teams that win the coin toss and get to kick first cannot be constantly better than teams that lose the coin toss. Thus, this finding, in a sense, implies that, winning the initial coin toss is also very important alongside the penalty-kicking and goal-saving skills of the two teams' players

[3] "On July 14, 1969, the armed forces of the Republic of El Salvador invaded the territory of the neighboring Republic of Honduras. The attack began a war that lasted only 100 hours, but left several thousand dead on both sides, turned 100,000 people into homeless and jobless refuges, destroyed half of El Salvador's oil refining and storage facilities, and paralyzed the nine-year old Central American Common Market" (Durham, 1979, p. 1).

[4] Until 1970, matches in major soccer tournaments that were tied after the extra time were either replayed in two days or decided by a coin toss. In fact, in the 1968 European football championship, the semi-final match between Italy and the Soviet Union was decided by a coin flip. Italy won the coin toss and proceeded to the final match against Yugoslavia, which too ended in a tie after the extra time. But this time the public outrage of soccer fans following the Italy-Soviet Union match and the fact that there was no further match left in the tournament led authorities to repeat the final match in two days; Italy won the match. The impracticality of repeat matches before the final match (causing a major delay in the tournament schedule) and strong public aversion to determine the outcomes of such highest level matches by a coin flip led FIFA in 1970 to replace both coin tosses and match replays by penalty kicks after the extra time instead.

[5] Since 2003, the team that wins the toss decides which team kicks first - needless to say, almost invariably all team captains that win the toss chose their team to kick first.

[6] In their corrigendum, Apesteguia and Palacios-Huerta (2011) state that the winning percentage of teams taking the first kick would be %64 in "the most complete specifications of columns 3 and 6" instead of %60.5, which was reported in their original paper.

in winning a critical major soccer elimination match. Thus, a situation with two teams that are totally balanced in terms of their players' shootout abilities and possess exactly the same number of opportunities to perform the same task is not necessarily sufficient to yield a 'sequentially fair' outcome where each team is expected to win the shootout with 50% probability whenever the score is tied after any round in a penalty shootout.[7]

Without a doubt this concern also played a role when, between 1993 and 2004, FIFA experimented with the Golden Goal and Silver Goal.[8] It was hoped that these parallel measures would produce more offensive flair and attacking play during extra time, and thus would effectively reduce the number of penalty shootouts which strongly favored the team that won the initial coin toss.[9]

A primary question to pose then would be the following: is it possible to devise a shootout mechanism in which each of the teams, totally-balanced in terms of skills, has exactly 50% chance to win the penalty shootout whenever the score is tied after any round in contrast to the current mechanism.[10] Thus, with a sequentially fair mechanism, the probability of winning is supposed to depend only on these teams' kicking skills of their kickers and goal-saving skills of their goalkeepers (i.e., goalies). The age-old Aristotelian Justice rests on

[7] A precursor of our concept of sequential fairness can be found in Che and Hendershott (2008), which uses a static one-shot version of it, ex-post fairness: In the National Football League (NFL), matches that end in a tie are determined by a sudden-death overtime, in which the first team that scores wins. Like in soccer penalty kicks, the initial coin toss - and the consequent opening possession - yields a significant advantage to the team which wins it, and thus overall the outcome fails to be ex-post fair. To minimize the impact of luck imposed by the initial coin toss, i.e., to make sure that the team that receives the opening possession has no real advantage, Che and Hendershott (2008, 2009) propose "auctioning off" or "dividing-and-choosing" the starting possession to potentially restore ex post fairness.

[8] The golden goal is a type of sudden-death ending for a match, which was introduced in 1993; the team that scores the first goal during extra time is the winner and the game ends when a golden goal is scored. The similar silver goal, which was announced in 2002, supplemented the golden goal during 2003 and 2004 as an alternative remedy especially in UEFA tournaments. With the silver goal, in extra time the team leading after the first fifteen minute half would win, but the game would no longer stop the instant a team scored.

[9] Nevertheless, it was widely thought that golden goal encouraged teams to play more defensively to safeguard against a loss. Teams often placed more emphasis on not conceding a goal rather than scoring a goal, and many golden-goal extra time periods remained scoreless. When he introduced the silver goal to the world press, "we believe that this will be good for clubs, players and fans," said UEFA communications director Mike Lee. "We have addressed the problems created by the golden goal which many in the game have identified. The new system will encourage positive football in the extra-time period, and produce a sensible and fairer ending to a game." Despite these high expectations, the silver goal lasted much shorter than its original golden counterpart. See http://www.theguardian.com/football/2003/apr/28/newsstory.sport10, http://www.telegraph.co.uk/sport/football/2373763/Silver-goal-loses-its-shine-for-rule-makers.html and http://www.uefa.com/news/newsid=71448.html.

[10] See Table 6 of Apesteguia and Palacios-Huerta (2010).

a two-part priciple: equals need to be treated equally and unequals unequally.[11] In other words, in our context, it would be unjust (1) for a team to win a shootout with a probability higher than 50% if the other team is equally skilled, and (2) for both teams to have an exactly 50% chance to win the penalty shootout whenever they are unequal in skills - i.e., the better team should have a higher probability to win the shootout. In this paper the answer to the above question turns out to affirmative; indeed, we find that there is a multitude of such sequentially fair mechanisms (we will elaborate more on the types of different sequentially fair mechanisms below).

First, what can be said about the kicking skills of players? The soccer players who take the kicks are among the highest paid professionals in the world and the task they have to perform - kick a ball once - is one of the simplest and effortless tasks they could possibly perform in soccer, albeit each such kick involves an element of risk and thus can turn out to be costly for the kicker if he misses it. It is possible that even the most skilled topnotch kickers may not be able to hit an exact spot with the ball from 12 yards (approximately 11 meters) every time at a sufficient speed to elude a high-caliber goalie.

The following quote by Italy's Roberto Baggio, who in the 1994 World Cup final's penalty shootout missed one of the most important penalty kicks in the history of soccer against Brazil, provides strong implications about plausible assumptions regarding players' preferences and various basic physical aspects of a penalty kick:[12]

"As for the penalty, I don't want to brag but I've only ever missed a couple of penalties in my career. And they were because the goalkeeper saved them not because I shot wide. That's just so you understand that there is no easy explanation for what happened at Pasadena. When I went up to the spot I was pretty lucid, as much as one can be in that kind of situation. I knew [the Brazilian goalie] Taffarel always dived so I decided to shoot for the middle, about halfway up, so he couldn't get it with his feet. It was an intelligent decision because Taffarel did go to his left, and he would never have got to the shot I planned.

Unfortunately, and I don't know how, the ball went up three metres and flew over the crossbar. ... I failed that time. Period. And it affected me for years. It is the worst moment of my career. I still dream about it. If I could erase a moment from my career, it would be that one."

[11] See Aristotle (1999, p. 76).

[12] Roberto Baggio played for top clubs such as Juventus, AC Milan and Inter Milan, winning many Italian League titles, Italian Cups and UEFA Cups, and was voted European Footballer of the Year in 1993. His five goals in the tournament had helped Italy to reach the final's match of the 1994 World Cup against Brazil. With a 0-0 tie in the match after the regular and extra time, the shootout was reached. When the shootout score was 3-2, it was Baggio's turn to kick. Baggio had to score to keep Italy's chances alive in the contest. He aimed for the middle and the ball sailed over the crossbar. Years later, Baggio referred to this miss in his 2001 autobiography 'Una Porta Nel Cielo' via the quote below.

As Apesteguia and Palacios-Huerta (2010) have also noted, players' kicks and their outcomes have "enormous consequences not only for their individual careers, but also for their team, their city and even their country as in a World Cup final, for instance." From a team's perspective a goal is preferred to a non-goal, and clearly there is no difference at all between a saved kick and a kick that is missed outright, i.e., a kick that goes out or hits the goal post. From Baggio's quote, we also infer that, from a player's perspective on the other hand, while scoring a goal is the best outcome and the goalie's save is to some extent a face-saving outcome, missing the penalty kick outright can be a devastating outcome for a kicker. Thus, a kick can be really costly for the kicker if he misses it outright.

One can not, however, posit whether a player's individual utility from his kick or his collective utility from his team's winning the shootout should outweigh one another. E.g., a player can still be happy to some extent if he scored his penalty kick while his team lost the shootout; likewise, a player can still be somewhat heartbroken and unhappy if he missed his penalty kick outright while his team won the shootout. This implies a reasonable amount of autonomy between a player's utilities from his individual perspective and his team (or collective) perspective.

We also infer from Baggio's quote that goalies may feel the need to dive. This is because, at the optimal speed-accuracy combinations of world-class kickers, the kicked ball typically takes around 0.3 seconds to reach the goal line (i.e., plane). [13] This 0.3-second window is less than the total of roughly 0.2 seconds reaction time of the goalie to clearly observe the kick direction of the ball first, plus the time during his dive to reach the expected arrival spot of the ball before it reaches the goal plane. Hence, a goalie cannot afford to wait until he clearly observes the kick direction. To be able to prevent a goal with non-trivial probability, the goalie must pick a side to dive - or alternatively stay in the middle -[14] at the time the ball is kicked. That way, the goalie can save the ball, albeit with less than 50% chance overall; to see that note that, to make a save, he has to be lucky enough to dive in the correct direction first and then be able to reach the ball as well, implying a less than 100% chance to save a goal even after diving in the correct direction - staying in the middle will likewise lead to a less than 50% chance to save a goal, and clearly less than a dive would.

We now turn to the primary question we intend to answer: is it possible to devise a sequentially fair shootout mechanism. We will use a recursive definition of sequential fairness: in a sequentially fair mechanism, at any round when the score is tied, the expected probability of each team winning the shootout is 50% between two totally-balanced teams and higher than 50% for the better team between two unbalanced teams, in any Markov

[13] See, for instance, Harford (2006), Chiappori et al (2002), and Palacios-Huerta (2003).

[14] See Bar-Eli et al (2007). But, as Baggio's quote also indicates, a shot aimed at the middle may be missed outright or hit the feet of the diving goalie and thus can be saved, even if the goalie dives.

(stationary) equilibrium between two fully-balanced teams.[15]

We first consider exogenous mechanisms, which, like the current mechanism, has a predetermined, random or fixed, kicking-order pattern by teams which does not depend on the score at any round; we also consider endogenous mechanisms of which endogenous kicking-order pattern takes into consideration whether one of the teams is behind in scoring or the score is tied after a round. We find that there is only one exogenous mechanism, namely the random-order mechanism - in which the kicking order before any round is determined by coin flip - that is sequentially fair. Any other exogenous mechanism, including the current mechanism and even the one in which each round the kicking order would reverse, namely the alternating order mechanism, turn out to fail sequential fairness.[16][17] Consequently, all of the other sequentially fair mechanisms we identify turn out to be endogenous.

Any sequentially fair endogenous mechanism we find is such that, when the score is tied at the end of any round, the kicking order can be determined arbitrarily (randomly or not), while when any team is ahead at the end of any round, that team goes first with a particular probability in the next round - where throughout the penalty kicks this probability remains fixed for any team that is ahead score-wise. Thus, there is a continuum of ex-post mechanisms. Sequentially fair mechanisms also satisfy the second part of the Aristotelian Justice principle: among two teams with unequal kicking skills, the better team will have a higher probability to win the shootout; any sequentially unfair mechanism, and thus the fixed order mechanism, fails to satisfy it.

Because of the continuum of the sequentially fair mechanisms we find, one needs to resort to other criteria beside sequential fairness to refine the set of mechanisms that can be deemed desirable. Among the sequentially fair mechanisms, some have higher goal probabilities than others. In our market-design section, we discuss the relative merits of different ex-post fair mechanisms in terms of these additional criteria. We show that there is a unique sequentially fair mechanism with the maximum goal efficiency, which we term the Behind-First mechanism, as follows: team one that won the coin toss kicks first in the next round as long as the score is tied or team one is behind in score; once the other team, team two, falls behind in score after some round, it starts kicking first in any round until team one falls

[15] A stationary Perfect Bayesian equilibrium is the one in which each kicker of each team adopts the same strategy given a particular score and order (to be made precise later). We use a two-round game with continuation rounds that can potentially go on forever until the tie is broken by one of the teams. Thus, it is an infinite-horizon stochastic hidden-action Markov game.

[16] The current mechanism is marred by multiple equilibria, vast majority of which not being sequentially fair.

[17] A more elaborate version of the alternating order mechanism, namely the Prouhet-Thue-Morse sequence (which was proposed by Palacios-Huerta, 2012, to replace the current mechanism) also turns out to fail sequential fairness. These mechanisms, however, are sequentially fair if they are employed in the sudden-death rounds, which are taken after the five regular kicks yield a tie.

behind in score, after which team one kicks first in any round until the sudden-death rounds. This mechanism also satisfies another important property, Instant Rectifiability, which all exogenous and most endogenous mechanisms fail to satisfy.

2 Related Literature

We will now provide a brief review of the relevant literature. Apart from the papers mentioned in the Introduction, our paper is related to the following strands of research as well. The first of these strands is on regular penalty kicks in soccer matches (Chiappori et al, 2002, Palacios-Huerta, 2003, and Bar-Eli et al, 2007), the second one is on incentives various different rules or their combinations - some partially involving penalty shootouts as well - would give to teams (Brocas and Carrillo, 2004, and Carrillo, 2007), and the third one is on economic design of sporting contests (Szymanski 2003's literature review, and Groh et al, 2012).

Chiappori et al (2002) studied soccer penalty kicks both theoretically and empirically to test mixed strategies, while Palacios-Huerta (2003) did so with a much more empirical focus. Chiappori et al (2002), considered regular penalty kicks in the French first league over a two-year period and in the Italian first league over a three-year period. Palacios-Huerta (2003) considered regular penalty kicks from Spanish, Italian, English and various other first leagues over a five-year period. As alluded to above, neither study considered any penalty kicks in penalty shootouts. Bar-Eli et al (2007), after studying mostly regular penalty kicks and some shootout penalty kicks in some championships, observed that goalies almost always jump right or left whereas it would also be optimal for goalies to stay in the goal's centre, at least with some probability.[18] They proposed an explanation for this behaviour via norm theory: goalies do so because it is the norm to jump.

Brocas and Carrillo (2004) show that, in terms of the incentives of teams to play offensively, the three-point victory rule (3PV) may not be more beneficial than 2PV, and the combination of 3PV with golden goal is more beneficial than 3PV alone. Carrillo (2007) considers having the penalty shootout before extra time where the shootout outcome counts only if the tie is preserved during extra time. He finds that this rule promotes offense (defense) for the team that loses (wins) the shootout. He also provides conditions under which this rule would dominate the current regime in terms of offensive play.

