Consistent Indices *

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Abstract

In many economically interesting decision making settings, it is useful to have a complete order over choices that does not refer to the particular preferences of an individual decision maker. I introduce an approach which requires, however, that rankings be consistent with comparisons of preferences. Applications in four settings are introduced: two in risk (the riskiness of gambles and portfolios), time preferences (the delay embeded in investment cashflows) and information acquisition (the appeal of information transactions). In all cases, a unique index is derived, and all indices share several favorable properties. Three of the indices have been introduced elsewhere, based on other approaches, but the index of delay is novel.

1 Introduction

In many decision problems agents base their actions on a simple index that summarizes the information that is available. This may happen due to difficulties in attaining and interpreting information, or due to an overabundance of useful information, as in the example of online restaurant star ratings [Luca, 2011]. Indices are also used to limit the discretion of agents when decision rights are being delegated [Turvey, 1963]. This happens in economically significant decisions. For example, a mutual fund manager may be given complete autonomy in choosing which bonds to purchase, as long as they are rated AAA. Similarly, credit decisions are frequently based on a credit rating, a number that is supposed to summarize relevant financial information about an individual. It is important therefore to be able to assess the validity of an index. A natural requirement is that the index should be "consistent" with its underlying motivation. For example: an index of riskiness should judge as riskier the alternative risk-averse individuals less prefer. For similar reasons, one agent should be deemed less risk averse than another if he is willing to accept riskier alternatives.

This paper formalizes the idea of consistency of an index in a given decision making problem. Several conditions are proposed, and they are shown to be satisfied uniquely by some index in four

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different decision making settings: two in risk, time preferences, and information acquisition. To be concrete, I start with one setting of risk (additive gambles), since it is better known and allows me to illustrate the general concepts before turning to other decision making environments. The paper is structured in the same spirit: the setting of risk is treated extensively, then the same techniques are used in the other environments.

The approach I take is somewhat non-standard. The starting point is a given *objective index of riskiness* – a function which assigns to each gamble some number, independently of any agent specific characteristics, with higher numbers being associated with more risk. Different functions induce different orders on lotteries, hence for a given index Q, I refer to the Q-riskiness of a gamble. I define the local aversion to Q-riskiness of an agent, as the inverse of the Q-riskiness of the most Q-risky "local" gamble that the agent is willing to accept. This definition is motivated by a certain kind of consistency as it implies that agents which are less Q-riskiness averse would accept (some) more Q-risky gambles. This approach is non-standard since instead of starting with an ordering over preferences and claiming that risk is *"what risk-averters hate"* [Machina and Rothschild, 2008], I start with an ordering over the objects of choice (an index of riskiness Q) and derive from it judgments on preferences (Q-riskiness aversion).

Recently, Aumann and Serrano [AS, 2008] presented an objective index of riskiness, based on a small set of axioms, including centrally a "duality axiom," which requires a certain kind of consistency. Roughly speaking, it asserts that (uniformly) less risk-averse individuals accept riskier gambles.[1] Subsequently Foster and Hart [FH, 2009] presented a different index of riskiness with an operational interpretation.[2] Their index identifies for every gamble the critical wealth level below which it becomes "risky" to accept the gamble.[3] This paper shows that for these two prominent indices of riskiness *local consistency* is satisfied. Roughly speaking, local consistency requires that, for small gambles, agents will not accept gambles that are more risky than ones that they reject. It is shown in Section 3 that for both indices the local aversion of agents coincides with the celebrated *absolute risk aversion* [Pratt, 1964, Arrow, 1965, 1971]. I show that agents accept or reject small gambles according to a (AS or FH) riskiness cutoff, and that this cutoff is equal to the inverse of their absolute risk aversion (ARA).[4]

This property of AS and FH is desirable, as it indicates that they fit well into the literature and, more importantly, that they "make sense." But this indication should not be regarded as definitive evidence in favor of these indices; I show that there exist many locally consistent indices of riskiness with the same property. Moreover, some of these indices are unreasonable in the sense that they are not monotonic with respect to stochastic dominance [Hanoch and Levy, 1969, Hadar and Russell, 1969, Rothschild and Stiglitz, 1970]. In fact, it is shown that local consistency, when

[1]Agent i uniformly no less risk-averse than agent j if whenever i accepts a gamble at some wealth, j accepts that gamble at any wealth.

[2]Homm and Pigorsch [2012] provide an operational interpretation of the Aumann–Serrano index of riskiness.

[3]Hart [2011] later demonstrated that both indices also arise from a comparison of acceptance and rejection of gambles.

[4]While the connection between riskiness and global (uniform) aversion to risk lies at the foundations of the axioms of AS and is demonstrated by Hart [2011], the connection with local risk aversion was not established in any paper known to the author.

combined with an additional technical condition,[5] ensures that the local aversion to an index is ordinally equivalent to ARA.

The discussion above suggests that while it may be a desirable property of an index of riskiness that the local aversion to it would coincide with ARA, and while this property follows from local consistency, it is not sufficient even for determining that an index is "reasonable." Recognizing this fact, in Section 4 I suggest an additional criterion, *global consistency*, and show that the unique index which satisfies it and the previous property is the Aumann-Serrano index of riskiness [Aumann and Serrano, 2008].[6] Roughly speaking, global consistency requires that if one agent is locally more averse to Q-riskiness than another at any two (maybe different) wealth levels, then the more Q-riskiness averse agent rejects gambles which are riskier than ones rejected by the other agent. This condition makes a requirement of consistency of the index which involves "large" gambles, using a partial order on preferences that is generated using the concept of local aversion to Q-riskiness.

According to Aumann and Serrano [2008], the fact that Arrow and Pratt addressed risk aversion, a subjective concept that depends on a person's utility function, but did not define the objective riskiness of gambles *"... is like speaking about subjective time perception ('this movie was too long') without having an objective measure of time ('3 hours') or about heat or cold aversion ('it's too cold in here') without an objective measure of temperature ('20 degrees Fahrenheit')."* Using their metaphor, a consistent index of riskiness excludes unintuitive conclusions such as a 3-hour movie being too long for a more patient agent, while a 6-hour movie is not for a less patient agent.

The approach of first assuming the existence of an index, and then requiring consistency proves to be a powerful tool in decision making problems where comparing agents' preferences is relatively easier than ranking the objects of choice. I provide three other applications of this approach in other decision making settings. Still in the realm of risk, in Section 4 I consider multiplicative gambles (like investment portfolios). I show that local consistency, combined with a technical condition similar to the one used before, ensure that the local aversion to an index of relative riskiness is ordinally equivalent to the Arrow-Pratt *relative risk aversion* order. Additionally, I show that adding the requirement of global consistency uniquely pins down the index of relative riskiness of Schreiber [2013].

Section 5 considers time preferences and the *delay* embedded in *investment cashflows.* Just like stochastic dominance in the risk setting, *time dominance* [Bøhren and Hansen, 1980, Ekern, 1981] generates a partial order on cashflows. But, if one wishes to compare the delay embedded in two general cashflows, difficulties analogous to the ones from the risk setting (re-)emerge, as well as a new one: cashflows could be considered at different points in time, and with a different perspective, judgments may potentially be altered. Similar to previous findings, I show that local consistency, combined with some technical conditions,[7] ensures that the local aversion to an index of delay is ordinally equivalent to the instantaneous discounting rate. Adding the requirement of

[5]The condition is *homogeneity* of degree 1.

[6]To be precise, one additional condition is required: homogeneity of degree 1.

[7]In addition to the condition from the previous setting, I add another consistency requirement: *translation invariance*. This condition excludes time inconsistent indices.

global consistency is then shown to pin down a novel index for the delay embedded in investment cashflows.

The new index of delay is closely related to a well-known measure of delay which is used in practice: the *internal rate of return* (IRR). I discuss this relation as well as the close connection of the index to the AS index of riskiness. This application is particularly important since, unlike the others, it generates a completely new index, in an environment which was not treated by the recent literature on indices. It therefore underscores a strength of the proposed methodology in that the indices emerge from the same requirements in each environment, and only minimal "creativity" is required in order to find the unique consistent index in a given decision problem.[8]

Section 6 treats the setting of information acquisition and the *appeal* of different *information transactions*. Comparing information structures proves to be a challenging undertaking. In his seminal paper, Blackwell [1953] proposed a partial order on information structures, which is in the spirit of stochastic dominance. The absence of a complete order on information structures has motivated the index of Cabrales et al. [2013], which completes this partial order. Later, these authors also proposed an index on the domain of *information transactions*, which are defined as information structures with an attached dollar price [Cabrales et al., 2012]. I show that the local distaste to both of their indices coincides with ARA, but also that any index which satisfies local consistency and some technical conditions has a similar property. Moreover, some of these indices are unreasonable in the sense that they are not monotonic with respect to Blackwell's partial order. To conclude the section, I show that the index of Cabrales et al. [2012] is the unique index which also satisfies global consistency.

Altogether, four applications of the new approach are reviewed in this paper (additive gambles, multiplicative gambles, time preferences, and information acquisition), demonstrating its generality and applicability for different settings. The approach could potentially be used in other settings in which indices are needed. A particular setting which seems promising in this regard is the measurement of inequality, which has many similarities to the setting of riskiness [Atkinson, 1970].

2 Preliminaries

In this section I provide the notation required for the next two sections which cover the risk setting.

A *gamble* g is a real-valued random variable with positive expectation and some negative values (i.e., $E[g] > 0$ and $P[g < 0] > 0$); for simplicity, I assume that g takes finitely many values. $\mathcal{G}$ is the collection of all such gambles. For any gamble $g \in \mathcal{G}$, $L(g)$ and $M(g)$ are respectively the maximal loss and gain from the gamble that occur with positive probability. Formally, $L(g) := \max supp(-g)$ and $M(g) := \max supp(g)$.

$\mathcal{G}_\epsilon$ is the class of gambles with support contained in an ϵ-ball around zero:

$$\mathcal{G}_\epsilon := \{g \in \mathcal{G} :\, \max\{M(g), L(g)\} \leq \epsilon\}.$$

[8]Creativity is, however, required in identifying suitable decision problems.

$[x_1,p_1; x_2, p_2...; x_n, p_n]$ represents a gamble which takes values $x_1, x_2, ..., x_n$ with respective probabilities of $p_1, p_2, ..., p_n$.

An *index of riskiness* is a function from the collection of gambles to the positive reals, $Q : \mathcal{G} \to \mathbb{R}_+$. Note that an index of riskiness is *objective*, in the sense that its value depends only on the gamble and not on any agent-specific attribute. An index of riskiness Q is *homogeneous* (of degree 1) if $Q(tg) = t \cdot Q(g)$ for all $t > 0$ and all gambles $g \in \mathcal{G}$.

$Q^{AS}(g)$, the *Aumann-Serrano index of riskiness* of gamble g, is implicitly defined by the equation

$$E\left[\exp\left(-\frac{g}{Q^{AS}(g)}\right)\right] = 1.$$

$Q^{FH}(g)$, the *Foster-Hart measure of riskiness* of g,[9] is implicitly defined by the equation

$$E\left[\log\left(1 + \frac{g}{Q^{FH}(g)}\right)\right] = 0.$$

Note that both Q^{AS} and Q^{FH} are homogeneous. Additionally, these indices are monotone with respect to first and second order stochastic dominance;[10] namely, if g is stochastically dominated by g' then $Q^{AS}(g) > Q^{AS}(g')$ and also $Q^{FH}(g) > Q^{FH}(g')$ [Aumann and Serrano, 2008, Foster and Hart, 2009].

Definition. *Full image.* An index of riskiness Q satisfies *full image* if for every $\epsilon > 0$, $\operatorname{Im} Q\left(\mathcal{G}_\epsilon\right) = \mathbb{R}_+$.

Full image says that even when the support of the gambles is limited to an ϵ-ball, the image of Q is all of $\mathbb{R}_+$. Both Q^{AS} and Q^{FH} satisfy full image. This is simply demonstrated by considering gambles of the form $g = [\epsilon, \frac{e^{c\epsilon}}{1+e^{c\epsilon}}; -\epsilon, \frac{1}{1+e^{c\epsilon}}]$ and $g' = [\epsilon, \frac{1}{2}; -\frac{\epsilon}{1+\epsilon \cdot c}, \frac{1}{2}]$, as $Q^{AS}(g) = \frac{1}{c}$ and $Q^{FH}(g') = \frac{1}{c}$.

In this paper, a *utility function* is a von Neumann–Morgenstern utility function for money. I assume that utility functions are strictly concave and twice continuously differentiable with a positive first derivative unless otherwise mentioned. The *Arrow-Pratt index of absolute risk aversion* (ARA), ρ, of u at wealth w is defined

$$\rho_u(w) := -\frac{u''(w)}{u'(w)}.$$

The *Arrow-Pratt index of relative risk aversion* (RRA), ϱ, of u at wealth w is defined

$$\varrho_u(w) := -w\frac{u''(w)}{u'(w)}.$$

[9] I also refer to Q^{FH} as an index of riskiness.

[10] A gamble g *first order stochastically dominates* h iff for every weakly increasing (not necessarily concave) utility function u and every $w \in \mathbb{R}$, $E\left[u\left(w+g\right)\right] \geq E\left[u\left(w+h\right)\right]$, with strict inequality for at least one such function. A gamble g *second order stochastically dominates* h iff for every weakly concave utility function u and every $w \in \mathbb{R}$, $E\left[u\left(w+g\right)\right] \geq E\left[u\left(w+h\right)\right]$, with strict inequality for at least one such function.

Note that $\rho_u(\cdot)$ and $\varrho_u(\cdot)$ are utility specific attributes and that both ρ and ϱ yield a complete order on utility-wealth pairs. That is, the risk aversion, as measured by ρ (or ϱ), of any two agents with two given wealth levels can be compared.

A gamble g is *accepted* by u at wealth w if $E\left[u(w+g)\right] > u(w)$, and is *rejected* otherwise. Given an index of riskiness Q, a utility function u and a wealth level w:

Definition 1. $R_Q(u,w) := \lim_{\epsilon \to 0^+} \sup \{Q(g) | \, g \in \mathcal{G}_\epsilon \text{ and } g \text{ is accepted by } u \text{ at } w\}$

Definition 2. $S_Q(u,w) := \lim_{\epsilon \to 0^+} \inf \{Q(g) | \, g \in \mathcal{G}_\epsilon \text{ and } g \text{ is rejected by } u \text{ at } w\}$

$R_Q(u,w)$ is the limit of the Q-riskiness of the *riskiest* accepted gamble according to Q, taking the support of the gambles to $\{0\}$ to capture the notion of "locality." $S_Q(u,w)$ is the limit of the Q-riskiness of the *safest* rejected gamble according to Q, again taking the support of the gambles to $\{0\}$.[11] Roughly speaking, $R_Q(u,w)$ is the Q-riskiness of the Q-riskiest "local gamble" that u accepts at w, and $S_Q(u,w)$ is the Q-riskiness of the Q-safest "local gamble" that is rejected by u at w.

