The core of aggregative cooperative games

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Abstract

We analyze cooperative games with externalities generated by aggregative normal-form games. We construct the characteristic function of a coalition and analyze the core for various beliefs a coalition has about the behavior of the outside players. We first show that the γ-core is non-empty, provided the payoff of a player is decreasing in the aggregate value of all players' strategies. We next define the class of linear aggregative games. We show that if a coalition S believes that the outsiders will form at least $\frac{n}{s} - 1$ coalitions, where n the number of all players and s the number of members of S, then it has no incentive to break from the grand coalition and the core is non-empty. Finally we allow a coalition to have probabilistic beliefs over the set of partitions the outsiders can form. We present sufficient conditions for the non-emptiness of the core in such an environment.

Keywords: aggregative game; cooperative game; externalities; core

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1 Introduction

The core is the most widely used solution concept in cooperative game theory. To define the core one needs to first define the characteristic function of a coalition. The characteristic function specifies the worth a group of players can attain if they act on their own, i.e., without cooperating with the outside players (i.e., the players not in the group). For a cooperative game with externalities, namely a game where the worth of a coalition depends on the actions of the outsiders, this task is not straightforward as the specification of the characteristic function relies on a prediction about the behavior of the non-members (in particular, their coalition structure).

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Different conjectures about the reaction of the outsiders lead to different coalitional worths and thus to different notions of core. The α and β cores (Aumann 1959) are based on min-max behavior on behalf of the non-members. The γ core (Chander & Tulkens 1997) is based on the assumption that the outsiders play individual best replies to the deviant coalition; the same approach can be followed under the additional assumption that the deviant coalition acts as a Stackelberg leader (Currarini & Marini 2003). The recursive core (Huang & Sjostrom 2003; Koczy 2007) is constructed under the assumption that the members of a coalition compute their value by looking recursively on the cores of the sub-games played among the outsiders.

Cooperative games with externalities are generated by normal-form games where players form coalitions and sign binding agrrements. The current paper focuses on cooperative games generated by aggregative normal-form games, i.e., games where the payoff of a player depends on his strategy and on the aggregate value of all players' strategies. Many economic models have this structure, such as common pool resource games, oligopoly models, rent seeking games, etc. We utilize the structure of these games in order to analyze various notions of core, each depending on certain beliefs a deviant coalition has about the behavior of the non-members. In particular: (i) we analyze the γ-core (Chander & Tulkens 1997), which is constructed under the assumption that a deviant coalition believes its opponents will all stay separate (i.e., form singleton coalitions); (ii) we generalize the "γ-beliefs" by allowing a deviant coalition to have sets of beliefs over the reaction of the outside players and determine the corresponding notion of core; (iii) we introduce the assumption that the members of a coalition assign various probability distributions on the set of partitions of the outsiders.

Our results can be summarized as follows:

(i) regarding normal-form games where the payoff of a player is decreasing in the aggregate value of all players' strategies and (weakly) concave in own strategy, it is shown that the γ-core is non-empty

(ii) the above result holds if we drop the above assumptions and instead assume that the payoff of each player satisfies some sort of linearity w.r.t. own strategy

(iii) regarding normal-form games where the payoff of a player satisfies the assumption descrived in point (ii) above, sets of beliefs are found under which a coalition has no incentive to deviate from the grand coalition; in particular, for a coalition S with s players this holds if S believes that the outside players will form *at least* $\frac{n}{s} - 1$ coalitions, where n is the number of players in the game.

(iv) we consider 3-player games where a deviant coalition has probabilistic beliefs about the coalitional behavior of the outsiders; we show that if a singleton coalition assigns a sufficiently low probability to the event that the outsiders form a single coalition then the core of the 3-player game is non-empty; we then generalize this result to any number n of players by using an induction argument on n.

The paper is organized as follows. Section 2 discusses the core for aggregative games where the payoffs are decreasing in the aggregate value of all player's strategies. Section 3 focuses on the class of aggregative games where the payoff of a player is linear in own strategy and analyzes beliefs that are set-valued (section 3.1) or probabilistic (sections 3.2 and 3.3). Section 4 concludes.