[18] Both Chiappori et al (2002) and Palacios-Huerta (2003) find that kickers kick it to the middle relatively rarely and goalies choose to stand in the middle even less often; thus a kicker kicking it to the middle and the goalie waiting in the middle is not a highly expected outcome – although Chiappori et al (2002) find that kickers achieve their highest average scoring probability by kicking to the middle (it is also noted by Chiappori et al, 2002, however, that kickers in regular penalty kicks of these non-elimination regular matches do not kick to the middle unless their team's score advantage is large enough).

The third strand focuses on topics of economic design of sporting contests such as the optimal number of entrants/teams in a race/league, the optimal structure of prizes (revenue sharing) for a tournament (league), and so on - see Szymanski (2003) for a review of this literature. Another interesting recent topic in this latter literature is on optimal seedings of teams or players in elimination tournaments - see, for instance, Groh et al (2012).

3 Model

Two soccer teams, which we refer to as Teams 1 ($T1$ for short) and 2 ($T2$ for short), are facing-off in a penalty shootout. Each team shall take n sequential rounds of penalty shots. Each round consists of one team kicking first, and after observing the outcome of that shot, the second team taking the next shot. If one team scores more goals than the other at the end of these n rounds, then the ahead team wins the game and eliminates the other team. We refer to this n rounds as the *regular rounds*. Throughout the paper we will assume that $n = 2$. This is sufficient to characterize sequential fairness and analyze the current scheme and other proposed mechanisms, such as the tennis tiebreak mechanism, 'alternating order.' With $n = 2$, the analysis is tractable and yet rich enough to capture the multi-round feature of penalty shootouts.[19]

If the teams are still tied at the end of the regular rounds, the format reverts to *sudden-death*; that is each team takes on additional round of shots and then if one team scores while the other one does not, then the ahead team wins the game; otherwise next round of sudden-death penalty shots are taken. We refer to the sudden-death rounds as $n+1, n+2, ...$

As potentially the game can continue forever, we assume that each team consists of an infinite number of kickers and each kicker takes at most one shot.[20]

A penalty kick consists of a probabilistic event with three outcomes: Either a goal is scored (G), the shot goes out wide (O), or the shot is saved by the goalie (S). The latter two outcomes lead to the same score for the team: a goal is not scored.

While each kicker is a strategic player, the goalie is modeled as a probabilistic machine. The goalie waits in the middle of the goal line prior to the shot. He jumps one side or the other with probabilities $\frac{1}{2} : \frac{1}{2}$ prior to the penalty shot, as he needs to react early to have any realistic chance to save the kick. So with probability $\frac{1}{2}$ he reaches to the same side of the goal as the kick goes. Hence, we model the goal line a one dimensional line segment $[0, 1]$,

[19] We have $n = 3$ results and can obtain results for $n > 3$. We skip those as the analysis will be extremely cumbersome and lengthy without providing any further insight.

[20] In reality, each soccer player can take at most one shot, unless all players in his team have already kicked penalty shots. As each team consists of 11 players, 11 shots need to be taken by each team before any player can kick a second shot. As $n = 5$, this happens very rarely.

where $x = 0$ refers to the center of the goal, and $x = 1$ refers to the goal pole on the side the kick goes.

Each kicker, who is a single round player in our game, has an action summarized as aiming to coordinate $x \in [0, 1]$ of the goal line. When a kicker aims at x, the exact spot the ball reaches on the goal line is determined by a continuous probability density function σ_x in a closed support $[\underline{\epsilon}_x, \overline{\epsilon}_x]$ for some $\overline{\epsilon}_x > x > \underline{\epsilon}_x \geq 0$. The spot the ball reaches, y, is observable by all other players, but not the intended spot, x. Both x and y are observable by the kicker himself. Moreover, given the shot is aimed at x, there is a $P_G(x) > \frac{1}{2}$ chance that a goal will be scored; and a $P_O(x)$ probability that, the shot will go out. Hence, the shot is saved by the goalie with probability $1 - P_G(x) - P_O(x)$.[21] We assume that P_G, P_O, and σ_x for all $x \in [0, 1]$ are all common knowledge.

We assume that P_G is twice continuously differentiable strictly concave function, single-peaked at $\overline{x} \in (0, 1)$. Coordinate $\overline{x}$ is the optimal spot for scoring a goal. We assume that neither aiming at exactly the middle, $x = 0$, or a goal post, $x = 1$.[22] Function P_O, on the other hand, is an increasing twice continuously differentiable convex function. Increasing P_O is immediate to motivate: the closer to the middle the ball is aimed, the lower is the chance the ball will go out. Single-peakednes of P_G is also easy to motivate: Whenever the ball is aimed at low x values, it can be saved with a higher chance by the diving goalie (he can save with his lower part of body or his hands as he dives). For higher x values, although the goalie's chances of saving the ball decreases as he may no longer reach it, its chances of going out increases. Hence, there is an optimal spot for the highest goal probability $\overline{x}$. Concavity of P_G and convexity of P_O are primarily assumed for the tractability of our analysis, and do not play any other major role for the interpretation of our results.

We assume that each kicker of both teams is identical in ability and has the same goal scoring and kicking out probability.

A **shootout mechanism** is a function, ϕ, which assigns a probability $\phi(h^{k-1}, g_{T1} : g_{T2})$ to $T1$ kicking first in Round k, given the sequence of first kicking teams in the first $k-1$ rounds is $h^{k-1} = (h_i^{k-1})_{i=1}^{k-1}$ where $h_i^{k-1} \in \{T1, T2\}$ is the team that kicked first in Round i and that $g_{T1} : g_{T2}$ is the score (i.e., the goals scored by $T1$ and $T2$, respectively) at the beginning of

[21] Actually P_G and P_O are summary functions obtained from the following process: The spot the ball reaches, y, is observable by all other players, but not the intended spot, x. If $y > 1$ then the ball goes out. So $P_O(x) = \int_{y=1}^{\overline{\epsilon}_x} \sigma_x(y)dy$. On the other hand, the goalkeeper can save the ball which arrives spot y with probability $S(y)$, which is a continuous function. Hence, $P_G(x) = \int_{\underline{\epsilon}_x}^{1} [1 - S(y)]\sigma_x(y)dy$. Hence, we assume that the family of densities $\{\sigma_x\}_{x\in[0,1]}$ and save probability function S have all the properties that need the below restrictions to hold for P_G and P_O.

[22] Even if the save probability could be very low when the kick exactly arrives at $y = 0$, the kick deviation probability σ_0 makes it a zero probability event that the ball will arrive exactly at $y = 0$. And once the ball deviates even a bit from the middle, a diving goalie can potentially save it with his lower part of his body or with his hands with a relatively high probability.

Round k. Thus, the probability of $T2$ kicking first in Round k is $1 - \phi(h^{k-1}; g_{T1} : g_{T2})$.

Each shootout mechanism ϕ induces a hidden action extensive-form game, which we will simply refer to as **the game**, such that the exact spot that each kicker aims the ball on the goal line is unobservable. Given the current state $(h^{k-1}; g_{T1} : g_{T2})$, for Rounds $k = 1, 2, ...$, the order of first kicking teams in the previous $k - 1$ Rounds h^{k-1}, and feasible scores $g_{T1} : g_{T2}$, the nature determines which team will kick next, with probability $\phi(h^{k-1}; g_{T1} : g_{T2})$ $T1$ kicking next. Then a kicker of the team takes the penalty shot, observing the state and the history of the outcomes of all the shots up to that point as goal, out, or save. The kicker aims to some spot $x \in [0, 1]$ to maximize his expected individual payoff (which we explain in the next paragraph). Then the nature determines with probability distribution $P_G(x), P_O(x), 1 - P_G(x) - P_O(x)$ whether the penalty kick results with a goal, goes out, or is saved, respectively. After the outcome of this shot is observed, the other team's kicker takes a penalty shot observing the history of the outcomes of the shots up to that point. We continue until the end of regular rounds $k = n$ similarly. If the score is tied, then we continue with the sudden-death rounds until the tie is broken at the end of a Sudden-Death Round $k > n$.

Each kicker aims to maximize his expected individual payoff in the game. Each kicker's payoff function consists of two additive components. The first one is the utility received when his teams wins or loses the shootout: V_W is the win payoff and $V_L < V_W$ is the loss payoff. This component of the payoff is common to each kicker of the same team. The second component of the individual payoff consists of an individual outcome based valuation: If the kicker scores a goal he gets the utility $U_G > 0$, if he kicks the ball out he receives a payoff $U_O < 0$, and if the goalie saves the kick he receives a payoff $U_S = 0$. This is a normalization that makes sure that scoring a goal is most desirable outcome, and kicking the ball out is less desirable than kicking the ball inside the goal frame and yet the goalie saves the ball. With this normalization, we can also drop a variable from our notation without affecting our analyses. Overall ex-post payoff of a kicker is then

$$u = V_t + U_p$$

where $t \in \{W, L\}$ refers to the overall team outcome, win or loss; and $p \in \{G, O, S\}$ refers to the kicker's penalty outcome, goal, out, or save.

An **information set** H_i that team i moves with positive probability consists of the exact spot the ball went for each of the previous kicks, the team of the kick, and whether the kick was scored as a goal, went out, or was saved by the goalie. This is the only thing observable by the team i kicker moving in information set H_i. Each information set has an associated round, order of kicking in the round as 1^{st} or 2^{nd}, and a current score difference between $T1$ and $T2$.

A **strategy** X_i is a function from the set of information sets that team i moves with positive probability to $[0,1]$, the spots that each player can target while taking the penalty shot.

This is a sequential hidden action game, as each player only observes where the ball goes and whether the kick was a goal, out, or a save in previous kicks, but not the intended spot the ball was kicked. Hence, as a kicker takes a penalty shot, he has a belief over intended spots of previous kicks. Formally, a **belief** $\mu(H_i)$ is a function that maps each information set H_i that team i moves with positive probability to a probability distribution over histories of actions taken that would lead to the same information set.

Our solution concept is a refinement of perfect Bayesian equilibrium, which we first define: A **perfect Bayesian equilibrium** in the game of shootout mechanism ϕ is a strategy profile - belief profile pair $\left[X=(X_1,X_2),\mu=(\mu(H_1),\mu(H_2))\right]$ such that

- for each team i, and information set H_i, $X_i(H_i)\in[0,1]$ maximizes the expected value over possible ex-post payoffs u of i's kicker among all spots in $[0,1]$ given $X_i(H_i')$ for all $H_i'\neq H_i$, X_j for $j\neq i$, and $\mu(H_i)$.
- for each team i and information set H_i, $\mu(H_i)$ is consistently derived by Bayes' rule from ϕ, X, P_G, P_O, $\mu(H_i')$ for all $H_i'\neq H_i$, and $\mu(H_j)$ for $j\neq i$.

Observe that each kicker is a one-shot player and he maximizes his individual expected payoff over his ex-post payoffs u defined in Equation **??**. Exact formulation of this expected payoff will be clear in our analysis.

Since, we are making a fairness analysis over different shootout mechanisms, we will focus on a symmetric equilibrium concept:

A **symmetric assessment** (X,μ) is defined as

- **In regular rounds:** $X_i(H_i)=X_j(H_j')$ and $\mu_i(H_i)=\mu_j(H_j')$ for any team $j=1,2$ where both information sets H_i and H_j' pertain to the same Regular Round $k\leq n$, and the same kicking order, 1^{st} or 2^{nd}, in the round while the score difference between $T1$ and $T2$ in H_i, s, and in H_j, s', satisfy $s=-s'$ if $j\neq i$ and $s=s'$ if $j=i$.
- **In sudden-death rounds:** $X_i(H_i)=X_j(H_j')$ and $\mu_i(H_i)=\mu_j(H_j')$ for any team $j=1,2$ where information sets H_i and H_j' may involve different Sudden Death Rounds $k>n$ and $k'>n$ respectively but they refer to the same kicking order, 1^{st} or 2^{nd}, while the score difference between $T1$ and $T2$ in H_i, s, and in H_j, s', satisfy $s=-s'$ if $j\neq i$ and $s=s'$ if $j=i$.

A symmetric assessment dictates two players in the same or different teams to aim exactly at the same intended spot and have exactly the same beliefs if they were in each other's shoes.

We distinguish regular and sudden-death rounds in the definition of a symmetric assessment because, every sudden-death round is identical if the game reaches it, while each regular round is different.

A **symmetric equilibrium** of a shootout mechanism ϕ is defined as a perfect Bayesian equilibrium that is a symmetric.

A symmetric equilibrium may not exist in general. It turns out that the mechanisms we will consider, the current mechanism and our new proposals, all have one or more symmetric equilibria.

We define the key design concept in our analysis as follows using symmetric equilibria: a mechanism ϕ is **sequentially fair** if at all symmetric equilibria, at any state $(h^{k-1}; g_{T1} : g_{T2})$ with $g_{T1} = g_{T2}$, - i.e., when they are tied at the beginning of Round k for any k -, each team has exactly 50% chance of winning.

Our desiderata are determining whether the current mechanism is sequentially fair, inspecting other plausible mechanisms, and characterizing the class of sequentially fair mechanisms.

4 The Current Scheme: The Fixed-Order Mechanism

The current shootout scheme is the fixed order mechanism, in which the first kicker is determined before Round 1 with an even lottery and then it continues with the same kicking order throughout. Formally, the **fixed order mechanism** ϕ is defined as follows:

$$\phi(\emptyset; 0:0) = 0.5 \quad \text{and} \quad \phi(h^{k-1}; g_{T1} : g_{T2}) = \begin{cases} 1 & \text{if } h_1^{k-1} = T1 \\ 0 & \text{if } h_1^{k-1} = T2 \end{cases}$$

for all Rounds $k > 0$, orders of first kicking teams in the previous $k-1$ rounds h^{k-1}, and feasible scores $g_{T1} : g_{T2}$ at the beginning of Round k.

We will characterize the symmetric equilibria of the fixed order mechanism. Therefore, without loss of generality assume that $T1$ wins the coin toss before Round 1, and kicks first throughout.

Since it is a sequential game with particular types of hidden actions, we can use backward induction in the regular rounds. To do that, we need to characterize the symmetric equilibria in the sudden-death rounds.

4.1 Analyzing the Sudden-Death Rounds of the Fixed Order Mechanism

At symmetric equilibria, if they exist, each $T1$ kicker will use exactly the same action when HE kicks in the sudden-death rounds, as $T1$ always goes first and the score is tied at the beginning of each sudden-death round.[23]

Similarly, by symmetry, each $T2$ kicker will use exactly the same action when his team is behind (which can be by one goal at most); and he will use exactly the same action when the score is even (which can happen if the preceding $T1$ kicker kicks out or HIS kick is saved).

On the other hand, $T1$ and $T2$ kickers may potentially use different actions at symmetric equilibria, as they kick in different orders: in each round $T1$ going first and $T2$ going second.

Hence, if a symmetric equilibrium exists, the probability of Team i winning at the beginning of each sudden-death round is the same for each $i = 1, 2$.

