The Bargaining Correspondence*

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Abstract

A new, more fundamental approach is proposed to the classical bargaining problem. The give-and-take feature in the negotiation process is explicitly modelled. A bargainer's compromise set consists of all allocations he/she is willing to accept as agreement. We focus on the relationship between the rationality principles adopted by players in making mutual concessions and the formation of compromise sets. The bargaining correspondence is then defined as the intersection of players' compromise sets. We study the non-emptyness, symmetry, efficiency and single-valuedness of the bargaining correspondence, and establish it's connection to the Nash bargaining solution. Our framework bridges the "Edgeworth-Nash gap," provides a novel foundation to the Nash bargaining solution, and opens doors for future research in bargaining theory.

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> Give me that which I want, and you shall have this which you want, is the meaning of every such offer. (Adam Smith)

1 Introduction

THE BARGAINING PROBLEM concerns how parties reconcile conflicting interests and reach a mutually acceptable agreement. A simple wage negotiation between Robinson Crusoe and Friday was considered by Edgeworth (1881), and he concluded that the terms of bargaining is indeterminate:

> This simple case brings clearly into view the characteristic evil of indeterminate contract, deadlock, undecidable opposition of interests, ... It is the interest of both parties that there should be some settlement, one of the contracts represented by the contract-curve between the limits. But which of these contracts is arbitrary in the absence of arbitration, the interests of the two *adversa pugnantia fronte* all along the contract-curve, ...

Edgeworth's view is that, beyond Pareto optimality and individual rationality, economic theory remains silent on how the agreement, if it is reached, is determined on the contract curve.

Built upon the expected utility theory newly developed by von Neumann and Morgenstern in late 1940s, Nash (1950) elegantly formalized the bargaining problem and provided the first definite answer on how the gains from trade would be divided. He first assumed that for every bargaining situation, there exists a unique utility allocation (solution) that is unanimously agreed by the parties as a "fair bargain," i.e., an allocation that gives each player what he/she expects to get. To locate this fair bargain, Nash suggested the following:

> One states as axioms several properties that it would seem natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely (Nash 1953, p. 129).

The axiomatic approach proposed by Nash does not impose any structure restrictions on the bargaining process. Instead, it appeals to general rationality postulates (axioms) in pinning down the allocation of fair bargain.[1] On top of the (weak) Pareto optimality and individual rationality axioms considered by Edgeworth, Nash proposed two additional axioms, and these four axioms uniquely determine the bargaining outcome. While many other solution concepts, axioms, and characterizations have been proposed since then, the whole literature on cooperative bargaining mainly follows Nash's framework.[2]

From Edgeworth's problem of indeterminacy to Nash's bargaining solution witnesses one of the greatest intellectual leaps in economic theory.[3] The contradictory views between

[1] The strategic approach, on the other hand, explicitly specifies the negotiation process in a multi-stage game, and predicts bargaining outcomes based on a suitable equilibrium concept.

[2] See Thomson (1994) for a comprehensive survey of the literature. For recent developments in the literature, see Thomson (2009).

[3] Myerson (1999) stated that Nash's bargaining solution was "virtually unanticipated in the literature," and Binmore (2005) argued that "Nash deserves his Nobel prize more for his bargaining solution than for his equilibrium concept, since his contribution to bargaining theory is entirely original, whereas his equilibrium idea had a number of precursors."

Edgeworth and Nash invite close scrutiny. A careful reading of Edgeworth's statement reveals that the problem of indeterminacy comes in two forms: (i) no agreement is reached, and (ii) an agreement is reached but cannot be predicted. Thus, to fully resolve the problem, the following two questions should be addressed. When facing a bargaining situation,

I. Existence of Agreement. Under what rationality principles, the parties reach a unique agreement, and

II. Characterization of Agreement. Under what additional rationality principles, the agreement in I. can be characterized (i.e., located or predicted).

Note the subtle difference between these two questions. The first question concerns the rational foundation of when the minds with competing interests meet, and the second question concerns where the minds meet. By resolving the first question, we conclude that the negotiation would not end in deadlock, but we may not have sufficient information to locate the agreement, which leads to the second question. The first question is more fundamental than the second one, as resolving the second question implies that either (a) the existence of agreement is also resolved under those rationality principles, or (b) the existence of agreement is assumed.

Having clarify the two distinct issues raised by Edgeworth, let us look what Nash's axiomatic approach has achieved. Nash's bargaining theory can be stated as follows: IF there exists a unique agreement for every bargaining problem, and the agreement satisfies Nash's system of axioms, then this agreement can be characterized (located). Hence Nash's theory is a characterization theory, not a theory of existence. The first issue, existence of agreement, is simply assumed away.[4] Therefore we cannot tell in Nash's theory how the parties consent that there must be an agreement for each problem in the first place. Indeed, so far there is no rational foundation for the existence of agreement in bargaining theory. In sum, Nash does not fully resolve Edgeworth's problem of indeterminacy. We term this the Edgeworth-Nash gap.

We propose a new, more fundamental approach to the bargaining problem, which allows us to address Edgeworth's two issues under a unified framework. Three essential characteristics of bargaining are featured in this framework. First, bargaining is a multiperson decision-making problem. Each bargainer's decision on whether to accept a particular compromise or not should be formally modelled. Second, bargaining is a give-and-take process. How much a bargainer is willing to concede depends on how much others concede. Third, the bargaining process is not artificially specified/restricted. It is modelled as a (cooperative) persuasion game. Bargainers meet at the bargaining table, communicate, negotiate, and bring every justifiable reason to convince each other to concede.

Define a bargainer's compromise set as the collection of all allocations he/she is willing to accept as agreement. We establish the relationship between the rationality principles adopted by bargainers in making mutual concessions and the formation of compromise sets.

[4] While it was not explicitly declared when Nash (1950) first proposed his axiomatic approach, the unique existence of the solution (agreement) was formally stated as the first fundamental axiom (assumption) in Nash (1953). This assumption is vital in Nash's framework. Without this assumption, Nash's axioms, which are properties defined on the assumed solution, would be meaningless, and Nash's approach would be logically unsound.

The bargaining correspondence is then defined as the intersection of bargainers' compromise sets. A rationality foundation is provided for the non-emptyness of the bargaining correspondence; i.e., for the existence of unanimous agreements. In particular, we show that pairwise concessions not only lead to pairwise agreements, but also lead to unanimous agreements. In other words, the rationality requirement for meeting of the minds is invariant with the number of bargainers involved, which contradicts the traditional view that reaching an agreement is more difficult in multilateral bargaining than in bilateral bargaining. We subsequently establish several properties on the bargaining correspondence, such as symmetry, efficiency and single-valuedness. A robust axiom named CIP$_{TU}$ is then proposed to describe how bargainers adjust their mutual concessions across the TU (Transferable Utility) problems, and another axiom is suggested to connect the bargaining correspondence to the Nash bargaining solution. Therefore our framework bridges the "Edgeworth-Nash gap," and provides a novel foundation to the Nash bargaining solution.

2 Nash's Axiomatic Approach

A bargaining problem (or a problem in short) among a collection of players (bargainers), $N = \{1, ..., n\}$, is represented by a pair (S, d), where $S \subset \mathbb{R}^n$ is the set of players' utility possibilities, and $d \in S$ is the disagreement point, which is the utility allocation that results if no agreement is reached by all parties. It is assumed that S is (i) compact, (ii) convex, (iii) comprehensive (x, $z \in S$ implies that $y \in S$ for all $x \le y \le z$), and (iv) $x > d$ for some $x \in S$.[5] Let Σ be the class of all n-person problems satisfying (i)-(iv). Define the set of individually rational utility allocations as $IR(S,d) \equiv \{x \in S | x \ge d\}$, the set of weakly Pareto optimal allocations as $WPO(S) \equiv \{x \in S | \forall x' \in \mathbb{R}^n \text{ and } x' > x \Rightarrow x' \notin S\}$, and the set of Pareto optimal allocations as $PO(S) \equiv \{x \in S | \forall x' \in \mathbb{R}^n,\ x' \ge x \text{ and } x' \neq x \Rightarrow x' \notin S\}$. Moreover, denote the ideal point of (S, d) as $b(S,d) \equiv (b_1(S,d), ..., b_n(S,d))$, where $b_i(S,d) = \max\{x_i | x \in IR(S,d)\}$; the midpoint of (S,d) is $m(S,d) \equiv \frac{1}{n} b(S,d) + (1 - \frac{1}{n})d$. Given $(S,d) \in \Sigma$, we define $(b_i(S,d), d_{-i}) \equiv (d_1, ..., d_{i-1}, b_i(S,d), d_{i+1}, ..., d_n)$ as player $i's$ dictatorial allocation. Let Π be the set of all permutations on $N = \{1, ..., n\}$, *i.e.*, all bijections $\pi : N \to N$. Given $x = (x_1, ..., x_n) \in \mathbb{R}^n$, let $\pi_{ij} x = (x_1, ..., x_{i-1}, x_j, x_{i+1}, ..., x_{j-1}, x_i, x_{j+1}, ..., x_n)$. A problem $(S,d) \in \Sigma$ is said to be symmetric if $(S,d) = (\pi S, \pi d)$ for all $\pi \in \Pi$.

