Simultaneous Auctions for Complementary Goods

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Abstract

This paper studies an environment of simultaneous, separate, first-price auctions for complementary goods. Agents observe private values of each good before making bids, and the complementarity between goods is explicitly incorporated in their utility. For simplicity, a model is presented with two first-price auctions and two bidders. We show that a monotone pure-strategy Bayesian Nash Equilibrium exists in the environment.

Keywords : simultaneous auctions, multi-object auction, first-price auction, equilibrium existence, monotone pure-strategies.

JEL Classification : D44, C72

1 Introduction

This paper studies an environment of simultaneous, separate, first-price auctions for complementary goods. Agents observe private values of each good before making bids, and the complementarity between goods is explicitly incorporated in their utility. For simplicity, a model is presented with two first-price auctions and two bidders. We show that a monotone pure-strategy Bayesian Nash Equilibrium exists in the environment. This model, in which the auction mechanism has an exogenous (albeit widely used) form and in which sellers do not interact strategically with one another, is intended to provide a tractable framework for applied research focusing on competition among buyers. Each buyer is trying to acquire a portfolio of objects that must be purchased by auction from several unrelated sellers. EBay's various sealed-bid auctions under same collectibles category which are run by different sellers with similar closing time can be an example of this environment. Also, less explicit, but equally compelling, auctions for acquisition of firms manufacturing a final good and an intermediate good input can be another example. Here, a buyer intends to create a vertically integrated firm and the

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two acquired firms can be viewed as complements when they are merged.

There are several branches of existing auction research that are similar to, but distinct from, what is being done here. Let us comment briefly on those contributions, and on how the present research is distinguished from them. A literature on multi-unit auctions, including Bresky (1999) and McAdams (2003), studies sale of multiple identical goods in an auction with specific pricing rules. As modeled, multiple identical objects resemble substitutable distinct objects rather than complementary ones modeled here. Package auctions(see Bernheim and Whinston (1986) and Ausubel and Milgrom (2002)) model multiple goods without assuming substitutability only. They suppose that an explicit bid can be offered for a package of objects, which is not possible when various objects are auctioned by distinct and unrelated sellers.

There are a few prior research contributions on simultaneous auctions, including Krishna and Rosenthal (1996) and Bikhchandani (1999). They either assume complete information on bidders' valuations or employ a single parameter for private value signals of multiple objects. However, such parameterization is too rigid to model heterogeneity of bidders' preferences on each item and complementarity. In this paper, we relax those restrictions so that the environment in which each bidder receives private independent signals for each heterogeneous object is permitted.

We follow Reny (2011) to prove the existence of a monotone pure-strategy Bayesian Nash equilibrium. Along with basic assumptions on the Bayesian games(G.1-G.6, p.509 Reny (2011)), *weak single crossing property* and *weak quasisupermodularity* are sufficient for the existence theorem in his paper.[1]

The remainder of this paper is as follows. In the next section, we present an environment of simultaneous first-price auctions for complementary goods in detail, and proposes a theorem for the existence of a monotone pure-strategy Bayesian Nash equilibrium. Section 3 provides the proof, and section 4 discusses an extension of the study to a continuum bid space setting.

2 Simultaneous first-price auctions for complementary goods

In this section, we define simultaneous first-price auctions for complementary goods with 2 non-identical objects and 2 bidders. At the end of the section, we propose a theorem for the existence of a monotone pure-strategy Bayesian Nash equilibrium.

2.1 Environment

Bidder $i \in \{1, 2\}$, $i \neq j$; i's opponent is denoted by j.

Object $k \in \{1, 2\}$, Values of objects(or bidder's type) are represented by vector $x_i = (x_{i1}, x_{i2})$. $x_{ik} \sim F(\cdot)$ and $x_{ik} \in X = [0, 1]$. Let $F(\cdot)$ be atomless and common knowledge among bidders ; x_{ik} is a random variable with a cumulative

[1] Reny (2011), Proposition 4.4(i) and Theorem 4.1

distribution function $F(\cdot)$, which has a support X and a probability distribution function $f : X \to [0,1]$. Each object k is distributed *i.i.d.* Therefore, a joint cumulative distribution function and a joint probability distribution function for two objects $(x_{i1},\, x_{i2})$ are given by $G(x_{i1},\, x_{i2}) = F(x_{i1}) \times F(x_{i2})$, and $g(x_{i1},\, x_{i2}) = f(x_{i1}) \times f(x_{i2})$, respectively.