Fact 1. *If Q satisfies full image then $R_Q(u,w) \geq S_Q(u,w)$ for every u and w.*

Proof. By the properties of the supremum, since

$$\{Q(g) | \, g \in \mathcal{G}_\epsilon \text{ and } g \text{ is accepted by } u \text{ at } w\} \cup \{Q(g) | \, g \in \mathcal{G}_\epsilon \text{ and } g \text{ is rejected by } u \text{ at } w\} = \mathbb{R}_+.$$

If the supremum of the first set is less than the infimum of the second, then intermediate points do not belong to either in violation of full-image. □

The inverse of R_Q and S_Q is a natural measure of the aversion to Q-riskiness.[12] The reason is that R_Q is high for utility-wealth pairs in which Q-risky gambles are accepted, so a reasonable definition of Q-riskiness aversion should imply that the aversion to Q-riskiness at such utility-wealth is low. Similarly, S_Q is low at a given utility-wealth pair when Q-safe gambles are rejected, so the local aversion to Q-riskiness must be high in this case.

The *local aversion of u to Q-riskiness at w* is therefore defined as

$$A_Q(u,w) := \frac{1}{R_Q(u,w)},$$

noting that unless otherwise mentioned, all of the results would hold for $\frac{1}{S_Q(u,w)}$ as well. Finally, u at w is *locally no less averse to Q-riskiness* than v at w' if $A_Q(u,w) \geq A_Q(v,w')$. As is shown below, this definition makes it possible to discuss the ordinal equivalence of the local aversion to Q-riskiness, which depends both on agents behavior and on the properties of the index Q, with orders such as ARA or RRA, which depend on the preferences exclusively, and are independent of the index.

[11] The nested supports assure monotonicity which assures the existence of the limit in the wide sense. Any nested sequence of compact supports that contain 0 in their interior, and which shrinks to $\{0\}$, will give the same results.

[12] For our purposes, $0 = \infty^{-1}$ and $\infty = 0^{-1}$.

3 The Local Aversion to an Index of Riskiness

Since no restrictions on Q were made (other then possibly homogeneity and full-image), at this point *locally no less averse to* Q-riskiness might look like an arbitrary class of orders on (u,w) pairs. However, I claim that its members are natural candidates for defining local risk aversion. The reason is that they *refine* the following *natural partial order [Yaari, 1969]*: u at w is locally no less risk averse than v at w' (written $(u,w) \gtrdot (v,w')$) if and only if there exists $\epsilon > 0$ such that for every $g \in \mathcal{G}_\epsilon$, if u accepts g at w then so does v at w'. An order O refines the natural partial order if for all g and h, $g \gtrdot h \implies gOh$.

Lemma 1. For every index of riskiness Q, A_Q refines the natural partial order.

Proof. Assume that $(u,w) \gtrdot (v,w')$. Then there exists $\epsilon' > 0$ such that for every $g \in \mathcal{G}_{\epsilon'}$ if u accepts g at w then so does v at w'. As in the definition of R_Q we have ϵ-ball supports with $\epsilon \to 0^+$, disregarding all ϵ-balls for $\epsilon \geq \epsilon'$ will not change the result. Note that for every $\epsilon < \epsilon'$

$$\{Q(g) | \, g \in \mathcal{G}_\epsilon \text{ and } g \text{ is accepted by } u \text{ at } w\} \subseteq \left\{Q(g) | \, g \in \mathcal{G}_\epsilon \text{ and } g \text{ is accepted by } v \text{ at } w'\right\}.$$

This means that the suprema in the definition of $R_Q(v,w')$ are taken on a superset of the corresponding sets in the definition of $R_Q(u,w)$, and therefore are higher. The result follows as weak inequalities are preserved in the limit. □

Next, I show that the local aversion to AS (FH) riskiness gives rise to a complete order which coincides with the one implied by the Arrow-Pratt ARA coefficient.

Lemma 2. For every utility function u and every w, $R_{Q^{AS}}(u,w) = S_{Q^{AS}}(u,w)$ and $A_{Q^{AS}}(u,w) = \rho_u(w)$.

Proof. First, observe that if u and v are two utility functions and there exists an interval $I \subseteq R$ such that $\rho_u(x) \geq \rho_v(x)$ for every $x \in I$ then for every wealth level w and lottery g such that $w + g \subset I$, if g is rejected by v at w it is also rejected by u for the same wealth level. Put differently, if g is accepted by u at w it is also accepted by v at the same wealth level. The reason is that the condition implies that in this domain, u is a concave transformation of v [Pratt, 1964], hence by the Jensen inequality $u(w) \leq \mathbb{E}[u(w+g)]$ implies that $v(w) \leq \mathbb{E}[v(w+g)]$.

Keeping in mind that $u'(x) > 0$ we have that $\rho_u(x)$ is continuous. Specifically,

$$\forall \, \delta > 0 \; \exists \, \epsilon > 0 \text{ s.t } x \in (w-\epsilon, w+\epsilon) \Rightarrow |\rho_u(x) - \rho_u(w)| < \delta \tag{3.0.1}$$

Recall that a CARA utility function with ARA coefficient of α rejects all gambles with AS riskiness greater than $\frac{1}{\alpha}$ and accepts all gambles with AS riskiness smaller than $\frac{1}{\alpha}$ [Aumann and Serrano, 2008]. Given an ϵ-environment of w in which $\rho_u \in (\rho_u(w) - \delta, \rho_u(w) + \delta)$, taking the

CARA functions with ARA of $\rho_u(w)+\delta$ and $\rho_u(w)-\delta$,[13] and applying the first observation (where I is $(w-\epsilon, w+\epsilon)$ completes the proof. □

Lemma 2 essentially shows that every utility function may be approximated locally using CARA functions, which are well-behaved with respect to the AS index. Given the ARA of u at a given wealth level, I take two CARA utility functions, one with slightly higher ARA, and the other with slightly lower ARA. For small environments around the given wealth level, ρ_u is almost constant, so the two CARA functions "sandwich" the utility function in terms of ARA. This implies that for small gambles, one CARA function accepts more gambles than u, and the other less gambles, in the sense of set inclusion. Since CARA functions accept and reject exactly according to an AS riskiness cutoff, and since cutoffs are close for similar ARA values, it follows that the local aversion to AS-riskiness is pinned down completely.

Lemma 3. For every utility function u and every w, $R_{Q^{FH}}(u,w) = S_{Q^{FH}}(u,w)$ and $A_{Q^{FH}}(u,w) = \rho_u(w)$.

Proof. According to Statement 4 in Foster and Hart [2009]:

$$-L(g) \leq Q^{AS}(g) - Q^{FH}(g) \leq M(g). \tag{3.0.2}$$

Therefore, if $g \in \mathcal{G}_\epsilon$ then:

$$\left|Q^{AS}(g) - Q^{FH}(g)\right| \leq \epsilon. \tag{3.0.3}$$

From Statement 3.0.3 one can deduce that $R_{Q^{FH}}(u,w) = R_{Q^{AS}}(u,w)$ and $S_{Q^{FH}}(u,w) = S_{Q^{AS}}(u,w)$. Lemma 2 completes the proof. □

The result of Lemma 3 is not surprising in light of Lemma 2, as Foster and Hart [2009] already noted that the Taylor expansions around 0 of the functions that define Q^{FH} and Q^{AS} differ only from the third term on. Roughly speaking, this means that for gambles with small supports Q^{AS} and Q^{FH} are close.

Theorem 1 summarizes the consistency results of Lemmata 1-3.

Theorem 1. *(i) For any index of riskiness Q, A_Q refines the natural partial order. (ii)* For every utility function u and every w, $A_{Q^{AS}}(u,w) = A_{Q^{FH}}(u,w) = \rho_u(w)$. Furthermore, $R_{Q^{AS}}(u,w) = S_{Q^{AS}}(u,w)$ and $R_{Q^{FH}}(u,w) = S_{Q^{FH}}(u,w)$.

Up until this point, I showed that the local aversion to AS and FH riskiness is equal to the ARA, the standard measure of local risk aversion. This means that one can start with a small set of axioms, namely Aumann and Serrano's [2008] or Foster and Hart's [2013], and define a complete order of riskiness over gambles. Then, the local aversion of agents to this riskiness index can be derived, and it will be equal to the well-known Arrow-Pratt coefficient. Hence, both AS and FH,

[13] In some cases, a smaller δ may be required to ensure that $\rho_u(w)-\delta$ is positive. This could be achieved by looking at a smaller environment of w.

together with ARA, satisfy the requirement suggested in the introduction that less risk averse agents (here, according to ARA) accept riskier gambles (according to AS or FH).

Theorem 1 might be interpreted as evidence that AS and FH were "well-chosen" in some sense. However, Theorem 2 shows that while according to the above results AS and FH satisfy one of the two requirements from the introduction, there are other indices that will give rise to the same order of local aversion. Moreover, some of these indices are not "reasonable" in the sense that they are not monotone with respect to first order stochastic dominance, in clear violation of the requirement that an index of riskiness should judge as riskier the alternative risk-averse individuals less prefer. Theorem 3 further identifies sufficient conditions on Q under which the local aversion to Q-riskiness yields the same order as the Arrow-Pratt (local) absolute risk aversion.

Axiom. *Local consistency.* $\forall u\ \forall w\ \ 0 < R_Q(u,w) \leq S_Q(u,w) < \infty$.

When Q satisfies full image, local consistency says that at a given wealth level, each agent has a unique Q-riskiness cutoff for acceptance and rejection of small gambles, and that it is proper (not all small gambles are accepted or rejected).[14]

To better understand the title local consistency, it is best to consider its violation. A violation of local consistency, $R_Q(u,w) > S_Q(u,w)$,[15] implies that the agent is willing to accept small gambles which are riskier, according to Q, than other small gamble that he rejects. This means, r that even for small gambles Q is not sufficient information to determine the agent's optimal behavior. In other words, the decisions of the agents are not consistent with the index, even on small domains.

Axiom. *Homogeneity.* $\forall \lambda > 0\ \forall g \in \mathcal{G},\ Q(\lambda \cdot g) = \lambda \cdot Q(g)$.

Although the homogeneity axiom could be replaced with a weaker condition,[16] I use it for its simplicity and since it appears in the original axiomatic characterization of the AS index.

Lemma 4. *If Q satisfies local consistency and homogeneity, Q satisfies full image.*

Proof. See appendix. □

Interestingly, the axioms imply a cardinal interpretation for A_Q; If Alice's aversion to Q-riskiness is twice as high as Bernie's, then her acceptance cutoff is half his cutoff. This interpretation applies, for example, to Q^{AS} or Q^{FH} and ρ; $\rho_u(w) = \frac{1}{2}\rho_v(w')$ if and only if u at w is willing to accept (small) gambles twice as risky (according to the index) as those that v is willing to accept at w'.

Definition. *Ordinally equivalent.* Given an index of riskiness Q, A_Q is *ordinally equivalent* to the index of absolute risk aversion ρ, if $\forall u, v\ \forall w, w'\ \ A_Q(u,w) > A_Q(v,w') \iff \rho_u(w) > \rho_v(w')$.

[14]Recall Fact 1 which states that full image implies $S_Q(u,w) \leq R_Q(u,w)$.

[15]In fact, there are other ways to violate the axiom, but this way is the most relevant in the current context.

[16]In Section 7 I show that the ordinal content of all the results is maintained if the requirement of homogeneity is replaced with three other requirements: *continuity*, *monotonicity with respect to first order stochastic dominance*, and *bounded ratios*. An index of riskiness Q satisfies bounded ratios if for every $\lambda > 1$ there exists $\delta(\lambda) > 1$ such that for every $g \in \mathcal{G}$ $Q(\lambda g) \geq \delta(\lambda) \cdot Q(g)$.

Theorem 2. *(i)* There exists a continuum of homogeneous and locally consistent riskiness indices for which the local aversion equals the local aversion to Q^{AS}-riskiness and Q^{FH}-riskiness at all u and w (and coincides with the Arrow-Pratt coefficient).[17] (ii) Moreover, some of these indices are not monotone with respect to stochastic dominance.

(i) is proved using the observation that for every $a > 0$ any combination of the form $Q_a(g) := Q^{FH}(g) + a \cdot \left|Q^{FH}(g) - Q^{AS}(g)\right|$ is an index of riskiness for which the local aversion equals the local aversion to Q^{FH}. The reason this holds is that for small supports, the second element in the definition is vanishingly small by Inequality 3.0.3, and so Q_a and Q^{FH} should be close. (ii) follows from Example 1.

Example 1. Take $Q_1(h) := Q^{FH}(h) + \left|Q^{FH}(h) - Q^{AS}(h)\right|$ and $g = [1, \frac{e}{1+e}; -1, \frac{1}{1+e}]$. $Q^{AS}(g) = 1$ and $Q^{FH}(g) \approx 1.26$, hence $Q_1(g) < 1.6$. Now take $g' = [1, 1-\epsilon; -1, \epsilon]$. For small values of ϵ, $Q^{AS}(g') \approx 0$ but $Q^{FH}(g') > 1$, so $Q_1(g') > 1.6$. Therefore, while g' first order stochastically dominates g, $Q_1(g) < Q_1(g')$.

Theorem 3. *If Q satisfies local consistency and homogeneity, then A_Q is ordinally equivalent to ρ.*

The proof of of the theorem extends the reasoning of Lemma 1.

Remark 1. Both axioms in Theorem 3 are essential: omitting either admits indices to which the local aversion is not ordinally equivalent to ρ.

Proof. Follows from following examples. □

Example 2. $Q \equiv 5$ satisfies local consistency, but it does not satisfy homogeneity of degree 1. The local aversion to this index induces the trivial order.

Example 3. $Q := \left|Q^{FH} - Q^{AS}\right|$ satisfies homogeneity, and has $R_Q \equiv S_Q \equiv 0$. It therefore induces the trivial order.

4 Consistent Indices of Riskiness

The findings of Theorem 3 indicate that, under weak conditions, the only "reasonable" order of local aversion is the one induced by the Arrow-Pratt ARA. But according to the other parts of the theorem this property is not enough to characterize a "reasonable" index of riskiness. These findings call for additional requirements from an index of riskiness.

[17]Omitting the homogeneity requirement would yield a trivial statement as, for example, an arbitrary change of the values of Q^{AS} for gambles taking values larger than some $M > 0$ will result in a valid index. The requirement that that the local aversion to the index coincides with the Arrow-Pratt coefficient, and not just with the order it implies, is a normalization that rules out, for example, the use of positive multiples of Q^{AS}.

Definition. *Globally no less averse to Q-riskiness.* Let Q be an index of riskiness. u is globally no less averse to Q-riskiness than v (written $u \succsim_Q v$) if for every w and w' the local aversion of u to Q-riskiness at w is no smaller than the local aversion of v to Q-riskiness at w' ($A_Q(u,w) \geq A_Q(v,w')$). u is globally more averse to Q-riskiness than v (written $u \succ_Q v$) if $u \succsim_Q v$ and not $v \succsim_Q u$.[18]

Axiom. *Global consistency. For every pair of utilities u and v, for every w and every g and h in $\mathcal{G}$, if $u \succ_Q v$, u accepts g at w, and $Q(g) > Q(h)$, then v accepts h at w.*

The axiom of global consistency is a weak requirement of consistency, in the sense that it poses no restriction for pairs of utilities which cannot be compared using the (very) partial order globally more averse to Q-riskiness. It is inspired by the duality axiom of AS. That it holds for small gambles follows immediately from the definitions. The content of the axiom comes from the fact that it places no restriction on the support of gambles, so that when two agents that can be compared by the partial order "globally more averse to Q-riskiness," the axiom requires that the less averse agent accepts Q-riskier gambles, and the requirement applies not only for small gambles.