A solution is a function $f : \Sigma \to \mathbb{R}^n$ such that for all $(S,d) \in \Sigma$, $f(S,d) \in S$. Nash proposed that f should satisfy the following four axioms:

Weak Pareto Optimality (WPO) For all $(S,d) \in \Sigma$, $f(S,d) \in WPO(S)$.

Symmetry (SYM) If $(S,d) \in \Sigma$ is symmetric, then $f_1(S,d) = ... = f_n(S,d)$.

Scale Invariance (SI) $G = (G_1, ..., G_n) : \mathbb{R}^n \to \mathbb{R}^n$ is a positive affine transformation if $G(x) = (a_1 x_1 + c_1, ..., a_n x_n + c_n)$ for some $a \in \mathbb{R}^n_{++}$ and $c \in \mathbb{R}^n$. SI requires that for any $(S,d) \in \Sigma$ and a positive affine transformation G, $f(G(S), G(d)) = G(f(S,d))$.

Independence of Irrelevant Alternatives (IIA) For all $(S,d), (T,d) \in \Sigma$,if $T \supset S$ and $f(T,d) \in S$, then $f(S,d) = f(T,d)$.

WPO is a collective rationality assumption requiring that the parties should fully utilize the surplus from the situation. The SYM axiom demends that the parties share the gain

[5] Given $x, y \in \mathbb{R}^n$, $x > y$ if $x_i > y_i$ for each i, and $x \ge y$ if $x_i \ge y_i$ for each i.

equally when facing a symmetric problem. Initially Nash (1950) appealed to equal bargaining ability to justify the SYM axiom, but admitted it is a mistake later on (Nash 1953, p. 137). Instead, he argued that the SYM axiom must be satisfied if the players are equally intelligent and equally rational, and all relevant factors are incorporated into the model. The SI axiom is assumed if the preferences for each player can be represented by a von Neumann-Morgernstern utility function. The IIA axiom attracts most criticisms. Given two problems (T, d) and (S, d) with $T \supset S$, IIA requires that if the parties unanimously agree that $f(T, d)$ is the fair bargain in (T, d), then $f(T, d)$ remains to be the fair bargain in (S, d).

Nash proved that the Nash solution defined below is the unique solution satisfying the above four axioms.

The Nash solution $Nash$: For each $(S, d) \in \Sigma$, $Nash(S, d) = \arg\max\{\Pi_{i=1}^{n}(x_i - d_i)|x \in IR(S, d)\}$.

The other two prominent solution concepts in the literature are the egalitarian and Kalai-Smorodinsky solutions:

The egalitarian solution E : For each $(S, d) \in \Sigma$, $E(S, d) = d + \lambda^*\mathbf{1}$, where $\mathbf{1} = (1, ..., 1)$ and $\lambda^* = \max\{\lambda|d + \lambda^*\mathbf{1} \in S\}$.

The Kalai-Smorodinsky solution KS : For each $(S, d) \in \Sigma$, $KS(S, d) = \lambda^* b(S, d) + (1 - \lambda^*)d$, where $\lambda^* = \max\{\lambda|\lambda b(S, d) + (1 - \lambda)d \in S\}$.

3 Compromise Sets and The Bargaining Correspondence

Bargaining, in a nutshell, is a multiperson decision-making problem, where players' interests are not fully aligned, yet a unanimous agreement has to be reached in order for them to benefit from the situation. Suppose the parties involved in a bargaining situation are highly rational with complete information. In the presence of their diverse interests, can we assert that a unique unanimous agreement could always be reached, as assumed in Nash (1950)? In other words, on what ground shall we argue that the parties with irreconcilable proposals or demands (and hence reach a bargaining impasse) are "irrational"? Moreover, does it become more difficult to reach a unanimous agreement when more people are involved?

To address the above issues, we have to look into how the bargaining stance is formed for each individual during the negotiation process. Here a bargaining stance is defined as the set of alternatives a player is willing to accept as possible final outcomes. At the bargaining table, the parties communicate and make proposals and counterproposals to each other. Suppose they face a problem (S, d). A (reasonable) proposal can simply be represented by an alternative $x \in S$, or a set of alternatives $A \subset S$. For example, player i may initially propose $(b_i(S, d), d_{-i})$ to others, which means that i is only willing to accept his/her dictatorial allocation as an agreement. Others, by appealing to whatever principles (fairness, efficiency, etc), may ask i to concede by accepting some other possible outcomes. Player i may or may not be persuaded by others. If i is persuaded by others, i revises the proposal to a new one. With free and sufficient communication, it reaches a stage at which all possible arguments/principles have been exhausted, and each player fully determines his/her bargaining stance. In other words, no further negotiation could make any player budge his/her bargaining stance. We call this final unbudged bargaining stance a player's compromise set. Formally speaking, let $C_i : \Sigma \longrightarrow 2^{\mathbb{R}^n}$ be a nonempty and closed (in the Hausdorff topology) correspondence such that for every $(S, d) \in \Sigma$, $C_i(S, d) \subset IR(S, d)$.

Then $C_i(S,d)$, named as $i's$ compromise set with respect to (S,d), is the set of feasible alternatives (compromises) deemed acceptable by player i when facing the problem (S,d). It has to be individually rational, as no player has an incentive to accept any payoff below what he/she can get at disagreement.[6] Denote a profile of compromise sets by $C = (C_1, ..., C_n)$. First let us list some examples of C :

Example 1 *For every $i \in N$ and $(S,d) \in \Sigma$, $C_i(S,d) = \{(b_i(S,d), d_{-i})\}$.*

Example 2 *For every $i \in N$ and $(S,d) \in \Sigma$, $C_i(S,d) = \{(b_1(S,d), d_{-1}), ..., (b_n(S,d), d_{-n})\}$.*

Example 3 *For every $i \in N$ and $(S,d) \in \Sigma$, $C_i(S,d) = \{x \in S | x_i \geq Nash(S,d)_i\}$.*

Example 4 *For every $i \in N$ and $(S,d) \in \Sigma$, $C_i(S,d) = \{x \in WPO(S) | x_i \geq E(S,d)_i\}$.*

Example 5 *For every $i \in N$ and $(S,d) \in \Sigma$, $C_i(S,d) = \{x \in S | x \geq m(S,d)\}$.*

Example 6 *Let $N = \{1,2\}$. For every $(S,d) \in \Sigma$, $C_1(S,d) = \{x \in S | \frac{x_1-d_1}{x_2-d_2} \geq \frac{Nash(S,d)_1-d_1}{Nash(S,d)_2-d_2}\}$ and $C_2(S,d) = \{x \in S | \frac{x_1-d_1}{x_2-d_2} \leq \frac{KS(S,d)_1-d_1}{KS(S,d)_2-d_2}\}$.*

In Example 1, each player only accepts his/her dictatorial allocation as a bargaining outcome. In other words, no one intends to make any compromises. Example 2 is a peculiar example in which players are willing to accept ANY dictatorial allocation. In Example 3, players view the Nash solution as a reasonable benchmark, and are willing to accept any allocation that gives him/her a payoff no less than what he/she can get at the Nash solution. In Example 4, each player would like to accept any efficient allocation that gives him/her a payoff no less than what he/she can obtain at the Egalitarian solution. In Example 5, all players are happy to accept any compromise that gives each one at least half of the ideal payoff. In Example 6, player 1 accepts any allocation with the relative gain no less than that at the Nash solution, and player 2 accepts any allocation with the relative gain no greater than that at the KS solution.

Given a profile of compromise sets C, the bargaining correspondence with respect to C is $B_c : \Sigma \longrightarrow 2^{\mathbb{R}^n}$ such that $B_c(S,d) = \cap_{i=1}^n C_i(S,d)$. We say the parties reach a unanimous agreement(s) when $B_c(S,d)$ is nonempty. A unique agreement is said to be reached among the parties under C when $B_c(S,d)$ is single-valued. For a given subset of agents, $I \subset N$, We use the notation $\cap_{i \in I} C_i$ to represent the intersection of $\{C_i\}_{i \in I}$. $|I|$ denotes the number of agents in I. Going back to the above examples, we observe that $B_c(S,d)$ is empty in Example 1, nonempty in Examples 2 and 5, and single-valued in Examples 3 and 4. Depending on the problem (S,d), $B_c(S,d)$ could be empty, a singleton, and nonempty with multiple elements in Example 6.

Bargaining is a give-and-take process. The insight from Adam Smith quoted in the beginning of the paper tells us that the concessions made by the parties are interdependent – whether a player would like to make a concession depends on whether others do the same. We would like to study under what rationality principles mutually adopted by the players, an agreement can be reached? Note that "mutual adoption of rationality principles" itself is a minimum requirement of the concept of "fairness" bargainers have in mind – if, by

[6] In other words, a bargainer does not make any "compromise" if his/her proposals are outside $IR(S,d)$.

appealing to some rationality principle, you persuade me to make concessions, then you should make the same concessions under the same principle when facing the same situation. First consider the following axiom:

Common Reasoning (CR) If $(S, d) = (\pi S, \pi d)$ for every $\pi \in \Pi$, then $C(S, d) = \pi(\pi C_1(S, d), ..., \pi C_n(S, d))$ for every $\pi \in \Pi$.