2 Aggregative games and the γ-core

We consider a set $N = \{1, 2, ..., n\}$ of players; X_i is the strategy set of player $i \in N$; u_i is i's payoff function. We will consider aggregative games, i.e., games where the payoff of a player depends on his strategy and on an aggregate of the strategies of all players. We take this aggregate to be simply the sum. Hence the payoff of player i is of the form $u_i(x_i, \sum_{k \in N} x_k)$.

Many economic phenomena are modelled via aggregative games, such us rent-seeking contests, common pool resource games, oligopoly games, etc. In this section we consider the class of aggregative games satisfying the following conditions.

A0 $u_i(x_i, \sum_{k \in N} x_k)$ *decreases in* $\sum_{k \in N} x_k$.

A1 $u_{p(i)}(x_{p(i)}, \sum_{k \in N} x_{p(k)}) = u_i(x_i, \sum_{k \in N} x_k)$, for all permutations p of players.

We examine frameworks where players form coalitions and sign contracts. If all players agree to cooperate and form the grand coalition then the latter's objective function is

$$u_N(x_1, x_2, ..., x_n) \equiv \sum_{i \in N} u_i(x_i, \sum_{k \in N} x_k)$$

We denote by $(x_1^*, x_2^*, ..., x_n^*)$ the choices of strategies the maximize the above sum. Then the worth of the grand coalition is

$$v(N) = \sum_{i \in N} u_i(x_i^*, \sum_{k \in N} x_k^*)$$

The formation of the grand coalition is potentially blocked by the formation of smaller coalitions. Let $S \subset N$ be such a coalition, with $|S| = s$ members. The payoff of S, which is the sum of its members payoffs, depends on the partition of the players not in S. In this section we assume that the members of S believe that the $n - s$ outside players stay separate, which is the scenario first introduced by Chander & Tulkens (1997). Under this approach, should S deviate from the grand coalition, a normal-form game under the partition $\{S, \{j\}_{j \notin S}\}$ will be played. The objective function S in this game is

$$u_S(x_1, x_2, ..., x_N) \equiv \sum_{i \in S} u_i(x_i, \sum_{k \in S} x_k + \sum_{j \notin S} x_j)$$

The payoff of player $j \notin S$ simply is $u_j(x_j, \sum_{k \in N} x_k)$. We denote by $(x_1^S, x_2^S, ..., x_n^S)$ the equilibrium choices in the above normal-form game. The worth of coalition S is

$$v(S) = \sum_{i \in S} u_i(x_i^S, \sum_{k \in N} x_k^S)$$

The resulting cooperative game is denoted by (N, v). An allocation in the game is a vector $r = (r_1, r_2, ..., r_n)$ satisfying $\sum_{k \in N} r_k = v(N)$. The γ-core is the set of all allocations that no coalition S can block given the "γ- beliefs" (Chander & Tulkens 1997).

In what follows we determine conditions for non-empty γ-core under the aggregative normal-form structure. We begin with two preliminary results.

Lemma 1 *Assume $A0$ and $A1$ hold and that $u_i(x_i, \sum_{k \in N} x_k)$ is weakly concave in x_i. Let $i \in S$ and $j \notin S$. Then $x_j^S \geq x_i^S$.*

Proof Observe that x_i^S and x_j^S satisfy respectively[1]

$$\frac{\partial u_i(x_i^S, \sum_{k \in N} x_k^S)}{\partial x_i} + \sum_{\substack{l \in S \\ l \neq i}} \frac{\partial u_l(x_l^S, \sum_{k \in N} x_k^S)}{\partial x_i} = 0 \tag{1}$$

$$\frac{\partial u_j(x_j^S, \sum_{k \in N} x_k^S)}{\partial x_j} = 0 \tag{2}$$

$A0$ implies that each term in the sum in (1) is negative. Hence by (1) we have that

$$\frac{\partial u_i(x_i^S, \sum_{k \in N} x_k^S)}{\partial x_i} > 0 \tag{3}$$

By assumption $u_i(x_i, \sum_{k \in N} x_k)$ is weakly concave in x_i. Hence if $x_i^S > x_j^S$ then by (3) we would have

$$\frac{\partial u_i(x_j^S, \sum_{k \in N} x_k^S)}{\partial x_i} > 0$$

and hence by $A1$

$$\frac{\partial u_j(x_j^S, \sum_{k \in N} x_k^S)}{\partial x_j} > 0$$

which violates (2). We conclude that $x_j^S \geq x_i^S$. ■

Lemma 2 *Assume the conditions of Lemma 1 hold. Let $i \in S$ and $j \notin S$. Then $u_j(x_j^S, \sum_{k \in N} x_k^S) \geq u_i(x_i^S, \sum_{k \in N} x_k^S)$.*

[1] We assume that the equilibrium strategies are in the interior of the strategy sets.