Lemma 5. *Q satisfies global consistency, homogeneity, and A_Q is ordinally equivalent to ρ if and only if Q is a positive multiple of Q^{AS}.*

Proof. The AS duality axiom states that if u is uniformly more averse to risk than v, u accepts g at w, and $Q(g) > Q(h)$, then v accepts h at w. That A_Q is ordinally equivalent to ρ, together with global consistency, imply the duality axiom. But the only indices that satisfy homogeneity and the duality axiom are positive multiples of Q^{AS} [Aumann and Serrano, 2008]. Finally, Theorem 1 and the discussion above imply that Q^{AS} satisfies the axioms, and the same holds for its positive multiples. □

The homogeneity axiom could be replaced with a weaker condition without changing the ordinal content of the index. I use it for its simplicity and since it appears in the original axiomatic characterization of the AS index, and discuss its removal in Section 7.

Theorem 4. *Q^{AS} is the unique index of riskiness that satisfies local consistency, global consistency and homogeneity, up to a multiplication by a positive number.*

Corollary 1. *Q^{FH}, the FH index of riskiness, does not satisfy global consistency.*

Example 4. Consider a gamble $g = [1, \frac{e}{1+e}; -1, \frac{1}{1+e}]$, $Q^{AS}(g) = 1$ and $Q^{FH}(g) \approx 1.26$, and a gamble $g' = [2, 1-\epsilon; -2, \epsilon]$. For small values of ϵ, $Q^{AS}(g') \approx 0$ but $Q^{FH}(g') > 2$. Hence $Q^{AS}(g) > Q^{AS}(g')$ yet $Q^{FH}(g) < Q^{FH}(g')$. Since the local aversion to Q^{FH}-riskiness is equal to the local aversion to Q^{AS} riskiness by Theorem 1, any two CARA utility functions with different ARA between $\frac{1}{Q^{AS}(g)}$ and $\frac{1}{Q^{AS}(g')}$ together with the two gambles violate global consistency.

[18] The above definition is different from the AS definition of uniformly more risk-averse. It is derived directly from the index Q and the utility function u. However, if A_Q is ordinally equivalent to ρ the two definitions are equivalent.

The discussion above suggests a general approach for deriving an index (of riskiness) based on consistency requirements in a given decision making problem. The rest of this section presents another application of this approach for the setting of multiplicative gambles.[19]

Define $\mathcal{U} := \{u : \mathbb{R}_+ \to \mathbb{R} | \, \varrho_u(w) > 1 \, \forall w > 0\}$, the set of (twice continuously differentiable) utility functions with relative risk aversion higher than the logarithmic utility function. Additionally, let $\mathcal{H} := \left\{g \in \mathcal{G} | Q^{FH}(g) < 1\right\}$ be the set of gambles with FH riskiness smaller than 1. The following is a result of FH:

Fact 2. $Q^{FH}(g) < 1 \iff \prod(1+g_i)^{p_i} > 1 \iff E\left[\log(1+g)\right] > 0$.

In what follows I will consider multiplicative gambles, so that now u *accepts* g at w if $u(w+gw) > u(w)$, and *rejects* g otherwise.[20] The interpretation of $Q^{FH}(g) < 1$ is that gambles of the form wg are accepted by a logarithmic utility function at wealth w. Repeatedly accepting gambles with $Q^{FH}(g) > 1$ would lead to bankruptcy with probability 1.

Adjusting the previous axioms to the current setting yields the following axioms for an index of riskiness $Q : \mathcal{H} \to \mathbb{R}_+$:

Axiom. *Scaling.* $\forall \alpha > 0 \;\; \forall g \in \mathcal{H}, \;\; Q\left((1+g)^\alpha - 1\right) = \alpha \cdot Q(g)$.

Similar to the homogeneity axiom, the scaling axiom embodies a cardinal interpretation.[21]

Definition. *Ordinally equivalent.* Given an index of riskiness Q, A_Q is *ordinally equivalent* to the index of relative riskiness ϱ if $\forall u, v \in \mathcal{U} \;\; \forall w, w' > 0, \;\; A_Q(u,w) > A_Q(v,w') \iff \varrho_u(w) > \varrho_v(w')$.

A_Q is ordinally equivalent to ϱ if local consistency and scaling hold. In the interest of brevity and concreteness, I omit the proof.

Axiom. *Global consistency. For every u and v in $\mathcal{U}$, for every $w > 0$ and every g and h in $\mathcal{H}$, if $u \succ_Q v$, u accepts g at w, and $Q(g) > Q(h)$, then v accepts h at w.*

Lemma 6. *(i) For any $g \in \mathcal{H}$ there is a unique positive number $S(g)$ such that $E\left[(1+g)^{-\frac{1}{S(g)}}\right] = 1$. (ii) Q satisfies global consistency, scaling, and A_Q is ordinally equivalent to ϱ if and only if Q is a positive multiple of S.*

Theorem 5. *S is the unique index of riskiness that satisfies local consistency, global consistency and scaling, up to a multiplication by a positive number.*

In previous sections, I introduced the concept of local aversion to an index (of riskiness), and used it to present a new consistency-motivated approach for deriving indices for decision problems. For the case of additive gambles, I showed that there is a unique "reasonable" order of local aversion to riskiness (for a vast class of indices of riskiness). With this result at hand, I showed that the only index in this class that can satisfy the property of global consistency is the AS index of riskiness.

[19]The following is inspired by Schreiber [2013].

[20]g can be interpreted as the return on some risky asset.

[21]Importantly, note that for every $\alpha > 0$ if $g \in \mathcal{H}$ then $(1+g)^\alpha - 1 \in \mathcal{H}$ by fact 2.

To demonstrated the generality of the new approach, I provided another application in the realm of risk. I showed that the index of relative riskiness suggested by Schreiber [2013] could be derived using the same technique. In the following sections, I show that the consistency-motivated approach may be used for deriving "objective" indices in other decision making settings. This proves to be a valuable tool in settings where "local preferences" are easy to compare while the objects of choice are not.

5 A Consistent Index of Delay

Similar to gambles, comparing cashflows which pay (require) different sums of money over several points in time is not a simple undertaking. Some pairs of cashflows may be compared using the partial order of *time-dominance* [Bøhren and Hansen, 1980, Ekern, 1981], which is the analogue of stochastic dominance in this setting. A cashflow c is *first-order time dominated* by c' if at any point in time the sum of money generated by c up to this point is lower then the sum that was generated by c'.[22] Bøhren and Hansen [1980] show that if c is first-order time dominated by c' then every agent with positive time preferences prefers c' to c. Positive time preferences mean that the agent prefers a dollar at time s to a dollar at time $s + \Delta$ for all $\Delta > 0$, or that the agent's discounting function is decreasing. They also show that if c is *second-order time dominated* by c' then every agent with a decreasing and convex discounting function prefers c' to c.[23]

Time dominance is, however, a partial order. With the goal of finding a complete order over investment cashflows, in this section I use the consistency motivated approach to derive a novel index for the delay embedded in an investment cashflow. The index I derive is new to the literature but it is related to the well-known internal rate of return. The index possesses several desirable properties similar to those of the AS index of riskiness. In particular, it is monotone with respect to time dominance.

5.1 Preliminaries

An *investment cashflow* is a sequence of outflows (investment) followed by inflows (return), and a sequence of times when they are conducted. Denote by $(x_n, t_n)_{n=1}^N$ such a cashflow. When x_n is positive the cashflow pays out x_n at time t_n, and when it is negative, an investment of $|x_n|$ is required at t_n. Assume, without loss of generality, that $t_1 < t_2 < ... < t_N$. Further, assume that $x_1 < 0$ and $\sum x_n > 0$, so that some investment is required, and the (undiscounted) return is greater than the investment. This property implies that an agent that does not discount the future will accept any investment cashflow, while a sufficiently impatient agent will reject it. Let $\mathcal{C}$ denote the collection of such cashflows, and $\mathcal{C}_{t,\epsilon}$ be the collection of cashflows with $t_1 \le t \le t_N$, and $t_N - t_1 < \epsilon$.

An *index of delay* is a function $T : \mathcal{C} \times \mathbb{R} \to \mathbb{R}_+$ from the product of the collection of cashflows and time to the positive reals. A cashflow c considered at time t is said to be more T*-delayed*

[22] The sum may be negative, representing a required investment.

[23] As the definition of second-order time domination requires some notation, I choose to omit it, noting that it is analogous to second order stochastic dominance from the risk setting.

then c' relative to t' if $T(c,t) > T(c',t')$. The second entry, t, represents the time at which the cashflow is considered. I allow this dependence on time to increase the generality by allowing time inconsistency, so that the index could rank two cashflows differently from different perspectives.[24]

The following definitions will prove useful.

Definition. *Homogeneity in payoffs. T is homogenous of degree 0 in payoffs* if $T\left((x_n,t_n)_{n=1}^N, t\right) = T\left((\lambda \cdot x_n,t_n)_{n=1}^N, t\right)$ for any cashflow, any $\lambda > 0$ and any t.

The interpretation of homogeneity of degree 0 in payoffs is that doubling all the sums of money (both investment and return) does not change the T-delay of the investment cashflow.

Definition. *Homogeneity in dates.* T is *homogenous of degree 1 in dates relative to t* if $T\left((x_n,t+\lambda\cdot(t_n-t))_{n=1}^N, t\right) = \lambda \cdot T\left((x_n,t_n)_{n=1}^N, t\right)$ for any cashflow and any $\lambda > 0$. T is *homogenous of degree 1 in dates*, if it is homogenous of degree 1 in dates relative to t for all t.

Homogeneity of degree 1 in dates relative to t means that doubling the periods between the flows of money while preserving their relative distance from t doubles the T-delay.

Definition. *translation invariance.* T is *translation invariant* if $T\left((x_n,t_n+\lambda))_{n=1}^N, t+\lambda\right) = T\left((x_n,t_n)_{n=1}^N, t\right)$ for any cashflow, any $\lambda > 0$ and any t.

Translation invariance of T means that T-delay is a time expression, like "in a week" or "a year before," and it does not depend on the current date. In contrast, the interpretation of expressions such as "this Tuesday" depends critically on whether they are said on Friday or Monday.

I consider a capital budgeting setting in which agent i discounts using a smooth schedule of positive instantaneous discounting rates, $r_i(t)$.[25,26] Similar to ρ in the risk setting, r induces a complete order on all agent and time-point pairs. The *net present value (NPV)* of an investment cashflow $c = (x_n,t_n)_{n=1}^N$ for the agent i at time t is

$$NPV(c,i,t) := \sum_n \mathrm{e}^{-\int_t^{t_n} r_k(s)ds} x_n.$$

If $NPV(c,i,t) > 0$ for some t, this inequality holds for any t. Agent i *accept* cashflow c at time t if $NPV(c,i,t) > 0$ and *rejects* it otherwise. c could be thought of as a suggested shift to a baseline cashflow.

The following two definitions are crucial for applying the consistency motivated approach from the previous sections in order to present axioms for an index of delay.

Definition. *Local T-delay aversion.* Agent i's *T-delay aversion at t* is equal to the inverse of $\lim_{\epsilon\to 0^+} \sup_{i \text{ accepts } c,\, c\in\mathcal{C}_{t,\epsilon}} T(c,t)$, and denoted by $A_T(i,t)$.

[24] Agents may present such dynamic inconsistency in their preferences in the presence of hyperbolic discount functions [Laibson, 1997].

[25] An alternative interpretation may be a social planner with such time preferences [Foster and Mitra, 2003].

[26] For a discussion of this condition see Bøhren and Hansen [1980] and references provided there.

Roughly speaking, this number is the inverse of the T-delay of the most T-delayed "local cashflow" with respect to t that is accepted by i.

Definition. *Globally no less T-delay averse.* i is *Globally no less T-delay averse* than j after t_o (denoted by $j \underset{T,t_o}{\blacktriangleleft} i$) if for every t, $t' \geq t_o$ the local aversion to T-delay of i at t is no smaller than the local aversion to T-delay of j at $t'\left(\inf_{t\geq t_0} A_T(i,t) \geq \sup_{t\geq t_o} A_T(j,t)\right)$.[27]

As in the previous settings, this definition generates a partial order over agents, based on their preferences and on the index of delay.

5.2 The Index

The following axioms are an adaptation of the axioms used in Theorem 3 for the current setting. They are used for presenting the analogue of this theorem, as well as the analogue of Theorem 2. Roughly speaking, Theorem 6 implies that there is only one reasonable order of local aversion to delay, and that it corresponds to the instantaneous discounting rate.

Definition. *Full image.* An index of delay T satisfies *full image* if for every $\epsilon > 0$ and t, $\text{Im } T(\mathcal{C}_{\epsilon,t}, t) = \mathbb{R}_+$.

Axiom. *Local consistency.* For all t and i, i's T-delay aversion at t is positive, finite, and no greater then the inverse of $\lim_{\epsilon\to 0^+ i} \inf_{\text{rejects } c,\, c\in\mathcal{C}_{t,\epsilon}} T(c,t)$.

The inequality in the other direction is implied by full image. So if T satisfies local consistency and full image, then for "local" decisions there exists a cut-off level of T-delay for acceptance and rejection.

Axiom. *Homogeneity.* $T(\cdot,\cdot)$ is homogenous of degree 1 in dates.

This axiom represents the notion that if each payment in the cashflow is conducted twice as late, then the entire cashflow is twice as delayed. This is a strong cardinal assumption and I discuss its removal in Section 7.

Lemma 7. *If T satisfies local consistency and homogeneity, T satisfies full image.*

Definition. *Ordinally equivalent.* Given an index of delay T, A_T is *ordinally equivalent* to the instantaneous discounting rate r if $\forall i,j$, $\forall t,t'$ $A_T(i,t) > A_T(j,t') \iff r_i(t) > r_j(t')$.

Lemma 8. *If T satisfies local consistency and homogeneity then $\forall i,j$, $\forall t$*
$A_T(i,t) > A_T(j,t) \iff r_i(t) > r_j(t)$.

Note that the two sides of the inequality are evaluated at the same t. For ordinal equivalence of A_T and r to hold, an additional condition must be required.

[27] Although in some settings it will make no sense to assume that different agents face different paths of interest rates [Debreu, 1972], the comparison is reasonable in many other settings [Bøhren and Hansen, 1980, Ekern, 1981]. It is also possible to think about inter-temporal comparisons for the same agent by shifting t_0 in one side of the formula.

Axiom. *Translation invariance.* T is translation invariant.

This is the only "new" requirement in the current setting; all other axioms are adaptions of the axioms from the risk settings to the current one. It requires a new kind of internal consistency, which was not relevant in the previous settings. The new requirement appears to be both natural and weak.

Theorem 6. *If T satisfies local consistency, homogeneity and translation invariance, then A_T is ordinally equivalent to r.*

Remark 2. All axioms in Theorem 6 are essential: omitting any admits indices to which the local aversion is not ordinally equivalent to r.

Example 5 demonstrates that without translation invariance the inference in part (i) is not necessarily correct. The following two definitions prove useful for the example as well as for the statement and proof of Theorem 8.

Definition. The *Internal rate of return* (IRR) of an investment cashflow $c = (x_n, t_n)_{n=1}^{N}$, written $\alpha(c)$, is the unique positive solution to the equation $\sum\limits_n \mathrm{e}^{-\alpha t_n} x_n = 0$.

Definition. For a cashflow c, $(D(c,t) =)\, D(c) := \frac{1}{\alpha(c)}$ is the inverse of the IRR of the cashflow.