CR stems from the idea that the parties, being equally rational, should adopt the same reasoning in formulating the compromise set. It requires that when facing a symmetric bargaining situation, the parties should propose the same compromises (from each's perspective). For example, given a two-person symmetric problem (S, d), if $(8, 2) \in C_1(S, d)$, then $(2, 8) \in C_2(S, d)$. CR is a weak mutually rational requirement. Indeed, all examples above (Examples 1-6) satisfy CR. CR alone, however, does not guarantee that an agreement can be reached. Even when an agreement is reached, it could be neither symmetric nor unique (Example 2).

The bargainers involved in the day-to-day negotiations commonly make interpersonal comparisons of utility to persuade others to concede (Shapley (1969) and Thomson (2009)). For example, Shapley (1969) made the following observation:

> At other times, a person may compare his gain against another's gain, or loss against loss. ... "My demand is more reasonable than yours," the bargainer may plead, "therefore you should give in."

This argument, however, is not sufficient for the parties to reach an agreement. Consider a two-person symmetric problem (S, d) with $S = \{x \in \mathbb{R}^n_+ | x_1 + x_2 = 10\}$ and $d = \mathbf{0}$. Suppose player 1 proposes $(8, 2)$ as an agreement. By CR, player 2 responds to player 1's proposal by submitting $(2, 8)$ as a counterproposal. Both demands are "equally reasonable", and the argument above does not have a bite – both players find no ground to against each other's proposal. In order to break the deadlock, someone has to make a concession. The following axiom captures this idea. Given $x^1 = (x^1_1, ..., x^1_n)$ and $x^2 = (x^2_1, ..., x^2_n)$ in $\mathbb{R}^n$, the join of x^1 and x^2 is $x^1 \vee x^2 = (\max(x^1_1, x^2_1), ..., \max(x^1_n, x^2_n))$, and the meet of x^1 and x^2 is $x^1 \wedge x^2 = (\min(x^1_1, x^2_1), ..., \min(x^1_n, x^2_n))$. The lattice spanned by x^1 and x^2 is denoted as $L(x^1, x^2) = \{y \in \mathbb{R}^n | x^1 \wedge x^2 \leq y \leq x^1 \vee x^2\}$. Given $x \in \mathbb{R}^n$, denote by $\underline{x} = (a, ..., a)$ with $a = \min\{x_1, ..., x_n\}$ and $\overline{x} = (b, ..., b)$ with $b = \max\{x_1, ..., x_n\}$.

Symmetric Pairwise Concession (SPC) For any pair of players $\{i, j\} \subset N$ and a problem $(S, d) \in \Sigma$, if $x \in C_i(S, d)$ and $\pi_{ij} x \in C_j(S, d)$ with $x \neq \pi_{ij} x$, then $C_i(S, d) \cap L(x, \pi_{ij} x) \backslash \{x\} \neq \varnothing$.

$x \in C_i(S, d)$ and $y \in C_j(S, d)$ are said to be a pair of symmetric demands (or compromises) between i and j if $y = \pi_{ij} x$. $L(x, \pi_{ij} x)$ consists of all compromises "between" x and $\pi_{ij} x$. SPC states that in response to player $j's$ symmetric compromise, player i is willing to make further concession accepting a compromise between x and $\pi_{ij} x$.

Lemma 1 *If C satisfies CR and SPC and $(S, d) \in \Sigma$ is symmetric, then for any pair of players $\{i, j\} \subset N$ and $x \in C_i(S, d)$, $C_i(S, d) \cap C_j(S, d) \cap L(x, \pi_{ij} x) \neq \varnothing$.*

When facing a symmetric problem, if players i and j adopt common reasoning, and are willing to make a concession between symmetric compromises (demands), then a pairwise agreement can be reached between i and j. Such an agreement may not give i and j the same payoff. Consider Example 2 with $|N| = 2$. In this example C satisfies CR and SPC, and we have $B_c(S,d) = \{(b_1(S,d), d_2), (d_1, b_2(S,d))\}$.

Could pairwise agreements lead to a unanimous agreement among all players? Unfortunately, the following example shows that an unanimous agreement may not be reached on a symmetric problem under CR and SPC.

Example 7 *Let $S = \{x \in \mathbb{R}^3_+ | \sum_{i=1}^3 x_i = 10\}$ and $d = \mathbf{0}$. $C(S,d)$ is such that*
$C_1(S,d) = \{(5,5,0), (5,0,5), (0,5,0), (0,0,5)\}$
$C_2(S,d) = \{(5,5,0), (0,5,5), (0,0,5), (5,0,0)\}$
$C_3(S,d) = \{(5,0,5), (0,5,5), (0,5,0), (5,0,0)\}$
It can be readily verified that $C(S,d)$ satisfies CR and SPC, $C_i(S,d) \cap C_j(S,d) \neq \varnothing$ for $i,\ j \in \{1,2,3\}$, but $\cap_{i=1}^3 C_i(S,d) = \varnothing$.

Example 7 is peculiar though. Consider player 1's compromise set. Player 1 is happy to share the total gain equally with either player 2 or player 3 ($(5,5,0) \in C_1(S,d)$ and $(5,0,5) \in C_1(S,d)$). By CR, $(0,5,5) \in C_2(S,d)$. In response to player 2's symmetric counterproposal $(0,5,5)$, however, player 1 is only willing to accept $(0,0,5)$, the worst allocation for both players in $L((5,0,5),(0,5,5))$.

Define $L^o(x, \pi_{ij}x) = L(x, \pi_{ij}x) \backslash \{y \in \mathbb{R}^n | y_i = \min\{x_i, x_j\}$ or $y_j = \min\{x_i, x_j\}\}$. Consider the following slightly stronger version of SPC.

Symmetric Pairwise Concession* (SPC*) For any pair of players $\{i,j\} \subset N$ and a problem $(S,d) \in \Sigma$, if $x \in C_i(S,d)$ and $\pi_{ij}x \in C_j(S,d)$ with $x \neq \pi_{ij}x$, then $C_i(S,d) \cap L^o(x, \pi_{ij}x) \neq \varnothing$.

SPC* requires that in response to player $j's$ symmetric demand, player i is willing to make further (strict) concession accepting a compromise between x and $\pi_{ij}x$ that gives both players payoffs strictly higher than $\min\{x_i, x_j\}$.

Lemma 2 *If C satisfies CR and SPC* and $(S,d) \in \Sigma$ is symmetric, then for any pair of players $\{i,j\} \subset N$ and $x \in C_i(S,d)$ with $x \neq \pi_{ij}x$, $C_i(S,d) \cap C_j(S,d) \cap \{y \in L^o(x, \pi_{ij}x) | y_i = y_j\} \neq \varnothing$.*

In a symmetric problem, let $(x, \pi_{ij}x)$ be a pair of symmetric demands between players i and j. If both players are willing to make strict concessions between any pair of symmetric demands, then a pairwise agreement can be reached between x and $\pi_{ij}x$. At this agreement, i and j obtain the same payoff. This result seems to be straightforward. What is surprising is that under CR and SPC*, an unanimous agreement always exists in every symmetric problem.

Proposition 1 *If C satisfies CR and SPC* and $(S,d) \in \Sigma$ is symmetric, then $B_c(S,d) \cap \{y \in \mathbb{R}^n | y_1 = ... = y_n\} \neq \varnothing$.*

Suppose players adopt common reasoning, and are willing to make concessions between symmetric demands. Then for a symmetric problem, a symmetric agreement could always be reached. The agreement is not unique, and it is possible to have other asymmetric agreements. To gain the intuition behind Proposition 1, let us describe how pairwise agreements could lead to a 3-wise agreement. Suppose players facing a symmetric problem follow the CR and SPC* principles in making their proposals/counterproposals. We know from Lemma 2 that there exists some $x \in S$ with $x_1 = x_2$ that is agreed upon between player 1 and player 2. As $x \in C_1(S,d)$, by CR, player 3 would make a counterproposal at $\pi_{13}x$. Clearly x is a unanimous agreement if $x_1 = x_3$. In the following, we consider the case when $x_1 < x_3$, and the case of $x_1 > x_3$ can be analyzed analogously. As $x \in C_1(S,d)$ and $\pi_{13}x \in C_3(S,d)$ with $x \neq \pi_{13}x$, by Lemma 2 again we conclude that there exists some $z \in S$ that is agreed upon between player 1 and player 3, where $z_1 = z_3 \in (x_1, x_3]$, and $z_i = x_i$ for $i \in N \backslash \{1,3\}$. As $z \in C_1(S,d)$, by CR, player 2 would propose $\pi_{12}z = (x_2, z_1, z_3, x_4, ..., x_n) = (x_1, z_1, z_1, x_4, ..., x_n)$. Applying Lemma 2 again we conclude that there exists some $y \in S$ that is agreed upon between players 1 and 2, where $y_1 = y_2 \in (x_1, z_1]$, $y_3 = z_1$, and $y_i = x_i$ for $i \in N \backslash \{1,2,3\}$. Comparing x and y, we observe that the difference in demands between players 1 and 2 and player 3 shrinks, as $x_1 < y_1 \leq y_3 = z_1 \leq x_3$. Through back-and-forth concessions in the negotiaion process, a 3-wise agreement which gives the three players equal payoff can be reached. Here the negotiation does not bump into a deadlock like that in Example 7, as the parties are willing to make strict pairwise concessions when demands are irreconcilable but equally reasonable.