Proof We have that

$$\begin{aligned}
u_j(x_j^S, \sum_{k\in N} x_k^S) &= u_j(x_j^S, \sum_{\substack{k\in N\\k\neq j}} x_k^S + x_j^S) \geq u_j(x_i^S, \sum_{\substack{k\in N\\k\neq j}} x_k^S + x_i^S) \\
&= u_j(x_i^S, \sum_{\substack{k\in N\\k\neq j,k\neq i}} x_k^S + x_i^S + x_i^S) \\
&\geq u_j(x_i^S, \sum_{\substack{k\in N\\k\neq j,k\neq i}} x_k^S + x_j^S + x_i^S) \text{ [as } x_j^S \geq x_i^S \text{ and A0 holds]} \\
&= u_i(x_i^S, \sum_{\substack{k\in N\\k\neq j,k\neq i}} x_k^S + x_i^S + x_j^S) \text{ [because of A1]} \\
&= u_i(x_i^S, \sum_{k\in N} x_k^S)
\end{aligned}$$

So the result is proved. ■

Proposition 1 *Assume the conditions of Lemmas 1-2 hold. Then the γ-core of (N, v) is non-empty.*

Proof Consider a coalition S with $|S| = s$ members. Given symmetry, the core is non-empty iff

$$\frac{v(N)}{n} \geq \frac{v(S)}{s}, \text{ for all } S \tag{4}$$

We have

$$\begin{aligned}
\frac{v(N)}{n} &= \frac{\sum_{i\in N} u_i(x_i^*, \sum_{k\in N} x_k^*)}{n} \geq \frac{\sum_{i\in N} u_i(x_i^S, \sum_{k\in N} x_k^S)}{n} \\
&= \frac{s u_i(x_i^S, \sum_{k\in N} x_k^S) + (n-s) u_j(x_j^S, \sum_{k\in N} x_k^S)}{n} \\
&\geq \frac{(s+n-s) u_i(x_i^S, \sum_{k\in N} x_k^S)}{n} \text{ [by Lemma 2]} \\
&= \frac{s u_i(x_i^S, \sum_{k\in N} x_k^S)}{s} = \frac{v(S)}{s}
\end{aligned}$$

So condition (4) holds. ■

3 Linear aggregative games

In this section we restrict ourselves to aggregative games satisfying the following condition:

$A2$ $u_i(x_i, \sum_{k \in N} x_k) = x_i \tilde{u}_i(\sum_{k \in N} x_k)$.

Let us give examples satisfying condition $A2$. Consider rent seeking contests with n players, each exerting an effort level in order to win a prize of value A. Let e_i denote the effort level of i and c the unit cost of effort. Then the payoff of player i is

$$u_i(e_i, \sum_{k=1}^{n} e_k) = \frac{e_i}{\sum_{k=1}^{n} e_k} A - c e_i = e_i \Big(\frac{1}{\sum_{k=1}^{n} e_k} A - c\Big) = e_i \tilde{u}_i(\sum_{k=1}^{n} e_k)$$

Common pool resource games are another example of aggregative games satisfying $A2$. Assume a set of n agents use a common resource (fisheries, forest, etc). If m_i is agent i's level of exploitation then the value i achieves is $m_i V(\sum_{k=1}^{n} m_k)$. Letting c denote the cost per unit of exploitation, the payoff of agent i is

$$u_i(m_i, \sum_{k=1}^{n} m_k) = m_i V(\sum_{k=1}^{n} m_k) - c m_i = m_i (V(\sum_{k=1}^{n} m_k) - c) = m_i \tilde{u}_i(\sum_{k=1}^{n} m_k)$$

We will use the term *linear aggregative games* to denote the family of aggregative games satisfying $A1$-$A2$ (recall that $A1$ is a symmetry condition) . The first result of this section shows that the γ-core of a cooperative game that corresponds to a linear aggregative game is non-empty without even assumming $A0$.