Example 5. Consider the index of delay $T(c,t) := (|t|+1) \cdot D(c)$, and an agent, i, with a constant discounting rate $r_i(t) \equiv C$. For $t > 0$, the T-delay aversion of the agent is strictly decreasing in t, but $r_i(\cdot)$ is constant by construction. That this index satisfies the other axioms, follows from the fact that D does, which is proved later.

Theorem 7. *(i) There exists a continuum of translation invariant, locally consistent homogenous indices of delay for which the local aversion equals to r. (ii) Moreover, some of these indices are not monotone with respect to time dominance.*

Proof. See appendix. □

As before, global consistency is an important part of the approach. The following axiom is an adaptation of the global consistency axiom from the risk setting to the current setting.

Axiom. *Global consistency.* Let $j \underset{T,t_o}{\blacktriangleleft} i$, $c = (x_n, t_n)_{n=1}^{N}$ and $c' = \left(x_n', t_n'\right)_{n=1}^{N'}$ and t_o is smaller than t, t_1 and t_1'. Then, if $T(c,t) < T(c',t)$, and i accepts c', then j accepts c.

Lemma 9. *T satisfies global consistency, homogeneity, translation invariance, and A_T is ordinally equivalent to r if and only if T is a positive multiple of D.*

Theorem 8. *D is the unique index of delay that satisfies local consistency, global consistency, homogeneity and translation invariance, up to a multiplication by a positive number.*

5.3 Properties of the Index D and a Comparison with Q^{AS}

This section discusses some properties of the index of delay D and demonstrates the close connection it has with the AS index of riskiness. The IRR is a counterpart of *the rate of return over cost* suggested by Fisher [1930] as a criterion for project selection almost a century ago. Later, some economists dismissed this criterion, arguing that the NPV was superior in comparing pairs of cashflows. Yet, others mentioned that this criterion has the benefit of objectivity, in that it does not require the value judgment of setting the future discounting rates [Turvey, 1963]. For example, Stalin and Nixon would agree on the IRR of an investment even though they might disagree on its NPV.[28]

Just like the AS-riskiness of a gamble depends *"on its distribution only—and not on any other parameters, such as the utility function of the decision maker or his wealth"* [Aumann and Serrano, 2008], D depends solely on the cashflow, and not on any agent specific properties. In this sense, D is an objective measure of delay. A related property of D is that it is also independent of the date when the cashflow is considered. That is, the delay embedded in an investments cashflow is independent of the time when it is considered.

D is homogenous of degree 0 in payoffs and unit free. This means, for example, that the D-delay of two cashflows denominated in different currencies may be compared without knowledge of the exchange rate. This stands in contrast to the AS index of riskiness which is homogenous of degree 1 in payoffs, but does not depend on timing. The property is analogous to the property of Q^{AS}, according to which "diluted" gambles inherit the riskiness of the original gamble. For $p \in (0, 1)$ a *p-dilution* of the gamble g takes the value of the gamble with probability p and 0 with probability $1 - p$, independently of the gamble. The reason why this analogy is correct is that in the current setting, times are the parallel of payoffs from the risk setting, while payoffs are the parallel of probabilities, as demonstrated by the remark in the end of this section.

Another property that D and Q^{AS} share is monotonicity. Q^{AS} is monotonic with respect to first and second order stochastic dominance. The analogous property for cashflows is *time-dominance* [Bøhren and Hansen, 1980, Ekern, 1981]. Proposition 3 of Bøhren and Hansen [1980] implies that D is monotonic with respect to time-dominance of any order.

If $Q^{AS}(g) = Q^{AS}(h)$, a compound gamble yielding (independently of g and h) g with probability p and h otherwise has AS riskiness equal to $Q^{AS}(g)$. If g and h are independent, $Q^{AS}(g + h)$ is also equal to $Q^{AS}(g)$. A similar property holds in the current setting. If c and c' are two investment cashflows such that $D(c) = D(c')$ and $c + c'$ is an investment cashflow,[29] then $D(c + c') = D(c')$. The slight difference between the conditions follows from the fact that unlike with gambles, the sum of two investment cashflows may not be an investment cashflow.

There are other similarities between the the measurement of delay and risk. *Value at Risk* (VaR)

[28] This resembles the point made by Hart [2011] that in general there are many pairs of agents and pairs of gambles such that each agent accepts a different gamble and rejects the other – our axioms only compare very specific pairs of agents.

[29] The interpretation of $c + c'$ is that all of the payoffs which are dictated by each of the cashflows takes place at the times they dictate. If both require a payoff at the same time point, the payoffs are added up.

is a family of indices commonly used in the financial industry [Aumann and Serrano, 2008]. VaR indices depend on a parameter called the *confidence level.* For example, the VaR of a gamble at the 95 percent confidence level is the largest loss that occurs with probability greater than 5 percent. Unlike the AS index, VaR is unaffected by tail events or rare-disasters, extremely negative outcomes that occur with low probability. In the context of project selection, *Turvey [1963]* mentions that *"the Pay-off Period, the number of years which it will take until the undiscounted sum of the gains realized from the investment equals its capital cost,"* was used by practitioners in the West and in Russia. He adds that *"[p]ractical men in industries with long-lived assets have perforce been made aware of the deficiencies of this criterion and have sought to bring in the time element."* The pay-off period criterion, unlike the index of delay, suffers from deficiencies similar to those of VaR. For example, shifting early or late payoffs does not change its value. In fact, recalling that times in the current setting are the parallel of payoffs in the risk setting, the lesson learned by the investors in long-lived assets should apply to investors in risky assets with distant tail events.

Q^{AS} is much more sensitive to the loss side of gambles than it is to gains. Analogously D is more sensitive to early flows than it is to later ones. This follows from the properties of the exponential function in the definition of the IRR. Additionally, both D and Q^{AS} are continuous in their respective spaces.

Finally, to clarify the analogies I made between probabilities and payoffs, and between payoffs and times, I present a reinterpretation of the AS index of riskiness in terms of the delay embedded in a (non-investment) cashflow.

Remark. Given a gamble $g := (g_j, p_j)$, a cashflow which requires an investment of one dollar at $t = 0$ and pays-out p_j at time g_j has a unique positive IRR whose inverse equal to $Q^{AS}(g)$.

To see this, recall that for a cashflow $c = (x_n, t_n)_{n=1}^{N}$ the IRR is the (unique) positive solution to the equation $\sum_n \mathrm{e}^{-\alpha t_n} x_n = 0$, when it exists. Noting that at $t = 0$, $e^{-\alpha t} = 1$ and that the above cashflow requires an investment of one dollar at $t = 0$, the corresponding equation could be written as

$$-1 + \sum_n \mathrm{e}^{-\alpha g_n} p_n = 0,$$

which could be expressed as

$$\mathbb{E}\left[e^{-\alpha g}\right] = 1.$$

But $Q^{AS}(g)$ was defined as the inverse of the unique positive α which solves the equation.

For general cashflows, multiple solutions to the equation defining the internal rate of return may exit. Interestingly, both Arrow and Pratt took interest in finding simple conditions that would rule out this possibility [Arrow and Levhari, 1969, Pratt and Hammond, 1979]. A corollary of the previous remark is that cashflows of the above form have a unique positive IRR.

6 A Consistent Index of the Appeal of Information Transactions

Similar to the previous settings, generating a sensible complete ranking of information structures is an illusive undertaking. In some settings, certain information may be vital, while in others it will not be very important. The implication is that it is not possible to rank all information structures so that higher ranked structures are preferred to lower ranked ones by all agents at every decision making problem. Some pairs of information structures may, however, be compared in this manner. In his seminal paper, Blackwell [1953] showed that one information structure is preferred to another by all agents in all settings if and only if the latter is a garbling of the prior.[30] That is, if one is a noisy version of the other. But this order is partial and cannot be used to compare many pairs of information structures.

The difficulty in generating a complete ranking which is independent of agents' preferences is discussed by Willinger [1989] in his paper which studies the relation between risk aversion and the value of information. Willinger [1989] discusses his choice of using the *expected value of information* (EVI) or *"asking price"* which was defined by LaValle [1968]. The EVI measures a certain decision maker's willingness to pay for certain information, and so, *"... the difficulty of defining a controversial continuous variable representing the 'amount of information' can be avoided."*

Cabrales et al. [2013] take an axiomatic approach to tackle this difficulty, in the spirit of Hart [2011]. Later, these authors took an approach in the spirit of AS, and axiomatically derived a different index for the appeal of *information transactions* [Cabrales et al., 2012]. In this section, I use the techniques from the previous sections to show that the local distaste for information is locally consistent with ARA for two prominent indices, and that the unique index that satisfies local consistency, global consistency, and a homogeneity axiom is the index of appeal of information transactions [Cabrales et al., 2012].

6.1 Preliminaries

This section follows closely Cabrales et al. [2012]. I consider agents with concave and twice continuously differentiable utility functions who have some initial wealth and face uncertainty about the state of nature. There are $K \in \mathbb{N}$ states of nature, $\{1, ..., K\}$,[31] over which the agents have the prior $p \in \Delta(K)$ which is assumed to have a full support.

The set of *investment opportunities* $B^* = \left\{ b \in \mathbb{R}^K | \sum_{k \in K} p_k b_k \leq 0 \right\}$, consists of all no arbitrage assets. In particular it includes the option of inaction. The reference to the members of B^* as no arbitrage investment opportunities attributes to p_k an additional interpretation as the price of an Arrow-Debreu security that pays 1 if the state k is realized and nothing otherwise. Hence, p plays a dual role in this setting.[32] When an agent with initial wealth w chooses investment $b \in B^*$ and state k is realized, his wealth becomes $w + b_k$.

[30] A simple proof is provided in Leshno and Spector [1992].

[31] With a slight abuse of notation, I also denote $\{1, ..., K\}$ by K. The meaning of K should be clear from the context.

[32] Cabrales et al. [2012] treat the more general case as well and disentangle the two roles of p, prices and prior.

Before choosing his investment, the agent has an opportunity to engage in an *information transaction* $a = (\mu, \alpha)$, where $\mu > 0$ is the cost of the transactions, and α is the information structure representing the information that a entails. To be more precise, α is given by a finite set of signals S_α and probability distributions $\alpha_k \in \Delta(S_\alpha)$ for every $k \in K$. When the state of nature is k, the probability that the signal s is observed equals $\alpha_k(s)$. Thus, the information structure may be represented by a stochastic matrix M_α, with K rows and $|S_\alpha|$ columns, and the total probability of the signals is given by the vector $p_\alpha := p \cdot M_\alpha$. For simplicity, assume that $p_\alpha(s) > 0$ for all s, so that each signal is observed with positive probability. Further, denote by q_k^s the probability the agent assigns to state k conditional on observing the signal s, using Bayes' law. Note that although my notation does not indicate it, $(q_k^s)_{k=1}^K = q^s \in \Delta(K)$ depends on α and the prior p.

The transaction a is said to be *excluding* if for every s there exists some k such that $q_k^s = 0$. This means that for every signal the agent receives, he knows that some states will not be realized (allowing him to generate infinite profit at no risk). Throughout, I will assume that information transactions are not excluding.

Agents are assumed to choose the optimal investment opportunity in B^* given their belief, q. Therefore, the expected utility of an agent with utility u, initial wealth w and beliefs q is

$$V(u, w, q) = \sup_{b \in B^*} \sum_k q_k u\left(w + b_k\right).$$

In case that the agent acquires no information, his beliefs are given by the prior p. Since the agent is risk averse, in such case his optimal choice is inaction. So,

$$V(u, w, p) = u(w).$$

Accordingly, an agents *accepts* an information transaction if

$$\sum_s p_\alpha(s) V(u, w - \mu, q^s) \geq V(u, w, p) = u(w)$$

and *rejects* it otherwise.

Denote by $\mathcal{A}$ the class of information transactions described above. Additionally, denote by $\mathcal{A}_\epsilon$ the sub-class of these information transactions such that $\|p - q^s\|_\infty < \epsilon$ for all s. An *index of appeal* of information transactions is a function from the class of information transactions to the positive reals $Q : \mathcal{A} \to \mathbb{R}_+$. The index of appeal A suggested by Cabrales et al. [2012] is defined by

$$A(a) = -\frac{1}{\mu} \ln\left(\sum_s p_\alpha(s) \exp\left(-d\left(p||q^s\right)\right)\right),$$

where

$$d\left(p||q\right) = \sum_k p_k \ln \frac{p_k}{q_k}$$

is the *Kulback-Leibler divergence* [Kullback and Leibler, 1951].

Cabrales et al. [2013] suggested the *entropy reduction* as a measure of informativeness of an information structure for investors. It is defined by

$$I_e(\alpha) = H(p) - \sum_s p_\alpha(s) \cdot H(q^s),$$

where,

$$H(q) = -\sum_{k \in K} q_k \ln(q_k).$$

In the current context, consider the index J_e, the *cost adjusted entropy reduction* defined by

$$J_e(\mu, \alpha) = \frac{I_e(\alpha)}{\mu}.$$

Finally, define the *local* distaste for Q-informativeness of an agent u with wealth w, as the the appeal of the most Q-appealing transaction that is rejected, and provides just a little information. More precisely, it is defined by $\lim_{\epsilon \to 0^+} \sup_{a \in \mathcal{A}_\epsilon,\, a \text{ is rejected by } u \text{ at } w} Q(a)$.[33]

6.2 The Index

Theorem 9 is the analog of Theorem 1 in the current context. It shows that the local distaste for the two indices discussed above coincides with ρ.[34]

Theorem 9. *(i) The local distaste for A of u with wealth w is equal to $\rho_u(w)$. Furthermore, it is equal to $\lim_{\epsilon \to 0^+} \inf_{a \in \mathcal{A}_\epsilon,\, a \text{ is accepted}} A(a)$. (ii) The local distaste for J_e of u with wealth w is equal to $\rho_u(w)$. Furthermore, it is equal to $\lim_{\epsilon \to 0^+} \inf_{a \in \mathcal{A}_\epsilon,\, a \text{ is accepted}} J_e(a)$.*

I now turn to proving the analogues of Theorems 2 and 3.

Axiom. *Local consistency. For all u and w, the local distaste for Q-informativeness of u at w is positive and finite. Furthermore, it is smaller or equal than $\lim_{\epsilon \to 0^+} \inf_{a \in \mathcal{A}_\epsilon,\, a \text{ is accepted}} Q(a)$.*

Axiom. *Homogeneity. For every information transaction $a = (\mu, \alpha)$ and every $\lambda > 0$, $Q(\lambda \cdot \mu, \alpha) = \frac{1}{\lambda} \cdot Q(a)$.*

The homogeneity axiom states that Q is homogenous of degree -1 in transaction prices. This axiom entails the cardinal content of the index. It is particularly interesting if the units of the index are interpreted as "information per dollar."

Theorem 10. *If Q satisfies local consistency and homogeneity of degree -1 in prices, then the local distaste for Q-informativeness is ordinally equivalent to ρ.*

[33] Note that in this setting the index is not independent of the prior p, even when the dependence is not made explicit by the notation I use.

[34] The relations between risk aversion and the taste for information have been discussed extensively in the literature [e.g. Willinger, 1989].

Remark 3. Both axioms in Theorem 10 are essential: omitting either admits indices to which the local distaste is not ordinally equivalent to ρ.