At a symmetric equilibrium, let us define V_{T1} to be the *value function of* $T1$, that is the expected utility it contributes by winning or losing to its all kickers, in the first sudden-death round. Denote by x the optimal kicking strategy for $T1$'s kicker. Define V_{T2}^B to be the value function of $T2$ in the first sudden-death round when $T2$ is currently behind by one goal, V_{T2}^E to be the value function of $T2$ in the first sudden-death round when the score is currently even. $T2$'s kicker's optimal kicking strategy in each scenario is y_B and y_E respectively.

We can write the following *Bellman* equation for V_{T1}, where recall that $P_G(x) \geq 0.5$ is the goal probability when the kick is aimed at x, V_W is the team victory payoff for each kicker, and V_L is the team loss payoff for each kicker:

$$\begin{aligned} V_{T1} = {} & P_G(x)P_G(y_B)V_{T1}^* && \text{(if both teams score, one more round is played)} \\ & + P_G(x)[1 - P_G(y_B)]V_W && \text{(if } T1 \text{ scores and } T2 \text{ cannot, then } T1 \text{ wins)} \\ & + [1 - P_G(x)]P_G(y_E)V_L && \text{(if } T2 \text{ scores and } T1 \text{ cannot, then } T1 \text{ loses)} \\ & + [1 - P_G(x)][1 - P_G(y_E)]V_{T1}^* && \text{(if both teams miss, one more round is played)} \end{aligned}$$

where V_{T1}^* is the continuation payoff attributed to $T1$ in case game goes to a second sudden-death round.

For $T2$, we have

$$V_{T2}^B = P_G(y_B)V_{T2}^* + [1 - P_G(y_B)]V_L \tag{1}$$
$$V_{T2}^E = P_G(y_E)V_W + [1 - P_G(y_E)]V_{T2}^* \tag{2}$$

where

$$V_{T2}^* = V_W + V_L - V_{T1}^*$$

is the continuation payoff attributed to $T2$ in our win-or-lose game. In the right-hand sides of Equations 1 and 2, the first addend refers to the event that will happen if $T2$'s kicker scores (game goes to a new round when $T2$ was behind in Equation 1, and $T2$ wins when game was tied in Equation 2, respectively, before $T2$'s kicker took his shot) and the second

[23]we will denote Team 1 kickers with upper-cased pronouns HE, HIM, or HIS, and Team 2 kickers with lower-cased pronouns he, him, or his, whenever appropriate.

addend refers to the event that will happen if $T2$'s kicker misses ($T1$ wins, if $T2$ was behind in Equation 1, and game goes to a new round when the score was tied in Equation 2, respectively, before $T2$'s kicker took his shot).

Next, we solve the decision problem faced by each kicker given other player's actions and beliefs. Recall that each kicker is a single-shot player maximizing his expected payoff, which is the sum of his individual kick payoff and his team's win-or-lose payoff. So at symmetric equilibrium, $T1$'s kicker in the first sudden-death round solves

$$\max_{x\in[0,1]} V_{T1} + P_G(x)U_G + P_O(x)U_O + [1 - P_G(x) - P_O(x)]0$$

where V_{T1} is given in Equation **??**, taking V^*_{T1} (HIS team's continuation payoff to be determined through the decisions of subsequent kickers in HIS team and kickers of the other team) together with y_E and y_B given. Recall that $U_G > 0$, $U_O < 0$, 0 are the individual kick payoffs HE gets if his kick becomes a goal with probability $P_G(x)$, goes out with probability $P_O(x)$, and is saved by the goalie with probability $1 - P_G(x) - P_O(x)$, respectively.

The objective function in Equation **??** is twice continuously differentiable. We seek an interior local solution to it and then we verify that it is indeed a global maximum.

The first order necessary condition for an interior local maximum x^* is written as

$$P'_G(x^*)\Big[P_G(y_E)[V^*_{T1} - V_L] + [1 - P_G(y_B)][V_W - V^*_{T1}] + U_G\Big] + P'_O(x^*)U_O = 0. \tag{3}$$

Second order condition is given by for all $x \in [0, 1]$,

$$P''_G(x)\Big[P_G(y_E)[V^*_{T1} - V_L] + [1 - P_G(y_B)][V_W - V^*_{T1}] + U_G\Big] + P''_O(x)U_O < 0, \tag{4}$$

as $P''_G < 0$, $P''_O \geq 0$, $U_O < 0$, $U_G > 0$, and $P_G(y_E)[V^*_{T1} - V_L] + [1 - P_G(y_B)][V_W - V^*_{T1}] \geq 0$ (since $V_W \geq V^*_{T1} \geq V_L$). Hence, the first order condition is also sufficient for global maxima.

Observe that from Equation 3, we get

$$P'_G(x^*) = -\frac{P'_O(x^*)U_O}{P_G(y_E)[V^*_{T1} - V_L] + [1 - P_G(y_B)][V_W - V^*_{T1}] + U_G} > 0 \tag{5}$$

(since by assumption $P'_O(\overline{x}) > 0$ for goal-optimal spot $\overline{x}$). As the optimal spot for a goal is characterized by the first order condition $P'_G(\overline{x}) = 0$ and as P_G is concave, we have $P_G(x)$ is weakly increasing for $x < \overline{x}$ and weakly decreasing for $x > \overline{x}$, which together with Equation 5 in turn implies that

$$x^* < \overline{x}.$$

Hence, $T1$'s kicker does not aim at the optimal spot, but acts more conservatively and aims at a more interior spot, although chance of such a kick being a goal is not maximized.

Similarly, we can write the $T2$'s kicker's decision problem when behind taking V_{T2}^* as given:

$$\max_{y_B \in [0,1]} V_{T2}^B + P_G(y_B)U_G + P_O(y_B)U_O + [1 - P_G(y_B) - P_O(y_B)]0$$

First order condition yields:

$$P_G'(y_B^*)[V_{T2}^* - V_L + U_G] + P_O'(y_B^*)U_O = 0. \tag{6}$$

$T2$'s kicker's decision problem when even taking V_{T2}^* as given can be written as follows:

$$\max_{y_E \in [0,1]} V_{T2}^E + P_G(y_E)U_G + P_O(y_E)U_O + [1 - P_G(y_E) - P_O(y_E)]0$$

First order condition yields:

$$P_G'(y_E^*)[V_W - V_{T2}^* + U_G] + P_O'(y_E^*)U_O = 0 \tag{7}$$

Second order conditions, as in Equation 4, shows the sufficiency of the first order conditions in either case. We skip them for brevity. Also like for $T1$, we get $y_B^* < \overline{x}$ and $y_E^* < \overline{x}$.

At equilibrium, $x = x^*$, $y_B = y_B^*$, and $y_E = y_E^*$, and hence we can solve them using Equations 3, 6, and 7. To do that we need to resolve the continuation values V_{T1}^* and V_{T2}^* for each team.

Hence, it is useful to note that in any symmetric equilibrium $V_{T1} = V_{T1}^*$. Therefore, by Equation **??**,

$$V_{T1}^* = \frac{P_G(x)[1 - P_G(y_B)]V_W + [1 - P_G(x)]P_G(y_E)V_L}{P_G(x)[1 - P_G(y_B)] + [1 - P_G(x)]P_G(y_E)} = \alpha V_W + (1 - \alpha)V_L \tag{8}$$

where the winning probability of $T1$, α is given by

$$\alpha = \frac{P_G(x)[1 - P_G(y_B)]}{P_G(x)[1 - P_G(y_B)] + [1 - P_G(x)]P_G(y_E)}. \tag{9}$$

A value for $\alpha > 0.5$ at a symmetric equilibrium will signal that the fixed order mechanism is biased in favor of the first kicking team in the sudden death rounds (and $\alpha < 0.5$ is vice versa for the second kicking team). On the other hand, the fixed order mechanism is a sequentially fair mechanism if and only if $\alpha = 0.5$ at every symmetric equilibrium. For $T2$ then we get by Equation 8,

$$V_{T2}^* = (1 - \alpha)V_W + \alpha V_L.$$

Hence, Equations 3, 6, and 7 become self-contained to solve for x^*, y_B^* and y_E^*. The following theorem characterizes the symmetric equilibrium strategy candidates solving these equations:[24]

[24] In our analysis, we did not have to model the beliefs of agents explicitly. We use the summary functions P_G and P_O, and the agents have to best respond to what the other players are doing at equilibrium. The beliefs will be crucial in equilibrium selection criterion, though, later in Section XXX.

Theorem 1 (Fixed-Order Mechanism, Sudden-Death Rounds) *(i) A symmetric equilibrium exists if and only if* $P'_G(0)[\frac{V_W - V_L}{2} + U_G] + P'_O(0)U_O \geq 0$.

(ii) When it exists, there are generically multiple symmetric equilibria with strategy profiles (x^*, y^*_B, y^*_E), *all of which are to the left of the goal-optimal spot, satisfying*

- $x^* = y^*_E$, *i.e.* $T1$ *kicker and* $T2$ *kicker when even kick at the same spot; and*
- *for every equilibrium with* (y^*_E, y^*_B), *there exists another equilibrium with* $(\hat{y_E}, \hat{y_B})$ *such that* $\hat{y}_E = y^*_B$ *and* $\hat{y}_B = y^*_E$.

It will be useful to quantify the term "generically" in the above theorem. The below lemma answers this question:

Lemma 1 *Suppose that in the sudden death rounds of the fixed-order mechanism, a symmetric equilibrium exists. Then, multiple symmetric equilibria exist if and only if there are multiple solutions* β *to the equation*

$$\beta = \frac{1 - P_G(y(1-\beta))}{2 - P_G(y(\beta)) - P_G(y(1-\beta))}, \tag{10}$$

where $y(\beta) = f^{-1}\big(- \frac{(V_W - V_L)\beta + U_G}{U_O}\big)$ *for* $f(.) = P'_O(.)/P'_G(.)$.

Moreover, there is an odd number of solutions with $\beta = \frac{1}{2}$ *always being a solution and others being located symmetrically around it. We also have* $y^*_B = y(\beta)$ *and* $x^* = y^*_E = y(1-\beta)$ *for any solution* β.

Thus, generically, the fixed order mechanism is not sequentially fair as the winning probability of $T1$ $\alpha \neq \frac{1}{2}$, whenever $y_B \neq y_E$.

Theorem 2 (Fixed-Order is not fair) *Generically, the fixed order mechanism is not sequentially fair.*

Its proof is immediately implied by Theorem 1.

4.1.1 Equilibrium Refinement

Next, we address the question which symmetric equilibrium is more likely to be observed when there are multiple symmetric equilibria in the fixed-order mechanism.

We use a selection criterion similar to Cho and Kreps (1987) 'Intuitive Criterion.'

Suppose there are multiple symmetric equilibria. Let the symmetric equilibrium with (x^*, y^*_E, y^*_B) be the one with highest x, i.e., the intended spot by $T1$'s kickers is the closest to the optimal spot among all symmetric equilibria. We will refer to this equilibrium as *the* *most aggressive one for* $T1$ for the following reason: As $x^* = y_E^* > y_B^*$, we have the winning probability of $T1$, $\alpha = \frac{1-P_G(y_B^*)}{2-P_G(y_E^*)-P_G(y_B^*)} > \frac{1}{2}$ by Equation 4; and moreover, such a winning probability for $T1$ is the highest among all symmetric equilibria.

Being the first mover, if $T1$ can credibly 'signal' $T2$ that they are indeed playing this most aggressive equilibrium, this would be the most beneficial for $T1$. In this case, we can use such a signaling through beliefs in the symmetric equilibrium to obtain a refinement. For example, if σ_{x^*}, the probability density function of the ball reaching to a particular spot on the goal line when it is aimed at x^* has the support set $[x^* - \underline{\epsilon_{x^*}}, x^* + \overline{\epsilon_{x^*}}]$. Suppose that this support is disjoint from such support sets of other equilibria. Then, whenever $T2$ kickers observe a kick spot in σ_{x^*}'s support, they can credibly deduce that indeed $T1$ is playing this aggressive equilibrium. Hence, beliefs of $T2$'s kickers in information sets that are never reached in a symmetric equilibrium can be fine-tuned so that less aggressive equilibria can be eliminated.

Definition 1 (Refinement Criterion) *If the most aggressive symmetric equilibrium for* $T1$ *involves aiming at* x^* *for each kicker, and the possible spots that the ball can go under* x^* *(as determined by the support of* σ_{x^*}*,* $[x^* - \underline{\epsilon}_{x^*}, x^* + \overline{\epsilon}_{x^*}]$*) is different from any of the spots that the ball can go under all other symmetric equilibria, then* $T1$ *can credibly enforce the most aggressive symmetric equilibrium.*

Hence, we get the following corollary:

Corollary 1 (Team 1 wins more often) *If the symmetric equilibria can be refined, then* $T1$*, the team that kicks first, wins with a higher probability than* $T2$ *in the sudden-death rounds of the fixed-order mechanism.*

Hence, in our pathological analysis with equal-strength players and similar goalies, fixed-order mechanism is biased toward the first mover. This is also supported by the data reported by Apesteguia and Palacios-Huerta (2010) from shout-outs in major tournaments in the world in their Table 6: In the sudden-death rounds $T1$ wins with probability around 59%.

5 Mechanism Design: Sequentially Fair Mechanisms

In the previous section we concluded that the currently used fixed-order mechanism is not sequentially fair. It turns out that even if we introduced a sequentially fair extension to the fixed order mechanism in the sudden-death rounds, it would still be sequentially unfair.

In fact a large class of intuitive mechanisms turn out to be sequentially unfair. Another example of such mechanisms is the alternating-order mechanism, a version of which is used in tennis tie-breaks. In the alternating-order mechanism, in every round, the kicking order reverses. So if $T1$ starts off the shootout, $T2$ kicks first in Round 2, $T1$ kicks first in Round 3, so on so forth. It turns out that even this mechanism is sequentially unfair.[25]

In fact, a large class of mechanisms, which we refer to as *exogenous mechanisms*, turn out to me sequentially unfair. A mechanism ϕ is **exogenous** if $\phi_(k : g_1 : g_2) = \rho(k)$ for some function ρ, i.e. who goes first in each round is determined independent of the current score but as a function of the round we are in. Hence, both fixed-order and alternating order mechanisms are exogenous.[26].

Another interesting exogenous mechanism is the *random-order* mechanism ϕ, which determines who goes first in every round using an even lottery, that it $\phi(k; g_1, g_2) = \frac{1}{2}$ for all k.[27]

Despite its impracticality, one may expect that this exogenous mechanism to be sequentially fair. Indeed, it turns out to be the case. However, the class of sequentially fair mechanisms are far richer than the random-order mechanism. There are some very practical mechanisms in this class.

Next, we characterize the sequentially fair mechanisms. Initially, we will focus on the regular rounds. We will assume that a mechanism that gives sequential fairness in the sudden-death rounds exists (and then show that actually there are many such mechanisms).