Bid $b_i = (b_{i1},\, b_{i2}) \in A = \{0, \frac{1}{n}, \frac{2}{n}, ..., \bar{u}\} \times \{0, \frac{1}{n}, \frac{2}{n}, ..., \bar{u}\}$, where $n \in \mathbb{Z}_+$ and $\bar{u} \in \mathbb{R}_+$. Each bidder i chooses bids b_{i1} and b_{i2} simultaneously from a two dimensional bid space A. A is a finite subset of $\mathbb{R}^2$. Therefore, A, the action space of this game is a Euclidean lattice with a coordinatewise partial order.

Utility Bidder i has a quasilinear ex-post utility $U_i(b_i,\, b_j,\, x_i,\, x_j)$.

$$U_i(b_i,\, b_j,\, x_i,\, x_j) = \begin{cases} w(x_{i1}, x_{i2}) - (b_{i1} + b_{i2}) & \text{if } b_{i1} > b_{j1} \text{ and } b_{i2} > b_{j2} \text{ , where } i \neq j \\ u(x_{i1}) - b_{i1} & \text{if } b_{i1} > b_{j1} \text{ and } b_{i2} < b_{j2} \\ u(x_{i2}) - b_{i2} & \text{if } b_{i1} < b_{j1} \text{ and } b_{i2} > b_{j2} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note that when ties occur, the object is allocated randomly to one of the two bidders with probability $\frac{1}{2}$. The ex-post utility U_i can be interpreted as follows. First, an agent realizes his utility from winning objects through a function $u : \mathbb{R} \to \mathbb{R}$ when he wins a single item(e.g. identity function), and $w : \mathbb{R}^2 \to \mathbb{R}$ when he wins both items. $w(x_{i1}, x_{i2})$ transforms the independent values of goods to utility that counts a synergy effect between two objects(e.g. $w(x_{i1}, x_{i2}) = x_{i1} + x_{i2} + \alpha$, where $\alpha > 0$). And then, the bidding values for the winning objects are subtracted from $u(\cdot)$ or $w(\cdot, \cdot)$. Additional assumptions on $u(\cdot)$ and $w(\cdot, \cdot)$ are listed below.

Let $\geq$ be a coordinatewise partial order on the values of objects and bids. For example, when $x_i, y_i \in \mathbb{R}^2$, $[x_i \geq y_i \Leftrightarrow (x_{i1} \geq y_{i1})$ and $(x_{i2} \geq y_{i2})]$.

(A1) $u(\cdot)$ and $w(\cdot, \cdot)$ are weakly increasing in their arguments.

(A2) Let λ be the complementary value from winning both goods. $\lambda(x_{i1}, x_{i2}) = w(x_{i1}, x_{i2}) - [u(x_{i1}) + u(x_{i2})] \geq 0$.
; This assumption implies that regardless of the values of objects, agents realize that utility from winning both objects is greater than or equal to sum of utilities of each good. (e.g. Goods are complementary, Acquiring all bike shops in New York City yields monopolistic positioning benefit in the business)

(A3)$[x_i \geq y_i \Rightarrow \lambda(x_i) \geq \lambda(y_i)]$. That is, the complementary value λ is weakly increasing in $x_i = (x_{i1}, x_{i2})$.

In summary, bidders expect extra synergy by obtaining both items, and the value of synergy is weakly increasing with the values of obtained goods.

In addition, since it is reasonable not to bid more than the maximum of utility that an agent can earn, we assign $w(1,1)$ to the maximum bid on one good, $\bar{u}$.

2.2 Monotone pure-strategy Bayesian Nash equilibrium

A *pure-strategy* for bidder i is a function $s_i : X^2 \to A$, and it is *monotone* if $x'_i \geq x_i$ implies $s_i(x'_i) \geq s_i(x_i)$, for all $x'_i, x_i \in X^2$. Let S_i be a set of monotone pure strategies of i. Based on the environment in section 2.1, bidder i's expected utility is written as follows.