Theorem 11. *(i) There exists a continuum of locally consistent homogenous indices of appeal for which the local distaste equals to ρ. (ii) Moreover, some of these indices are not monotone with respect to Blackwell dominance.*

Proof. See appendix. □

I now turn to the task of characterizing a consistent index for the appeal of information transactions. For an index Q, say that Q-informativeness *is globally more attractive for* agent i than to agent j (written $j \underset{Q}{\lessapprox} i$) if the supremum over w of the local distaste for Q-informativeness of i is smaller than the infimum over w of the local distaste for Q-informativeness of j.

Axiom. *Global consistency. For any w and $a, b \in \mathcal{A}$, if $j \underset{Q}{\lessapprox} i$, $A(a) < A(b)$ and j accepts a at w, then i accepts b at w.*

Lemma 10. *Q satisfies global consistency, homogeneity of degree -1 in prices, and the local distaste to Q-informativeness is ordinally equivalent to ρ if and only if Q is a positive multiple of A.*

Proof. Ordinal equivalence to ρ implies that if $j \underset{Q}{\lessapprox} i$ then j is uniformly more risk averse than i. Combined with this fact, global consistency and homogeneity of degree -1 in prices imply the two axioms that are uniquely satisfied by positive multiples of A, according to Theorem 4 in Cabrales et al. [2012]. That the local distaste to A is ordinally equivalent to ρ follows from Theorem 9. That the other axioms are satisfied is shown in Cabrales et al. [2012]. The same holds for positive multiples of A. □

Theorem 12. *A is the unique index that satisfies local consistency, global consistency and homogeneity of degree -1 in prices, up to a multiplication by a positive number.*

Proof. Follows from the lemma and the previous theorem. □

Corollary 2. *J_e, the cost adjusted entropy reduction index, does not satisfy global consistency.*

Example 6. (Based on Example 2 of Cabrales et al. [2012]). Let $K = \{1, 2, 3\}$ and fix a uniform prior. Consider the information structures

$$\alpha_1 = \begin{bmatrix} 1-\epsilon_1 & \epsilon_1 \\ 1-\epsilon_1 & \epsilon_1 \\ \epsilon_1 & 1-\epsilon_1 \end{bmatrix}, \ \alpha_2 = \begin{bmatrix} 1-\epsilon_2 & \epsilon_2 \\ 0.1 & 0.9 \\ \epsilon_2 & 1-\epsilon_2 \end{bmatrix},$$

and the information transactions $a_1 = (1, \alpha_1)$ and $a_2 = (1, \alpha_2)$. It can be shown that

$$A(a_1) \approx -\log\left(\frac{2}{3}\epsilon_1^{1/3} + \frac{1}{3}\epsilon_1^{2/3}\right),$$

and

$$A(a_2) \approx -\log\left(\epsilon_2^{1/3}\right).$$

This means that the ordering of the two transactions according to A depends on the choices of $\epsilon_1, \epsilon_2 > 0$. Even when they are both small, their relative magnitude matters.

In contrast, the cost adjusted entropy reduction index, J_e, ranks a_2 higher than a_1 for small $\epsilon_1, \epsilon_2 > 0$. To see this, note that

$$J_e\left(a_1\right) \approx \ln 3 - 0.462,$$

and

$$J_e\left(a_2\right) \approx \ln 3 - 0.550.$$

This means that there exists a choice of small enough ϵ_1, ϵ_2 such that $A(a_1) < A(a_2)$ and $J_e(a_1) > J_e(a_2)$. Hence, there exists two CARA functions with different ARA coefficients (between $A(a_1)$ and $A(a_2)$), which both accept a_2 but reject a_1, demonstrating that J_e violates global consistency.

6.3 Properties of the Index A

The setting of information transactions is somewhat different than other settings that are discussed in this paper, in that the index depends on the prior, and is therefore not completely objective. Cabrales et al. [2013] showed that in their setting this limitation cannot be avoided. The fact that in the setting presented here the prior and the prices (which are observable) coincide is comforting in this regard.

An important property of the index A, is that it is monotonic with respect to *Blackwell's [1953] partial ordering* of information structures [Cabrales et al., 2012]. According to Blackwell's order, one information structure is more informative than another if the latter is a garbling of the prior. Blackwell [1953] proved that one information structure is more informative than another according to this partial ordering if and only if every decision maker prefers it to the other. Cabrales et al. [2012] show that if α is more informative than β in the sense of Blackwell, then $A(\mu, \alpha) > A(\mu, \beta)$ for every $\mu > 0$ and every prior p.[35] As Blackwell's ordering is the parallel of stochastic dominance and time dominance, this property is analogous to the properties of the indices presented in previous sections. Other desirable properties of the index include continuity and monotonicity in prices. For an extensive discussion of the properties of this index see Cabrales et al. [2012].

Finally, the cardinal interpretation of the index A is relatively more compelling, as the homogeneity axiom can be interpreted as stating that the index measures information per dollar payed. If this interpretation is taken seriously, then the index may be used in practice for comparing different portfolio managers, charging a fixed fee.

[35] Recall that A depends on the prior p, even though this fact is not reflected in the notation I use.

7 Ordinality

In the previous sections the indices that were generated had some cardinal content, driven by the various homogeneity (scaling) axioms. In some settings, however, these axioms may be hard to justify. In what follows I show that the ordinal content of the indices remains when these axioms are omitted and replaced with weaker conditions. Theorem 13 shows that Lemmata 5, 6, 9 and 10 still hold when homogeneity (scaling) is replaced by monotonicity with respect to first order stochastic (time) dominance (or with respect to price), and continuity.

Recall that two indices are *ordianlly equivalent* if they induce the same order over the objects they rank.

Theorem 13. *(i) If Q is a continuous index of riskiness that satisfies monotonicity with respect to first order stochastic dominance and global consistency, and A_Q is ordinally equivalent to ρ then Q is ordinally equivalent to Q^{AS}. (ii) If Q is a continuous index of relative riskiness that satisfies monotonicity with respect to first order stochastic dominance and global consistency, and A_Q is ordinally equivalents to ϱ, then Q is ordinally equivalent to S. (iii) If T is a continuous index of delay that satisfies monotonicity with respect to first order time dominance, translation invariance and global consistency, and A_T is ordinally equivalent to r, then T is ordinally equivalent to D. (iv) If Q is a continuous index of the appeal of information transactions that satisfies monotonicity in price and global consistency, and the local distaste to Q is ordinally equivalent to ρ, then Q is ordinally equivalent to A.*

Under the conditions of the theorem, which are quite standard with the exception of the ordinal equivalence, all indices generate the same ranking. The uniqueness, however, is lost, and with it the cardinal interpretation of the indices. To complete the discussion, I present an additional condition which allows dropping the ordinal equivalence requirement. Theorem 14 shows that Theorems 4, 5, 8 and 12 still hold when homogeneity (scaling) is replaced by monotonicity with respect to first order stochastic (time) dominance (or with respect to price), continuity and *bounded ratios.*

Definition 3. *Bounded ratios.* (i) An index of riskiness Q satisfies *bounded ratios* if for every $\lambda > 1$ there exists $\delta(\lambda) > 1$ such that for every $g \in \mathcal{G}$ $Q(\lambda g) \geq \delta(\lambda) \cdot Q(g)$. (ii) An index of relative riskiness Q satisfies *bounded ratios* if for every $\lambda > 1$ there exists $\delta(\lambda) > 1$ such that for every $g \in \mathcal{G}$ $Q\left((1+g)^{\lambda} - 1\right) \geq \delta(\lambda) \cdot Q(g)$. (iii) An index of delay T satisfies *bounded ratios* if for every t and $\lambda > 1$ there exists $\delta(\lambda) > 1$ such that for every $c \in \mathcal{C}$, $c = (x_i, t_i)_{i=1}^{N}$, $T\left((x_i, t + \lambda \cdot (t_i - t))_{i=1}^{N}, t\right) \geq \delta(\lambda) \cdot T\left((x_i, t_i)_{i=1}^{N}, t\right)$. (iv) An index of the appeal of information transactions Q satisfies *bounded ratios* if for every $\lambda > 1$ there exists $\delta(\lambda) > 1$ such that for every $a \in \mathcal{A}$, $a = (\mu, \alpha)$, $Q(\lambda \cdot \mu, \alpha) \geq \frac{1}{\delta(\lambda)} \cdot Q(a)$.

This property is clearly satisfied by homogenous indices (or indices which satisfy scaling, in the case of relative riskiness), with $\delta(\lambda) = \lambda$. In fact this condition is more general, as is demonstrated by considering indices that are homogenous of degree k for $k > 1$. The order of quantifiers is crucial for the requirement to suffice for what follows. In the setting of riskiness, for example, given $\lambda > 1$, $\delta(\lambda)$ is *the same for all* g.

Theorem 14. *(i) If Q is a continuous index of riskiness that satisfies monotonicity with respect to first order stochastic dominance, bounded ratios, local consistency and global consistency, then Q is ordinally equivalent to Q^{AS}. (ii) If Q is a continuous index of relative riskiness that satisfies monotonicity with respect to first order stochastic dominance, bounded ratios, local consistency and global consistency, then Q is ordinally equivalent to S. (iii) If T is a continuous index of delay that satisfies monotonicity with respect to first order time dominance, translation invariance, bounded ratios, local consistency and global consistency, then T is ordinally equivalent to D. (iv) If Q is a continuous index of the appeal of information transactions that satisfies (monotonicity in price), bounded ratios, local consistency and global consistency, then Q is ordinally equivalent to A.*

Remark. In part (iv) of the theorem, monotonicity in price is parenthesized as this property follows directly from bounded ratios. I state it in the theorem in order to maintain consistency.

Remark. The requirement of bounded ratios is essential: omitting it admits indices which induce different orders.

I only provide an example for (i). Examples for the other settings could be constructed in the same spirit.

Example 7. Let $Q(g) = 1 + E[g]$. It is positive by the assumptions on $\mathcal{G}$. It is continuous and monotonic, since the expectation operator is. It is locally consistent since the local aversion to Q-riskiness always equals 1, and globally consistent since the partial order globally more averse to Q-riskiness is trivial.

To see that bounded ratios is violated, assume the for some $\lambda > 1$ the corresponding value is $\delta(\lambda)$. Clearly, this number is lower than λ as

$$\lambda \cdot Q(g) = \lambda(1 + E[g]) > 1 + \lambda E[g] = Q(\lambda g) \geq \delta(\lambda) \cdot Q(g).$$

Choose $h \in \mathcal{G}_\epsilon$, for $0 < \epsilon < \frac{1-\delta(\lambda)}{(\delta(\lambda)-\lambda)}$, and observe that

$$\frac{Q(\lambda h)}{Q(h)} = \frac{1 + E[\lambda h]}{1 + E[h]} < \delta(\lambda) \iff 1 - \delta(\lambda) < \delta(\lambda) \cdot E[h] - E[\lambda h] = E[h](\delta(\lambda) - \lambda) \iff \frac{1 - \delta(\lambda)}{(\delta(\lambda) - \lambda)} > E[h]$$

where the last inference uses the fact that $\delta(\lambda) < \lambda$. But since $h \in \mathcal{G}_\epsilon$, its expectation must be lower than ϵ, hence lower than $\frac{1-\delta(\lambda)}{(\delta(\lambda)-\lambda)}$.

8 Some Further Remarks

(a) *Rating agencies and simple decision rules.* Let us interpret gambles as risky assets and the riskiness index as an objective rating, similar to the ones that rating agencies produce. The consistency results of Sections 3 and 4 imply that for small investments individuals are able to form a simple investment strategy, based only on the objective rating of the risky asset and their own preferences, that will result in decisions similar to the ones an informed expected utility maximizer would produce.

In the context of additive gambles, the rating could be Q^{AS} and a possible strategy strategy will reject gambles riskier than $\frac{1}{\rho_u(w)}+\epsilon$ and accept gambles that are less risky than $\frac{1}{\rho_u(w)}-\epsilon'$. This result is especially interesting for environments in which attaining and interpreting information about the risky asset is costly [see for example Dellavigna and Pollet, 2009].

The same reasoning applies when a social planner delegates the right to decide about small projects to a bureaucrat. If projects are completed over short periods of time, and the bureaucrat's discretion is limited by a simple decision rule based on the IRR and the discounting schedule $r(t)$, the resulting decisions will resemble the optimal ones.

(b) *Cardinal interpretation.* If we take seriously the cardinal content of an index (of riskiness) Q, then the local aversion to the Q-riskiness also carries some cardinal content. For example, from the AS riskiness index, we get a cardinal interpretation for ρ; $\rho_u(w)=\frac{1}{2}\rho_v(w')$ if and only if u at w is willing to accept (small) gambles twice as risky as those that v is willing to accept at w'. The same applies to the other indices presented here.

The cardinal content of the index of the appeal of information transactions seems to be the most compelling. The reason is that the units of the index may be interpreted as "information per dollar," which makes homogeneity of degree -1 in prices a natural requirement.

(c) *Local consistency.* Theorem 1 shows that for $Q\in\left\{Q^{AS},Q^{FH}\right\}$, for the class of utility functions treated in this paper $R_Q\equiv S_Q$. But this property does not have to hold for all utility functions and all risk indices, as the following example illustrates:

$$u(w):=|w|^{\frac{1}{3}}\cdot sgn(w),$$

where $sgn(\cdot)$ is the sign function.
u at wealth 0 accepts any gamble of the form $g=[\epsilon,\frac{e^{c\epsilon}}{1+e^{c\epsilon}};-\epsilon,\frac{1}{1+e^{c\epsilon}}]$ with $c>0$, for which $Q^{AS}(g)=c$, hence $\frac{1}{R_{Q^{AS}}(u,0)}=0$. On the other hand, for small $\delta>0$, u at wealth 0 rejects gambles of the form $g'=[2\epsilon,\frac{1}{3}+\delta;-\epsilon,\frac{2}{3}-\delta]$ for small fixed $\delta>0$.[36] $Q^{AS}(g')\underset{\epsilon\to 0}{\longrightarrow}0$, and therefore $\frac{1}{S_{Q^{AS}}(u,0)}=\infty$. Comparing u with any CARA utility function proves that even the ordinal content of S_Q and R_Q may differ.

Local consistency was used throughout this paper. Accordingly, similar issues arise when restrictions on preferences are relaxed in the other settings.

[36] δ does not depend on ϵ.

(d) *Risk lovers.* In their concluding remarks Aumann and Serrano [2008] suggested extending their approach to gambles with negative expectation, which will apply to risk lovers. The current results highlight this need, since the local aversion to Q-riskiness as it is currently defined is unable to accommodate risk loving behavior.[37]

Similarly, the index of relative riskiness S was constructed here using the class of gambles with FH-riskiness lower than 1. It is easy to see that the same criticism applies to it, as it cannot handle multiplicative gambles with geometric mean smaller than 1, and hence it cannot accommodate utility functions with relative risk aversion lower than that of the logarithmic utility.

The index of delay D also suffers from the same flaws. D cannot accommodate investment cashflows in which the return in smaller than the investments, nor borrowing. Finally, the index A of the appeal of information transaction cannot accommodate transactions that reduce the information available to the agent nor negative prices.

(e) *Monotonicity.* A common property of all the indices for which axioms were provided in this paper is that they are monotone with respect to some intuitive partial order. In the risk settings it is stochastic-dominance, while in the other cases it is time-dominance, or Blackwell's partial order.