We introduce a class of mechanisms that will be crucial in our analysis of sequentially fair mechanisms. A mechanism ϕ is **uneven score symmetric** if for all $(k; g_1 : g_2)$ such that $g_1 \neq g_2$ and $k \leq n$, we have $\phi(k; g_1 : g_2) = 1 - \phi(k, g_2 : g_1)$. That is, as long as the score is not tied, the probability of who goes first is the same for $T1$ and $T2$ whenever they are in each other's shoes. E.g., when $T1$ is ahead $3 : 2$ in (the beginning of) Round 4, and when $T2$ is ahead in Round 4 with score $2 : 3$, in Round 4 $T1$'s probability of kicking first in the first case is the same as $T2$'s probability of kicking first in the second case.

It turns out that such mechanisms fully characterize the sequentially fair mechanisms in the regular rounds.

Theorem 3 (Sequentially fair mechanisms) *Suppose both teams have equal chance of winning in sudden-death rounds. Then a mechanism ϕ is sequentially fair if and only if ϕ is uneven score symmetric in regular rounds.*

[25] Since there are 5 regular rounds in the shootout, Team 1 goes first in three rounds and Team 2 goes first in two rounds in this mechanism. One may think that this unevenness causes sequential unfairness. However, even if we flipped a coin to determine who goes first in Round 5, sequential unfairness would still prevail.

[26] Even the version of the alternating order mechanism in which the 5th round order is determined randomly is exogenous

[27] However, an uneven random order mechanism where the probability of who goes first does not depend on the current score is sequentially unfair.

Interestingly, there is only one sequentially fair exogenous mechanism: The random-order mechanism is also uneven-score symmetric, and hence, sequentially fair.

The theorem makes another interesting point. We do not need to treat both teams symmetrically all the time to obtain sequential fairness. In fact, when the score is tied, it does not matter which team kicks first.

This feature opens the door for some interesting practical mechanisms to be sequentially fair. Two examples of such mechanisms are **ahead first** and **behind first** mechanisms. In ahead first [behind first] mechanism, the team who is ahead [behind] in score after a round kicks first, and otherwise the order of the teams does not change after a round.

These mechanisms are quite practical and we will discuss the reasons behind their practicality in the next section.

There are also many other uneven-score symmetric mechanisms in which lotteries play a significant role. For example, a lottery mechanism that forces the behind team to go first in 75% of the time and $T1$ to go first 60% of the time when the score is tied is also sequentially fair. Moreover, among the deterministic mechanisms, ahead first and behind first are not the only two that are sequentially fair. For example, a mechanism which forces $T1$ to go first in Round 1, then behind team to go first in Round 2, and if the score is tied in Round 2, $T2$ to go first.

5.1 Sequential Fairness in Sudden-Death Rounds

The class of sequentially fair mechanisms is larger when sudden-death rounds are considered additionally.

First we introduce a practical sequentially fair mechanism for the sudden-death rounds.

As we concluded in the previous section, fixed order mechanism fails sequential fairness in the sudden-death rounds miserably. So is there a simple and deterministic mechanism that is sequentially fair in the sudden-death rounds? The answer is affirmative, and the alternating order mechanism *is* sequentially fair in sudden-death rounds, although it is not in regular rounds. The intuition is straightforward: Under alternating order, we can have uneven scores, such as $T1$ being ahead, in an intermediate regular round. Hence, it cannot satisfy uneven-score symmetry as required in a sequentially fair mechanism. On the other hand, in the sudden-death rounds, the score is never uneven at the beginning of a round. Hence, exogeneity of the alternating order does not prevent sequential fairness.[28]

[28] One can argue that the situation is similar for the fixed-order mechanism. So why is it not sequentially fair in the sudden-death rounds? However, the game is an infinite game. The recursive nature of the infinite game can lead to existence of multiple symmetric equilibria for the fixed-order mechanism, in most of which either the first kicking team or the second kicking team has an advantage.

Theorem 4 *The alternating-order mechanism is sequentially fair in sudden-death rounds.*

In fact there are many other mechanisms that are sequential fair:

Theorem 5 *Take any mechanism ϕ and any sequentially fair mechanism φ. Construct a mechanism ψ such that for a given sudden-death Round k, for all $n < \ell < k$ and feasible scores $g_1 : g_2$, $\psi(\ell; g_1 : g_2) = \phi(\ell; g_1 : g_2)$ and for all $\ell \geq k$ and $\ell \leq n$ and feasible scores $g_1 : g_2$, $\psi(\ell; g1 : g_2) = \varphi(\ell; g_1 : g_2)$. Then ϕ is also sequentially fair.*

That is, we can replace the continuation of any mechanism in the sudden-death rounds after some round with a sequentially-fair mechanism, and regardless of what the initial part of the mechanism looks like, the newly constructed mechanism becomes sequentially fair.

6 Market Design and Practical Criteria

Sequential fairness is capable of ruling out many mechanisms, including the mechanism used worldwide currently. Interestingly, it also rules out seemingly a very fair exogenous mechanism in which teams would alternate the kicking order in each round - just like tennis players alternate their serve sequence in their tiebreaks (i.e., the first player takes the first serve, then the second player serves twice, then the first player serves twice, and so on until one of the players wins the set). Nevertheless, a case could easily be made for the latter mechanism over the only sequentially fair exogenous mechanism, namely random mechanism, especially in the sudden-death rounds.

In terms of endogenous mechanisms, however, sequential fairness doesn't pose much restriction. In any such mechanism, when the score is even at the end of a round, it doesn't matter which team kicks first in the next round. In addition, when the score is not even at the end of a round, as long as the same probability is used in determining the team to kick first, whether the winning team or the losing team kicks first wouldn't matter. Thus, one needs a relevant property to help refine the set of sequentially fair mechanisms.

It is not hard to make a case that requiring a high goal efficiency for a penalty shootout mechanism is a desirable property since most soccer fans want to see higher penalty shootout scores - or simply more goals in a match to some extent regardless of how they take place. Thus, a very crucial question is 'does one of the sequentially fair mechanisms have an advantage over others' in terms of goal efficiency?' To that end, we consider the following property:

Maximum Goal Efficiency: *A maximum-goal-efficiency mechanism $\phi \in SF$ is such that there is no other $\phi' \in SF$ which has a higher goal-efficiency whenever the score is even or one of the teams is ahead (or behind) in score at a Round k.*

As a matter of fact it turns out that there is no other sequentially fair mechanism which has a higher goal efficiency than the mechanism below:

Behind-First mechanism: *$T1$ kicks first in Round k as long as the score is tied or $T1$ is behind in score in Round $k-1$; once $T2$ falls behind in score after some Round $k' > k$, $T2$ kicks first until $T1$ falls behind in score after some Round $k'' > k'$, after which $T1$ kicks first, ...*

Then we have the following result:

Theorem 6 *The Behind-First mechanism is the unique sequentially fair mechanism which satisfies maximum goal efficiency.*

In fact, as already hinted above, in practice it would be ideal to combine the behind-first mechanism in regular rounds with the alternating order mechanism (or the Prouhet-Thue-Morse sequence in general) in the sudden-death rounds.

Another relevant concern one can have would be 'whether a mechanism could increase score rectifiability for the team that is behind in score'. Consider the following property:

Instant Rectifiability: *In any $\phi \in SF$, whenever any Ti is behind in score after Round k, Ti should have the chance to make the score discrepancy smaller before Tj where $i \neq j$ has a chance to make the score discrepancy larger in Round $k+1$.*

Thus, instant rectifiability intends that the team that is behind in score catches up with the team that ahead in score as soon as possible before a larger score deficit may arise. A larger score deficit may put the losing team in a totally non-rectifiable position especially towards the end of a shootout which would deprive one of that team's kickers from contributing at all while he could still contribute to his team under another mechanism that satisfies insant rectifiability; e.g., consider the fixed order mechanism suppose that $T1$ is up 4-3 in the beginning of Round 5 - in that case, if $T1$ scores $T2$'s kicker who is supposed to kick at Round 5 won't be able to help his team by kicking. Further, in a mechanism which violates insant rectifiability, this issue could even be more profound and subtle, due to the forward-looking rectifiability concerns of other kickers who will kick before their team may fall behind in score. In the next paragraph, we will try to illustrate how insant rectifiability can also alleviate any such concerns.

Consider a situation in which the third round (R3) ends with a tie; for simplicity, say, the score is 3-3 by then. Consider $T1$'s and $T2$'s kickers who will kick first and second in the fourth round (R4) respectively - whether it is the current mechanism or the behind-first mechanism.

First, consider the current mechanism. $T2$'s kicker at R4 knows that, if he misses his kick, $T1$'s last kicker (at R5) can finish the shootout by scoring and $T2$'s last kicker can't even get an opportunity to contribute to this process on behalf of his team, as elaborate above. So, if he misses, $T2$'s kicker at R4 is likely to put his team and his team's kicker in R5 in a less rectifiable situation compared to that in which $T1$'s kicker at R4 would put his team and his team kicker in R5.

Now, consider the behind-first mechanism or any other similar mechanism which satisfies insant rectifiability. $T2$'s kicker at R4 knows that, even if he misses his kick, right after that his team's last kicker can have a chance to tie the game and put $T1$'s last kicker in a less rectifiable position than he would be in the current mechanism. Thus, in that case, it is true that in the behind-first mechanism $T2$'s last kicker will be in a non-rectifiable position as well, but, compared to the situation with the current mechanism, at least he will be in a position to contribute on his team's behalf instead of facing the possibility of not being able to kick at all. Thus, in the behind-first mechanism, putting his team in a less rectifiable position will shift to both kickers at R5 together, from the likelihood of $T2$'s kicker in R4 putting his team in a less rectifiable position alone by missing his kick.

Clearly, the behind-first mechanism satisfies insant rectifiability as well. In this paper, we chose not to get into a rather complicated equilibrium analysis along the lines of instant rectifiability, simply because our results, based on one simple assumption about a kicker's individual utility ranking in the case of not scoring - where he prefers the goalie saving his shot to his missing his kick outright - already capture the fairness difference between the current fixed order mechanism and our sequentially fair mechanisms. Surely, this type of a rectifiability-based and forward-looking analysis would magnify the fairness differences between these mechanisms, but would not really be needed at the level analysis that suffices for our purposes in this paper.

7 Concluding Remarks

In the recent decades, market design has made many tangible contributions in solving a wide variety of real-life problems most notably in areas such as interns-to-hospitals matching, school choice, kidney exchanges, in which monetary transfers are typically not allowed. In this paper we have explored a new application and designed a practical solution to another challenging real-life problem. To that end, we have considered and analysed sequentially fair mechanisms and other supporting criteria in soccer tiebreaks, i.e., in soccer penalty shootouts.

It has turned out that there is only one such exogenous mechanism and all other sequentially fair mechanisms are endogenous. Because of this multitude of sequentially fair mechanisms, we had to resort to other criteria to refine the set of desirable mechanisms further. We showed that there is a unique sequentially fair mechanism with maximum goal efficiency, i.e., the behind-first mechanism. In that mechanism, team one that won the coin toss kicks first in the next round as long as the score is tied or team one is behind in score; once the other team, team two, falls behind in score after some round, it kicks first until team one falls behind in score, after which team one kicks first until the continuation rounds. The behind-first mechanism also satisfies another relevant property, namely instant rectifiability. Recall that the only significant assumption we used in our analysis is that, as far as a kicker's individual utility is concerned, one should prefer scoring to not scoring and, when it comes to not scoring, he should prefer the goalie saving his shot to his missing his kick outright – without positing as to whether a player's individual utility from his kick or his collective utility from his team's winning the shootout should outweigh one, etc.

Note that these sequentially fair mechanisms and the additional criteria or properties we have considered can help beyond soccer too. For instance, one can easily apply the same idea to tennis tiebreaks in the same manner as well. In addition, ice hockey and field hockey as well as water polo, handball, cricket and rugby also have their tiebreak or penalty shootout mechanisms same as or similar to that of soccer. More generally, all of the sequential player draft mechanisms in major professional leagues in the U.S. such as National Football League (NFL), National Basketball Association (NBA), Major League Baseball (MLB), National Hockey League (NHL), can be considered some special cases of generalized behind-first mechanisms where the more disadvantaged teams (in terms of their league record in the previous season) go first.

In addition, real life teems with tournament competitions which are often at the heart of competitions not only in penalty shootout and draft mechanisms in sports but also in many internal promotions of individuals or in relative support of sales/production teams in firms and other organizations[29] (including funding of some departments vs others in organizations - e.g., academic departments in a university), multi-stage patent races,[30] and many other situations. Like the current fixed order shootout mechanism in soccer, some sequential tournaments may be conducive to a first mover advantage and some others to a second mover advantage, which may impede efficiency and/or fairness of these tournaments.[31] Further analysis of related specific exogenous and endogenous tiebreak mechanims may help the design of new tournament structures with more desirable efficiency or fairness characteristics in the above-mentioned competitions of real life as well.

[29] See, for instance, Prendergast (1999) for a review of this literature.

[30] See Fudenberg et al (1983) and Harris and Vickers (1985) in particular. See Reinganum (1989) for a thorough survey of that literature.

[31] See, for instance, Harris and Vickers (1985), among others.

References

[1] Abdulkadiroğlu, A. and T. Sönmez. 2013. "Matching Markets: Theory and Practice." In D. Acemoglu, M. Arello, and E. Dekel, Editors, *Advances in Economics and Econometrics*, Vol. 1, Cambridge, 3-47.

[2] Apesteguia, J., and I. Palacios-Huerta. 2010. "Psychological Pressure in Competitive Environments: Evidence from a Randomized Natural Experiment." *American Economic Review*, 100, 2548-2564.

[3] Apesteguia, J., and I. Palacios-Huerta. 2011. "Corrigendum: Psychological Pressure in Competitive Environments: Evidence from a Randomized Natural Experiment." *American Economic Review*, 101, 1636-1636.

[4] Aristotle. 1999. *Nicomachean Ethics*, translated by W.D. Ross, Batoche books.

[5] Baggio, R. 2001. *Una Porta Nel Cielo: Un'autobiografia*, (*A Goal in The Sky: An Autobiography* in English), Limina; see the following website for the translated quote used in this paper: http://www.theguardian.com/sport/2002/may/19/worldcupfootball2002.football.

[6] Bar-Eli, M., O.H. Azar, I. Ritov, Y. Keidar-Levin, and G. Schein. 2007. "Action Bias among Elite Soccer Goalkeepers: The Case of Penalty Kicks." *Journal of Economic Psychology*, 28, 606–621

[7] Brocas, I. and J.D. Carrillo. 2004. "Do the 'Three-Point Victory' and 'Golden Goal' Rules Make Soccer More Exciting?" *Journal of Sports Economics*, 5, 169-185.

[8] Carrillo, J.D. 2007. "Penalty Shoot-outs: Before or After Extra Time?" *Journal of Sports Economics*, 8, 505-518.

[9] Che, Y-K. 2010. "Discussion of 'Matching Markets: Theory and Practice' by Abdulkadiroglu and Sönmez," prepared for the 2010 World Congress of Econometric Society.