Proposition 2 *Assume A1-A2 hold. Then the γ-core of (N, v) is non-empty.*

Proof Let S be a coalition with $|S| = s$ members. Define $x = \sum_{i \in S} x_i$. Then by $A1$-$A2$ we can write the objective function of S as

$$\begin{aligned} u_S(x_1, x_2, ...x_n) &\equiv \sum_{i \in S} u_i(x_i, \sum_{k \in N} x_k) = \sum_{i \in S} x_i \tilde{u}_i(\sum_{k \in N} x_k) \\ &= x \tilde{u}_i(\sum_{k \in N} x_k) = u_i(x, \sum_{k \in N} x_k) \end{aligned}$$

Hence coalition S chooses simply the sum its members' strategies. Therefore it is as if we have a normal-form game with $n - s + 1$ symmetric players. Let x^S denote the equilibrium choice of S and x_j^S the choice of $j \notin S$. By symmetry, $x^S = x_j^S$, all $j \notin S$. Let $X^S = (n - s + 1)x^S$. Then, by the above, $v(S) = u_i(x^S, X^S)$, any $i \in S$.

Using again $A1$-$A2$, we can write the objective function of the grand coalition as

$$u_N(x_1, x_2, ..., x_n) = \sum_{i \in N} x_i \tilde{u}_i(\sum_{k \in N} x_k) = u_i(\sum_{i \in N} x_i, \sum_{k \in N} x_k) = u_i(X, X)$$

where $X = \sum_{i \in N} x_i$. Hence the grand coalition selects simply X. Let X^* denote its optimal choice. Then, $v(N) = u_i(X^*, X^*)$, any i.

We then have

$$\begin{aligned} \frac{v(N)}{n} &= \frac{u_i(X^*, X^*)}{n} \geq \frac{u_i(X^S, X^S)}{n} \\ &= \frac{X^S}{n}\tilde{u}_i(X^S) = \frac{n-s+1}{n}x^S\tilde{u}_i(X^S) \\ &= \frac{n-s+1}{n}u_i(x^S, X^S) \geq \frac{1}{s}u_i(x^S, X^S) \\ &= \frac{v(S)}{s} \end{aligned}$$

where the first inequality holds because X^* maximizes the value of the grand coalition and the last inequality holds because $(n-s+1)/n \geq 1/s$. ■

3.1 Set-valued beliefs

The γ-core is defined under the assumption that the members of a deviant coalition believe the outside players will stay separate, i.e, they form singleton coalitions. In this subsection we examine more general coalitional beliefs. Consider a candidate deviant coalition S and let l denote the number of coalitions of the outside players. We will determine a range of the values of l that guarantee S does not break off from the grand coalition.

Notice that if coalition S (as before it has s members) believes that the $n-s$ outsiders form l coalitions then, given $A1$-$A2$, we have a normal-form game with $l+1$ symmetric players. The next result identifies a threshold $l(s)$ such that if S believes that the outsiders will form at least $l(s)$ coalitions, then it has no incentive to deviate from the grand coalition.

Proposition 3 *Assume $A1-A2$ hold. If a coalition S with s members believes that the outsiders will form at least $l(s) = \frac{n}{s} - 1$ coalitions then the core is non-empty.*

Proof Consider the normal-form game played among k outside coalitions and coalition S. By $A1-A2$, we have a symmetric normal-form game. Denote by z^S the Nash equilibrium strategy of each of the $l+1$ players and $Z^S = (l+1)z^S$. The worth of S then is $v(S) = u_i(z^S, Z^S)$. We then have

$$\begin{aligned} \frac{v(N)}{n} &= \frac{u_i(X^*, X^*)}{n} \geq \frac{u_i(Z^S, Z^S)}{n} = \frac{Z^S}{n}\tilde{u}_i(Z^S) \\ &= \frac{l+1}{n}z^S\tilde{u}_i(Z^S) = \frac{l+1}{n}u_i(z^S, Z^S) \end{aligned}$$

Notice finally that $\frac{l+1}{n} \geq \frac{1}{s}$ if $l \geq \frac{n}{s} - 1$. ■

For singleton coalitions, $l(1) = n-1$, i.e., as in the γ-core scenario. For $s > 1$, we have that $n - s > l(s)$. Hence for $s > 1$, a coalition does not break from the grand coalition not only when the outsiders form $n-s$ singleton coalitions, but also for cases where they form fewer coalitions.