(f) *Empirical Applications.* Recently, attempts are being made to apply the AS and FH indices in the empirical setting. The most prominent example is Kadan and Liu [Forthcoming] who observe that tail events and rare disasters are unaccounted for by traditional performance evaluation measures [Barro, 2006]. They propose a reinterpretation of the AS and FH indices as performance measures, and illustrate the applicability of these measures by using them to evaluate popular anomalies and investment strategies, and by applying them to the selection of mutual funds.

The findings of this paper may turn out to be useful in the empirical setting as well. First, the index S of relative risk aversion, seems to be a more natural choice for portfolio selection. Similarly to the FH measure it amplifies the weight of rare disasters, but unlike the FH measure it is continuous in its domain, thus avoiding numerous difficulties in estimation. Second, the paper suggests a method for generating indices for different environments. In cases where some property must be measured, but no acceptable index exists, this approach may be useful for generating an index as required.

9 Discussion

Pratt [1964] explains that $\rho_u(x)$ can be interpreted as *"a measure of local risk aversion (risk aversion in the small)."* Moreover, he states that while he does not introduce a simple measure to compare *"risk aversion in the large,"* global risks are considered. Specifically, he shows that u is globally more averse to risk than v if and only if u is locally more averse to risk than v at all wealth levels.

[37] To see this, note that A_Q is positive by definition.

In contrast, Aumann and Serrano [2008] start by defining some global properties of utility functions. They define two partial orders that represent global risk aversion; "*no less risk averse*," Pratt's order, and "*uniformly no less risk averse*," a stronger property. Their *duality axiom* is a consistency requirement which links between global risk aversion and riskiness.

The current paper takes a different approach. For an index of riskiness Q, I start by introducing the concept of *local aversion to Q-riskiness*, so riskiness is defined prior to the definition of its (local) aversion. I then show that for two prominent riskiness indices the concept coincides with the Arrow-Pratt local aversion to risk. However, I find that this property is not satisfied uniquely by these indices, and provide examples for "unreasonable" indices with this property.

Using the concept of local aversion to a riskiness index, I present a novel approach for deriving indices (of riskiness), motivated by consistency. For the case of additive gambles, I show that there is a unique "reasonable" order of local aversion (still, for a general class of indices of riskiness). With this result at hand, I show that the only index (in this class) that satisfies the property of global consistency and homogeneity is the AS index of riskiness.

This methodology is quite general. I provide three other applications for different decision making problems. Still in the context of risk, I show that the index of relative riskiness suggested by Schreiber [2013] could be derived using the same techniques. In the context of information acquisition, I use the same methodology to derive the index of appeal of information transactions of Cabrales et al. [2012].

Apart from indices suggested elsewhere, I derive a novel index for the delay embedded in investment cashflows. This index shares many desirable properties with the other indices; it is continuous, monotonic with respect to time-dominance, and is more sensitive to early periods than later ones. The local aversion to this index corresponds to the instantaneous discounting rate.

Future work should use the methodology proposed in this paper for deriving other objective indices for different decision making problems. A particular setting that seems promising in this regard is the measurement of inequality, which has many similarities to the setting of riskiness [Atkinson, 1970].

References

Kenneth J. Arrow. *Aspects of the Theory of Risk-Bearing*. Helsinki: Yrjö Jahnssonin Säätio, 1965.

Kenneth J. Arrow. *Essays in the Theory of Risk Bearing*. Chicago: Markham, 1971.

Kenneth J. Arrow and David Levhari. Uniqueness of the internal rate of return with variable life of investment. *The Economic Journal*, 79(315):pp. 560–566, 1969.

Anthony B. Atkinson. On the measurement of inequality. *Journal of economic theory*, 2(3):244–263, 1970.

Robert J. Aumann and Roberto Serrano. An economic index of riskiness. *Journal of Political Economy*, 116(5):pp. 810–836, 2008.

Robert J. Barro. Rare disasters and asset markets in the twentieth century. *The Quarterly Journal of Economics*, 121(3):823–866, 2006.

David Blackwell. Equivalent comparisons of experiments. *The Annals of Mathematical Statistics*, 24(2):pp. 265–272, 1953.

Øyvind Bøhren and Terje Hansen. Capital budgeting with unspecified discount rates. *The Scandinavian Journal of Economics*, 82(1):pp. 45–58, 1980.

Antonio Cabrales, Oliver Gossner, and Roberto Serrano. The appeal of information transactions. *Working paper*, 2012.

Antonio Cabrales, Olivier Gossner, and Roberto Serrano. Entropy and the value of information for investors. *The American Economic Review*, 103(1):360–377, 2013.

Gerard Debreu. *Theory of value: An axiomatic analysis of economic equilibrium*, volume 17. Yale University Press, 1972.

Stefano Dellavigna and Joshua M. Pollet. Investor inattention and friday earnings announcements. *The Journal of Finance*, 64(2):709–749, 2009.

Steinar Ekern. Time dominance efficiency analysis. *The Journal of Finance*, 36(5):1023–1033, 1981.

Irving Fisher. The theory of interest. *New York: Kelley, Reprint of the 1930 Edition*, 1930.

Dean P. Foster and Sergiu Hart. An operational measure of riskiness. *Journal of Political Economy*, 117(5):pp. 785–814, 2009.

Dean P Foster and Sergiu Hart. A wealth-requirement axiomatization of riskiness. *Theoretical Economics*, 8(2):591–620, 2013.

James E. Foster and Tapan Mitra. Ranking investment projects. *Economic Theory*, 22:469–494, 2003. ISSN 0938-2259.

Josef Hadar and William R. Russell. Rules for ordering uncertain prospects. *The American Economic Review*, 59(1):pp. 25–34, 1969.

Giora Hanoch and Haim Levy. The efficiency analysis of choices involving risk. *The Review of Economic Studies*, 36(3):pp. 335–346, 1969.

Sergiu Hart. Comparing risks by acceptance and rejection. *Journal of Political Economy*, 119(4): pp. 617–638, 2011.

Ulrich Homm and Christian Pigorsch. An operational interpretation and existence of the Aumann-Serrano index of riskiness. *Economics Letters*, 114(3):265 – 267, 2012.

Ohad Kadan and Fang Liu. Performance evaluation with high moments and disaster risk. *Journal of Financial Economics*, Forthcoming.

S. Kullback and R.A. Leibler. On information and sufficiency. *The Annals of Mathematical Statistics*, 22(1):79–86, 1951.

David Laibson. Golden eggs and hyperbolic discounting. *The Quarterly Journal of Economics*, 112 (2):443–478, 1997.

Irving H. LaValle. On cash equivalents and information evaluation in decisions under uncertainty: Part i: Basic theory. *Journal of the American Statistical Association*, 63(321):pp. 252–276, 1968.

Moshe Leshno and Yishay Spector. An elementary proof of blackwell's theorem. *Mathematical Social Sciences*, 25(1):95 – 98, 1992.

Michael Luca. Reviews, reputation, and revenue: The case of yelp. com. Technical report, Harvard Business School, 2011.

Mark Machina and Michael Rothschild. Risk. in the new palgrave dictionary of economics, edited by steven n. durlauf and lawrence e. blume, 2008.

Carl J. Norstrøm. A sufficient condition for a unique nonnegative internal rate of return. *Journal of Financial and Quantitative Analysis*, 7(03):1835–1839, 1972.

John W. Pratt. Risk aversion in the small and in the large. *Econometrica*, 32(1/2):pp. 122–136, 1964.

John W. Pratt and John S. Hammond. Evaluating and comparing projects: Simple detection of false alarms. *The Journal of Finance*, 34(5):1231–1242, 1979.

Michael Rothschild and Joseph E. Stiglitz. Increasing risk: I. a definition. *Journal of Economic Theory*, 2(3):225–243, September 1970.

Amnon Schreiber. Economic indices of absolute and relative riskiness. *Economic Theory*, pages 1–23, 2013.

Ralph Turvey. Present value versus internal rate of return-an essay in the theory of the third best. *The Economic Journal*, 73(289):93–98, 1963.

Marc Willinger. Risk aversion and the value of information. *Journal of Risk and Insurance*, pages 104–112, 1989.

Menahem E. Yaari. Some remarks on measures of risk aversion and on their uses. *Journal of Economic Theory*, 1(3):315–329, 1969.

10 Appendix - Proofs

10.1 Lemmata 4 and 7

I provide a proof of Lemma 4. The proof of Lemma 7 is analogous.

Proof. For some u and w, $R_Q(u,w) = c$, $0 < c < \infty$. Hence for some small positive ϵ', for every $0 < \epsilon < \epsilon'$ there exists gambles in $\mathcal{G}_\epsilon$ with Q-riskiness greater than $\frac{c}{2}$. Since multiplying by $0 < \lambda < 1$ keeps the gambles in $\mathcal{G}_\epsilon$, there are gambles with any level of Q-riskiness lower than $\frac{c}{2}$ in $\mathcal{G}_\epsilon$. Since for $\lambda > 1$, $\epsilon < \epsilon'$ implies that $\frac{\epsilon}{\lambda} < \epsilon'$, the same applies to $\mathcal{G}_{\frac{\epsilon}{\lambda}}$. But, using homogeneity, this means that $\mathcal{G}_\epsilon$ includes gambles with any level of Q-riskiness lower than $\lambda \cdot \frac{c}{2}$. Since $\lambda > 1$ was arbitrary, the proof is complete. $\square$

10.2 Theorem 2

Proof. In one direction, $\rho_u(w) > \rho_v(w')$ implies that $(u,w) \gtrdot (v,w')$ [Yaari, 1969], so Lemma 1 implies that $A_Q(u,w) \geq A_Q(v,w')$.

To see that $A_Q(u,w) \neq A_Q(v,w')$, define $c := \left(\frac{\rho_u(w)+\rho_v(w')}{2}\right)^{-1}$. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of gambles such that $g_n \in \mathcal{G}_{\frac{1}{n}}$ and $Q^{AS}(g_n) = c$. For a small $\delta > 0$ let $h_n = (1+\delta)g_n$. By Theorem 1, for large values of n, g_n and h_n will be rejected by u at w and accepted by v at w', so

$$S_Q(v,w') \geq R_Q(v,w') \geq (1+\delta) \cdot S_Q(u,w) > S_Q(u,w) \geq R_Q(u,w),$$

where the strict inequality follows from the fact that $S_Q(u,w) > 0$ by the local consistency axiom, the first and the last ineqality follow from the local consistency axiom, and the second inequality follows from the definitions of R_Q and S_Q and homogeneity, by the properties of g_n and h_n. This proves that $A_Q(u,w) > A_Q(v,w')$.

In the other direction, if $A_Q(u,w) > A_Q(v,w')$ then, by Lemma 4, there exists a sequence of gambles $\{k_n\}_{n=1}^{\infty}$ such that $k_n \in \mathcal{G}_{\frac{1}{n}}$ and $Q(k_n) = c'$, where $c' := \left(\frac{A_Q(u,w)+A_Q(v,w')}{2}\right)^{-1}$. For a small $\delta > 0$ let $l_n = (1+\delta)g_n$. A similar argument shows that

$$S_{Q^{AS}}(v,w') = R_{Q^{AS}}(v,w') \geq (1+\delta) \cdot S_{Q^{AS}}(u,w) > S_{Q^{AS}}(u,w) = R_{Q^{AS}}(u,w),$$

where the strict inequality follows from the fact that $S_{Q^{AS}}(u,w) > 0$ by Lemma 2, the equalities follow from the same lemma, and the weak inequality follows from the definitions of $R_{Q^{AS}}$ and $S_{Q^{AS}}$ and the homogeneity of Q^{AS}, by the properties of g_n and l_n. Using Lemma 2 once again, this implies that $\rho_u(w) > \rho_v(w')$. $\square$

10.3 Theorem 3

Proof. (i) I first show that for every $a > 0$ any combination of the form $Q_a(g) := Q^{FH}(g) + a \cdot \left|Q^{FH}(g) - Q^{AS}(g)\right|$ is an index of riskiness for which the local aversion equals the local aversion to Q^{FH}. The reason is that for small supports, the second element in the definition is vanishingly small by Inequality 3.0.3, and so Q_a and Q^{FH} should be close.

Fix $a > 0$. First, note that

$$\forall g \in \mathcal{G} \; 0 < Q^{FH}(g) \leq Q^{FH}(g) + a \cdot \left| Q^{FH}(g) - Q^{AS}(g) \right|,$$

so $Q_a(g) \in \mathbb{R}_+$. Additionally, for every $\delta > 0$ there exists $\epsilon > 0$ small enough such that for every $g \in \mathcal{G}_\epsilon$,

$$Q^{FH}(g) \leq Q^{FH}(g) + a \cdot \left| Q^{FH}(g) - Q^{AS}(g) \right| \leq Q^{FH}(g) + \delta. \tag{10.3.1}$$

Inequality 10.3.1 stems from the small support combined with Inequality 3.0.3. It tells us that the local aversion to Q_a-riskiness cannot be different from $A_{Q^{\text{FH}}}$ which equals $A_{Q^{\text{AS}}}$ according to Theorem 1. The proof of (i) is completed by recalling that $Q^{FH} \neq Q^{AS}$ and that both indices are homogenous and locally consistent.[38]

(ii) Follows from example 1. □

10.4 Lemma 6 and Theorem 5

Lemma 11. $g \in \mathcal{H} \iff \log(1+g) \in \mathcal{G}$.

Proof. In one direction, $g \in \mathcal{H} \Rightarrow g \in \mathcal{G}$ and $Q^{FH}(g) < 1$. Since $Q^{FH}(g) \geq L(g)$ it follows that $\log(1+g)$ is well-defined. As $g \in \mathcal{G}$, it assumes a negative value with positive probability and therefore so does $\log(1+g)$. Finally, $Q^{FH}(g) < 1$ implies that $E\left[\log(1+g)\right] > 0$. Hence, $\log(1+g) \in \mathcal{G}$.

In the other direction, if $\log(1+g) \in \mathcal{G}$ we have that $\log(1+g)$ assumes a negative value with positive probability and therefore so does g. In addition, we have $\sum p_i \log(1+g_i) > 0$. Hence, by Fact 2, $g \in \mathcal{H}$. □

Lemma 12. *For every* $g \in \mathcal{H}$ the *equation* $E\left[(1+g)^{-\frac{1}{S}}\right] = 1$ *has a unique positive solution.*

Proof. Note that for every $g \in \mathcal{H}$ and $S > 0$, we have $E\left[(1+g)^{-\frac{1}{S}}\right] = E\left[e^{-\frac{\log(1+g)}{S}}\right]$. Consequentially, Lemma 11 and Theorem A in AS imply that the unique positive solution for the equation is $S(g) = Q^{AS}\left(\log(1+g)\right)$. □

Claim 1. For all $g \in \mathcal{H}$, If $u \in \mathcal{U}$ has a constant RRA then $\varrho_u(w) - 1 < \frac{1}{S(g)}$ if and only if $E\left[u(w + wg)\right] > u(w) \; \forall w > 0$.