When facing an asymmetric problem, a stronger rationality principle is required for the parties to achieve a unanimous agreement. Consider the following axiom:

Pairwise Concession (PC) For any pair of players $\{i,j\} \subset N$ and a problem $(S,d) \in \Sigma$, let $x \in C_i(S,d)$ and $y \in C_j(S,d)$ with $x \neq y$.

(i) If $y = \pi_{ij}x$, then $C_i(S,d) \cap L^o(x, \pi_{ij}x) \neq \varnothing$

(ii) If $y \neq \pi_{ij}x$ and $x \notin C_j(S,d)$, then $C_i(S,d) \cap L(x,y) \backslash \{x, x \wedge y\} \neq \varnothing$.

The PC axiom extends the rationale of SPC* into asymmetric demands. Like SPC*, it requires that when facing symmetric demands, a bargainer has to make a strict concession. When demands are asymmetric and irreconcilable, a bargainer has to concede and propose a revised demand, and this new demand cannot be the worst outcome in $L(x,y)$. The following proposition provides a sufficient condition for the existence of unanimous agreements.

Proposition 2 *If C satisfies CR and PC, then $B_c(S,d) \neq \varnothing$ for every $(S,d) \in \Sigma$. Moreover, $B_c(S,d) \cap \{y \in \mathbb{R}^n | y_1 = ... = y_n\} \neq \varnothing$ when (S,d) is symmetric.*

When the parties mutually adopt CR and PC as rationality principles in making concessions during the negotiation process, a unanimous agreement exists. Furthermore, a symmetric agreement exists in a symmetric bargaining situation. It is conventional wisdom that reaching a unanimous agreement becomes more difficult when more people are involved in negotiation. For example, Myerson (1997) wrote:

> So far we have only considered bargaining problems that involve two parties. Multilateral bargaining may become even more complicated,... If any party to a multilateral agreement can upset the agreement between anyone else, however, then the need to have well-coordinated expectations about what each party can reasonably demand is even greater than in bilateral bargaining problems.

Arguably it could take longer to coordinate and reach an agreement when more people are involved in a bargaining situation. Proposition 2, however, shows that the existence of unanimous agreements is independent with the number of bargainers involved. In other words, whenever a unanimous agreement cannot be reached among $N > 2$ players, we could boil the issue of deadlock down to that there is a pair of players who do not make (strict) concessions between irreconcilable demands. The mutual rationality requirement for "meeting of the minds" does not become more stringent from bilateral to multilateral bargaining.

Proposition 2 establishes the existence of agreements. For each problem, however, there could be multiple agreements. For symmetric problems, some asymmetric agreements may exist. Example 5 is one such example. The multiplicity of agreements arises when some bargainers make excessive concessions. The following axiom is built on the rationality that each player has an incentive to make sure that the concession is not excessive:

Exact Concession (EC) For every $i \in N$ and $(S, d) \in \Sigma$, if $\cap_{-i} C_j(S, d)) \neq \varnothing$ and $a = \max\{x_i | x \in \cap_{-i} C_j(S, d))\}$, then $C_i(S, d) \cap \{x \in S | x_i < a\} = \varnothing$.

The EC axiom states that each party should not accept less than what others are willing to concede. It is straightforward to see the following (the proof is omitted):

Lemma 3 *If C satisfies EC, then $B_c(S, d)$ is either empty or single-valued for every $(S, d) \in \Sigma$.*

Combining Lemma 3 with Proposition 2 we obtain the following corollary:

Corollary 1 *If C satisfies CR, PC, and EC, then $B_c(S, d)$ is single-valued for every $(S, d) \in \Sigma$ with $B_c(S, d) \in \{y \in \mathbb{R}^n | y_1 = ... = y_n\}$ whenever (S, d) is symmetric.*

Hence CR, PC and EC guarantee that there exists a unique agreement for every problem. This agreement is symmetric whenever the problem is symmetric, which is the SYM axiom postulated by Nash. There was a fantastic, unsettled debate around 1960 between Thomas Schelling and John Harsanyi on whether the SYM axiom used by Nash in bargaining theory, or even more broadly, any symmetry assumption made in game theory, is justifiable. Harsanyi justified the SYM axiom as follow:

> The bargaining problem has an obvious determinate solution in at least one special case: viz., in situations that are completely symmetric with respect to the two bargaining parties. In this case it is natural to assume that the two parties will tend to share the net gain equally since neither would be prepared to grant the other better terms than the latter would grant him (Harsanyi 1956, p. 147).

Schelling argued that symmetry should not be imposed as a constraint of rationality:

> What I am going to argue is that though symmetry is consistent with the rationality of the players, it can not be demonstrated that asymmetry is inconsistent with their rationality, while the inclusion of symmetry in the *definition* of rationality begs the question... Both players, being rational, must recognize that the only kind of "rational" expectation they can have is a fully shared expectation of an *outcome* (Schelling 1959, p. 219).

He continued to give an example showing that an asymmetric outcome could arise from the parties' mutually rational expectations:

> Specifically, suppose that two players may have \$100 to divide as soon as they agree explicitly on how to divide it; and they quite readily agree that A shall have \$80 and B shall have \$20; and we know that dollar amounts in this particular case are proportionate to utilities, and the players do too: can we demonstrate that the players have been irrational (Schelling 1959, p. 220)?

Harsanyi responded by asserting that symmetry is a necessary premise for the existence of a unique outcome:

> On the contrary, the symmetry postulate has to be satisfied, as a matter of sheer logical necessity, by any theory whatever that assigns a unique outcome to the bargaining process (Harsanyi 1961, p. 188).

In Schelling's example, Harsanyi argued that, if $(80, 20)$ is an agreement, then the exact opposite allocation, $(20, 80)$, would also be another outcome predicted by the theory. Harsanyi (1961) then concluded that "Schelling cannot avoid the symmetry postulate if he is to propose any definite theory of bargaining at all."

Our results reconcile the two distinct views. If CR and PC (or SPC*) are the only two rationality principles the two bargainers adopt in making mutual concessions, then Schelling is right that an asymmetric agreement such as $(80, 20)$ could arise from a symmetric problem. But Harsanyi is also correct: by CR, $(20, 80)$ must be another plausible agreement. However, Harsanyi's statement that SYM is a *necessary postulate* for the existence of a unique agreement, is wrong. First, in Nash's approach, the existence of unique agreement is already assumed, implicitly in Nash (1950) and explicitly in Nash (1953). Second, on top of the CR and PC axioms, if the bargainers also make no excessive concessions, then the agreement must be symmetric in symmetric problems (Corollary 1). Thus, under our framework, SYM is an outcome derived from the rationality of the bargainers' expectations, not a restriction of rationality.[7]

Next we study the efficiency of agreement. Consider the following axiom:

Concession Monotonicity (CM) For every $i \in N$ and $(S, d) \in \Sigma$, if $x \in C_i(S, d)$, then $y \in C_i(S, d)$ for every $y \in S$ such that $y_i > x_i$ and $y_j = x_j$ for every $j \neq i$.

CM is an axiom on self-interest. If an alternative is acceptable by a bargainer, then any other alternative that gives this bargainer a higher payoff while keeping the payoffs of others unchanged should be accepted by him/her. However, suppose another alternative is such that each and every bargainer can get a higher payoff, CM does not require the bargainer to accept this alternative.

Lemma 4 *Let $(S, d) \in \Sigma$. If $B_c(S, d)$ is single-valued and C satisfies PC and CM, then $B_c(S, d) \in WPO(S) \cap IR(S, d)$.*

[7] Note that CR itself is a kind of symmetry assumption on the players' behavior (but obviously much weaker than SYM). Hence we do not completely accomplish Schelling's goal of abandoning ANY symmetry assumption in game theory. Our view is that this direction is unrealistic.

If the agreement is unique, and the bargainers follow PC and CM in making mutual concessions, then this agreement is efficient. Note that WPO is a collective rationality requirement. The result shows that it comes from each bargainer pursuing his or her self-interest. But uniqueness of agreement is crucial as well. Without uniqueness, we may have multiple agreements and some of them are inefficient ones.

Several properties we have studied so far can be summarized in the following definition:

Definition 1 *A profile of compromise sets C is said to be regular if for every $(S, d) \in \Sigma$,*
(i) $B_c(S, d)$ is single-valued
(ii) $B_c(S, d)$ is symmetric whenever (S, d) is symmetric
(iii) $B_c(S, d) \in WPO(S) \cap IR(S, d)$

The sufficient condition for a profile of compromise sets C to be regular can be readily provided:

Corollary 2 *If C satisfies CR, PC, EC and CM, then C is regular.*

4 The Concession Invariance Principle

In the previous secton, we provide a rational foundation for a profile of compromise sets to be regular. It resolves the first issue of contract indeterminacy – an agreement exists in every problem. It also partially resolves the second issue – the agreement is uniquely determined for symmetric problems. Not too much can be said for asymmetric problems, except that there exists a unique efficient agreement. In the following, we take regularity as a basic requirement, and suggest a rationality postulate to pin down a unique agreement in TU (Transferable Utility) games. We then impose an additional axiom to characterize the Nash bargaining solution.

