Stochastic stability in coalitional bargaining problems

Ryoji Sawa*

Center for Cultural Research and Studies, University of Aizu

March 4, 2014

Abstract This paper examines a dynamic process of n-person coalitional bargaining problems. We study the stochastic evolution of social conventions by embedding a static bargaining setting in a dynamic process; Over time agents revise their coalitions and surplus distributions in the presence of stochastic payoff shocks which lead agents to make a suboptimal choice. Under a logit specification of choice probabilities, we find that the stability of a core allocation decreases in the wealth of the richest player, and that stochastically stable allocations are core allocations which minimize the wealth of the richest.

Keywords: Stochastic stability; Coalitions; Logit-response dynamics; Bargaining.

JEL Classification Numbers: C72, C73, C78.

*Address: Tsuruga, Ikki-machi, Aizu-Wakamatsu City, Fukushima 965-8580, Japan., telephone: +81-242-37-2500, e-mail: *rsawa@u-aizu.ac.jp*. The author is grateful to William Sandholm, Marzena Rostek and Marek Weretka for their advice and suggestions. The author also thanks Pierpaolo Battigalli, George Mailath, Jonathan Newton, Akira Okada, Daisuke Oyama, Satoru Takahashi, Yuichi Yamamoto, H. Peyton Young, Dai Zusai, and seminar participants at Econometric Society North American Summer Meeting, Japan Economic Association 2013 Spring Meeting, Temple University, University of Aizu, University of Kansas and University of Wisconsin-Madison for their comments and suggestions.

1 Introduction

We study stochastic stability of a non-cooperative model of n-person coalitional bargaining. In the model, a characteristic function describes the surplus available to different coalitions. A coalition forms when its agents agree on how to share the surplus it generates. At any given period, the surplus is divided as per the outcome of a bargaining game. We are interested in which coalitions are formed and how the surplus is most likely divided among the agents in the long run.

The present paper characterizes allocations that are stochastically stable against both individual deviations and coalitional deviations. The notion of stochastic stability was introduced by Foster and Young (1990), Kandori et al. (1993) and Young (1993). It is a method to assess the robustness of equilibria by checking its resilience to stochastic shocks. We apply a version of the method here. As in the literature, our approach can be viewed as performing a stability test for allocations in a coalitional bargaining game by adding stochastic noise.

To assess robustness against deviations, we study a stochastic dynamic process which follows the dynamic of Sawa (2013). In each period, some agents form a tentative team and randomly choose a surplus distribution which each agent in the team will weigh up shortly. In the unperturbed updating process, an agent agrees to the distribution if it yields her higher payoff than the current allocation. The distribution will be accepted if all agents in the team agree. To this unperturbed dynamic process, we add stochastic noise that leads agents to agree to a distribution according to the logit choice rule. Coalitional deviations sometimes occur even if not all members of the team benefit, and this occurs with a probability that declines in the total payoff deficits to team members. As a consequence, the stochastic process visits every allocations repeatedly, and predictions can be made concerning the relative amounts of time that the process spends at each. We examine the behavior of this system as the level of stochastic noise becomes small, defining the stochastically stable allocations to be those which are observed with positive frequency in the long run as noise vanishes.

We find that stochastically stable allocations are core allocations whenever the set of interior points in the core is not empty.[1] Moreover, we find the following characterizations. The stability of a core allocation decreases in the wealth of the richest player, and the stochastically stable allocations are core allocations which minimize the wealth of the richest. We view this result interesting because equity consideration winds up playing an important role even with myopic payoff-maximizing players.

The related studies in the literature which examines stochastic stability in coalitional bargaining games are Agastya (1999) and Newton (2012). Both papers consider a perturbed dynamic of games and characterize allocations which are stochastically stable against perturbations. One of differences from ours is that those two papers assume a sort of central authority. Agents submit their claims to it, and then it decides which coalitions to be formed and how demands are ra-

[1] The set of interior points is called *strict core* in the paper.

tioned. While, we assume that agents randomly meet and decide whether to form a coalition by themselves.