[10] Che, Y-K. and T. Hendershott. 2008. "How to Divide the Possession of a Football?" *Economics Letters*, 99, 561-565.

[11] Che, Y-K. and T. Hendershott. 2009. "The NFL Should Auction Possession in Overtime Games" *Economists' Voice* 6.9, Article 5.

[12] Chiappori, P.-A., S. Levitt, and T. Groseclose. 2002. "Testing Mixed-Strategy Equilibria When Players Are Heterogeneous: The Case of Penalty Kicks in Soccer ." *American Economic Review*, 92, 1138-1151.

[13] Durham, W. 1979. *Scarcity and Survival in Central American-Ecological. Origins of the Soccer War*, Stanford University Press.

[14] Edmans, A., D. Garcia, and O . Norli. 2007. "Sports Sentiment and Stock Returns." Journal of Finance, 62, 1967-1998.

[15] Fudenberg, D., G. Richard, J. Stiglitz, and J. Tirole. 1983. "Preemption, Leapfrogging and Competition in Patent Races," *European Economic Review*, 22, 3-31.

[16] Groh, C., B. Moldovanu, A. Sela, and U. Sunde. 2012. "Optimal Seedings in Elimination Tournaments." *Economic Theory*, 49, 59-80.

[17] Harford, T. 2006. "World Cup Game Theory: What Economics Tells Us about Penalty Kicks." The Undercover Economist (http://www.slate.com/articles/arts/the_undercover_economist/2006/06/world_cup_game_theory

[18] Harris, C. and J. Vickers. 1985. "Perfect Equilibrium in a Model of a Race." *Review of Economic Studies*, 52, 193-209.

[19] Economic Sciences Nobel Prize Committee, Royal Swedish Academy. 2012. "Stable Allocations and the Practice of Market Design." *Scientific Background on the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012* (http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/2012/advanced.html)

[20] Ozanian, M. 2013. "Soccer's Most Valuable Teams: At $3.3 Billion, Real Madrid Knocks Manchester United From Top Spot." *Forbes* (http://www.forbes.com/sites/mikeozanian/2013/04/17/soccers-most-valuable-teams-real-madrid-dethrones-manchester-united-from-top-spot-at-3-3-billion/).

[21] Palacios-Huerta, I. 2003. "Professionals Play Minimax." *Review of Economic Studies*, 70, 395-415.

[22] Palacios-Huerta, I. 2012. "Tournaments, Fairness and the Prouhet-Thue-Morse Sequence." *Economic Inquiry*, 50, 848-849.

[23] Prendergast, C. 1999. "The Provision of Incentives in Firms." *Journal of Economic Literature*, 37, 7–63.

[24] Reinganum, J.F. 1989. "The Timing of Innovation: Research, Development, and Diffusion." Chapter 14 in R. Schmalensee and R. Willig, Editors, *Handbook of Industrial Organization*, Elsevier, Volume 1, 849-908.

[25] Sönmez, T. and M. U. Ünver. 2011. "Matching, Allocation, and Exchange of Discrete Resources." In J. Benhabib, A. Bisin, and M. Jackson, Editors, *Handbook of Social Economics*, Vol. 1A, North-Holland, 781-852.

[26] Sönmez, T. and M. U. Ünver. 2013. "Market Design for Kidney Exchange." In Z. Neeman, A. Roth, N. Vulkan, Editors, *Oxford Handbook of Market Design*, Oxford.

[27] Szymanski, S. 2003. "The Economic Design of Sporting Contests." *Journal of Economic Literature*, 41, 1137-1187.

[28] Zygband, P. and H. Collignon. 2011. "The Sports Market." AT Kearney (http://www.atkearney.com/paper/-/asset_publisher/dVxv4Hz2h8bS/content/the-sports-market/10192).

A Proofs

Proof of Theorem 1. We drop "*" superscripts for convenience. We rewrite Equations 3, 6, and 7 for first order conditions as:

$$P'_G(x)[P_G(y_B)V_{T1} + (1 - P_G(y_B))V_W - P_G(y_E)V_L - (1 - P_G(y_E))V_{T1} + U_G] + P'_O(x)U_O = 0$$

$$P'_G(y_B)[V_{T2} - V_L + U_G] + P'_O(y_B)U_O = 0$$
$$P'_G(y_E)[V_W - V_{T2} + U_G] + P'_O(y_E)U_O = 0$$

We first prove that $x = y_E$.

Claim 1. $x = y_E$.

Proof of Claim 1. Define

$$\Delta = P_G(y_B)V_{T1} + (1 - P_G(y_B))V_W - P_G(y_E)V_L - [1 - P_G(y_E)]V_{T1} - V_W + V_{T2}.$$

From the first-order conditions of x and y_E, $x \geq y_E$ if and only if $\Delta \geq 0$. Recall that the winning probability of $T1$ in equilibrium, α is given in Equation **??**. Hence,

$$\begin{aligned}\Delta &= P_G(y_B)(V_{T1} - V_W) + P_G(y_E)(V_{T1} - V_L) + V_{T2} - V_{T1} \\ &= P_G(y_B)(1 - \alpha)(V_L - V_W) + P_G(y_E)\alpha(V_W - V_L) + (1 - 2\alpha)(V_W - V_L) \\ &= [-P_G(y_B)(1 - \alpha) + P_G(y_E)\alpha + 1 - 2\alpha](V_W - V_L) \\ &= [1 - P_G(y_B) + (P_G(y_E) + P_G(y_B) - 2)\alpha](V_W - V_L)\end{aligned}$$

We substitute α from Equation **??** as follows:

$$\begin{aligned}
\Delta &= [1 - P_G(y_B) + (P_G(y_E) + P_G(y_B) - 2)\frac{P_G(x)(1-P_G(y_B))}{P_G(x)(1-P_G(y_B)) + (1-P_G(x))P_G(y_E)}](V_W - V_L) \\
&= (1 - P_G(y_B))[1 + \frac{(P_G(y_E) + P_G(y_B) - 2)P_G(x)}{P_G(x)(1-P_G(y_B)) + (1-P_G(x))P_G(y_E)}](V_W - V_L) \\
&= \left[\frac{(1-P_G(y_B))(V_W - V_L)}{P_G(x)(1-P_G(y_B)) + (1-P_G(x))P_G(y_E)}\right] \\
&\quad \times [P_G(x)(1-P_G(y_B)) + (1-P_G(x))P_G(y_E) + (P_G(y_E) + P_G(y_B) - 2)P_G(x)] \\
&= \frac{(1-P_G(y_B))(V_W - V_L)}{P_G(x)(1-P_G(y_B)) + (1-P_G(x))P_G(y_E)}[P_G(y_E) - P_G(x)]
\end{aligned}$$

Suppose $x > y_E$, then as both $x, y_E < \overline{x}$ and P_G is increasing on the left of $\overline{x}$, we have $P_G(x) > P_G(y)$. But then $\Delta < 0$, contradicting that $x > y_E$. Supposition $x < y_E$ leads to a similar contradiction. Therefore, we must have $x = y_E$. $\diamondsuit$

Given $x = y_E$, α can be simplified as

$$\alpha = \frac{P_G(x)(1-P_G(y_B))}{P_G(x)(1-P_G(y_B)) + (1-P_G(x))P_G(y_E)} = \frac{1-P_G(y_B)}{2 - P_G(y_B) - P_G(y_E)},$$

and $\alpha = \frac{1}{2}$ iff $x = y_B$. Then the first-order condition w.r.t. y_B can be simplified as:

$$\begin{aligned}
P'_G(y_B)[V_{T2} - V_L + U_G] + P'_O(y_B)U_O = 0 \\
\Longrightarrow P'_G(y_B)[(1-\alpha)(V_W - V_L) + U_G] + P'_O(y_B)U_O = 0 \\
\Longrightarrow P'_G(y_B)[(V_W - V_L)\frac{1-P_G(y_E)}{2 - P_G(y_B) - P_G(y_E)} + U_G] + P'_O(y_B)U_O = 0
\end{aligned} \tag{11}$$

Similarly, the first-order condition w.r.t. y_E can be simplified as:

$$\begin{aligned}
P'_G(y_E)[V_W - V_{T2} + U_G] + P'_O(y_E)U_O = 0 \\
\Longrightarrow P'_G(y_E)[\alpha(V_W - V_L) + U_G] + P'_O(y_E)U_O = 0 \\
\Longrightarrow P'_G(y_E)[(V_W - V_L)\frac{1-P_G(y_B)}{2 - P_G(y_B) - P_G(y_E)} + U_G] + P'_O(y_E)U_O = 0
\end{aligned} \tag{12}$$

Now we are ready to prove part (i). First we show that $P'_G(0)[\frac{V_W - V_L}{2} + U_G] + P'_O(0)U_O \geq 0$ implies the existence of equilibrium. Define $H(z) \equiv P'_G(z)[\frac{V_W - V_L}{2} + U_G] + P'_O(z)U_O$. $H(z)$ is continuously differentiable with $H'(z) < 0$ as $P''_G(z) < 0$ and $P''_O(z) \geq 0$. Then $H'(0) = P'_G(0)[\frac{V_W - V_L}{2} + U_G] + P'_O(0)U_O \geq 0$ and $H'(\overline{x}) = P'_O(\overline{x})U_O < 0$ implies that there exists some $a \in [0, \overline{x})$ such that $H'(a) = 0$. It can be readily seen that $(x, y_B, y_E) = (a, a, a)$ solves the two first-order conditions, and hence constitutes an equilibrium.

On the other hand, assume now $P'_G(0)[\frac{V_W - V_L}{2} + U_G] + P'_O(0)U_O = H(0) < 0$. As $H'(z) < 0$, $H(z) < 0$ for every $z \in [0, 1]$. Suppose to the contrary that there exists an equilibrium

(x, y_B, y_E). Clearly $y_B \neq y_E$, for otherwise $\frac{1-P_G(y_E)}{2-P_G(y_B)-P_G(y_E)} = \frac{1}{2}$ and the first-order condition of y_B becomes $H(y_B) < 0$. Suppose $y_B > y_E$. Then the first-order condition w.r.t. y_E in Equation 12 becomes:

$$P'_G(y_E)[(V_W - V_L)\frac{1 - P_G(y_B)}{2 - P_G(y_B) - P_G(y_E)} + U_G] + P'_O(y_E)U_O$$
$$< P'_G(y_E)[\frac{V_W - V_L}{2} + U_G] + P'_O(y_E)U_O = H(y_E) < 0,$$

a contradiction! Then $y_B < y_E$; and similarly, the first order condition for y_B is negative, leading to a contradiction. Therefore, an equilibrium exists if and only if $P'_G[\frac{V_W-V_L}{2} + U_G] + P'_O(0)U_O = H(0) \geq 0$.

Generically there are multiple solutions (y_E, y_B) and whenever one exists then $(\hat{y_E}, \hat{y_B})$ satisfying $\hat{y_E} = y_B$ and $\hat{y_B} = y_E$ also lead to a symmetric equilibrium. ■

Proof of Lemma 1. The first order conditions are given by Equations 11 and 12 for y_B and y_E in the proof of Theorem 1, respectively (dropping the superscript "$*$"). We get y_B=y(β)and $y_E = y(1-\beta)$, since $f = P'_O/P'_G$is an invertible differentiable increasing function in the region $[0, \bar{x}]$by assumption that $P_O$is convex and increasing and $P_G$is strictly concave and increasing in the interval $[0, \bar{x}]$. Thus, circularly, plugging in $y_B$and y_E, we get Equation 10. Optimal spots $y_B$and $x = y_E$are multiple valued if and only if $\beta$is multiple valued. $\beta = \frac{1}{2}$always solves Equation 10, and if $\beta = \alpha$is a solution then $\beta = 1 - \alpha$is also a solution. Thus, there are an odd number of solutions. ■

Proof of Theorem 3. We solve it by backward induction. As both teams have equal chance of winning in sudden-death rounds, the value function is $\frac{V_W-V_L}{2}$ for each team at the end of the regular rounds. In Round 2, whether the last-kicking team is currently even or behind, the optimal kicking strategy is always ξ, where ξ is determined by the following first-order conditon:

$$P'_G(\xi)[\frac{V_W - V_L}{2} + U_G] + P'_O(\xi)U_O = 0$$

Next we look into the optimal kicking strategy for the first-kicking team in Round 2. Consider two cases:

Case I. When $T2$ kicks first in Round 2. When the score is currently even, the value function for $T2$ is $\frac{V_W+V_L}{2}$.

When $T2$ is currently behind, the value function for $T2$ is

$$V_{T2,P2,B} = P_G(y_{2B})P_G(\xi)V_L + P_G(y_{2B})(1 - P_G(\xi))\frac{V_W + V_L}{2} + (1 - P_G(y_{2B}))V_L$$

The optimal kicking strategy, y_{2B}, satisfies the following first-order condition:

$$\begin{aligned} P_G'(y_{2B})[P_G(\xi)V_L + (1-P_G(\xi))\frac{V_W+V_L}{2} - V_L + U_G] + P_O'(y_{2B})U_O = 0 \\ \implies P_G'(y_{2B})[(1-P_G(\xi))\frac{V_W-V_L}{2} + U_G] + P_O'(y_{2B})U_O = 0 \end{aligned}$$

On the other hand, when $T2$ is currently ahead, the value function for $T2$ is

$$V_{T2,P2,A} = P_G(y_{2A})V_W + (1-P_G(y_{2A}))[(1-P_G(\xi))V_W + P_G(\xi)\frac{V_W+V_L}{2}]$$

The optimal kicking strategy, y_{2A}, satisfies the following first-order condition:

$$\begin{aligned} P_G'(y_{2A})[V_W - (1-P_G(\xi))V_W - P_G(\xi)\frac{V_W+V_L}{2} + U_G] + P_O'(y_{2A})U_O = 0 \\ \implies P_G'(y_{2A})[P_G(\xi)\frac{V_W-V_L}{2} + U_G] + P_O'(y_{2A})U_O = 0 \end{aligned}$$

As $P_G(\xi) > \frac{1}{2}$, $y_{2A} > y_{2B}$.