For two-person bargaining problems, let C be such that $C_1(S, d) = C_2(S, d) = m(S, d)$ for every symmetric $(S, d) \in \Sigma$, and $C_1(S, d) = C_2(S, d) = \{(b_1(S, d), d_2)\}$ otherwise. C is regular, but C is hardly sensible in portraying bargainers' behavior. The unique unanimous agreement associated with C is symmetric in a symmetric problem, but jumps to player 1's dictorial allocation for an asymmetric one. Given that in bargaining every single party has the ability to unilaterally block any agreement reached by others, we should expect the parties to compromise and meet in the middle. The question is where the parties perceive "the middle" is? When facing a bargaining situation, fairness could be the major concern the parties have in mind. As argued in Shapley (1969), a bargainer may contemplate whether an agreement is a "fair division" based on how much he/she sacrifices compared to others at this agreement. In other words, a bargainer would convince others to concede further by saying that "I give up more than you do in an effort to reach an agreement, therefore you should give way." But how do bargainers evaluate their "sacrifices"? Consider the example depicted below. Suppose player 2 is willing to make a compromise at x^*. The sacrifice made by player 2 in reaching an agreement can be measured by area A, the set of all alternatives giving player 2 higher payoffs than x^* but he/she gives in. What if player 2 is willing to make a compromise at y^* instead of x^*? The area $A + B$ consists of all alternatives that give player 2 higher payoffs than y^*. However, player 2 makes unnecessary, inefficient sacrifice in B. In order to offer payoff y_1^* to player 1, player 2 does not have to sacrifice that much by accepting y_2^* as his/her payoff. Instead, player 2 could propose x^* as a compromise, improving his/her own payoff without asking player 1 to make a further concession compared to the alternative proposal y^*. Accordingly, the effective sacrifice made by player 2 is still measured by area A.

We formalize this idea as follows. Let $i \in N$ and $(S,d) \in \Sigma$. Define $\varphi_{i,S} : S \longrightarrow \mathbb{R}$ by $\varphi_{i,S}(x) = \max\{z_i | (z_i, x_{-i}) \in S\}$. Let $Vol : B_{\mathbb{R}^n} \longrightarrow \mathbb{R}_+$ denote the Lebesgue measure (n-dimensional volume) on $\mathbb{R}^n$, where $B_{\mathbb{R}^n}$ stands for the Borel $\sigma-$algebra on $\mathbb{R}^n$. Define $\mu_{i,(S,d)} : 2^S \longrightarrow \mathbb{R}_+$ by

$$\mu_{i,(S,d)}(A) = \begin{cases} \sup_{x \in A \cap IR(S,d)} Vol(\{y \in IR(S,d) | y_i \geq \varphi_{i,S}(x)\}) & A \cap IR(S,d) \neq \varnothing \\ 0 & A \cap IR(S,d) = \varnothing \end{cases}.$$

Then $\mu_{i,(S,d)}(C_i(S,d))$ is a proper measure the parties may take to fathom the (maximal) concession made by player i. Denote by $\mu^{C(S,d)} = (\mu_{1,(S,d)}(C_1(S,d)), ..., \mu_{n,(S,d)}(C_n(S,d)))$.

A problem $(S,d) \in \Sigma$ is said to be a TU problem if S is a standard orthogonal $n-$simplex in $\mathbb{R}^n$; i.e., S can be expressed in the following form: $S = \{x \in \mathbb{R}^n | \sum_{i=1}^n \frac{x_i - b_i}{a_i} \leq 1,\ a > \mathbf{0},\ x \geq b\}$. Let $\Sigma^{TU} \subset \Sigma$ be the collection of all TU problems in Σ. If two vectors u and v in $\mathbb{R}^n$ are parallel, we write $u//v$. Consider the following rationality principle:

Concession Invariance Principle in TU Problems ($\mathbf{CIP}_{TU}$) : For every pair of problems (S,d) and (T,e) in Σ^{TU}, $\mu^{C(S,d)}//\mu^{C(T,e)}$.

Pick a problem (S,d) in Σ^{TU}, and let $C(S,d)$ be the players' compromise sets when facing (S,d). For each player $i \in N$, given the concessions made by other players, $C_{-i}(S,d)$, i must perceive that the concession made by him/her, $C_i(S,d)$, is a justifiable, fair one. When some alternatives are added to or eliminated from S, and now they face a new problem (T,e) in Σ^{TU}, how would the parties renegotiate their compromise sets? $\mathbf{CIP}_{TU}$ suggests that the parties would proportionally adjust their mutual concessions in adapting to the new bargaining situation. That is to say, the ratio of concessions measure by μ between two TU problems remains constant across parties. It also implies that, if for some TU problem all parties make equal concessions ($\mu_{1,(S,d)}(C_1(S,d)) = ... = \mu_{n,(S,d)}(C_n(S,d))$ for some (S,d) in Σ^{TU}), then they make equal concessions in all TU problems.

For symmetric problems, the following lemma shows that the parties make the same degree of concession under regularity and EC.

Lemma 5 *Let $(S,d) \in \Sigma$ be a symmetric problem. If C is regular and satisfies EC, then* $\mu_{1,(S,d)}(C_1(S,d)) = ... = \mu_{n,(S,d)}(C_n(S,d))$.

Observe that for a symmetric problem $(S,d) \in \Sigma$, CR directly implies $\mu_{1,(S,d)}(C_1(S,d)) = ... = \mu_{n,(S,d)}(C_n(S,d))$. Lemma 5 does not resort to CR for the parties to make equal concessions. Instead, it states that in a symmetric problem, the parties make equal concessions when (i) they concent to reach a unique symmetric agreement, and (ii) they do not make excessive concessions. With this lemma in hand, we are ready to show the following:

Proposition 3 *If C is regular and satisfies EC and CIP_{TU}, then $B_c(S,d) = m(S,d)$ for every $(S,d) \in \Sigma^{TU}$.*

In TU problems, the midpoint is the unique efficient agreement at which the parties make exact and equal concessions. All well-known solutions in bargaining literature except the proportional solutions coincide with the midpoint allocation in TU games. The coincidence arises as the characterizations of those solution concepts make use of either the SI axiom or the Midpoint Domination (MD) axiom (Moulin (1983)). The expected utility theorem is the cornerstone of Nash's bargaining theory, and the von Neumann-Morgenstern utilities are unique up to positive affine transformations. Thus SI was naturally introduced by Nash as a desirable property a solution should have. On the other hand, MD is a fairness principle stemmed from random dictatorship: the parties should get no less than the average of their dictatorial payoffs. Here the midpoint is the unique agreement outcome in TU games as the parties make equal concessions in symmetric problems, and extend these concessions proportionally to TU problems.

What if the parties use a measure different from μ to evaluate the concessions made by each other? How would it change the result? Consider the following two other measures the parties may have in mind. Let $\widetilde{Vol} : B_{\mathbb{R}^{n-1}} \longrightarrow \mathbb{R}_+$ denote the $(n-1)$-dimensional Lebesgue measure on $\mathbb{R}^{n-1}$. Define $\widetilde{\mu}_{i,(S,d)} : 2^S \longrightarrow \mathbb{R}_+$ by

$$\widetilde{\mu}_{i,(S,d)}(A) = \begin{cases} \sup_{x \in A \cap IR(S,d)} \widetilde{Vol}(\{y \in IR(S,d) \cap WPO(S) | y_i \geq \varphi_{i,S}(x)\}) & A \cap IR(S,d) \neq \varnothing \\ 0 & A \cap IR(S,d) = \varnothing \end{cases}.$$

$\widetilde{\mu}$ differs from μ in that only efficient allocations are included in calculating a party's degree of concession. Alternatively, a party may compare the payoff he/she is willing to accept against the highest possible payoff he/she could obtain: define $\widetilde{\widetilde{\mu}}_{i,(S,d)} : 2^S \longrightarrow \mathbb{R}_+$ by

$$\widetilde{\widetilde{\mu}}_{i,(S,d)}(A) = \begin{cases} \sup_{x \in A \cap IR(S,d)} \frac{\varphi_{i,S}(x) - d_i}{b_i(S,d) - d_i} & A \cap IR(S,d) \neq \varnothing \\ 0 & A \cap IR(S,d) = \varnothing \end{cases}.$$

All three measures are intuitively appealing, and different bargainers may use different measures in extending the concessions made in symmetric problems to TU problems. It turns out that Proposition 3 still holds – no matter which of the three measures we use in defining $\mathbf{CIP}_{TU}$ – the agreement is still the midpoint in TU problems. Hence $\mathbf{CIP}_{TU}$ is a robust rationality principle in TU problems.

This robustness property, however, disappears when one tries to extend $\mathbf{CIP}_{TU}$ to non-TU problems. Depending on which measure we use, the concession invariance principle, when it is applied to all problems rather than just TU problems, could lead us to the Equal Area solution, the Equal Length or Equal Surface solution, or the Kalai-Smorodinsky solution. Given that the Nash solution is the most prominent solution concept in the literature, in the following a characterization of the Nash solution under our new framework is provided. First we propose an axiom:

Contraction Inclusion (CI) Given (S,d) and (T,e) in Σ with $d = e$, if $S \subset T$ then $C_i(T,d) \cap S \subset C_i(S,d)$ for every $i \in N$.