3.2 Probabilistic beliefs

In this section we generalize the analysis of the previous section by assuming that a deviant coalition assigns a probability distribution on the set of partitions the outsiders can form. We restrict attention to linear aggregative games. As before, let S be a coalition with $|S| = s$ members. Denote by Π_{n-s} the set of partitions that the $n-s$ outsiders can form. The members of S assign a probability distribution $h_{n,s}$ over Π_{n-s}, so for $\pi \in \Pi_{n-s}$, $h_{n,s}(\pi)$ is the probability assigned to partition π.

Recall that assumptions A1-A2 imply that what matters for a deviant coalition S is only the number of coalitions it faces. So it will be convenient to define a new probability distribution on the set $\{1, 2, ..., n-s\}$. We denote this new distribution by $f_{n,s}(.)$. Then

$$f_{n,s}(l) = \sum_{\pi:|\pi|=k} h_{n,s}(\pi), \quad l = 1, 2, ..., n-s$$

The value of $f_{n,s}(l)$ is interpreted as the probability of the event that the outsiders form l coalitions, $l = 1, 2, ..., n-s$. A generic partition with l members will be called l-partition and will be denoted by $\pi_l = \{S_1, S_2, ..., S_l\}$.

Consider the l-partition $\pi_k = \{S_1, S_2, ..., S_k\}$ of the outsiders. Let $x_k(l)$ be the strategy of a generic (outside) player k. For $S_j \in \pi_l$ define

$$x_{S_j}(l) = \sum_{l \in S_j} x_k(l) \quad \text{and} \quad x_{-S}(l) = \sum_{S_j \in \pi_l} x_{S_j}(l)$$

Using A2, and fixing a strategy $x_S = \sum_{k \in S} x_k$ for the deviant coalition S, we can write

$$u_{S_j}(x_{S_j}(l), x_{-S}(l) + x_S) = \sum_{k \in S_j} u_k(x_k(l), x_{-S}(l) + x_S)$$

Moreover let

$$u_S(x_S, x_{-S}(l) + x_S) = \sum_{k \in S} u_k(x_k, x_{-S}(l) + x_S)$$

The maximization problem facing coalition S then is

$$\max_{x_S} \sum_{l=1}^{n-s} f_{n,s}(l) u_S(x_S, x_{-S}(l) + x_S) \tag{5}$$

where $x_{S_j}(l)$ (for $S_j \in \pi_l$ and $l = 1, 2, ..., n-s$) is chosen via

$$\max_{x_{S_j}(l)} u_{S_j}(x_{S_j}(l), x_{-S}(l) + x_S) \tag{6}$$

The solution of (5)-(6) is denoted by x_S^f, $x_{S_j}^f(l)$ and $x_{-S}^f(l) = \sum_{S_j \in \pi_l} x_{S_j}^f(l)$, $l = 1, 2, ..., n-s$. The characteristic function of S is denoted by $v^{f_{n,s}}(S)$ or simply $v^f(S)$. By the above analysis we have,

$$v^f(S) = \sum_{l=1}^{n-s} f_{n,s}(l) u_S(x_S^f, x_{-S}^f(l) + x_S^f) \tag{7}$$

Finally, the value of the grand coalition is computed as in the previous sections. A cooperative game in this section is denoted by (N, v^f). The core is the set of all allocations that no coalition S can block, given distribution $f_{n,s}$. We denote this core by $\mathcal{C}^f$.

In this section we assume the following (in addition to $A0 - A2$).

B1 X_i *is compact and convex.*

B2 $u_i(x_i, \sum_{j \in N} x_j)$ *is twice continuously differentiable in* x_i.

Return to (6) and fix l. The maximization problems in (6) define l best replies of the outside coalitions S_j, $j = 1, 2, ..., l$. The solution of the system of the equations defined by the l best replies is denoted by $\widehat{x}_{S_j}(x_S, l)$, $j = 1, 2, ..., l$. Notice that the latter is *not* the reduced-form solution (as everything is defined w.r.t x_S) and hence it does not yet depend on $f_{n,s}$. Let

$$\widehat{x}_{-S}(x_S, l) = \sum_{S_j \in \pi_l} \widehat{x}_{S_j}(x_S, l)$$

Remark 0 *Assumptions A1-A2 and B1-B2 imply that* $\widehat{x}_{-S}(x_S, l)$ *increases in* l.