Proof. As positive affine transformations of the utility function do not change acceptance and rejection, it is enough to treat functions of the form $u(w) = -w^{1-\alpha}$. Now observe that:

[38] An alternative proof could use indices of the form:$(Q^{FH})^{\alpha}(Q^{AS})^{1-\alpha}$, $\alpha \in (0,1)$. This form may prove to be useful in empirical work, since it enables some flexibility in the estimation. In addition, it allows us to put some weight on the FH measure that "punishes" heavily for rare disasters [Barro, 2006].

$$E\left[u(w+wg)\right] > u(w) \iff E\left[-w^{1-\alpha}(1+g)^{1-\alpha}\right] > -w^{1-\alpha} \iff E\left[(1+g)^{1-\alpha}\right] < 1 \iff$$

$$\iff E\left[e^{(1-\alpha)\cdot\log(1+g)}\right] < 1 \iff Q^{AS}\left(\log(1+g)\right) < \frac{1}{\alpha-1} \iff \alpha - 1 < \frac{1}{S(g)}.$$

□

Lemma 13. *For every $u, v \in \mathcal{U}$, if $\inf_{x} \varrho_u(x) \geq \sup_{x'} \varrho_v(x')$ then for every w, if u accepts g at w so does v.*

Proof. Without loss of generality, assume that $v(w) = u(w) = 0$ and that $v'(w) = u'(w) = 1$. For every $t > 1$

$$\log v'(tw) = \log v'(tw) - \log v'(w) = \int_1^t \frac{\partial \log v'(sw)}{\partial s} ds = \int_1^t w\frac{v''(sw)}{v'(sw)} ds =$$

$$= \int_1^t \frac{1}{s} \cdot \left(sw\frac{v''(sw)}{v'(sw)}\right) ds \geq \int_1^t \frac{1}{s} \cdot \left(sw\frac{u''(sw)}{u'(sw)}\right) ds = \log u'(tw)$$

$$\log v'(\frac{w}{t}) = \log v'(\frac{w}{t}) - \log v'(w) = \int_1^t \frac{\partial \log v'(\frac{w}{s})}{\partial s} ds = \int_1^t -\frac{w}{s^2}\frac{v''(\frac{w}{s})}{v'(\frac{w}{s})} ds =$$

$$= \int_1^t \frac{1}{s} \cdot \left(-\frac{w}{s}\frac{v''(\frac{w}{s})}{v'(\frac{w}{s})}\right) ds \leq \int_1^t \frac{1}{s} \cdot \left(-\frac{w}{s}\frac{u''(\frac{w}{s})}{u'(\frac{w}{s})}\right) ds = \log u'(\frac{w}{t})$$

This means that for every $t > 0$:

$$v(tw) = v(tw) - v(w) = \int_1^t wv'(sw)ds \geq \int_1^t wu'(sw)ds = u(tw)$$

And so, if $E\left[u(w+wg)\right] > u(w) = 0$ then necessarily $E\left[v(w+wg)\right] > v(w) = 0$ as $E\left[v(w+wg)\right] \geq E\left[u(w+wg)\right]$. □

Lemma 14. For every $u \in \mathcal{U}$ and every $w > 0$, $R_S(u,w) = S_S(u,w)$ and $A_S(u,w) = \varrho_u(w) - 1$.

The proof of Lemma 14 is analogous to the proof of Lemma 2 and is therefore omitted. Recalling that the CRRA utility function with parameter α is often expressed as

$$-w^{1-\alpha} = -w^{-(\alpha-1)},$$

this transformation of $\varrho_u(\cdot)$ seems particularly natural.

Proof. (Of Lemma 6). (i) follows from Lemma 12. Turning to (ii), first observe that for every $\alpha > 0$ $S\left((1+g)^{\alpha}-1\right) = Q^{AS}\left(\log(1+g)^{\alpha}\right) = Q^{AS}\left(\alpha \cdot \log(1+g)\right) = \alpha \cdot Q^{AS}\left(\log(1+g)\right) = \alpha \cdot S(g)$, so S satisfies Scaling. By Lemma 14, $A_S(u,w) = \varrho_u(w) - 1$ (and S satisfies local consistency).

To see that S satisfies global consistency, observe that the fact that A_S is ordinally equivalent to ϱ implies that if $v \succ u$ then there exist $\lambda \geq 1$ with $\inf_w \varrho_v(w) \geq \lambda \geq \sup_{w'} \varrho_u(w')$. Therefore, by Lemma 13 if v accepts g at w so does an agent with a CRRA utility function with RRA equals λ. Furthermore, by Claim 1, if $S(h) < S(g)$ this agent will accept h at any wealth level. Applying Lemma 13 again implies that u accepts h at w.

For uniqueness, assume that $\hat{Q}$ satisfies the requirements. By Lemma 11 $\hat{P}(g) := \hat{Q}(e^g - 1)$ is an index of riskiness $\hat{P} : \mathcal{G} \to \mathbb{R}_+$. For every $\alpha > 0$, we have $\hat{P}(\alpha g) = \hat{Q}(e^{\alpha g} - 1) = \hat{Q}\left((1 + e^g - 1)^{\alpha} - 1\right) = \alpha \cdot \hat{Q}(e^g - 1) = \alpha \cdot \hat{P}(g)$, so $\hat{P}$ satisfies homogeneity. From global consistency and the fact that $A_{\hat{Q}}$ is ordinally equivalent to ϱ one gets that S and $\hat{Q}$ order lotteries in the same manner (using CRRA functions). Hence, $\hat{P}$ and Q^{AS} also agree on the order of lotteries. Since both $\hat{P}$ and Q^{AS} are homogenous, we have that $\hat{P} = \lambda \cdot Q^{AS}$ for some $\lambda > 0$. This in turn, implies that $\hat{Q} = \lambda \cdot S$, for some $\lambda > 0$. □

The theorem follows from the previous lemmata and the observation that A_Q is ordinally equivalent to ϱ if local consistency and scaling are satisfied.

10.5 Lemma 8 and Theorem 6

Lemma 15. *Let $c = (x_n, t_n)_{n=1}^N$ be an investment cashflow. If $r_k(s) < r_j(s)$ for all $s \in [t_1, t_N]$ then, for all t, $\sum_n e^{-\int_t^{t_n} r_k(s)ds} x_n \leq 0$ implies that $\sum_n e^{-\int_t^{t_n} r_j(s)ds} x_n < 0$.*

Proof. Denote by n^* the highest index with $x_n < 0$. Then

$$\sum_n e^{-\int_t^{t_n} r_k(s)ds} x_n = \sum_{n \leq n^*} e^{-\int_t^{t_n} r_k(s)ds} x_n + \sum_{n > n^*} e^{-\int_t^{t_n} r_k(s)ds} x_n = -\sum_{n \leq n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n| + \sum_{n > n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n|, \tag{10.5.1}$$

and

$$-\sum_{n \leq n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n| + \sum_{n > n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n| \leq 0 \iff e^{\int_t^{t_{n^*}} r_k(s)ds} \cdot \left(-\sum_{n \leq n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n| + \sum_{n > n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n|\right) \leq 0, \tag{10.5.2}$$

and similar statements hold when r_k is replaced with r_j. But,

$$e^{\int_t^{t_{n^*}} r_k(s)ds} \cdot \left(-\sum_{n \leq n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n| + \sum_{n > n^*} e^{-\int_t^{t_n} r_k(s)ds} |x_n|\right) = -\sum_{n \leq n^*} e^{-\int_{t_{n*}}^{t_n} r_k(s)ds} |x_n| + \sum_{n > n^*} e^{-\int_{t_{n*}}^{t_n} r_k(s)ds} |x_n| >$$

$$-\sum_{n \leq n^*} e^{-\int_{t_{n*}}^{t_n} r_j(s)ds} |x_n| + \sum_{n > n^*} e^{-\int_{t_{n*}}^{t_n} r_j(s)ds} |x_n|$$

as positives are only multiplied by smaller numbers and negatives are multiplied by greater (positive) numbers. ☐

Lemma 16. *If $c = (x_n, t_n)_{n=1}^{N}$ is an investment cashflow then there exists a unique positive number r such that $\sum_n e^{-rt_n} x_n = 0$. Furthermore, if $\tilde{r}(t) > r > \hat{r}(t)$ for all $t \in [t_1, t_N]$, then the NPV of c is negative using $\tilde{r}$, and is positive using $\hat{r}$.*

For general cashflows, multiple solutions to the equation defining the internal rate of return may exit. Interestingly, both Arrow and Pratt took interest in finding simple conditions that would rule out this possibility [Arrow and Levhari, 1969, Pratt and Hammond, 1979]. Lemma 16 generalizes the result of Norstrøm [1972] who had shown that investment cashflows have a unique positive IRR in the discrete setting.

Proof. Define the function $f(\alpha) := \sum_n e^{-\alpha t_n} x_n$. Observe that $f(\cdot)$ is continuous, and satisfies $f(0) > 0$ and $f(\alpha) < 0$ for large values of α. Hence, continuity implies the existence of a solution. Lemma 15 implies its uniqueness, and the second part of the claim. ☐

Proof. (of the theorem) Using the smoothness of r_i and r_j, one can see that if $r_i(t) > r_j(t')$, then there exist environments of t and t' such that for every s and s' in these environments $r_i(s) > \mu > \nu > r_j(s')$ for some positive numbers μ and ν. Using homogeneity, translation invariance, and Lemma 15, an argument analogous to one proving Theorem 2, proves that $A_T(i,t) \geq \frac{\mu}{\nu} \cdot A_T(j,t') > A_T(j,t')$.

In the other direction, assume that $r_i(t) = r_j(t')$ and, by way of contradiction (and without loss of generality), $A_T(i,t) > A_T(j,t')$. Using Lemma 7, let $\{c_k\}$ be a sequence of cashflows with $T(c_k, t) = \frac{1}{2} \cdot \left(\frac{1}{A_T(i,t)} + \frac{1}{A_T(j,t')} \right)$ and $c_k \in \mathcal{C}_{t,\frac{1}{k}}$, and denote by $\{h_k\}$ the cashflow with the same payoffs, but with each t_n replaced by $t + (1-\delta)(t_n - t)$ for small $\delta > 0$. For all $\epsilon > 0$, if there exists a sub-sequence of cashflows in $\{h_k\}$ with IRR is greater than $(1+\epsilon) \cdot r_i(t)$, by Lemma 16 it must be the case that they are almost always accepted by i, in contradiction to the definition of $A_T(i,t)$. Additionally, if there exists a sub-sequence of (translated) cashflows in $\{c_k + (t'-t)\}$ with IRR smaller than $(1-\epsilon) \cdot r_j(t')$,[39] it must be the case that they are almost always rejected by j, in violation of local consistency. Together, we get that

$$\liminf_{k \to \infty} \alpha(c_k) \geq r_j(t') = r_i(t) \geq \limsup_{k \to \infty} \alpha(h_k).$$

But

$$(1-\delta) \limsup_{k \to \infty} \alpha(h_k) = \limsup_{k \to \infty} \alpha(c_k),$$

[39] I abuse notation slightly by denoting the translated cashflow as $c_k + (t'-t)$, but this should cause no confusion.

So, using the fact that $\limsup_{k\to\infty}\alpha(c_k)$ is positive and finite,[40] these equations imply that

$$\limsup_{k\to\infty}\alpha(c_k) < \liminf_{k\to\infty}\alpha(c_k),$$

a contradiction. The proof of the Lemma is completely analogous. □

10.6 Theorem 7

Proof. To prove (i) I first identify one such index. The construction draws upon the findings of previous sections. First, denote by $\mathcal{C}^1$ the class of investment cashflows with $|t_N - t_1| = 1$. Restricting attention to this class of cashflows, I define a function from $\mathcal{C}^1$ to $\mathcal{G}$, the class of gambles, $\mathcal{T} : \mathcal{C}^1 \to \mathcal{G}$,

$$\mathcal{T}(c) = \left[1, \frac{e^{\frac{1}{D(c)}}}{1+e^{\frac{1}{D(c)}}}; -1, \frac{1}{1+e^{\frac{1}{D(c)}}}\right].$$

Observe that $Q^{AS}(\mathcal{T}(\cdot)) \equiv D(\cdot)$. Now, given a cashflow $c = (x_n, t_n)_{n=1}^N$, let $\alpha_c := |t_N - t_1|$. Given t, define $\hat{c}_t := \left(x_n, t + \frac{1}{\alpha_c}(t_n - t)\right)_{n=1}^N$. By construction, $\hat{c}_t$ is a member of $\mathcal{C}^1$. This allows defining a new index $Z(c,t) : \mathcal{C} \to \mathbb{R}_+$ in the following way:

$$Z(c,t) := Q^{FH}(\alpha_c \cdot \mathcal{T}(\hat{c}_t)).$$

Z is homogenous and translation invariant since Q^{FH} is homogenous, and $\mathcal{T}$ was constructed to assure these properties.

Noting that for $c \in \mathcal{C}_{t,\epsilon}$

$$|D(c,t) - Z(c,t)| = \left|Q^{AS}(\alpha_c \cdot \mathcal{T}(\hat{c}_t)) - Q^{FH}(\alpha_c \cdot \mathcal{T}(\hat{c}_t))\right| \le 2\alpha_c \le 2\epsilon,$$

one observes that the local aversion to Z is equal to the local aversion to D, and that if D is locally consistent so is Z.

D satisfies all the requirements of the theorem (proved later on) and the local aversion to D equals to r. This implies that combinations of the form $W_a(\cdot,\cdot) = Z(\cdot,\cdot) + a\,|D(\cdot,\cdot) - Z(\cdot,\cdot)|$ also satisfy the requirements of (i). To see that $D \ne Z$, it is enough to consider a cashflow c with $\alpha_c = 1$ and $D(c) = 1$. For this cashflow $Z(c,t) \approx 1.26$. The fact that the local aversion to both Z and D equals to r implies that the same holds for W_a, which completes the proof of this part.

(ii) Follows from example 8. □

Example 8. Consider $W_1(\cdot,\cdot)$ and a cashflow c with $\alpha_c = 1$ for which $D(c) = 1$. This implies that $Z(c,t) \approx 1.26$, hence $W_1(c,t) < 1.6$. Now consider another cashflow, c', with $\alpha_{c'} = 1$, which first order time dominates c and has $D(c') = \epsilon$ for a small ϵ.[41] Since $Z(c,t) \ge 1$ from the properties of

[40] The fact that $0 < \limsup_{n\to\infty}\alpha(c_n) < \infty$ follows from the fact that $\{c_n\}$ is almost always accepted by one agent and rejected by the other.

[41] This could be achieved by increasing x_N.

Q^{FH} and $\mathcal{T}$, $W_1(c',t) > 1.6$. Therefore, while c' first order time dominates c, $W_1(c,t) < Q_1(c',t)$.

10.7 Remark 2

Proof. The proof follows from Example 5 and the following examples. □

Example 9. $Q \equiv 5$ satisfies local consistency and translation invariance, but it does not satisfy homogeneity of degree 1. The local aversion to this index induces the trivial order.

Example 10. $T := t_1 - t_2$ satisfies homogeneity and translation invariance, as $t + \lambda(t_1 - t) - (t + \lambda(t_2 - t)) = \lambda(t_1 - t_2)$ and $t_1 - t_2 = (t_1 + \lambda) - (t_2 + \lambda)$. Local consistency is, however, violated. Finally, $A_T \equiv \infty$.

10.8 Lemma 9 and Theorem 8

I prove the theorem, but it is easy to see that only minor adaptations are required to prove the Lemma.