Let μ^{s_j} be an opponent's bids distribution constructed by the opponent's monotone pure-strategy $s_j(x_j)$ and the type distribution $G(x_j)$. Define three probabilities in terms of winning cases. $P_1^{s_j}(b_i)$ and $P_2^{s_j}(b_i)$ denote probabilities for winning object 1 *only* and object 2 *only*, respectively, when the opponent plays s_j. $P_3^{s_j}(b_i)$ indicates a probability that bidder i wins both objects with b_i.

$$\begin{aligned}
P_1^{s_j}(b_i) &= \mu^{s_j}([0,b_{i1}) \times (b_{i2},\bar{u}]) + \frac{1}{2}\mu^{s_j}([0,b_{i1}) \times [b_{i2},b_{i2}]) \\
&\quad + \frac{1}{2}\mu^{s_j}([b_{i1},b_{i1}] \times (b_{i2},\bar{u}]) + \frac{1}{4}\mu^{s_j}([b_{i1},b_{i1}] \times [b_{i2},b_{i2}]);^{2} \\
P_2^{s_j}(b_i) &= \mu^{s_j}((b_{i1},\bar{u}] \times [0,b_{i2})) + \frac{1}{2}\mu^{s_j}([b_{i1},b_{i1}] \times [0,b_{i2})) \\
&\quad + \frac{1}{2}\mu^{s_j}((b_{i1},\bar{u}] \times [b_{i2},b_{i2}]) + \frac{1}{4}\mu^{s_j}([b_{i1},b_{i1}] \times [b_{i2},b_{i2}]); \\
P_3^{s_j}(b_i) &= \mu^{s_j}([0,b_{i1}) \times [0,b_{i2})) + \frac{1}{2}\mu^{s_j}([b_{i1},b_{i1}] \times [0,b_{i2})) \\
&\quad + \frac{1}{2}\mu^{s_j}([0,b_{i1}) \times [b_{i2},b_{i2}]) + \frac{1}{4}\mu^{s_j}([b_{i1},b_{i1}] \times [b_{i2},b_{i2}]).
\end{aligned} \tag{2}$$

Then, if an agent with type x_i bids b_i, the bidder i's expected utility is

$$V_i(b_i,x_i,s_j) = P_3^{s_j}(b_i)[w(x_{i1},x_{i2}) - (b_{i1}+b_{i2})] + P_1^{s_j}(b_i)[u(x_{i1}) - b_{i1}] + P_2^{s_j}(b_i)[u(x_{i2}) - b_{i2}]. \tag{3}$$

To make proofs tractable in section 3, we modify the representation of the expected utility. First, we define $Q_1^{s_j}, Q_2^{s_j}$ and $Q_3^{s_j}$ as follows.

$$\begin{aligned}
Q_1^{s_j}(b_{i1}) &= \mu^{s_j}([0,b_{i1}) \times [0,\bar{u}]) + \frac{1}{2}\mu^{s_j}([b_{i1},b_{i1}] \times [0,\bar{u}]); \\
Q_2^{s_j}(b_{i2}) &= \mu^{s_j}([0,\bar{u}] \times [0,b_{i2})) + \frac{1}{2}\mu^{s_j}([0,\bar{u}] \times [b_{i2},b_{i2}]); \\
Q_3^{s_j}(b_{i1},b_{i2}) &= P_3^{s_j}(b_i).
\end{aligned} \tag{4}$$

$Q_1^{s_j}(b_{i1})$ denotes a probability that bidder i with a bid b_{i1} wins the object 1 regardless of i's acquiring object 2. Similarly, $Q_2^{s_j}(b_{i2})$ describes a probability that bidder i with a bid b_{i2} wins the object 2. Note that $Q_1^{s_j}$ only depends on b_{i1} and is weakly increasing in b_{i1}, and $Q_2^{s_j}$ only depends on b_{i2} and is weakly increasing in b_{i2}. $Q_3^{s_j}$, the probability of winning both objects is weakly increasing in (b_{i1}, b_{i2}) with the coordinatewise partial order.

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[2] $\mu^{s_j}([0,b_{i1}) \times (b_{i2},\bar{u}]) = \text{probability}(0 \leq b_{j1} < b_{i1},\ b_{i2} < b_{j2} \leq \bar{u})$. The terms with coefficients $\frac{1}{2}$ and $\frac{1}{4}$ count the probabilities of events that ties occur for one and both of the two objects, respectively.