Consider a bargaining situation and the compromises accepted by each party as agreement. Suppose now some alternatives become infeasible. Facing this new bargaining situation, how would the parties readjust their acceptable compromises? CI states that the parties would be willing to accept those compromises that were previously accepted as agreement as long as they are still available.

Proposition 4 *If C is regular and satisfies EC, CIP_{TU} and CI, then $B_c = Nash$.*

Suppose the parties, who make no excessive concessions, consent to reach a unique efficient (symmetric) agreement for every (symmetric) problem, and proportionally adjust their concessions in TU problems. Moreover, when facing a new bargaining situation with less alternatives, the parties accept previously accepted compromises as long as they are still available. Then the agreement is the one predicted by Nash.

Several remarks are in order. First, the rational foundation of the Nash solution provided here is very different from that provided by Nash himself. In Nash's characterization, the agreements are uniquely determined by PO and SYM in symmetric problems. SI is then assumed to generalize the agreements in symmetric problems to TU problems, and IIA is imposed to extend those agreements to non-TU problems. In our characterization, PO and SYM, two properties implied in the regularity of C, are derived through rationality principles adopted by the parties in making mutual concessions. CIP_{TU} portrays how the parties make proportional mutual adjustments in their concessions to reach agreements in TU problems, and CI extend those agreements to non-TU problems.

Second, Nash's characterization and ours complement each other. Given a bargaining situation, consider the following two questions:

(i) What is the "just agreement" an impartial arbitrator would recommend to the parties? What rationality principles should the arbitrator follow to make this recommendation?

(ii) What would be the agreement reached by the parties? What rationality principles do they follow in making mutual concessions to reach this agreement?

Note that SI excludes interpersonal comparisons of utility, and it is very unlikely the parties would have this principle in mind, or even use this principle as an argument to convince each other to make concessions during the negotiation process. Thus Nash's characterization is more suitable to answer the first question. On the other hand, our characterization is more suitable to address the second one.

Third, while the agreement is uniquely determined by the Nash solution, the profile of compromises, C, is not unique. The most trivial example satisfying Proposition 4 is the profile C with $C_i(S,d) = Nash(S,d)$ for every $i \in N$ and $(S,d) \in \Sigma$. It can be readily seen that there are uncountably many other examples. This implies that when we see two groups of bargainers reach the same agreement for the same problem, we cannot jump to a conclusion that both groups share exactly the same reasonings or follow the same "fairness" principles in reconciling their differences. The reached agreement provides only censored data – no further information can be inferred except that each bargainer's compromise set contains this agreement.

5 Concluding Remarks

This paper proposes an alternative framework to understand the bargaining problem. We study the formation of bargainers' compromise sets during negotiations, and establish the existence of agreement. A rational foundation for the Nash solution under this new framework is also provided. Two lines of future research seem to be promising. One may take the framework developed here to characterize other well-known bargaining solutions. This direction may bring new insight into old solution concepts, and new solution concepts may emerge along the way. The second line of research could be to embed this new approach into various types of modelling in bargaining to address several important issues, such as bargaining with uncertain disagreement (Chun and Thomson (1990)), bargaining with a variable population (Lensberg and Thomson (1989)), bargaining with non-expected utility preferences (Rubinstein, Safra, and Thomson (1992)), and bargaining in committees (Laruelle and Valenciano (2007)).

6 Appendix: Proofs

Proof of Lemma 1. Let C satisfy CR and SPC. Pick any pair of agents $\{i,j\}$, a symmetric problem $(S,d) \in \Sigma$, and $x \in C_i(S,d)$. By CR, $C(S,d) = \pi_{ij}(\pi_{ij}C_1(S,d),...,\pi_{ij}C_n(S,d))$. Therefore $C_j(S,d) = \pi_{ij}C_i(S,d)$. As $x \in C_i(S,d)$, $\pi_{ij}x \in C_j(S,d)$. If $x = \pi_{ij}x$, then $x \in C_i(S,d) \cap C_j(S,d) \cap L(x,\pi_{ij}x)$ and the lemma holds. Assume now $x \neq \pi_{ij}x$. Suppose to the contrary that $C_i(S,d) \cap C_j(S,d) \cap L(x,\pi_{ij}x) = \varnothing$. Let $\rho : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0,\infty)$ denote the Euclidean metric. Then the distance between two sets E and F in $\mathbb{R}^n$ is defined as $\rho(E,F) = \inf\{\rho(y,z) : y \in E,\ z \in F\}$. Since $C_i(S,d) \cap L(x,\pi_{ij}x)$ and $C_j(S,d) \cap L(x,\pi_{ij}x)$ are two nonempty disjoint compact sets, $\rho(C_i(S,d)\cap L(x,\pi_{ij}x), C_j(S,d)\cap L(x,\pi_{ij}x)) = \epsilon > 0$. By the compactness of $C_i(S,d)\cap L(x,\pi_{ij}x)$, there exists $\widetilde{x} \in C_i(S,d)\cap L(x,\pi_{ij}x)$ with $\rho(\widetilde{x}, C_j(S,d)\cap L(x,\pi_{ij}x)) = \epsilon$. By CR, $\pi_{ij}\widetilde{x} \in C_j(S,d)\cap L(x,\pi_{ij}x)$. By SPC, $C_i(S,d)\cap L(\widetilde{x},\pi_{ij}\widetilde{x})\backslash\{\widetilde{x}\} \neq \varnothing$. Pick $z \in C_i(S,d) \cap L(\widetilde{x},\pi_{ij}\widetilde{x})\backslash\{\widetilde{x}\}$. $z \neq \pi_{ij}\widetilde{x}$, for otherwise $z = \pi_{ij}\widetilde{x} \in C_i(S,d) \cap C_j(S,d) \cap L(x,\pi_{ij}x)$, contradicting the assumption that $C_i(S,d) \cap C_j(S,d) \cap L(x,\pi_{ij}x) = \varnothing$. By CR again, $\pi_{ij}z \in C_j(S,d) \cap L(x,\pi_{ij}x)$. It can be readily seen that $\rho(z,\pi_{ij}z) < \epsilon$. Accordingly, $\rho(C_i(S,d) \cap L(x,\pi_{ij}x), C_j(S,d) \cap L(x,\pi_{ij}x)) < \epsilon$, a contradiction. ■

Proof of Lemma 2. Let C satisfy CR and SPC*. Pick any pair of agents $\{i,j\}$, a symmetric problem $(S,d) \in \Sigma$, and $x \in C_i(S,d)$ with $x \neq \pi_{ij}x$. Define $\mathcal{A} = \{y \in L^o(x,\pi_{ij}x) | y_i = y_j\}$. By CR, it suffices to show that $C_i(S,d) \cap \mathcal{A} \neq \varnothing$. Suppose to the contrary that $C_i(S,d) \cap \mathcal{A} = \varnothing$. By CR, $\pi_{ij}x \in C_j(S,d)$. By SPC*, there exists $\widetilde{x} \in C_i(S,d) \cap L^o(x,\pi_{ij}x)$. Observe that $C_i(S,d) \cap L(\widetilde{x},\pi_{ij}\widetilde{x}) \subset C_i(S,d) \cap L^o(x,\pi_{ij}x)$ and $\widetilde{\mathcal{A}} = \{y \in L(\widetilde{x},\pi_{ij}\widetilde{x}) | y_i = y_j\} \subset \mathcal{A}$. Hence $C_i(S,d) \cap \mathcal{A} = \varnothing$ implies $C_i(S,d) \cap L(\widetilde{x},\pi_{ij}\widetilde{x}) \cap \widetilde{\mathcal{A}} = \varnothing$. Let $\rho : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0,\infty)$ denote the Euclidean metric. Since $C_i(S,d) \cap L(\widetilde{x},\pi_{ij}\widetilde{x})$ and $\widetilde{\mathcal{A}}$ are two nonempty disjoint compact sets, $\rho(C_i(S,d) \cap L(\widetilde{x},\pi_{ij}\widetilde{x}), \widetilde{\mathcal{A}}) = \epsilon > 0$. By the compactness of $C_i(S,d) \cap L(\widetilde{x},\pi_{ij}\widetilde{x})$, there exists $\widetilde{\widetilde{x}} \in C_i(S,d) \cap L(\widetilde{x},\pi_{ij}\widetilde{x})$ with $\rho(\widetilde{\widetilde{x}}, \widetilde{\mathcal{A}}) = \epsilon$. By CR, $\pi_{ij}\widetilde{\widetilde{x}} \in C_j(S,d) \cap L(\widetilde{x},\pi_{ij}\widetilde{x})$. $\widetilde{\widetilde{x}} \neq \pi_{ij}\widetilde{\widetilde{x}}$, for otherwise $\rho(\widetilde{\widetilde{x}}, \widetilde{\mathcal{A}}) = 0$, which contradicts $\rho(\widetilde{\widetilde{x}}, \widetilde{\mathcal{A}}) = \epsilon > 0$. By SPC*, $C_i(S,d) \cap L^o(\widetilde{\widetilde{x}},\pi_{ij}\widetilde{\widetilde{x}}) \neq \varnothing$. Pick $z \in C_i(S,d) \cap L^o(\widetilde{\widetilde{x}},\pi_{ij}\widetilde{\widetilde{x}}) \subset C_i(S,d) \cap L(\widetilde{x},\pi_{ij}\widetilde{x})$. It can be readily seen that $\rho(z, \widetilde{\mathcal{A}}) < \epsilon$, which contradicts the fact that $\rho(C_i(S,d) \cap L(\widetilde{x},\pi_{ij}\widetilde{x}), \widetilde{\mathcal{A}}) = \epsilon$. Therefore we must have $C_i(S,d) \cap \mathcal{A} \neq \varnothing$. ■