Case II. When $T1$ kicks first in Round 2. When the score is currently even, the value function for $T1$ is $\frac{V_W+V_L}{2}$. When $T1$ is currently behind, the value function for $T1$ is

$$V_{T1,P2,B} = P_G(x_{2B})P_G(\xi)V_L + P_G(x_{2B})(1-P_G(\xi))\frac{V_W+V_L}{2} + (1-P_G(x_{2B}))V_L$$

The optimal kicking strategy, x_{2B}, satisfies the following first-order condition:

$$\begin{aligned} P_G'(x_{2B})[P_G(\xi)V_L + (1-P_G(\xi))\frac{V_W+V_L}{2} - V_L + U_G] + P_O'(x_{2B})U_O = 0 \\ \implies P_G'(x_{2B})[(1-P_G(\xi))\frac{V_W-V_L}{2} + U_G] + P_O'(x_{2B})U_O = 0 \end{aligned}$$

On the other hand, when $T1$ is currently ahead, the value function for $T1$ is

$$V_{T1,P2,A} = P_G(x_{2A})V_W + (1-P_G(x_{2A}))[(1-P_G(\xi))V_W + P_G(\xi)\frac{V_W+V_L}{2}]$$

The optimal kicking strategy, x_{2A}, satisfies the following first-order condition:

$$\begin{aligned} P_G'(x_{2A})[V_W - (1-P_G(\xi))V_W - P_G(\xi)\frac{V_W+V_L}{2} + U_G] + P_O'(x_{2A})U_O = 0 \\ \implies P_G'(x_{2A})[P_G(\xi)\frac{V_W-V_L}{2} + U_G] + P_O'(x_{2A})U_O = 0 \end{aligned}$$

As $P_G(\xi) > \frac{1}{2}$, $x_{2A} > x_{2B}$.

Next we study the second team's optimal kicking strategy in Round 1. When $T1$ does not score in Round 1, the value function for $T2$ is

$$V_{T2,P1,E} = P_G(y_{1E})[\phi(T1;0:1)(V_W+V_L-V_{T1,P2,B})+(1-\phi(T1;0:1))V_{T2,P2,A}]+(1-P_G(y_{1E}))\frac{V_W+V_L}{2},$$

where

$$V_{T1,P2,B} = P_G(x_{2B})P_G(\xi)V_L + P_G(x_{2B})(1-P_G(\xi))\frac{V_W+V_L}{2} + (1-P_G(x_{2B}))V_L$$
$$= \frac{V_W+V_L}{2} - [1-P_G(x_{2B})(1-P_G(\xi))]\frac{V_W-V_L}{2}$$

$$V_{T2,P2,A} = P_G(y_{2A})V_W + (1-P_G(y_{2A}))[(1-P_G(\xi))V_W + P_G(\xi)\frac{V_W+V_L}{2}]$$
$$= \frac{V_W+V_L}{2} + [1-(1-P_G(y_{2A}))P_G(\xi)]\frac{V_W-V_L}{2}$$

Therefore

$$V_{T2,P1,E} = \frac{V_W+V_L}{2} + P_G(y_{1E})\{\phi(T1;0:1)[1-P_G(x_{2B})(1-P_G(\xi))]$$
$$+(1-\phi(T1;0:1))[1-(1-P_G(y_{2A}))P_G(\xi)]\}\frac{V_W-V_L}{2}$$

The optimal kicking strategy, y_{1E}, satisfies the following first-order condition:

$$P'_G(y_{1E})\{\alpha_1\frac{V_W-V_L}{2} + U_G\} + P'_O(y_{1E})U_O = 0, \text{ where}$$

$$\alpha_1 = \phi(T1;0:1)[1-P_G(x_{2B})(1-P_G(\xi))] + (1-\phi(T1;0:1))[1-(1-P_G(y_{2A}))P_G(\xi)].$$

When $T1$ scores in Round 1, the value function for $T2$ is

$$V_{T2,P1,B} = P_G(y_{1B})\frac{V_W+V_L}{2} + (1-P_G(y_{1B}))[(1-\phi(T1;1:0))V_{T2,P2,B} + \phi(T1;1:0)(V_W+V_L-V_{T1,P2,A})],$$

where

$$V_{T2,P2,B} = P_G(y_{2B})P_G(\xi)V_L + P_G(y_{2B})(1-P_G(\xi))\frac{V_W+V_L}{2} + (1-P_G(y_{2B}))V_L$$
$$= \frac{V_W+V_L}{2} - [1-P_G(y_{2B})(1-P_G(\xi))]\frac{V_W-V_L}{2}$$

$$V_{T1,P2,A} = P_G(x_{2A})V_W + (1-P_G(x_{2A}))[(1-P_G(\xi))V_W + P_G(\xi)\frac{V_W+V_L}{2}]$$
$$= \frac{V_W+V_L}{2} + [1-(1-P_G(x_{2A}))P_G(\xi)]\frac{V_W-V_L}{2}$$

We substitute the equations of $V_{T2,P2,B}$ and $V_{T1,P2,A}$ into $V_{T2,P1,B}$ as follows:

$$V_{T2,P1,B} = \frac{V_W+V_L}{2} - (1-P_G(y_{1B}))[(1-\phi(T1;1:0))[1-P_G(y_{2B})(1-P_G(\xi))]$$
$$+\phi(T1;1:0)[1-(1-P_G(x_{2A}))P_G(\xi)]]\frac{V_W-V_L}{2}$$

The optimal kicking strategy, y_{1B}, satisfies the following first-order condition:

$$P'_G(y_{1B})\{[(1-\phi(T1;1:0))[1-P_G(y_{2B})(1-P_G(\xi))] \\ +\phi(T1;1:0)[1-(1-P_G(x_{2A}))P_G(\xi)]]\frac{V_W-V_L}{2}+U_G\}+P'_O(y_{1B})U_O=0$$

Given that $y_{2B}=x_{2B}$ and $x_{2A}=y_{2A}$, the first-order condition can be rewritten as

$$P'_G(y_{1B})\{\alpha_2\frac{V_W-V_L}{2}+U_G\}+P'_O(y_{1B})U_O=0, \text{ where}$$

$$\alpha_2=(1-\phi(T1;1:0))[1-P_G(x_{2B})(1-P_G(\xi))]+\phi(T1;1:0)[1-(1-P_G(y_{2A}))P_G(\xi)].$$

Therefore $y_{1B}=y_{1E}$ iff $\alpha_1=\alpha_2$ iff

$$\begin{aligned}
&\phi(0,1)[1-P_G(x_{2B})(1-P_G(\xi))]+(1-\phi(0,1))[1-(1-P_G(y_{2A}))P_G(\xi)] \\
&=(1-\phi(T1;1:0))[1-P_G(x_{2B})(1-P_G(\xi))]+\phi(T1;1:0)[1-(1-P_G(y_{2A}))P_G(\xi)] \\
&\Longleftrightarrow (1-\phi(T1;0:1)-\phi(T1;1:0))[1-(1-P_G(y_{2A}))P_G(\xi)] \\
&=(1-\phi(T1;0:1)-\phi(T1;1:0))[1-P_G(x_{2B})(1-P_G(\xi))] \\
&\Longleftrightarrow \phi(T1;0:1)+\phi(T1;1:0)=1 \text{ or } (1-P_G(y_{2A}))P_G(\xi)=P_G(x_{2B})(1-P_G(\xi))
\end{aligned}$$

Finally, we solve for $T1$'s optimal kicking strategy in Round 1. The value function for $T1$ is

$$\begin{aligned}
V_{T1}&=P_G(x_1)[V_W+V_L-V_{T2,P1,B}]+(1-P_G(x_1))[V_W+V_L-V_{T2,P1,E}] \\
&=V_W+V_L-P_G(x_1)V_{T2,P1,B}-(1-P_G(x_1))V_{T2,P1,E} \\
&=\frac{V_W+V_L}{2}+[P_G(x_1)(1-P_G(y_{1B}))\alpha_2-(1-P_G(x_1))P_G(y_{1E})\alpha_1]\frac{V_W-V_L}{2}
\end{aligned}$$

The optimal kicking strategy, x_1, satisfies the following first-order condition:

$$P'_G(x_1)\{[(1-P_G(y_{1B}))\alpha_2+P_G(y_{1E})\alpha_1]\frac{V_W-V_L}{2}+U_G\}+P'_O(x_1)U_O=0$$

Therefore

$$x_1 \gtreqless y_{1E} \Longleftrightarrow (1-P_G(y_{1B}))\alpha_2 \gtreqless (1-P_G(y_{1E}))\alpha_1$$

On the other hand, we have

$$\begin{aligned}
V_{T1}=\frac{V_W+V_L}{2} &\Longleftrightarrow P_G(x_1)(1-P_G(y_{1B}))\alpha_2=(1-P_G(x_1))P_G(y_{1E})\alpha_1 \\
&\Longleftrightarrow (1-P_G(y_{1B}))\alpha_2=(1-P_G(y_{1E}))\alpha_1
\end{aligned}$$

This condition holds when $\phi(T1;0:1)+\phi(T1;1:0)=1$. When $\phi(T1;0:1)+\phi(T1;1:0)\neq 1$, generically this condition does not hold. ■

Proof of Theorem 4. Without loss of generality assume T1 kicks first in the first sudden-death round (i.e., in Round $n+1$). In a symmetric equilibrium, denote by x_I the optimal kicking strategy for the first kicker in each sudden-death round, and x_B (x_E) the optimal kicking strategy for the second kicker in each sudden-death round when the score is behind (tied). Let V_{T1} (V_{T2}) denote T1's (T2's) value function at the beginning of the first sudden-death round (Round $n+1$). Then

$$\begin{aligned} V_{T1} &= [P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))]V_{T2} \\ &\quad + P_G(x_I)(1-P_G(x_B))V_W + (1-P_G(x_I))P_G(x_E)V_L \\ V_{T2} &= [P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))]V_{T1} \\ &\quad + P_G(x_I)(1-P_G(x_B))V_L + (1-P_G(x_I))P_G(x_E)V_W \end{aligned}$$

We substitute V_{T2} into the equation of V_{T1} as follows:

$$\begin{aligned} V_{T1} &= [P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))]^2 V_{T1} \\ &\quad + \{[P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))]P_G(x_I)(1-P_G(x_B)) + (1-P_G(x_I))P_G(x_E)\}V_L \\ &\quad + \{[P_G(x_I)P_G(x_B) + (1-P_G(x_I))(1-P_G(x_E))](1-P_G(x_I))P_G(x_E) + P_G(x_I)(1-P_G(x_B))\}V_W \end{aligned}$$

Then V_{T1} can be solved as:

$$V_{T1} = \gamma V_W + (1-\gamma)V_L, \text{ where}$$

$$\gamma = \frac{1-(1-P_G(x_I))P_G(x_E)}{2-(1-P_G(x_I))P_G(x_E) - P_G(x_I)(1-P_G(x_B))}.$$

As this is a zero-sum game, we have $V_{T2} = (1-\gamma)V_W + \gamma V_L$.

The optimal kicking strategy, x_I, satisfies the following first-order condition:

$$P'_G(x_I)\{[P_G(x_B) - (1-P_G(x_E))]V_{T2} + (1-P_G(x_B))V_W - P_G(x_E)V_L + U_G\} + P'_O(x_I)U_O = 0.$$

Similarly, the optimal kicking strategies x_B and x_E are determined by the following conditions:

$$\begin{aligned} P'_G(x_B)\{V_{T1} - V_L + U_G\} + P'_O(x_B)U_O = 0 \\ P'_G(x_E)\{V_W - V_{T1} + U_G\} + P'_O(x_E)U_O = 0 \end{aligned}$$

We are going to claim that all three kicking strategies are equivalent, i.e., $x_I = x_B = x_E$, which in turn implies that $V_{T1} = V_{T2} = \frac{V_W+V_L}{2}$ as $\gamma = \frac{1}{2}$, and sequential fairness is established.

First we compare x_I and x_E. Define

$$\Delta_{IE} = [P_G(x_B) - (1-P_G(x_E))]V_{T2} + (1-P_G(x_B))V_W - P_G(x_E)V_L - (V_W - V_{T1})$$

By comparing the first-order conditions of x_I and x_E, we observe that

$$\Delta_{IE} \gtreqless 0 \text{ if and only if } x_I \gtreqless x_E.$$

Substituting the equations of V_{T2} and V_{T1} into Δ_{IE} gives us

$$\begin{aligned}\Delta_{IE} &= V_{T1} - V_{T2} - P_G(x_B)(V_W - V_{T2}) + P_G(x_E)(V_{T2} - V_L) \\ &= [2\gamma - 1 - P_G(x_B)\gamma + P_G(x_E)(1-\gamma)](V_W - V_L) \\ &= [(2 - P_G(x_B) - P_G(x_E))\gamma - 1 + P_G(x_E)](V_W - V_L).\end{aligned}$$

Plugging in the expression of γ and doing some simplifications, we have

$$\Delta_{IE} = \frac{P_G(x_I)(1 - P_G(x_B)) - P_G(x_B)(1 - P_G(x_E))}{2 - (1 - P_G(x_I))P_G(x_E) - P_G(x_I)(1 - P_G(x_B))}(V_W - V_L)$$

We can then conclude that

$$x_I \gtreqless x_E \text{ if and only if } x_I \gtreqless x_B$$

Next we compare x_I and x_B. Define

$$\Delta_{IB} = [P_G(x_B) - (1 - P_G(x_E))]V_{T2} + (1 - P_G(x_B))V_W - P_G(x_E)V_L - (V_{T1} - V_L)$$

By comparing the first-order conditions of x_I and x_B, we observe that

$$\Delta_{IB} \gtreqless 0 \text{ if and only if } x_I \gtreqless x_B.$$

By the same token, we can simplify Δ_{IB} as

$$\Delta_{IB} = \frac{P_G(x_E)(1 - P_G(x_I)) - P_G(x_B)(1 - P_G(x_E))}{2 - (1 - P_G(x_I))P_G(x_E) - P_G(x_I)(1 - P_G(x_B))}(V_W - V_L)$$

Therefore

$$x_I \gtreqless x_B \text{ if and only if } x_E \gtreqless x_B.$$

Finally we compare x_E and x_B. Define

$$\Delta_{EB} = V_W - V_{T1} - (V_{T1} - V_L)$$

Δ_{EB} can be simplified as

$$\Delta_{EB} = \frac{P_G(x_E)(1 - P_G(x_I)) - P_G(x_I)(1 - P_G(x_B))}{2 - (1 - P_G(x_I))P_G(x_E) - P_G(x_I)(1 - P_G(x_B))}(V_W - V_L)$$

Accordingly,

$$x_E \gtreqless x_B \text{ if and only if } x_E \gtreqless x_I.$$

Combining all three observations (inequalities) above, we conclude that in a symmetric equilibrium we must have $x_I = x_E = x_B$. ■

Proof of Theorem 5. Take any mechanism ϕ and any sequentially fair mechanism φ. Construct a mechanism ψ such that for a given Sudden-Death Round k, for all $n < \ell < k$ and feasible scores $g_1 : g_2$, $\psi(\ell; g_1 : g_2) = \phi(\ell; g_1 : g_2)$ and for all $\ell \geq k$ and $\ell \leq n$ and feasible scores $g_1 : g_2$, $\psi(\ell; g1 : g_2) = \varphi(\ell; g_1 : g_2)$.