The redefined winning probability functions (4) imply $[P_3^{s_j}(b_i) = Q_3^{s_j}(b_i),\ P_1^{s_j}(b_i) = Q_1^{s_j}(b_{i1}) - Q_3^{s_j}(b_i),\ P_2^{s_j}(b_i) = Q_2^{s_j}(b_{i2}) - Q_3^{s_j}(b_i)]$. Substituting these to the expected utility function (3), we have

$$
\begin{aligned}
V_i(b_i, x_i, s_j) &= Q_1^{s_j}(b_{i1})[u(x_{i1}) - b_{i1}] + Q_2^{s_j}(b_{i2})[u(x_{i2}) - b_{i2}] \\
&\quad + Q_3^{s_j}(b_{i1}, b_{i2})[w(x_{i1}, x_{i2}) - (b_{i1} + b_{i2}) - (u(x_{i1}) - b_{i1}) - (u(x_{i2}) - b_{i2})] \\
&= Q_1^{s_j}(b_{i1})[u(x_{i1}) - b_{i1}] + Q_2^{s_j}(b_{i2})[u(x_{i2}) - b_{i2}] \qquad (5) \\
&\quad + Q_3^{s_j}(b_{i1}, b_{i2})[w(x_{i1}, x_{i2}) - u(x_{i1}) - u(x_{i2})] \\
&= Q_1^{s_j}(b_{i1})[u(x_{i1}) - b_{i1}] + Q_2^{s_j}(b_{i2})[u(x_{i2}) - b_{i2}] + Q_3^{s_j}(b_{i1}, b_{i2})\lambda(x_i).
\end{aligned}
$$

Under the auction environment in section 2.1, $(s_1, s_2) \in S_1 \times S_2$ is a *monotone pure-strategy Bayesian Nash equilibrium*, if for every bidder $i \in \{1, 2\}$ and every $x_i \in X^2$,

$$V_i(s_i(x_i), x_i, s_j) \geq V_i(b_i, x_i, s_j) \tag{6}$$

for all $b_i \in A$.

Theorem 1. *A monotone pure-strategy Bayesian Nash equilibrium exists in simultaneous first-price auctions for complementary goods, which are specified in section 2.1.*

3 Proof of Theorem 1

Under the auction environment in section 2.1, we have an atomless cumulative distribution function for object values with a two-dimensional compact support. Therefore, by propositon 3.1 in Reny (2011), assumptions G.1-G.5 on the Bayesian game (p.509 Reny (2011)) are satisfied. The assumption G.6 is also satisfied by the fact that the ex-post utility U_i is bounded, jointly measurable and continuous in $b_i \in A$ for every $x_i \in X^2$. With a finite action space A definied in section 2.1, the action space is a lattice. Then, by proposition 4.4(i) in Reny (2011), if the interim utility $V_i(b_i, x_i, s_j)$ satisfies *weak single crossing property* and *weak quasisupermodularity*, **Theorem 1** is proved. **Lemma 1** and **Lemma 2** in the following sections show that the two conditions are satisfied in the auction environment of section 2.1.

3.1 Weak single crossing property of $V_i(b_i, x_i, s_j)$

Definition 1. $V_i(b_i, x_i, s_j)$ satisfies *weak single crossing* if for all monotone pure strategies s_j of the opponent, for all $b'_i \geq b_i$, and for all $x'_i \geq x_i$,

$$V_i(b'_i, x_i, s_j) \geq V_i(b_i, x_i, s_j) \text{ implies } V_i(b'_i, x'_i, s_j) \geq V_i(b_i, x'_i, s_j). \tag{7}$$

Let $b'_i = (b_{i1}^h, b_{i2}^h)$, $b_i = (b_{i1}^l, b_{i2}^l)$, $x'_i = (x_{i1}^h, x_{i2}^h)$ and $x_i = (x_{i1}^l, x_{i2}^l)$, where $b_{ik}^h \geq b_{ik}^l$ and $x_{ik}^h \geq x_{ik}^l$ for $k \in \{1, 2\}$.