Before proceeding to the proof of Proposition 1, we first establish a lemma. Given $m \in \{1,...,n-2\}$, define the following property:

Property P^m : For any $I \subset N$ with $|I| = m$, $\cap_{i\in I}C_i(S,d) \cap \{y \in \mathbb{R}^n | y_i = y_j \text{ for } i,j \in I\} \neq \varnothing$. Moreover, for every $x \in \cap_{i\in I}C_i(S,d) \cap \{y \in \mathbb{R}^n | y_i = y_j \text{ for } i,j \in I\}$ and $t \in N\backslash I$ with $x_t \neq x_{i^*}$, where $i^* \in I$, $\cap_{i\in I\cup\{t\}}C_i(S,d) \cap \{y \in \mathbb{R}^n | \min\{x_{i^*}, x_t\} < y_i = y_j \leq \max\{x_{i^*}, x_t\}$ for $i,j \in I \cup \{t\}, y_k = x_k$ for $k \in N\backslash(I \cup \{t\})\} \neq \varnothing$.

Lemma 6 *Suppose C satisfies CR and $(S,d) \in \Sigma$ is a symmetric problem. Then P^m implies P^{m+1} for $m = \{1,...,n-2\}$.*

Proof of Lemma 6. Assume the premise of the lemma holds. Pick $W \subset N$ with $|W| = m+1$. Without loss of generality, assume $W = \{1,...,m+1\}$. First we show that $\cap_{i\in W} C_i(S,d) \cap \{y \in \mathbb{R}^n | y_i = y_j \text{ for } i,j \in W\} \neq \varnothing$. By P^m, there exists $x \in \cap_{i \in W\backslash\{1\}} C_i(S,d) \cap \{y \in \mathbb{R}^n | y_i = y_j \text{ for } i,j \in W\backslash\{1\}\}$. If $x_1 = x_2$, by CR and the fact that (S,d) is symmetric, $x \in C_1(S,d)$. Therefore $\cap_{i\in W} C_i(S,d) \cap \{y \in \mathbb{R}^n | y_i = y_j \text{ for } i,j \in W\} \neq \varnothing$. If $x_1 \neq x_2$, then we apply P^m and conclude that $\cap_W C_i(S,d) \cap \{y \in \mathbb{R}^n | \min\{x_2, x_1\} < y_i = y_j \leq \max\{x_2, x_1\} \text{ for } i,j \in W, y_k = x_k \text{ for } k \in N\backslash W\} \neq \varnothing$. Hence $\cap_{i\in W} C_i(S,d) \cap \{y \in \mathbb{R}^n | y_i = y_j \text{ for } i,j \in W\} \neq \varnothing$ in both cases.

Next we prove the second part of P^{m+1}. Pick any $x \in \cap_{i\in W} C_i(S,d) \cap \{y \in \mathbb{R}^n | y_i = y_j$ for $i,j \in W\}$, and $t \in N\backslash I$ with $x_t \neq x_1$. Without loss of generality, let $t = m+2$. Consider two subcases:

(i) $x_1 < x_{m+2}$. Define $E = \{y \in \mathbb{R}^n | x_1 \leq y_1 = ... = y_{m+1} \leq y_{m+2} \leq x_{m+2},\ y_i = x_i \text{ for } i = m+3,...,n\}$. E is compact and nonempty ($x \in E$). Then the set $G \equiv \cap_{i=1}^{m+1} C_i(S,d) \cap E$ is compact and nonempty. Define a continuous mapping $f : G \longrightarrow \mathbb{R}$ by $f(y) = y_{m+2} - y_1$. f attains its minimum on G. Let $\epsilon = \min_{y\in G} f(y)$ and $z \in \arg\min_{y \in G} f(y)$. $\epsilon \geq 0$ as $y_{m+2} - y_1 \geq 0$ for every y in G. We claim that $\epsilon = 0$. Suppose to the contrary that $\epsilon > 0$. Since $z \in \arg\min_{y\in G} f(y)$, $z_{m+2} - z_1 = z_{m+2} - z_2 = \epsilon$. Applying property P^m to the triple $(I = \{2,...,m+1\}, z, t = m+2)$, there exists $z' = (z_1, a, ..., a, x_{m+3}, ..., x_n) \in \cap_{i=2}^{m+2} C_i(S,d)$, where $\min\{z_2, z_{m+2}\} = z_2 = z_1 < a \leq \max\{z_2, z_{m+2}\} = z_{m+2}$. Applying P^m again to the triple $(I = \{2,...,m+1\}, z', t = 1)$, there exists $z'' = (b, ..., b, a, x_{m+3}, ..., x_n) \in \cap_{i=1}^{m+1} C_i(S,d)$, where $\min\{z_1, a\} = z_1 < b \leq \max\{z_1, a\} = a$. Observe that $b > z_1 \geq x_1$ and $b \leq a \leq z_{m+2} \leq x_{m+2}$. Therefore $z'' \in E$, which in turn implies that $z'' \in G$. Then $f(z'') = a - b < z_{m+2} - z_1 = \epsilon$, a contradiction! Accordingly we must have $\epsilon = 0$. $\epsilon = 0$ implies $z_1 = ... = z_{m+2}$. By CR and the fact that $z \in C_1(S,d)$, $z \in C_{m+2}(S,d)$. Hence $z \in \cap_{i=1}^{m+2} C_i(S,d)$. Moreover, the above procedure also shows that $x_1 < z_1 \leq x_{m+2}$. Accordingly, $\cap_{i=1}^{m+2} C_i(S,d) \cap \{y \in \mathbb{R}^n | \min\{x_1, x_{m+2}\} = x_1 < y_i = y_j \leq \max\{x_1, x_{m+2}\} = x_{m+2} \text{ for } i,j \in W \cup \{m+2\}, y_k = x_k \text{ for } k \in N\backslash(W \cup \{m+2\})\} \neq \varnothing$.

(ii) $x_1 > x_{m+2}$. Simply replace E by $E' = \{y \in \mathbb{R}^n | x_1 \geq y_1 = ... = y_{m+1} \geq y_{m+2} \geq x_{m+2}, y_i = x_i \text{ for } i = m+3,...,n\}$ and $f(y) = y_{m+2} - y_1$ by $f'(y) = y_1 - y_{m+2}$, and repeat the steps in (i). ■

Proof of Proposition 1. The proof is by induction. Assume the premise of the proposition holds. By Lemma 2, property P^m holds for $m = 1$. By iteratively applying Lemma 6, property P^m holds for $m = 2,...,n-1$. Then by P^{n-1}, $\cap_{i=1}^{n-1} C_i(S,d) \cap \{y \in \mathbb{R}^n | y_1 = ... = y_{n-1}\} \neq \varnothing$. Let $x \in \cap_{i=1}^{n-1} C_i(S,d) \cap \{y \in \mathbb{R}^n | y_1 = ... = y_{n-1}\}$. If $x_1 = x_n$, then it can be readily seen that $x \in C_n(S,d)$ and the proof is complete. If $x_1 \neq x_n$, property P^{n-1} again implies that $\cap_{i=1}^{n} C_i(S,d) \cap \{y \in \mathbb{R}^n | y_1 = ... = y_n\} \neq \varnothing$. ■

To prove Proposition 2, we first establish two lemmas.

Lemma 7 *Suppose C satisfies PC. For any pair of players $\{i,j\} \subset N$ and $(S,d) \in \Sigma$, $C_i(S,d) \cap C_j(S,d) \cap L(x,y) \neq \varnothing$ for every $x \in C_i(S,d)$ and $y \in C_j(S,d)$.*

The proof is analogous to that of Lemma 1 and hence is omitted.

Given $m \in \{1,...,n-1\}$, define the following property:

Property Q^m : For any $I \subset N$ with $|I| = m$, $\cap_{i \in I} C_i(S,d) \neq \varnothing$. Moreover, for every $I \subset N$ with $|I| = m$ and $t \in N \backslash I$, $x \in \cap_{i \in I} C_i(S,d)$ and $y \in C_t(S,d)$, $\cap_{i \in I \cup \{t\}} C_i(S,d) \cap L(x,y) \neq \varnothing$.