Remark 0 follows by Acemoglu and Jensen (2013). They derive conditions under which the entry of an extra player in an aggregative game increases the aggregate value of the players' equilibrium strategies. The application of their result in our framework is immediate by observing that for for each l, x_S is a constant. Hence, using $A2$, for each k it is as if we have a game with the k outside coalitions.

In the next subsection we analyze the core of a three-player game. This case will be used later on to analyze the general n-player case.

3.3 The 3-player case

Consider a three-player game, i.e., $N = \{1, 2, 3\}$. Let $(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ denote the Nash equilibrium profile in the corresponding normal-form game and let (x_1^*, x_2^*, x_3^*) denote the strategy profile that maximizes the sum $\sum_{i \in N} u_i(x_i, \sum_{j \in N} x_j)$. Given the symmetry assumption, $\overline{x}_i = \overline{x}$ and $x_i^* = x^*$, all i.

A singleton coalition assigns probability $f_{3,1}(1)$ to the event that players 2 and 3 form a coalition and probability $1 - f_{3,1}(1)$ to the event that they stay separate. For simplicity, let $f_{3,1}(1) = f$.

Proposition 4 *Assume that f is sufficiently low and that there exists a strategy profile that Pareto dominates the Nash equilibrium of the normal-form game. Then $\mathcal{C}^f \neq \emptyset$.*

Proof The core is non-empty if for all S

$$\frac{v(N)}{n} \geq \frac{v^f(S)}{s} \tag{8}$$

Consider first a singleton coalition, say $S = \{1\}$. Given the notation defined earlier, x_S^f is the choice of S; $x_{-S}^f(1)$ is the choice of the outsiders under partition $\{2, 3\}$; and $x_{-S}^f(2)$ the choice of the outsiders under partition $\{\{2\}, \{3\}\}$. For simplicity, we write

$$x^f(1) = x_S^f + x_{-S}^f(1), \quad x^f(2) = x_S^f + x_{-S}^f(2)$$

The worth of the singleton coalition S then is

$$v^f(S) = fu_1(x_S^f, x^f(1)) + (1-f)u_1(x_S^f, x^f(2))$$

By $A2$ the value of the grand coalition is

$$v(N) = \sum_{i \in N} u_i(x^*, 3x^*) = 3x_i^* \tilde{u}_i(x^*, 3x^*) = u_i(3x^*, 3x^*), \quad i = 1, 2, 3 \tag{9}$$

By (8), a singleton coalition will not deviate from the grand coalition (setting $i = 1$ in (9)) if

$$u_1(x^*, 3x^*) \geq fu_1(x_S^f, x^f(1)) + (1-f)u_1(x_S^f, x^f(2)) \tag{10}$$

Since $x_{-S}^f(2) > x_{-S}^f(1)$ (applying Remark 0) we have by B0 that

$$u_1(x_S^f, x^f(1)) > u_1(x_S^f, x^f(2))$$

Hence when seen as a function of f, $v^f(S)$ is minimized at $f = 0$ and maximized at $f = 1$. Moreover,

$$v^{f=0}(S) = u_1(\overline{x}, 3\overline{x}) < u_1(x^*, 3x^*)$$

where the inequality is due to the assumption that the Nash equilibrium outcome is Pareto dominated. By continuity, for small enough values of f, say $f \leq f^*$, inequality (10) holds.

Consider now a coalition consisting of two players. We can apply the results of the previous sections (there we show that the γ core is non-empty) and readily conclude that for such coalitions, expression (8) holds. ■

3.4 The n-player case

We will use the result of the previous section in order to discuss the case of n players. Let m be a positive integer. Define the numbers $\overline{n} = n + m$ and $\overline{s} = s + m$. With a slight abuse of notation instead of $v(S)$ we will write $v(s)$, where $s = |S|$. We have the following two intermediate results.

Lemma 1 The equality $v^n(s) = v^{\overline{n}}(\overline{s})$ holds.