The related studies in coalitional bargaining models with rational agents are Okada (1996) and Compte and Jehiel (2010). Those studies resemble ours in that the proposer is randomly chosen.[2] One of differences is that the proposer rationally chooses her proposal in the two studies, while a proposal is randomly chosen to be assessed in ours. Despite the differences, it is interesting to see that an egalitarian outcome is favored by all of those studies and ours.

The paper is organized as follows. Section 2 contains the basic coalitional bargaining model. Section 3 describes the dynamic of the bargaining model. In Section 4, we characterize stochastically stable allocations. We compare our result with the existing literature of coalitional bargaining models in Section 5.

2 Model

We consider a non-cooperative model of multi-player coalitional bargaining situations examined by several studies, for example, Chatterjee et al. (1993). There is a set of players $N = \{1, \ldots, n\}$. Let $\mathcal{R}$ be the class of all subsets of N. Any $J \in \mathcal{R}$ may form a team and the surplus that such a team generates is given by production function $v\colon \mathcal{R} \to \mathbb{R}_+$ with $v(\varnothing) = 0$. Surplus $v(J)$ can be distributed to members if all members of J agree on a surplus distribution. We assume that there exists small $\Delta > 0$ such that $v(J)/\Delta \in \mathbb{Z}$ for all $J \in \mathcal{R}$.[3]

Let $S_i = \{0, \Delta, 2\Delta, \ldots\}$ denote the set of player i's claim s_i. For a team $J \in \mathcal{R}$, its surplus distribution has to satisfy the following feasibility constraints:

$$\begin{cases} \sum_{i \in J} s_i \leq v(J) & \text{if } |J| \geq 2, \\ s_i = v(J) & \text{if } J = \{i\}. \end{cases} \tag{1}$$

The second constraint implies that player i earns what she can produce by her own when she does not form a team with other players. Player i gets payoffs equal to $u(s_i)$ where $u(\cdot)$ is concave, strictly increasing and satisfies that $u(0) = 0$.[4] Let the set of feasible surplus distributions for team J be denoted by

$$S^J = \left\{ s_J \in \prod_{i \in J} S_i : s_J \text{ satisfies (1).} \right\}.$$

Note that $S^J = \{v(\{i\})\}$ if $J = \{i\}$.

We assume that there can be more than one team, but an agent can participate in exactly one

[2] Another strand of studies assumes that the player who rejects an offer becomes the next player to make an offer. See Chatterjee et al. (1993), for example.

[3] This assumption guarantees that surplus can be distributed without loss for all $J \in \mathcal{R}$.

[4] We assume that surplus $v(\cdot)$ is transferable, but utility may not. Utility is transferable when function u is linear over payments, and it is non-transferable otherwise.

team.[5] Let $\mathcal{M}$ denote the set of existing teams. Since each player participates one team, $\mathcal{M}$ must be a partition of N.[6] The space of claim profiles with $\mathcal{M}$ is defined as

$$\mathscr{S}_{\mathcal{M}} = \left\{ (s, \mathcal{M}) \colon s_M \in S^M \ \forall M \in \mathcal{M} \right\}.$$

Note that $(s, \mathcal{M})$ in $\mathscr{S}_{\mathcal{M}}$ satisfies the feasibility constraints for all existing teams. The space of feasible strategy profiles is given by

$$\mathscr{S} = \bigcup_{\mathcal{M} \in \text{part}(N)} \mathscr{S}_{\mathcal{M}},$$

where $\text{part}(N)$ denotes the set of partitions of N.

We employ equilibrium concepts due to Sawa (2013). A pair $(s, \mathcal{M})$ is an *$\mathcal{R}$-stable equilibrium* if for all $J \in \mathcal{R}$ and all $s_J \in S^J$,

$$s_i \geq s'_i \quad \text{for some } i \in J.$$

It is a *strict $\mathcal{R}$-stable equilibrium* if the strict inequality holds in the above expression for all $J \in \mathcal{R}$ and all $s_J \in S^J$.[7]

Our model of team formation is similar to Okada (1996) and is more general than the model of Compte and Jehiel (2010) which restrict the number of (multi-player) teams to at most one. We will compare the stochastically stable outcomes resulting from myopic players with outcomes resulting from perfectly rational players shown in Okada (1996) and Compte and Jehiel (2010).