Lemma 8 *Q^m implies Q^{m+1} for $m = \{1,...,n-2\}$.*

Proof of Lemma 8. Fix some $m \in \{1,...,n-2\}$ and assume Q^m holds. Pick any $I \subset N$ with $|I| = m+1$ and let $j \in I$. By Q^m, $\cap_{i \in I \backslash \{j\}} C_i(S,d) \neq \varnothing$. Let $x \in \cap_{i \in I \backslash \{j\}} C_i(S,d)$ and $y \in C_j(S,d)$. By Q^m again, $\cap_{i \in I} C_i(S,d) \cap L(x,y) \neq \varnothing$. Hence $\cap_{i \in I} C_i(S,d) \neq \varnothing$ for any $I \subset N$ with $|I| = m+1$. This establishes the first part of Q^{m+1}. Next we show the second part of Q^{m+1} is also true. Pick any $I \subset N$ with $|I| = m+1$ and $t \in N \backslash I$. Pick $x \in \cap_{i \in I} C_i(S,d)$ and $y \in C_t(S,d)$. Suppose to the contrary that $\cap_{i \in I \cup \{t\}} C_i(S,d) \cap L(x,y) = \varnothing$. Let $\rho : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0,\infty)$ denote the Euclidean metric. Since $\cap_{i \in I} C_i(S,d) \cap L(x,y)$ and $C_t(S,d) \cap L(x,y)$ are two nonempty disjoint compact sets, $\rho(\cap_{i \in I} C_i(S,d) \cap L(x,y), C_t(S,d) \cap L(x,y)) = \epsilon > 0$. By compactness, there exist $\widetilde{x} \in \cap_{i \in I} C_i(S,d) \cap L(x,y)$ and $\widetilde{y} \in C_t(S,d) \cap L(x,y)$ with $\rho(\widetilde{x},\widetilde{y}) = \epsilon$. Let $j \in I$. By Q^m, $\cap_{i \in \{t\} \cup I \backslash \{j\}} C_i(S,d) \cap L(\widetilde{x},\widetilde{y}) \neq \varnothing$. Let $z \in C_t(S,d) \cap L(\widetilde{x},\widetilde{y}) \subset C_t(S,d) \cap L(x,y)$. $z \neq \widetilde{y}$ for otherwise $\widetilde{y} \in \cap_{i \in I \cup \{t\}} C_i(S,d) \cap L(x,y)$ and $\rho(\cap_{i \in I} C_i(S,d) \cap L(x,y), C_t(S,d) \cap L(x,y)) = 0$. Then $\rho(\widetilde{x},z) < \rho(\widetilde{x},\widetilde{y}) = \rho(\cap_{i \in I} C_i(S,d) \cap L(x,y), C_t(S,d) \cap L(x,y)) = \epsilon$, a contradiction. ■

Proof of Proposition 2. The statement $B_c(S,d) \cap \{y \in \mathbb{R}^n | y_1 = ... = y_n\} \neq \varnothing$ when (S,d) is symmetric is already established in Proposition 1. Here we prove the nonemptyness of $B_c(S,d)$. Suppose C satisfies PC. Let $(S,d) \in \Sigma$. By Lemma 7, property Q^1 holds. Iteratively invoking Lemma 8 concludes that Q^{n-1} holds. By Q^{n-1}, $B_c(S,d) \neq \varnothing$. ■

Proof of Lemma 4. Suppose to the contrary that the unique agreement is $x \notin WPO(S)$. Then there exists $y \in \mathbb{R}^n$ with $y > x$ such that $(y_i, x_{-i}) \in S$ for every $i \in N$. By CM, $(y_i, x_{-i}) \in C_i(S,d)$ for every i. Following the proof of Proposition 2, there exists a unanimous agreement $x^* \in L(x,y) \backslash \{x\}$, which contradicts the uniqueness of agreement. ■

Proof of Lemma 5. Suppose C is regular and satisfies EC, and $(S,d) \in \Sigma$ is symmetric. By the regularity of C, $B_c(S,d) = C_i(S,d) \cap (\cap_{-i} C_j(S,d)) = (a,...,a) \in PO(S) \cap IR(S,d)$. By EC, $x_i \geq a$ for every $x \in C_i(S,d)$, $i = 1,...,n$. Then $\varphi_{i,(S,d)}(x) \geq a$ for every $x \in C_i(S,d)$, $i = 1,...,n$. Accordingly, $\mu_{i,(S,d)}(C_i(S,d)) = \sup_{x \in C_i(S,d) \cap IR(S,d)} Vol(\{y \in IR(S,d) | y_i \geq \varphi_{i,(S,d)}(x)\}) = Vol(\{y \in IR(S,d) | y_i \geq a\})$. As (S,d) is symmetric, $Vol(\{y \in IR(S,d) | y_i \geq a\}) = Vol(\{y \in IR(S,d) | y_j \geq a\})$, $i,j \in N$. Hence, $\mu_{1,(S,d)}(C_1(S,d)) = ... = \mu_{n,(S,d)}(C_n(S,d))$. ■

The proof of Proposition 3 makes use of the following result:

Lemma 9 *Let $Conv(v_0,...,v_n)$ denote a n-simplex in $\mathbb{R}^n$ with vertices $v_0,...,v_n$. Then* $Vol(Conv(v_0,...,v_n)) = \frac{1}{n!} \left| \det[v_1 - v_0, ..., v_n - v_0] \right|$.

Proof of Lemma 9. See, for example, Stein (1966). ■

Proof of Proposition 3. Suppose C is regular and satisfies EC and CIP_{TU}. Pick any $(S,d) \in \Sigma^{TU}$. Then $IR(S,d)$ is a standard orthogonal $n-$simplex with $IR(S,d) = \{x \in$

$\mathbb{R}^n | \sum_{i=1}^n \frac{x_i - d_i}{b_i(S,d) - d_i} \leq 1,\ x \geq d\}$. By the regularity of C, $B_c(S,d)$ is single-valued and belongs to $PO(S) \cap IR(S,d)$. Let $B_c(S,d) \equiv y$. Then $\sum_{i=1}^n \frac{y_i - d_i}{b_i(S,d) - d_i} = 1$. By EC, $\mu_{i,(S,d)}(C_i(S,d)) = \sup_{x \in C_i(S,d) \cap IR(S,d)} Vol(\{z \in IR(S,d) | z_i \geq \varphi_{i,(S,d)}(x)\}) = Vol(\{z \in IR(S,d) | z_i \geq y_i\})$. It can be readily verified that $\{z \in IR(S,d) | z_i \geq y_i\} = Conv(v_0, v_1, ..., v_n)$, where

$$v_j = \begin{cases} (d_1, ..., d_{i-1}, y_i, d_{i+1}, ..., d_n) & j = 0 \\ (d_1, ..., d_{i-1}, b_i(S,d), d_{i+1}, ..., d_n) & j = i \\ (d_1, ..., d_{j-1}, d_j + (b_j(S,d) - d_j)(1 - \frac{y_i - d_i}{b_i(S,d) - d_i}), d_{j+1}, ..., d_{i-1}, y_i, d_{i+1}, ..., d_n) & 0 < j < i \\ (d_1, ..., d_{i-1}, y_i, d_{i+1}, ..., d_{j-1}, d_j + (b_j(S,d) - d_j)(1 - \frac{y_i - d_i}{b_i(S,d) - d_i}), d_{j+1}, ..., d_n) & j > i \end{cases}$$

Accordingly,

$$v_j - v_0 = \begin{cases} (0, ..., 0, b_i(S,d) - y_i, 0, ..., 0) & j = i \\ (0, ..., 0, (b_j(S,d) - d_j)(1 - \frac{y_i - d_i}{b_i(S,d) - d_i}), 0, ..., 0) & j \neq i \end{cases}$$

By Lemma 9,

$$\begin{aligned} \mu_{i,(S,d)}(C_i(S,d)) &= Vol(Conv(v_0, v_1, ..., v_n)) = \frac{1}{n!} |\det[v_1 - v_0, ..., v_n - v_0]| \\ &= \frac{1}{n!}(b_i(S,d) - y_i)\Pi_{j \neq i}[(b_j(S,d) - d_j)(1 - \frac{y_i - d_i}{b_i(S,d) - d_i})] \\ &= \frac{1}{n!}(\frac{b_i(S,d) - y_i}{b_i(S,d) - d_i})^n \Pi_{k=1}^n (b_k(S,d) - d_k) \end{aligned}$$

By Lemma 5 and CIP$_{TU}$, we must have

$$\frac{b_1(S,d) - y_1}{b_1(S,d) - d_1} = ... = \frac{b_n(S,d) - y_n}{b_n(S,d) - d_n}$$

Combined this with the condition $\sum_{i=1}^n \frac{y_i - d_i}{b_i(S,d) - d_i} = 1$, we conclude that $y_i = m_i(S,d) = \frac{1}{n} b_i(S,d) + (1 - \frac{1}{n}) d_i$, and $B_c(S,d) = m(S,d)$. ■

Proof of Proposition 4. Suppose C is regular and satisfies EC, CIP$_{TU}$ and CI. Pick $(S,d) \in \Sigma$ and identify its Nash solution $Nash(S,d)$. Following Nash(1950)'s proof we can construct a problem (T,d) such that T has the following properties: (i) $T \supset S$, (ii) T is a standard orthogonal $n-$simplex in $\mathbb{R}^n$, and (iii) $Nash(S,d) = m(T,d)$. By Proposition 3, $B_c(T,d) = \cap_{i=1}^n C_i(T,d) = m(T,d) = Nash(S,d)$. By CI, $Nash(S,d) \in C_i(S,d)$ for every i. Then $Nash(S,d) \in B_c(S,d)$. By the regularity of C, $Nash(S,d) = B_c(S,d)$. ■

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