Lemma 1. *With b_i, b'_i, x_i and x'_i, (7) holds. That is,*

$$V_i(b_{i1}^h, b_{i2}^h, x_i, s_j) - V_i(b_{i1}^l, b_{i2}^l, x_i, s_j) \geq 0 \tag{8}$$

implies

$$V_i(b_{i1}^h, b_{i2}^h, x_i', s_j) - V_i(b_{i1}^l, b_{i2}^l, x_i', s_j) \geq 0. \tag{9}$$

Proof. By (5), (8) gives

$$\begin{aligned} & Q_1^{s_j}(b_{i1}^h)[u(x_{i1}^l) - b_{i1}^h] + Q_2^{s_j}(b_{i2}^h)[u(x_{i2}^l) - b_{i2}^h] \\ & +Q_3^{s_j}(b_{i1}^h, b_{i2}^h)[w(x_{i1}^l, x_{i2}^l) - u(x_{i1}^l) - u(x_{i2}^l)] \\ \geq \quad & Q_1^{s_j}(b_{i1}^l)[u(x_{i1}^l) - b_{i1}^l] + Q_2^{s_j}(b_{i2}^l)[u(x_{i2}^l) - b_{i2}^l] \\ & +Q_3^{s_j}(b_{i1}^l, b_{i2}^l)[w(x_{i1}^l, x_{i2}^l) - u(x_{i1}^l) - u(x_{i2}^l)] \end{aligned} \tag{10}$$

$$\begin{aligned} \Leftrightarrow \quad & [Q_3^{s_j}(b_{i1}^h, b_{i2}^h) - Q_3^{s_j}(b_{i1}^l, b_{i2}^l)]\lambda(x_i) \\ \geq \quad & [Q_1^{s_j}(b_{i1}^l)[u(x_{i1}^l) - b_{i1}^l] - Q_1^{s_j}(b_{i1}^h)[u(x_{i1}^l) - b_{i1}^h]] \\ & +[Q_2^{s_j}(b_{i2}^l)[u(x_{i2}^l) - b_{i2}^l] - Q_2^{s_j}(b_{i2}^h)[u(x_{i2}^l) - b_{i2}^h]] \end{aligned} \tag{11}$$

We want to show that (11) gives (9). Rewriting (9),

$$\begin{aligned} & [Q_3^{s_j}(b_{i1}^h, b_{i2}^h) - Q_3^{s_j}(b_{i1}^l, b_{i2}^l)]\lambda(x_i') \\ \geq \quad & [Q_1^{s_j}(b_{i1}^l)[u(x_{i1}^h) - b_{i1}^l] - Q_1^{s_j}(b_{i1}^h)[u(x_{i1}^h) - b_{i1}^h]] \\ & +[Q_2^{s_j}(b_{i2}^l)[u(x_{i2}^h) - b_{i2}^l] - Q_2^{s_j}(b_{i2}^h)[u(x_{i2}^h) - b_{i2}^h]] \end{aligned} \tag{12}$$

To prove the lemma, it is sufficient to show that $[(LHS(12) \geq LHS(11))$ and $(RHS(11) \geq RHS(12))]$.

$$LHS(12) - LHS(11) \quad = \quad [Q_3^{s_j}(b_{i1}^h, b_{i2}^h) - Q_3^{s_j}(b_{i1}^l, b_{i2}^l)][\lambda(x_i') - \lambda(x_i)] \geq 0. \tag{13}$$

The inequality comes from the fact that both terms are nonnegative. $Q_3^{s_j}$ is weakly increasing in its argument. Also, $\lambda(x_i)$ is weakly increasing in x_i by (A3) of section 2.1, and $x_i' \geq x_i$.

$$\begin{aligned} RHS(11) - RHS(12) \quad = \quad & Q_1^{s_j}(b_{i1}^l)[u(x_{i1}^l) - u(x_{i1}^h)] - Q_1^{s_j}(b_{i1}^h)[u(x_{i1}^l) - u(x_{i1}^h)] \\ & +Q_2^{s_j}(b_{i2}^l)[u(x_{i2}^l) - u(x_{i2}^h)] - Q_2^{s_j}(b_{i2}^h)[u(x_{i2}^l) - u(x_{i2}^h)] \\ = \quad & [Q_1^{s_j}(b_{i1}^l) - Q_1^{s_j}(b_{i1}^h)][u(x_{i1}^l) - u(x_{i1}^h)] \\ & +[Q_2^{s_j}(b_{i2}^l) - Q_2^{s_j}(b_{i2}^h)][u(x_{i2}^l) - u(x_{i2}^h)] \\ \geq \quad & 0 \end{aligned} \tag{14}$$

The inequality comes from the fact that $Q_1^{s_j}, Q_2^{s_j}$ and u are weakly increasing in their arguments. ♦

By Lemma 1, $V_i(b_i, x_i, s_j)$ satisfies weak single crossing property.