Proof Appears in the Appendix. ■

Lemma 2 *Fix n. The function $v^n(s)$ is increasing in s.*

Proof We will use induction on the number of players n. Let first $n = 2$, i.e, $N = \{1, 2\}$. We need to show that

$$v^2(N) > v^2(S) > v(\emptyset)$$

By $A1$ we can write the payoff of the grand coalition as $v(N) = u_i(x^*, x^*)$, where x^* is the optimal choice of N. On the other, consider a singleton coalition, say $S = \{1\}$. If it forms we are in a two-player normal-form game. Denote by $(\widehat{x}_1, \widehat{x}_2)$ the equilibrium strategies in this game. Then $v(S) = u_i(\widehat{x}_1, \widehat{x}_1 + \widehat{x}_2)$. By Acemoglu and Jensen (2013), we have that $\widehat{x}_1 + \widehat{x}_2 > x^*$. Hence,

$$v^2(N) = u_i(x^*, x^*) \geq u_i(\widehat{x}_1, \widehat{x}_1) > u_i(\widehat{x}_1, \widehat{x}_1 + \widehat{x}_2) = v^2(S) > 0$$

where the first inequality is due to the fact that x^* is the stategy that maximizes the value of the grand coalition and the second inequality is due to $B0$.

Assume now that in a game with $n > 2$ players we have $v^n(s) > v^n(s-1)$, i.e, the induction hypothesis. We will prove that $v^{n+1}(s) > v^{n+1}(s-1)$. By Lemma 1 and the induction hypothesis we have that $v^{n+1}(s) = v^n(s-1) > v^n(s-2) = v^{n+1}(s-1)$. ■

We will now use Proposition 3 and Lemmas 1-2 tp show the following.

Proposition 4 *Assume the conditions of Proposition 3 hold. Then the core of the n-player game is non-empty.*

Proof We will use induction to show

$$\frac{v^n(n)}{n} \geq \frac{v^n(s)}{s} \tag{11}$$

Base: Proposition 3 establishes the base case ($n = 3$).

Induction hypothesis: For all $S : |S| = s \leq n$, $\dfrac{v^n(n)}{n} \geq \dfrac{v^n(s)}{s}$.

Induction step: We will show that for all $S : |S| = s \leq n + 1$,

$$\frac{v^{n+1}(n+1)}{n+1} \geq \frac{v^{n+1}(s)}{s}$$

By Lemma 1 we have that $v^{n+1}(s) = v^{n+1}((s-1)+1) = v^n(s-1)$ and also $v^{n+1}(n+1) = v^n(n)$. So we have to show that

$$\frac{v^n(n)}{n+1} \geq \frac{v^n(s-1)}{s} \tag{12}$$

From the Induction hypothesis we have

$$v^n(n) \geq \frac{n}{s-1} v^n(s-1)$$

and thus

$$(s-1)v^n(n) \geq nv^n(s-1) \tag{13}$$

Using Lemma 2,

$$v^n(n) > v^n(s-1) \tag{14}$$

Adding (13) and (14) we have

$$sv^n(n) > (n+1)v^n(s-1)$$

which implies that (12) holds. So we have the proof for $n+1$ and thus the proposition is proved. ■

Appendix

Proof of Lemma 1 We first observe that

$$f_{n,s}(k) = f_{\overline{n},\overline{s}}(k) \tag{15}$$

where the first probability function corresponds to the case where a coalition has s members while the game has n players; the second corresponds to the case where the coalition has $\overline{s} = s + m$ members and the game has $\overline{n} = n + m$ players. Let x_s^n and $x_{\overline{s}}^{\overline{n}}$ denote the choices of a coalition in these two scenarios. Denote the choices of the outsiders in a similar fashion. We observe that due to no synergies and (15), we have $x_s^n = x_{\overline{s}}^{\overline{n}}$ and $x_{-s}^n = x_{-\overline{s}}^{\overline{n}}$. Therefore

$$\begin{aligned} v^n(s) &= \sum_{k=1}^{n-s} f_{n,s}(k) u_s(x_s^n, x_{-s}^n(k)) \\ &= \sum_{k=1}^{\overline{n}-\overline{s}} f_{\overline{n},\overline{s}}(k) u_s(x_{\overline{s}}^{\overline{n}}, x_{-s'}^{\overline{n}}(k)) \\ &= v^{\overline{n}}(\overline{s}) \end{aligned}$$

■

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