2.1 The Core

We say that coalition J *blocks* allocation $s \in \mathscr{S}$ if there exists $s' \in S^J$ such that

$$s'_i > s_i \ \forall i \in J.$$

Similarly, coalition J *weakly blocks* allocation $s \in \mathscr{S}$ if there exists $s' \in S^J$ such that

$$s'_i \geq s_i \ \forall i \in J.$$

Note that if s cannot be weakly blocked by J, then there exists at least one player in J who would be strictly worse off if J were to form a team and implement s'. Now, we define the *core* and the *strict core*.

Definition 2.1. *The* core *consists of the feasible allocations with $\mathcal{M} = \{N\}$ that cannot be blocked by any coalition $J \in \mathcal{R}$. The* strict core *consists of the feasible allocations with $\mathcal{M} = \{N\}$ that cannot be weakly blocked by any coalition $J \in \mathcal{R}$.*

[5] An agent forms a singleton team when she does not form a team with others.
[6] $\mathcal{M}$ is a partition of N if $N = \cup\mathcal{M}$, and $M \cap M' = \varnothing \ \forall M, M' \in \mathcal{M}, M \neq M'$.
[7] A $\mathcal{R}$-stable equilibrium with $\mathcal{R}$ being the power set of N is a strong equilibrium due to Aumann (1959).

A strict core allocation is a strict $\mathcal{R}$-stable equilibrium, but a non-strict core needs not be. In the present model, notice that a strict core allocation s satisfies the following set of inequalities:

$$\sum_{i \in J} s_i \geq v(J) + \Delta \quad \forall J \in \mathcal{R}. \tag{2}$$

Let $\mathscr{C}_\Delta$ denote the set of strict core allocations given Δ. In what follows, we assume that $\mathscr{C}_\Delta$ is non-empty.

For example, a strictly convex interaction with Δ sufficiently small has non-empty $\mathscr{C}_\Delta$. Production function v is *strictly convex* if for all $J, J' \in N$ with $J \cap J' \notin \{J, J'\}$,[8]

$$v(J \cup J') > v(J) + v(J') - v(J \cap J'). \tag{3}$$

An interaction is strictly convex if its production function is strictly convex. For sufficiently small Δ, a strictly convex interaction has at least one strict core allocation. The next example illustrates this point.

Example 1. Let $N = \{1,2\}$ and $\Delta = 2$, and consider a strictly convex production function $v(1) = 2$, $v(2) = 0$, $v(\{1,2\}) = 4$. Note that $S^{\{1,2\}} = \{(4,0),(2,2),(0,4)\}$. $(4,0)$ and $(2,2)$ are core allocations, but not strict. And they are not strict $\mathcal{R}$-stable equilibria. The non-existence of strict core is resolved with finer grids. For instance, for $\Delta = 1$, $(3,1) \in S^{\{1,2\}}$ is a strict core allocation.

3 Dynamic

We apply the stochastic stability approach to coalitional bargaining problems described in the previous section. In this approach, we embed a static interaction in a dynamic process in which agents randomly form coalitions and jointly revise their strategies based on improvements in their payoffs in the presence of stochastic payoff shocks. We examine the limiting probability distribution over strategy profiles as the level of stochastic shocks approaches zero.

The dynamic interaction proceeds as follows. Let s^t denote a profile of players' claims in period t, and $\mathcal{M}^t$ the set of the existing teams in t. At the beginning of each period t, $J \in \mathcal{R}$ is randomly chosen, and then a payment proposal $s = \{s_i\}_{i \in N}$ to share the surplus $v(J)$ is randomly chosen. We assume that proposal s given J satisfies the feasibility constraint (1). Each player in J is asked whether she accepts or rejects proposal s. A player accepts with probability $\Psi^\eta(s^t, s)$. If they all accept, players in J form a team and each team member $i \in J$ gets payoffs $u(s_i)$. If coalition J forms a team, then any existing team having some $i \in J$ will be dissolved, i.e. members other than i in such an existing team will form singleton teams. If at least one player in J rejects s, the state will remain $(s^t, \mathcal{M}^t)$.