3.2 Weak quasisupermodularity of $V_i(b_i, x_i, s_j)$

Definition 2. $V_i(b_i, x_i, s_j)$ is *weakly quasisupermodular* if for all monotone pure strategies s_j of the opponent, all $b_i, b_i' \in A$, and every $x_i \in X^2$,

$$V_i(b_i, x_i, s_j) \geq V_i(b_i \wedge b_i', x_i, s_j) \text{ implies } V_i(b_i \vee b_i', x_i, s_j) \geq V_i(b_i', x_i, s_j). \tag{15}$$

If $b_i' \geq b_i$, where $\geq$ is a coordinatewise partial order, (15) holds trivially. Therefore, we focus on a case $[(b_{i1} > b_{i1}')$ and $(b_{i2} < b_{i2}')]$[3]. Denote $b_i = (b_{i1}^h, b_{i2}^l)$ and $b_i' = (b_{i1}^l, b_{i2}^h)$, where $b_{ik}^h > b_{ik}^l$ for $k \in \{1, 2\}$.

Substitute b_i, b_i', to (15). That is,

$$V_i(b_{i1}^h, b_{i2}^l, x_i, s_j) - V_i(b_{i1}^l, b_{i2}^l, x_i, s_j) \geq 0 \tag{16}$$

implies

$$V_i(b_{i1}^h, b_{i2}^h, x_i, s_j) - V_i(b_{i1}^l, b_{i2}^h, x_i, s_j) \geq 0. \tag{17}$$

(16) gives

$$\begin{aligned} & Q_1^{s_j}(b_{i1}^h)[u(x_{i1}) - b_{i1}^h] + Q_2^{s_j}(b_{i2}^l)[u(x_{i2}) - b_{i2}^l] \\ & +Q_3^{s_j}(b_{i1}^h, b_{i2}^l)[w(x_{i1}, x_{i2}) - u(x_{i1}) - u(x_{i2})] \\ \geq \quad & Q_1^{s_j}(b_{i1}^l)[u(x_{i1}) - b_{i1}^l] + Q_2^{s_j}(b_{i2}^l)[u(x_{i2}) - b_{i2}^l] \\ & +Q_3^{s_j}(b_{i1}^l, b_{i2}^l)[w(x_{i1}, x_{i2}) - u(x_{i1}) - u(x_{i2})] \end{aligned} \tag{18}$$

$$\Leftrightarrow Q_1^{s_j}(b_{i1}^h)[u(x_{i1}) - b_{i1}^h] - Q_1^{s_j}(b_{i1}^l)[u(x_{i1}) - b_{i1}^l] \geq [Q_3^{s_j}(b_{i1}^l, b_{i2}^l) - Q_3^{s_j}(b_{i1}^h, b_{i2}^l)]\lambda(x_i). \tag{19}$$

Rewriting (17),

$$Q_1^{s_j}(b_{i1}^h)[u(x_{i1}) - b_{i1}^h] - Q_1^{s_j}(b_{i1}^l)[u(x_{i1}) - b_{i1}^l] \geq [Q_3^{s_j}(b_{i1}^l, b_{i2}^h) - Q_3^{s_j}(b_{i1}^h, b_{i2}^h)]\lambda(x_i). \tag{20}$$

Since LHS of (19) and (20) are identical, a sufficient condition for weak quasisupermodularity is [RHS (19) $\geq$ RHS (20)].

$$\begin{aligned} RHS(19) - RHS(20) = \quad & \lambda(x_i) \cdot \\ & [[Q_3^{s_j}(b_{i1}^h, b_{i2}^h) - Q_3^{s_j}(b_{i1}^h, b_{i2}^l)] - [Q_3^{s_j}(b_{i1}^l, b_{i2}^h) - Q_3^{s_j}(b_{i1}^l, b_{i2}^l)]]. \end{aligned} \tag{21}$$

With the assumption(A2) $\lambda(x_i) \geq 0$ in section 2.1 , for weak quasisupermodularity, it is sufficient to show that the next term of $\lambda(x_i)$ in (21) is positive. That is, we want to show that

$$[Q_3^{s_j}(b_{i1}^h, b_{i2}^h) - Q_3^{s_j}(b_{i1}^h, b_{i2}^l)] - [Q_3^{s_j}(b_{i1}^l, b_{i2}^h) - Q_3^{s_j}(b_{i1}^l, b_{i2}^l)] \geq 0. \tag{22}$$

[3] A proof for weak quasisupermodularity of this case similarly applies to the opposite case $[(b_{i1} < b_{i1}')$ and $(b_{i2} > b_{i2}')]$.