Now in the Sudden-Death Round k and after, whenever the game reaches to this round, the probability of winning is given as $\frac{1}{2}$ for each team. By backward induction, consider Round $k-1$. Consider the team that kicks second. Without loss of generality suppose it is $T2$, and $T1$ goes first in Round $k-1$. We can reuse the same first order conditions for both teams that we used in the proof of Theorem 1 setting

$$V_{T1} = V_{T2} = \frac{V_W + V_L}{2}$$

is the continuation value under the sequentially fair mechanism in Round k. Suppose x is $T1$'s kicker's optimal spot, y_E is $T2$'s kicker's optimal spot when they are still tied and y_B is $T1$'s kicker's optimal spot when $T1$ is ahead (by one goal). We remind the first order conditions Equations 3, 6, and 7 as:

$$P_G'(x)[P_G(y_B)V_{T1} + (1 - P_G(y_B))V_W - P_G(y_E)V_L - (1 - P_G(y_E))V_{T1} + U_G] + P_O'(x)U_O = 0$$

$$P_G'(y_B)[V_{T2} - V_L + U_G] + P_O'(y_B)U_O = 0$$

$$P_G'(y_E)[V_W - V_{T2} + U_G] + P_O'(y_E)U_O = 0$$

We rewrite T2's kicker's first order conditions plugging in $V_{T1} = V_{T2}$:

$$P_G'(y_B)[\frac{V_W - V_L}{2} + U_G] + P_O'(y_B)U_O = 0$$

$$P_G'(y_E)[\frac{V_W - V_L}{2} + U_G] + P_O'(y_E)U_O = 0$$

The last two equations yield $y_B = y_E$ (each has a unique solution by assumptions). Given that $T1$'s equation yields:

$$P_G'(x)[\frac{V_W - V_L}{2} + U_G] + P_O'(x)U_O = 0$$

As $T1$ has the same first order conditions as $T2$, we get $x = y_B = y_E$. So each team's winning probability is the same, $\frac{1}{2}$ in Round k, as well. The mechanism ψ is sequentially fair starting from Round k. We repeat this argument for each Sudden-Death Round $\ell = k-2, k-3, ..., n+1$ and obtain the desired result. ■

Proof of Theorem 6. From the proof of Theorem 3, we observe that any sequentially fair mechanism must satisfy the condition $\phi(T1; 0:1) + \phi(T1; 1:0) = 1$. Moreover, under this condition, the three optimal kicking strategies in the first round are the same: $x_1 = y_{1E} = y_{1B}$, and they are determined by the following first-order condition:

$$P'_G(x_1)\{\alpha_1 \frac{V_W - V_L}{2} + U_G\} + P'_O(x_1)U_O = 0, \text{ where}$$

$$\alpha_1 = \phi(T1; 0:1)[1 - P_G(x_{2B})(1 - P_G(\xi))] + (1 - \phi(T1; 0:1))[1 - (1 - P_G(y_{2A}))P_G(\xi)].$$

Hence the higher the value of α_1, the higher the goal efficiency. As $x_{2B} < \xi$ and $y_{2A} < \xi$, $1 - P_G(x_{2B})(1 - P_G(\xi)) > 1 - (1 - P_G(y_{2A}))P_G(\xi)$. Therefore maximum goal efficiency is achieved when $\phi(T1; 0:1) = 1$, i.e., when ϕ is the Behind-First mechanism. ■

B Online Appendix: Analysis of the Regular Rounds of the Fixed-Order Mechanism

Let $\Omega > \frac{V_W + V_L}{2}$ denote $T1$'s value function after two rounds.

Second Round, Second Kick

When the score is currently even, the value function for the second kicker in $T2$ is

$$V_{T2,P2,E} = P_G(y_{2E})V_W + (1 - P_G(y_{2E}))(V_W + V_L - \Omega)$$

The optimal kicking strategy, y^*_{2E}, satisfies the following first-order condition:

$$P'_G(y^*_{2E})[\Omega - V_L + U_G] + P'_O(y^*_{2E})U_O = 0$$

When $T2$ is currently behind, the value function for the second kicker in $T2$ is

$$P_G(y_{2B})(V_W + V_L - \Omega) + (1 - P_G(y_{2B}))V_L$$

The optimal kicking strategy, y^*_{2B}, satisfies the following first-order condition:

$$P'_G(y^*_{2B})[V_W - \Omega + U_G] + P'_O(y^*_{2B})U_O = 0$$

Accordingly, we have $y^*_{2B} < y^*_{2E}$.

Second Round, First Kick

When the score is currently even, the value function for the second kicker in $T1$ is

$$\begin{aligned} V_{T1,P2,E} = {} & [P_G(x_{2E})P_G(y^*_{2B}) + (1 - P_G(x_{2E}))(1 - P_G(y^*_{2E}))]\Omega \\ & + P_G(x_{2E})(1 - P_G(y^*_{2B}))V_W + (1 - P_G(x_{2E}))P_G(y^*_{2E})V_L \end{aligned}$$

The optimal kicking strategy, x^*_{2E}, satisfies the following first-order condition:

$$\begin{aligned} & P'_G(x^*_{2E})[(P_G(y^*_{2B}) - 1 + P_G(y^*_{2E}))\Omega + (1 - P_G(y^*_{2B}))V_W - P_G(y^*_{2E})V_L + U_G] + P'_O(x^*_{2E})U_O = 0 \\ & \implies P'_G(x^*_{2E})[(1 - P_G(y^*_{2B}))(V_W - \Omega) + P_G(y^*_{2E})(\Omega - V_L) + U_G] + P'_O(x^*_{2E})U_O = 0 \end{aligned}$$

When $T1$ is currently behind, the value function for the second kicker in $T1$ is

$$V_{T1,P2,B} = P_G(x_{2B})[P_G(y^*_{2E})V_L + (1 - P_G(y^*_{2E}))\Omega] + (1 - P_G(x_{2B}))V_L$$

The optimal kicking strategy, x^*_{2B}, satisfies the following first-order condition:

$$P'_G(x^*_{2B})[(1 - P_G(y^*_{2E}))(\Omega - V_L) + U_G] + P'_O(x^*_{2B})U_O = 0$$

When $T1$ is currently ahead, the value function for the second kicker in $T1$ is

$$V_{T1,P2,A} = P_G(x_{2A})V_W + (1 - P_G(x_{2A}))[P_G(y^*_{2B})\Omega + (1 - P_G(y^*_{2B}))V_W]$$

The optimal kicking strategy, x_{2A}^*, satisfies the following first-order condition:

$$P_G'(x_{2A}^*)[P_G(y_{2B}^*)(V_W - \Omega) + U_G] + P_O'(x_{2A}^*)U_O = 0$$

First Round, Second Kick
When the score is currently even, the value function for the first kicker in $T2$ is

$$\begin{aligned} V_{T2,P1,E} &= P_G(y_{1E})(V_W + V_L - V_{T1,P2,B}) + (1 - P_G(y_{1E}))(V_W + V_L - V_{T1,P2,E}) \\ &= V_W + V_L - P_G(y_{1E})V_{T1,P2,B} - (1 - P_G(y_{1E}))V_{T1,P2,E}, \end{aligned}$$

where

$$\begin{aligned} V_{T1,P2,B} &= P_G(x_{2B}^*)[P_G(y_{2E}^*)V_L + (1 - P_G(y_{2E}^*))\Omega] + (1 - P_G(x_{2B}^*))V_L \\ &= \Omega - [P_G(x_{2B}^*)P_G(y_{2E}^*) + (1 - P_G(x_{2B}^*))](\Omega - V_L) \end{aligned}$$

and

$$\begin{aligned} V_{T1,P2,E} &= [P_G(x_{2E}^*)P_G(y_{2B}^*) + (1 - P_G(x_{2E}^*))(1 - P_G(y_{2E}^*))]\Omega \\ &\quad + P_G(x_{2E}^*)(1 - P_G(y_{2B}^*))V_W + (1 - P_G(x_{2E}^*))P_G(y_{2E}^*)V_L \\ &= \Omega + P_G(x_{2E}^*)(1 - P_G(y_{2B}^*))(V_W - \Omega) - (1 - P_G(x_{2E}^*))P_G(y_{2E}^*)(\Omega - V_L) \end{aligned}$$

Therefore

$$\begin{aligned} V_{T2,P1,E} &= V_W + V_L - P_G(y_{1E})V_{T1,P2,B} - (1 - P_G(y_{1E}))V_{T1,P2,E} \\ &= V_W + V_L - \Omega + P_G(y_{1E})[P_G(x_{2B}^*)P_G(y_{2E}^*) + (1 - P_G(x_{2B}^*))](\Omega - V_L) \\ &\quad - (1 - P_G(y_{1E}))[P_G(x_{2E}^*)(1 - P_G(y_{2B}^*))(V_W - \Omega) - (1 - P_G(x_{2E}^*))P_G(y_{2E}^*)(\Omega - V_L)] \end{aligned}$$

The optimal kicking strategy, y_{1E}^*, satisfies the following first-order condition:

$$\begin{aligned} &P_G'(y_{1E}^*)\{[P_G(x_{2B}^*)P_G(y_{2E}^*) + (1 - P_G(x_{2B}^*)) - (1 - P_G(x_{2E}^*))P_G(y_{2E}^*)](\Omega - V_L) \\ &+ P_G(x_{2E}^*)(1 - P_G(y_{2B}^*))(V_W - \Omega) + U_G\} + P_O'(y_{1E}^*)U_O = 0 \end{aligned}$$

When $T2$ is currently behind, the value function for the first kicker in $T2$ is

$$\begin{aligned} V_{T2,P1,B} &= P_G(y_{1B})(V_W + V_L - V_{T1,P2,E}) + (1 - P_G(y_{1B}))(V_W + V_L - V_{T1,P2,A}) \\ &= V_W + V_L - P_G(y_{1B})V_{T1,P2,E} - (1 - P_G(y_{1B}))V_{T1,P2,A}, \end{aligned}$$

where

$$V_{T1,P2,E} = \Omega + P_G(x_{2E}^*)(1 - P_G(y_{2B}^*))(V_W - \Omega) - (1 - P_G(x_{2E}^*))P_G(y_{2E}^*)(\Omega - V_L)$$

and

$$
\begin{aligned}
V_{T1,P2,A} &= P_G(x^*_{2A})V_W + (1 - P_G(x^*_{2A}))[P_G(y^*_{2B})\Omega + (1 - P_G(y^*_{2B}))V_W] \\
&= \Omega + [P_G(x^*_{2A}) + (1 - P_G(x^*_{2A}))(1 - P_G(y^*_{2B}))](V_W - \Omega).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
V_{T2,P1,B} &= V_W + V_L - P_G(y_{1B})V_{T1,P2,E} - (1 - P_G(y_{1B}))V_{T1,P2,A} \\
&= V_W + V_L - \Omega - P_G(y_{1B})[P_G(x^*_{2E})(1 - P_G(y^*_{2B}))(V_W - \Omega) - (1 - P_G(x^*_{2E}))P_G(y^*_{2E})(\Omega - V_L)] \\
&- (1 - P_G(y_{1B}))(P_G(x^*_{2A}) + (1 - P_G(x^*_{2A}))(1 - P_G(y^*_{2B})))(V_W - \Omega)
\end{aligned}
$$

The optimal kicking strategy, y^*_{1B}, satisfies the following first-order condition:

$$
\begin{aligned}
&P'_G(y^*_{1B})\{(1 - P_G(x^*_{2E}))P_G(y^*_{2E})(\Omega - V_L) - [P_G(x^*_{2E})(1 - P_G(y^*_{2B})) - P_G(x^*_{2A}) \\
&-(1 - P_G(x^*_{2A}))(1 - P_G(y^*_{2B}))](V_W - \Omega) + U_G\} + P'_O(y^*_{1B})U_O = 0
\end{aligned}
$$

First Round, First Kick

The value function for the first kicker in $T1$ is

$$
\begin{aligned}
V_{T1} &= P_G(x_1)[V_W + V_L - V_{T2,P1,B}] + (1 - P_G(x_1))[V_W + V_L - V_{T2,P1,E}] \\
&= V_W + V_L - P_G(x_1)V_{T2,P1,B} - (1 - P_G(x_1))V_{T2,P1,E} \\
&= \Omega + P_G(x_1)\{P_G(y^*_{1B})[P_G(x^*_{2E})(1 - P_G(y^*_{2B}))(V_W - \Omega) - (1 - P_G(x^*_{2E}))P_G(y^*_{2E})(\Omega - V_L)] \\
&+ (1 - P_G(y^*_{1B}))(P_G(x^*_{2A}) + (1 - P_G(x^*_{2A}))(1 - P_G(y^*_{2B})))(V_W - \Omega)] \\
&- (1 - P_G(x_1))\{P_G(y^*_{1E})[P_G(x^*_{2B})P_G(y^*_{2E}) + (1 - P_G(x^*_{2B}))](\Omega - V_L) \\
&- (1 - P_G(y^*_{1E}))[P_G(x^*_{2E})(1 - P_G(y^*_{2B}))(V_W - \Omega) - (1 - P_G(x^*_{2E}))P_G(y^*_{2E})(\Omega - V_L)]\}
\end{aligned}
$$

C Online Appendix: Three Regular Round Sequentially Fair Mechanisms

Let us define $V_{i,j,s}$ to be the value function for the kicker who is the j^{th} kicker to kick in Round k when the state is $s = (s_1, s_2)$, where s_i is the score for the team who kicks i^{th} in Round k. Denote by $x_{i,j,s}$ the optimal kicking strategy for this kicker.

Third Round, Second Kick

Whether the team is currently even or behind, the optimal kicking strategy is always x^*, where x^* is determined by the following first-order conditon:

$$
P'_G(x^*)[\frac{V_W - V_L}{2} + U_G] + P'_O(x^*)U_O = 0
$$

Third Round, First Kick

When the score is currently even ($s = (2,2)$, $(1,1)$ or $s = (0,0)$), the value function for the team is $\frac{V_W+V_L}{2}$.

When $s = (0,1)$, the value function for the kicker is

$$\begin{aligned} V_{3,1,(0,1)} &= P_G(x_{3,1,(0,1)})P_G(x^*)V_L + P_G(x_{3,1,(0,1)})(1-P_G(x^*))\frac{V_W+V_L}{2} + (1-P_G(x_{3,1,(0,1)}))V_L \\ &= \frac{V_W+V_L}{2} - [1-P_G(x_{3,1,(0,1)})(1-P_G(x^*))]\frac{V_W-V_L}{2} = \frac{V_W+V_L}{2} - \alpha_{3,1,(0,1)}\frac{V_W-V_L}{2} \end{aligned}$$

The optimal kicking strategy, $x_{3,1,(0,1)}$, satisfies the following first-order condition:

$$P'_G(x_{3,1,(0,1)})[(1-P_G(x^*))\frac{V_W-V_L}{2} + U_G] + P'_O(x_{3,1,(0,1)})U_O = 0$$

Similarly, we have $V_{3,1,(0,1)} = V_{3,1,(1,2)}$ and $x_{3,1,(0,1)} = x_{3,1,(1,2)}$.