Note that a state of the Markov chain of this interaction consists of $(s, \mathcal{M}) \in \mathscr{S}$.[9] A transition

[8] A similar production function is assumed in several studies, e.g. Okada (1996) and Okada (2011).

[9] Transitions depend on not only a current claim profile but also a set of existing teams. If a coalition forms a new team, it will affect other players in dissolved teams. Which players will be affected depends on $\mathcal{M}$.

from $(s, \mathcal{M})$ occurs when coalition J forms a new team (if $J \notin \mathcal{M}$), or J redistributes its surplus (if $J \in \mathcal{M}$). Formally, transition $((s, \mathcal{M}), (s', \mathcal{M}'))$ is said to be feasible if the following conditions are satisfied.

(i) If $\mathcal{M} \neq \mathcal{M}'$, then there exists $J \in \mathcal{M}'$ such that for all $M \in \mathcal{M}$

$$\begin{aligned} &\{i\} \in \mathcal{M}' \ \ \forall i \in M \setminus J && \text{if } J \cap M \neq \varnothing, \\ &M \in \mathcal{M}', \text{ and } s_i = s'_i \ \ \forall i \in M && \text{if } J \cap M = \varnothing. \end{aligned}$$

(ii) $\sum_{i \in M'} s'_i \leq v(M') \quad \forall M' \in \mathcal{M}'$.

(iii) If $\mathcal{M} = \mathcal{M}'$, then there exists $J \in \mathcal{M}'$ such that

$$s_i = s'_i \quad \forall i \in M',\ M' \in \mathcal{M}' \setminus \{J\}.$$

(iv) $s'_i = v(\{i\}) \quad \forall \{i\} \in \mathcal{M}'$.

Condition (i) is a feasibility constraint for the transition from $\mathcal{M}$ to $\mathcal{M}'$. It says that when a new team J is formed, an existing team M must be dissolved if at least one player leaves M to join J. Otherwise, the existing team should remain. For the remaining teams, their distributions must be unaffected. Conditions (ii)-(iv) are constraints on surplus distribution s' given $\mathcal{M}'$. Condition (ii) is a set of the feasibility constraints given by Equation (1) for all teams in $\mathcal{M}'$. Condition (iii) is the case of a surplus redistribution ($\mathcal{M} = \mathcal{M}'$). It says that at most one team redistributes the surplus in a transition. Note that Condition (iii) also applies to the cases that someone rejects the proposal (then, $\mathcal{M} = \mathcal{M}'$ holds). Finally, Condition (iv) is the feasibility constraint for players forming singleton teams, including those players whose team is dissolved in the transition.

Let $R_{(s,\mathcal{M}),(s',\mathcal{M}')}$ be a set of coalitions potentially leading from $(s, \mathcal{M})$ to $(s', \mathcal{M}')$. It is given by

$$R_{(s,\mathcal{M}),(s',\mathcal{M}')} = \begin{cases} \{J \in \mathcal{R} :\ J \text{ satisfies (ii).}\} & \text{if } \mathcal{M} \neq \mathcal{M}', \text{ and (i)–(iv) are satisfied,} \\ \{J \in \mathcal{R} :\ J \text{ satisfies (iii).}\} & \text{if } \mathcal{M} = \mathcal{M}', \text{ and (i)–(iv) are satisfied,} \\ \varnothing & \text{if some (i)–(iv) is violated.} \end{cases}$$

The last case denotes infeasible transitions. We write transition from $(s, \mathcal{M})$ to $(s', \mathcal{M}')$ as $((s, \mathcal{M}), (s', \mathcal{M}'))$.