Lemma 2. *Recall* $X^2 = [0,1] \times [0,1]$ *and* $A = \{0, \frac{1}{n}, \frac{2}{n}, ..., \bar{u}\} \times \{0, \frac{1}{n}, \frac{2}{n}, ..., \bar{u}\}, n \in \mathbb{Z}_+$ *in the auction environment of section 2.1. In this environment, the inequality (22), which is a sufficient condition of weak quasisupermodularity, holds for all monotone pure-strategies of the opponent.*

Proof. Let $x_j \in X \times X$ be object values of the opponent j and $g(x_j) = \text{probability}(x_j)$. First, we define $q_i(b_i, b_j)$, i's ex-post winning probability function for two objects, as follows.

$$q_i(b_i, b_j) = \begin{cases} 1 & \text{if } b_i > b_j \\ 1/2 & \text{if } [(b_{i1} > b_{j1}) \text{ and } (b_{i2} = b_{j2})] \text{ or } [(b_{i1} = b_{j1}]) \text{ and } (b_{i2} > b_{j2})] \\ 1/4 & \text{if } b_i = b_j \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

Suppose that the opponent j plays an arbitrary monotone pure-strategy $s_j(x_j) = (s_{j1}(x_j), s_{j2}(x_j))$. Then, given s_j, the bidder i's interim probability for winning both objects $Q_3^{s_j}(b_i)$ is

$$Q_3^{s_j}(b_i) = \int_{x_j \in X^2} g(x_j) \cdot q_i(b_i, s_j(x_j)) \; dx_j. \tag{24}$$

As a result, the LHS of (22) can be expressed as follows.

$$\begin{aligned} & [Q_3^{s_j}(b_{i1}^h, b_{i2}^h) - Q_3^{s_j}(b_{i1}^h, b_{i2}^l)] - [Q_3^{s_j}(b_{i1}^l, b_{i2}^h) - Q_3^{s_j}(b_{i1}^l, b_{i2}^l)] \\ = & \int_{x_j \in X^2} g(x_j)[\{q_i(b_{i1}^h, b_{i2}^h, s_j(x_j)) - q_i(b_{i1}^h, b_{i2}^l, s_j(x_j))\} \\ & \qquad - \{q_i(b_{i1}^l, b_{i2}^h, s_j(x_j)) - q_i(b_{i1}^l, b_{i2}^l, s_j(x_j))\}] \; dx_j \\ = & \int_{x_j \in X^2} g(x_j)[H(b_{i1}^h, b_{i2}^l, b_{i2}^h, s_j(x_j)) - D(b_{i1}^l, b_{i2}^l, b_{i2}^h, s_j(x_j))] \; dx_j, \\ & \text{where } H(b_{i1}^h, b_{i2}^l, b_{i2}^h, s_j(x_j)) = q_i(b_{i1}^h, b_{i2}^h, s_j(x_j)) - q_i(b_{i1}^h, b_{i2}^l, s_j(x_j)) \\ & \text{and } D(b_{i1}^l, b_{i2}^l, b_{i2}^h, s_j(x_j)) = q_i(b_{i1}^l, b_{i2}^h, s_j(x_j)) - q_i(b_{i1}^l, b_{i2}^l, s_j(x_j)). \end{aligned} \tag{25}$$

Note that when the agent i increases the second object's bid from b_{i2}^l to b_{i2}^h, H represents the increase in the ex-post winning probability for both objects, while bidding high for the first object. D describes the same effect, when i bids low for the first object.