When $s = (1,0)$, the value function for the kicker is

$$\begin{aligned} V_{3,1,(1,0)} &= P_G(x_{3,1,(1,0)})V_W + (1-P_G(x_{3,1,(1,0)}))[(1-P_G(x^*))V_W + P_G(x^*)\frac{V_W+V_L}{2}] \\ &= \frac{V_W+V_L}{2} + [1-(1-P_G(x_{3,1,(1,0)}))P_G(x^*)]\frac{V_W-V_L}{2} = \frac{V_W+V_L}{2} + \alpha_{3,1,(1,0)}\frac{V_W-V_L}{2} \end{aligned}$$

The optimal kicking strategy, $x_{3,1,(1,0)}$, satisfies the following first-order condition:

$$P'_G(x_{3,1,(1,0)})[P_G(x^*)\frac{V_W-V_L}{2} + U_G] + P'_O(x_{3,1,(1,0)})U_O = 0$$

Similarly, we have $V_{3,1,(1,0)} = V_{3,1,(2,1)}$ and $x_{3,1,(1,0)} = x_{3,1,(2,1)}$.

Second Round, Second Kick

Denote by $\phi_3(s)$ the prob. that the first-kicking team in Round 2 kicks first in Round 3 when the state at the end of Round 2 is s.

When $s = (0,0)$, the value function for the kicker is

$$\begin{aligned} V_{2,2,(0,0)} &= P_G(x_{2,2,(0,0)})[\phi_3(0,1)(V_W+V_L-V_{3,1,(0,1)}) + (1-\phi_3(0,1))V_{3,1,(1,0)}] + (1-P_G(x_{2,2,(0,0)}))\frac{V_W+V_L}{2} \\ &= \frac{V_W+V_L}{2} + P_G(x_{2,2,(0,0)})\alpha_{2,2,(0,0)}\frac{V_W-V_L}{2}, \end{aligned}$$

where

$$\alpha_{2,2,(0,0)} = \phi_3(0,1)\alpha_{3,1,(0,1)} + (1-\phi_3(0,1))\alpha_{3,1,(1,0)}$$

The optimal kicking strategy, $x_{2,2,(0,0)}$, satisfies the following first-order condition:

$$P'_G(x_{2,2,(0,0)})\{\alpha_{2,2,(0,0)}\frac{V_W-V_L}{2} + U_G\} + P'_O(x_{2,2,(0,0)})U_O = 0$$

When $s=(1,0)$, the value function for the kicker is

$$\begin{aligned}V_{2,2,(1,0)} &= P_G(x_{2,2,(1,0)})\frac{V_W+V_L}{2} + (1-P_G(x_{2,2,(1,0)}))[\phi_3(1,0)(V_W+V_L-V_{3,1,(1,0)}) + (1-\phi_3(1,0))V_{3,1,(0,1)}] \\ &= \frac{V_W+V_L}{2} - (1-P_G(x_{2,2,(1,0)}))\alpha_{2,2,(1,0)}\frac{V_W-V_L}{2},\end{aligned}$$

where

$$\alpha_{2,2,(1,0)} = \phi_3(1,0)\alpha_{3,1,(1,0)} + (1-\phi_3(1,0))\alpha_{3,1,(0,1)}.$$

The optimal kicking strategy, $x_{2,2,(1,0)}$, satisfies the following first-order condition:

$$P_G'(x_{2,2,(1,0)})\{\alpha_{2,2,(1,0)}\frac{V_W-V_L}{2} + U_G\} + P_O'(x_{2,2,(1,0)})U_O = 0$$

When $s=(1,1)$, the value function for the kicker is

$$\begin{aligned}V_{2,2,(1,1)} &= P_G(x_{2,2,(1,1)})[\phi_3(1,2)(V_W+V_L-V_{3,1,(1,2)}) + (1-\phi_3(1,2))V_{3,1,(2,1)}] + (1-P_G(x_{2,2,(1,1)}))\frac{V_W+V_L}{2} \\ &= \frac{V_W+V_L}{2} + P_G(x_{2,2,(1,1)})\alpha_{2,2,(1,1)}\frac{V_W-V_L}{2},\end{aligned}$$

where

$$\alpha_{2,2,(1,1)} = \phi_3(1,2)\alpha_{3,1,(1,2)} + (1-\phi_3(1,2))\alpha_{3,1,(2,1)}$$

The optimal kicking strategy, $x_{2,2,(1,1)}$, satisfies the following first-order condition:

$$P_G'(x_{2,2,(1,1)})\{\alpha_{2,2,(1,1)}\frac{V_W-V_L}{2} + U_G\} + P_O'(x_{2,2,(1,1)})U_O = 0$$

When $s=(2,1)$, the value function for the kicker is

$$\begin{aligned}V_{2,2,(2,1)} &= P_G(x_{2,2,(2,1)})\frac{V_W+V_L}{2} + (1-P_G(x_{2,2,(2,1)}))[\phi_3(2,1)(V_W+V_L-V_{3,1,(2,1)}) + (1-\phi_3(2,1))V_{3,1,(1,2)}] \\ &= \frac{V_W+V_L}{2} - (1-P_G(x_{2,2,(2,1)}))\alpha_{2,2,(2,1)}\frac{V_W-V_L}{2},\end{aligned}$$

where

$$\alpha_{2,2,(1,0)} = \phi_3(2,1)\alpha_{3,1,(2,1)} + (1-\phi_3(2,1))\alpha_{3,1,(1,2)}.$$

The optimal kicking strategy, $x_{2,2,(2,1)}$, satisfies the following first-order condition:

$$P_G'(x_{2,2,(2,1)})\{\alpha_{2,2,(1,0)}\frac{V_W-V_L}{2} + U_G\} + P_O'(x_{2,2,(2,1)})U_O = 0$$

When $s=(0,1)$, the value function for the kicker is

$$\begin{aligned}V_{2,2,(0,1)} &= P_G(x_{2,2,(0,1)})V_W + (1-P_G(x_{2,2,(0,1)}))[\phi_3(0,1)(V_W+V_L-V_{3,1,(0,1)}) + (1-\phi_3(0,1))V_{3,1,(1,0)}] \\ &= \frac{V_W+V_L}{2} + \alpha_{2,2,(0,1)}\frac{V_W-V_L}{2},\end{aligned}$$

where

$$\alpha_{2,2,(0,1)} = P_G(x_{2,2,(0,1)}) + (1 - P_G(x_{2,2,(0,1)}))[\phi_3(0,1)\alpha_{3,1,(0,1)} + (1 - \phi_3(0,1))\alpha_{3,1,(1,0)}].$$

The optimal kicking strategy, $x_{2,2,(0,1)}$, satisfies the following first-order condition:

$$P'_G(x_{2,2,(0,1)})\{[1-[\phi_3(0,1)\alpha_{3,1,(0,1)}+(1-\phi_3(0,1))\alpha_{3,1,(1,0)}]]\frac{V_W - V_L}{2}+U_G\}+P'_O(x_{2,2,(0,1)})U_O = 0$$

When $s = (2,0)$, the value function for the kicker is

$$\begin{aligned} V_{2,2,(2,0)} &= P_G(x_{2,2,(2,0)})[\phi_3(2,1)(V_W + V_L - V_{3,1,(2,1)}) + (1 - \phi_3(2,1))V_{3,1,(1,2)}] + (1 - P_G(x_{2,2,(2,0)}))V_L \\ &= \frac{V_W + V_L}{2} - \alpha_{2,2,(2,0)}\frac{V_W - V_L}{2}, \end{aligned}$$

where

$$\alpha_{2,2,(2,0)} = P_G(x_{2,2,(2,0)})[\phi_3(2,1)\alpha_{3,1,(2,1)} + (1 - \phi_3(2,1))\alpha_{3,1,(1,2)}] + 1 - P_G(x_{2,2,(2,0)}).$$

The optimal kicking strategy, $x_{2,2,(2,0)}$, satisfies the following first-order condition:

$$P'_G(x_{2,2,(2,0)})\{[1-[\phi_3(2,1)\alpha_{3,1,(2,1)}+(1-\phi_3(2,1))\alpha_{3,1,(1,2)}]]\frac{V_W - V_L}{2}+U_G\}+P'_O(x_{2,2,(2,0)})U_O = 0$$

Second Round, First Kick

When $s = (0,0)$ or $s = (1,1)$, the value function for the team is $\frac{V_W+V_L}{2}$.

When $s = (0,1)$, the value function for the kicker is

$$\begin{aligned} V_{2,1,(0,1)} &= P_G(x_{2,1,(0,1)})(V_W + V_L - V_{2,2,(1,1)}) + (1 - P_G(x_{2,1,(0,1)}))(V_W + V_L - V_{2,2,(0,1)}) \\ &= \frac{V_W + V_L}{2} - \alpha_{2,1,(0,1)}\frac{V_W - V_L}{2}, \end{aligned}$$

where

$$\alpha_{2,1,(0,1)} = P_G(x_{2,1,(0,1)})P_G(x_{2,2,(1,1)})\alpha_{2,2,(1,1)} + (1 - P_G(x_{2,1,(0,1)}))\alpha_{2,2,(0,1)}.$$

The optimal kicking strategy, $x_{2,1,(0,1)}$, satisfies the following first-order condition:

$$P'_G(x_{2,1,(0,1)})\{[\alpha_{2,2,(0,1)} - P_G(x_{2,2,(1,1)})\alpha_{2,2,(1,1)}]\frac{V_W - V_L}{2} + U_G\} + P'_O(x_{2,1,(0,1)})U_O = 0$$

When $s = (1,0)$, the value function for the kicker is

$$\begin{aligned} V_{2,1,(1,0)} &= P_G(x_{2,1,(1,0)})(V_W + V_L - V_{2,2,(2,0)}) + (1 - P_G(x_{2,1,(1,0)}))(V_W + V_L - V_{2,2,(1,0)}) \\ &= \frac{V_W + V_L}{2} + \alpha_{2,1,(1,0)}\frac{V_W - V_L}{2}, \end{aligned}$$

where

$$\alpha_{2,1,(1,0)} = P_G(x_{2,1,(1,0)})\alpha_{2,2,(2,0)} + (1 - P_G(x_{2,1,(1,0)}))(1 - P_G(x_{2,2,(1,0)}))\alpha_{2,2,(1,0)}.$$

The optimal kicking strategy, $x_{2,1,(1,0)}$, satisfies the following first-order condition:

$$P'_G(x_{2,1,(1,0)})\{[\alpha_{2,2,(2,0)} - (1 - P_G(x_{2,2,(1,0)}))\alpha_{2,2,(1,0)}]\frac{V_W - V_L}{2} + U_G\} + P'_O(x_{2,1,(1,0)})U_O = 0$$

First Round, Second Kick

When $s = (0,0)$, the value function for the kicker is

$$\begin{aligned} V_{1,2,(0,0)} &= P_G(x_{1,2,(0,0)})[\phi_2(0,1)(V_W + V_L - V_{2,1,(0,1)}) + (1 - \phi_2(0,1))V_{2,1,(1,0)}] + (1 - P_G(x_{1,2,(0,0)}))\frac{V_W + V_L}{2} \\ &= \frac{V_W + V_L}{2} + P_G(x_{1,2,(0,0)})\alpha_{1,2,(0,0)}\frac{V_W - V_L}{2}, \end{aligned}$$

where

$$\alpha_{1,2,(0,0)} = \phi_2(0,1)\alpha_{2,1,(0,1)} + (1 - \phi_2(0,1))\alpha_{2,1,(1,0)}.$$

The optimal kicking strategy, $x_{1,2,(0,0)}$, satisfies the following first-order condition:

$$P'_G(x_{1,2,(0,0)})\{\alpha_{1,2,(0,0)}\frac{V_W - V_L}{2} + U_G\} + P'_O(x_{1,2,(0,0)})U_O = 0.$$

When $s = (1,0)$, the value function for the kicker is

$$\begin{aligned} V_{1,2,(1,0)} &= P_G(x_{1,2,(1,0)})\frac{V_W + V_L}{2} + (1 - P_G(x_{1,2,(1,0)}))[\phi_2(1,0)(V_W + V_L - V_{2,1,(1,0)}) + (1 - \phi_2(1,0))V_{2,1,(0,1)}] \\ &= \frac{V_W + V_L}{2} - (1 - P_G(x_{1,2,(1,0)}))\alpha_{1,2,(1,0)}\frac{V_W - V_L}{2}, \end{aligned}$$

where

$$\alpha_{1,2,(1,0)} = \phi_2(1,0)\alpha_{2,1,(1,0)} + (1 - \phi_2(1,0))\alpha_{2,1,(0,1)}.$$

The optimal kicking strategy, $x_{1,2,(1,0)}$, satisfies the following first-order condition:

$$P'_G(x_{1,2,(1,0)})\{\alpha_{1,2,(1,0)}\frac{V_W - V_L}{2} + U_G\} + P'_O(x_{1,2,(1,0)})U_O = 0.$$

First Round, First Kick

The value function for the kicker is

$$\begin{aligned} V_{1,1,(0,0)} &= P_G(x_{1,1,(0,0)})[V_W + V_L - V_{1,2,(1,0)}] + (1 - P_G(x_{1,1,(0,0)}))[V_W + V_L - V_{1,2,(0,0)}] \\ &= \frac{V_W + V_L}{2} + [P_G(x_{1,1,(0,0)})(1 - P_G(x_{1,2,(1,0)}))\alpha_{1,2,(1,0)} \\ &\quad - (1 - P_G(x_{1,1,(0,0)}))P_G(x_{1,2,(0,0)})\alpha_{1,2,(0,0)}]\frac{V_W - V_L}{2} \end{aligned}$$

The optimal kicking strategy, $x_{1,1,(0,0)}$, satisfies the following first-order condition:

$$P'_G(x_{1,1,(0,0)})\{[(1-P_G(x_{1,2,(1,0)}))\alpha_{1,2,(1,0)}+P_G(x_{1,2,(0,0)})\alpha_{1,2,(0,0)}]\frac{V_W-V_L}{2}+U_G\}+P'_O(x_{1,1,(0,0)})U_O=0$$

Therefore

$$x_{1,1,(0,0)} \gtreqless x_{1,2,(0,0)} \iff (1-P_G(x_{1,2,(1,0)}))\alpha_{1,2,(1,0)} \gtreqless P_G(x_{1,2,(0,0)})\alpha_{1,2,(0,0)}$$

On the other hand, we have

$$\begin{aligned} V_{1,1,(0,0)} = \frac{V_W+V_L}{2} &\iff P_G(x_{1,1,(0,0)})(1-P_G(x_{1,2,(1,0)}))\alpha_{1,2,(1,0)} = (1-P_G(x_{1,1,(0,0)}))P_G(x_{1,2,(0,0)})\alpha_{1,2,(0,0)} \\ &\iff (1-P_G(x_{1,2,(1,0)}))\alpha_{1,2,(1,0)} = (1-P_G(x_{1,1,(0,0)}))\alpha_{1,2,(0,0)} \end{aligned}$$

The condition holds if $\phi_2(1,0)+\phi_2(0,1)=1$.