Proof. I first check that D satisfies the axioms. Homogeneity is clearly satisfied as

$$\sum_n e^{-rt_n} x_n = 0 \iff e^{rt} \sum_n e^{-rt_n} x_n = 0 \iff \sum_n e^{-r(t_n - t)} x_n = 0 \iff \sum_n e^{-\frac{r}{\lambda} \cdot \lambda(t_n - t)} x_n = 0 \ \forall t \ \forall \lambda > 0.$$

Translation invariance is also satisfied as

$$\sum_n e^{-rt_n} x_n = 0 \iff e^{rt} \sum_n e^{-rt_n} x_n = 0 \ \forall t.$$

For local consistency, I use the smoothness of $r_i(\cdot)$ to deduce that for every small $\epsilon > 0$ there exists $\delta > 0$ such that if $s \in (t - \delta, t + \delta)$ then $r_i(t) - \epsilon < r_i(s) < r_i(t) + \epsilon$. This fact, together with Lemmata 15 and 16, implies that the axiom is satisfied and that $A_T(i,s) = r_i(s)$.

Finally, to see that global consistency is satisfied, consider an agent that discounts at the constant rate ν, with $\sup r_j(t) \le \nu \le \inf r_i(t)$, where the supremum and infimum are taken on the relevant domain. Label this agent ν. Lemma 15 implies that ν accepts any cashflow accepted be j, Lemma 16 implies that he also accepts cashflows with higher IRR, and another application of Lemma 15 implies that i accepts these cashflows.

I now turn to show that the only indices that satisfy the five axioms are positive multiples of D. This is done in two steps. In the first step, I show that indices that satisfy the axioms agree with the order induced by D. Then, I show that they are also multiples of this index.

For the first step, assume by way of contradiction that there exists another index, Q, that satisfies the axioms but does not agree with D on the ordering of two cashflows at some given time points. There are three possibilities:

1. $Q(c,t) > Q(c',t')$ and $D(c,t) < D(c',t')$ for time points t and t' and cashflows c and c'.

2. $Q(c,t) > Q(c',t')$ and $D(c,t) = D(c',t')$ for time points t and t' and cashflows c and c'.
3. $Q(c,t) = Q(c',t')$ and $D(c,t) < D(c',t')$ for time points t and t' and cashflows c and c'.

There is no loss of generality in treating just the first case, since using the first degree homogeneity c may be shifted slightly in a way that would preserve the strict inequality, but break the equality in the right direction, leading to the first case. To obtain a contradiction, choose r_1 and r_2 such that

$$D(c,t) < \frac{1}{r_2} < \frac{1}{r_1} < D(c',t'),$$

and consider two agents that discount with the constant rates r_1 and r_2, and are labeled accordingly r_1 and r_2 (with a slight abuse of notation). Using Lemma 16 both r_1 and r_2 accept c and rejects c'. Together with homogeneity and translation invariance, Theorem 6 implies that $r_1 \underset{Q,t_o}{\blacktriangleleft} r_2$ for all t_o. But this means that Q violates global consistency, as r_2, the impatient agent, accepts c, the Q-delayed cashflow, but r_1 does not accept c' which is less Q-delayed. Thus, Q and D must agree on the ordering of any two cashflows at any given time point.

For the second step, choose an arbitrary cashflow $c_0 = (x_n, t_n)_{n=1}^N$, a point in time t_o, and an index that satisfies the axioms, T. For any cash flow c and time t, there exists a positive number $\lambda > 0$ such that $T\left((x_n, t_o + \lambda \cdot (t_n - t_o))_{n=1}^N, t_o\right) = T(c,t)$. The first step implies that $D\left((x_n, t_o + \lambda \cdot (t_n - t_o))_{n=1}^N, t_o\right) = D(c,t)$. But $D\left((x_n, t_o + \lambda \cdot (t_n - t_o))_{n=1}^N, t_o\right) = \lambda \cdot D(c_0, t_o)$, and also $T\left((x_n, t_o + \lambda \cdot (t_n - t_o))_{n=1}^N, t_o\right) = \lambda \cdot T(c_0, t_o)$. Altogether this means that $T(c,t) = \frac{T(c_0,t_o)}{D(c_0,t_o)} D(c,t)$ for every c. □

10.9 Theorem 9

Proof. (i) The proof is similar to the proof of Theorem 1. First, note that if $a_n = (\mu_n, \alpha_n) \in \mathcal{A}_{\frac{1}{n}}$ are accepted it must be the case that $\mu_n \underset{n\to\infty}{\longrightarrow} 0$. To see this, assume by way of contradiction that there is a sub-sequence of such transactions where the price does not converge to 0, without loss of generality $a_n = (\mu_n, \alpha_n)$. Denote $\mu = \liminf_{n\to\infty} \mu_n$. Then, there exits N such that for all $n > N$ $l_n := \left(\frac{\mu}{2}, \alpha_n\right)$ is accepted. Lemma 2 of Cabrales et al. [2012] proves that as ϵ approaches 0, so does the scale of the optimal investment $\|b^n\|$. Therefore, for ϵ small enough, $w - \frac{\mu}{2} + b_k^n$ is in a δ-environment of $w - \frac{\mu}{2} < w$ for all k, a contradiction.

For the second step, from the discussion above it follows that for ϵ small enough, $w - \mu_n + b_k^n$ is in a δ-environment of w for all k, if $a = (\mu, \alpha) \in \mathcal{A}_\epsilon$ is accepted. $\rho_u(w)$ is continuous, and so for every $\gamma > 0$ there exists a $\delta > 0$ small enough such that $x \in (w - \delta, w + \delta)$ implies $|\rho_u(x) - \rho_u(w)| < \gamma$.

For the final step, choose a small positive number η, and consider the CARA agents with absolute risk aversion coefficients $\rho_u(w) + \eta$ and $\rho_u(w) - \eta > 0$. For a small enough environment of w, I,

$$\rho_u(w) - \eta \le \inf_{x\in I} \rho_u(x) \le \sup_{x\in I} \rho_u(x) \le \rho_u(w) + \eta.$$

This, in turn, implies, using Theorem 3 of Cabrales et al. [2012] and a slightly modified version of their Theorem 2, that the local distaste for A of u with wealth w is equal to $\rho_u(w)$, and is equal to $\lim_{\epsilon\to 0^+} \inf_{a\in\mathcal{A}_\epsilon,\, a\, is\, accepted} A(a)$.

(ii) Cabrales et al. [2013] showed that $a=(\mu,\alpha)$ is accepted by an agent with log utility function if and only if $I_e(\alpha) > \log\left(\frac{w}{w-\mu}\right)$. Using a Taylor approximation yields

$$\log\left(\frac{w}{w-\mu}\right) = \log(w) - \log(w-\mu) \approx \frac{1}{w}\mu + \frac{\mu^2}{2w^2}.$$

As shown above, if $a_n=(\mu_n,\alpha_n)\in\mathcal{A}_{\frac{1}{n}}$ are accepted it must be the case that $\mu_n \underset{n\to\infty}{\longrightarrow} 0$. It is therefore the case that for n large enough (when posteriors are close to the prior), a_n is accepted by agents with log utility function if

$$J_e(a_n) = \frac{I_e(\alpha_n)}{\mu_n} > \frac{1}{w} + O(\mu_n) \underset{n\to\infty}{\longrightarrow} \frac{1}{w} = \rho_{\log}(w),$$

and rejected if

$$J_e(a_n) = \frac{I_e(\alpha_n)}{\mu_n} < \frac{1}{w} + O(\mu_n) \underset{n\to\infty}{\longrightarrow} \frac{1}{w} = \rho_{\log}(w).$$

For any $x\in\mathbb{R}_+$ $\frac{1}{x}\equiv w\in\mathbb{R}_+$ satisfies $\rho_{\log}(w)=x$, and so by properly translating the log utility function (and changing all but an environment of the baseline wealth level of the agent), one can use a "sandwich" argument of the form used above to complete the proof. □

10.10 Theorem 10

Proof. The proof uses the same techniques used above. If $\rho_u(w)>\rho_v(w')$ then there exists some $\gamma>0$ such that $\rho_u(w)>(1+\gamma)\cdot\rho_v(w')$. Following the arguments used before, for $\epsilon>0$ small enough, if u accepts $a=(\mu,\alpha)\in\mathcal{A}_\epsilon$ then v accepts $((1+\frac{\gamma}{2})\cdot\mu,\alpha)$. Together with local consistency and homogeneity this implies that the local distaste for Q-informativeness of u at w is greater than the local distaste for Q-informativeness of v at w'.

In the other direction, assume $\rho_u(w)=\rho_v(w')$, and by way of contradiction assume that the local distaste for Q-informativeness of u at w is not equal to the local distaste for Q-informativeness of v at w'. Without loss of generality, assume that the local distaste for Q-informativeness of u at w is smaller than the local distaste for Q-informativeness of v at w' . This means that there exists a sequence $\{a_n\}_{n=1}^\infty$ of information transactions, such that for every n, $a_n=(\mu_n,\alpha_n)$ satisfies (a) $a_n\in\mathcal{A}_{\frac{1}{n}}$, (b) For some small $\gamma>0$, $((1+\gamma)\cdot\mu_n,\alpha_n)$ is accepted by u at w, and (c) a_n is rejected by v at w'. But this implies that A violates local consistency, a contradiction, and so the local distaste for Q-informativeness of u at w is equal to the local distaste for Q-informativeness of v at w'. □

10.11 Theorem 11

Proof. For (i), let $\delta := \frac{1}{2}\min_i \{\min\{p_i, 1-p_i\}\}$. Define

$$B(a) = \begin{cases} A(a) & \|p - q^s\| < \delta \ \forall s \\ \frac{1}{\mu} \cdot f(\alpha) & else \end{cases}$$

for some positive f. Then B satisfies the required properties since for local transactions (ones with posteriors close to the prior) it is equal to A, and since both A and $\frac{1}{\mu}f(\alpha)$ are homogenous and changes in the price do not change the distance of the posteriors from the prior (and hence the rule that governs B). Choosing $f \equiv 1$ (or many other choices) completes the proof of (ii). □

10.12 Remark 3

Proof. Follows from following examples. □

Example 11. $Q \equiv 5$ satisfies local consistency, but it does not satisfy homogeneity of degree -1. The local distaste to this index induces the trivial order.

Example 12. $Q := \frac{1}{\mu}$ satisfies homogeneity, but for all agents the local distaste to Q is infinite.

10.13 Theorem 13

Proof. (i) From Theorem D of Aumann and Serrano [2008] it is enough to show that the conditions imply the duality axiom, which is implied by the combination of global consistency and the requirement that A_Q is ordinally equivalent to ρ.

(ii) Global consistency and the ordinal equivalence of A_Q and ϱ imply that CRRA functions may be used to order gambles. Monotonicity and continuity imply that no "ties" are created, and that no new "ties" are generated. That is, if $S(g) > S(h)$, monotonicity assures that for some small ϵ, $S(g) > S(g+\epsilon) > S(h)$, where $g+\epsilon$ represents a gamble which translates g by ϵ for any realization of g. Then, monotonicity assures that g is regarded as strictly more risky than $g+\epsilon$, assuring that g is riskier than h. If $S(g) = S(h)$, but for some other index, Q, their riskiness is different, say $Q(g) > Q(h)$, then monotonicity, continuity and the ordinal equivalence of A_Q and ϱ, imply that global consistency is violated. To see this, choose ϵ small enough such that $Q(g+\epsilon) > Q(h)$. From monotonicity $S(g+\epsilon) < S(h)$. But this situation was excluded above.

(iii) Translation invariance, global consistency and the ordinal equivalence of A_T and r imply that cashflows are ranked by individuals with constant discounting. This allows replicating the previous proof with a slight modification for the current setting. Here, if one cashflow is earlier than another, shift all investment to be ϵ earlier, and all return periods to be ϵ later, with ϵ small enough such that the order is preserved. The rest of the proof is identical.

(iv) Just in like previous parts, observe that information transactions are ranked by CARA agent, and then shift their appeal slightly using continuity and monotonicity. □

10.14 Theorem 14

Lemma 17. *Let Q be a continuous index of riskiness which satisfies bounded ratios and local consistency, then Q satisfies full image.*

Proof. I slightly modify the proof of Lemma 4. For some u and w, $R_Q(u,w) = c$, $0 < c < \infty$. Hence for some small positive ϵ', for every $0 < \epsilon < \epsilon'$ there exists gambles in $\mathcal{G}_\epsilon$ with Q-riskiness greater than $\frac{c}{2}$. Since multiplying by λ^k, $k \in \mathbb{N}$, $0 < \lambda < 1$, keeps the gambles in $\mathcal{G}_\epsilon$, there are gambles with arbitrarily small levels of Q-riskiness lower than $\frac{c}{2}$ in $\mathcal{G}_\epsilon$. Continuity assures that $\mathcal{G}_\epsilon$ includes gambles with any level of Q-riskiness lower than $\frac{c}{2}$.

Since for λ^k, $k \in \mathbb{N}$, $\lambda > 1$, $\epsilon < \epsilon'$ implies that $\frac{\epsilon}{\lambda^k} < \epsilon'$, the same applies to $\mathcal{G}_{\frac{\epsilon}{\lambda^k}}$. But, using homogeneity, this means that $\mathcal{G}_\epsilon$ includes gambles with arbitrarily large levels of Q-riskiness. Continuity assures that $\mathcal{G}_\epsilon$ includes gambles with any level of Q-riskiness. □

Proof. (of the theorem) I only prove (i). Other proofs are similar. By Theorem 13 it suffices to show that A_Q is ordinally equivalent to ρ. To prove this, I slightly modify the proof of Theorem 2 as follows.

In one direction, $\rho_u(w) > \rho_v(w')$ implies that $(u,w) \gtrdot (v,w')$, so Lemma 1 implies that $A_Q(u,w) \geq A_Q(v,w')$. To see that $A_Q(u,w) \neq A_Q(v,w')$, define $c := \left(\frac{\rho_u(w)+\rho_v(w')}{2}\right)^{-1}$. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of gambles such that $g_n \in \mathcal{G}_{\frac{1}{n}}$ and $Q^{AS}(g_n) = c$. For a $\lambda > 1$ close to 1, let $h_n = \lambda \cdot g_n$. For large values of n, g_n and h_n will be rejected by u at w and accepted by v at w', so

$$S_Q(v,w') = R_Q(v,w') \geq \delta(\lambda) \cdot S_Q(u,w) > S_Q(u,w) = R_Q(u,w),$$

where the strict inequality follows from the fact that $\delta(\lambda) > 1$ by bounded ratios and since $S_Q(u,w) > 0$ by the local consistency axiom. This proves that $A_Q(u,w) > A_Q(v,w')$.

In the other direction, if $A_Q(u,w) > A_Q(v,w')$ then, by continuity, there exists a sequence of gambles $\{k_n\}_{n=1}^{\infty}$ such that $k_n \in \mathcal{G}_{\frac{1}{n}}$ and $Q(k_n) = c'$, where $c' := \left(\frac{A_Q(u,w)+A_Q(v,w')}{2}\right)^{-1}$.

For $\lambda > 1$ close to 1, let $l_n = \lambda \cdot g_n$. A similar argument shows that

$$S_{Q^{AS}}(v,w') = R_{Q^{AS}}(v,w') \geq \lambda \cdot S_{Q^{AS}}(u,w) > S_{Q^{AS}}(u,w) = R_{Q^{AS}}(u,w),$$

where the strict inequality follows from the fact that $S_{Q^{AS}}(u,w) > 0$ by Lemma 2. Using Lemma 2 again, this implies that $\rho_u(w) > \rho_v(w')$. □