For the proof, we need to show that the second term of the integral in (25) is always positive. That is, $\forall s_j(x_j), \forall \vec{b_i} \in \{(b_{i1}^l, b_{i2}^l, b_{i1}^h, b_{i2}^h) | \; b_{ik}^h > b_{ik}^l, \; k \in \{1, 2\}\}$,

$$H(b_{i1}^h, b_{i2}^l, b_{i2}^h, s_j(x_j)) - D(b_{i1}^l, b_{i2}^l, b_{i2}^h, s_j(x_j)) \geq 0.$$

Let $\beta_j = (\beta_{j1}, \beta_{j2})$ be an arbitrary two-dimensional bid vector generated by $s_j(x_j)$ and $x_j \in X^2$. Table 1 shows values of $H(\cdot) - D(\cdot)$ for every 25 cases determined by the values of $(b_{ik}^l, b_{ik}^h, \beta_{jk})_{k \in \{1,2\}}$. In the table, every cell is composed of a difference between two brackets. The first bracket corresponds to the value of $H(\cdot)$, and the second bracket corresponds to the value of $D(\cdot)$. For instance, a cell at the 3rd row and the 3rd column in the table contains $[\frac{1}{4} - 0] - [0 - 0]$. In this cell, the first bracket $[\frac{1}{4} - 0]$ corresponds

Table 1: $H(b^h_{i1}, b^l_{i2}, b^h_{i2}, s_j(x_j)) - D(b^l_{i1}, b^l_{i2}, b^h_{i2}, s_j(x_j))$

$H - D$	$b^l_{i2} < b^h_{i2} < \beta_{j2}$	$b^l_{i2} < b^h_{i2} = \beta_{j2}$	$b^l_{i2} < \beta_{j2} < b^h_{i2}$	$b^l_{i2} = \beta_{j2} < b^h_{i2}$	$\beta_{j2} < b^l_{i2} < b^h_{i2}$
$b^l_{i1} < b^h_{i1} < \beta_{j1}$	$[0-0]-[0-0]$	$[0-0]-[0-0]$	$[0-0]-[0-0]$	$[0-0]-[0-0]$	$[0-0]-[0-0]$
$b^l_{i1} < b^h_{i1} = \beta_{j1}$	$[0-0]-[0-0]$	$[\frac{1}{4}-0]-[0-0]$	$[\frac{1}{2}-0]-[0-0]$	$[\frac{1}{2}-\frac{1}{4}]-[0-0]$	$[\frac{1}{2}-\frac{1}{2}]-[0-0]$
$b^l_{i1} < \beta_{j1} < b^h_{i1}$	$[0-0]-[0-0]$	$[\frac{1}{2}-0]-[0-0]$	$[1-0]-[0-0]$	$[1-\frac{1}{2}]-[0-0]$	$[1-1]-[0-0]$
$b^l_{i1} = \beta_{j1} < b^h_{i1}$	$[0-0]-[0-0]$	$[\frac{1}{2}-0]-[\frac{1}{4}-0]$	$[1-0]-[\frac{1}{2}-0]$	$[1-\frac{1}{2}]-[\frac{1}{2}-\frac{1}{4}]$	$[1-1]-[\frac{1}{2}-\frac{1}{2}]$
$\beta_{j1} < b^l_{i1} < b^h_{i1}$	$[0-0]-[0-0]$	$[\frac{1}{2}-0]-[\frac{1}{2}-0]$	$[1-0]-[1-0]$	$[1-\frac{1}{2}]-[1-\frac{1}{2}]$	$[1-1]-[1-1]$

to $H(b^h_{i1}, b^l_{i2}, b^h_{i2}, s_j(x_j)) = q_i(b^h_{i1}, b^h_{i2}, s_j(x_j)) - q_i(b^h_{i1}, b^l_{i2}, s_j(x_j))$, when $[b^l_{i1} < b^h_{i1} = \beta_{j1}]$ and $[b^l_{i2} < b^h_{i2} = \beta_{j2}]$.

From the table, we observe that every value of cells are positive. Therefore, with $g(x_j) \geq 0$, the value of the integral in (25) is positive and the inequality (22) holds. ♦

By Lemma 2, the interim utility $V_i(b_i, x_i, s_j)$ is weakly quasisupermodular. Together with Lemma 1, this completes the proof of Theorem 1.

4 Conclusion

This paper has shown the existence of monotone pure-strategy Bayesian Nash equilibrium in simultaneous first-price auctions, where the objects have synergy in bidders' utility. Under the complementarity assumptions, weak single crossing property and weak quasisupermodularity of interim utility are satisfied, so Reny (2011)'s existence theorem is applicable to this environment.

As future research, it is meaningful to think about the environment in a continuum action space setting. As Athey (2001) argues in the paper, it would support a differential equation approach to verify existence conditions. She provides the extension with a continuum action space by showing ties do not occur in the game. Reny (2011) also presents the extension through a better-reply secure game and a limit of a pure-strategy of ϵ equilibria.

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