3.1 The Coalitional Logit Dynamic

Following Sawa (2013), we introduce stochastic shocks and formally describe a perturbed dynamic behavior. We focus on a logit-response dynamic of Blume (1993). To describe the logit choice rule, suppose that the current claim profile is given by s, that a randomly chosen coalition is J, and that $s'_J \in S^J$ is proposed as the surplus distribution. The probability that agent i in coalition J agrees with s'_J is given by

$$\Psi_i^\eta(s,s') = \frac{\exp\left[\eta^{-1}u_i(s')\right]}{\exp\left[\eta^{-1}u_i(s')\right] + \exp\left[\eta^{-1}u_i(s)\right]}. \tag{4}$$

where $s' = (s'_J, s_{-J})$ and $\eta \in (0,\infty)$ denotes the noise level of the logit choice rule. Note that agent i takes into account other agents' new strategies, i.e. s'_J, in Equation (4). The probability that all members in J agree is given by $\prod_{i\in J}\Psi_i^\eta$.

The logit-response dynamic is a Markov chain on the state space $\mathscr{S}$ with stationary transition probabilities. The probability for transition $((s,\mathcal{M}),(s',\mathcal{M}'))$ is given by

$$P^\eta_{(s,\mathcal{M}),(s',\mathcal{M}')} = \sum_{J\in R_{(s,\mathcal{M}),(s',\mathcal{M}')}} q_J\, q_{s'}(J,s) \prod_{i\in J}\Psi_i^\eta(s,s'). \tag{5}$$

Note that the unperturbed dynamic is obtained in the limit as η approaches zero, i.e.

$$P^0_{(s,\mathcal{M}),(s',\mathcal{M}')} = \sum_{J\in R_{(s,\mathcal{M}),(s',\mathcal{M}')}} q_J\, q_{s'}(J,s) \prod_{i\in J}\Psi_i^0(s,s'), \tag{6}$$

where[10]

$$\Psi_i^0(s,s') = \begin{cases} 0 & u_i(s) > u_i(s') \\ \alpha & u_i(s) = u_i(s') \\ 1 & u_i(s) < u_i(s'). \end{cases}$$

3.2 Limiting Stationary Distributions and Stochastic Stability

The Markov chain induced by P^η is irreducible and aperiodic for $\eta > 0$, and so admits a unique stationary distribution, denoted by π^η. Let $\pi^\eta(s)$ denote the probability that π^η places on state s. $\pi^\eta(s)$ represents the fraction of time in which state s is observed over a long time horizon. It is also the probability that state s will be observed at any given time t, provided that t is sufficiently large. Thus, the agents' behavior is nicely summarized by π^η in the long-run. We say that state s is $\mathcal{R}$-stochastically stable if the limiting stationary distribution places positive probability on s.

Definition 3.1. *State s is $\mathcal{R}$-*stochastically stable *if* $\lim_{\eta\to 0}\pi^\eta(s) > 0$.

To determine which states will be observed most frequently in the long run, we now introduce several definitions in order to compute the unlikeliness of transitions. Given a state s, define an s-tree to be a directed graph T with a unique path from any state $s' \in \mathscr{S}$ to s. An edge of a s-tree, denoted by $(s',s'') \in T(s)$, represents a transition from s' to s'' in the dynamic.

[10] An unperturbed dynamic with $\alpha = 1/2$ is the limiting dynamic as η approaches zero. Our analysis will not differ for all $\alpha \in (0,1)$ because the set of recurrent classes in the unperturbed dynamic does not differ for $\alpha \in (0,1)$.

We define the cost of edge, or transition, (s, s') as follows.

$$c_{s,s'} = \begin{cases} \min_{J \in R_{s,s'}} \left[\sum_{i \in J} \max\{u_i(s) - u_i(s'), 0\}\right] & \text{if } R_{s,s'} \neq \varnothing, \\ \infty & \text{if } R_{s,s'} = \varnothing. \end{cases} \tag{7}$$

In words, the cost of a transition is the sum of payoff losses of agents revising in that transition.

The next lemma shows that cost $c_{s,s'}$ is equal to the exponential rate of decay of the corresponding transition probability, $P^{\eta}_{s,s'}$.[11]

Lemma 3.2. *If $R_{s,s'} \neq \varnothing$, then*

$$-\lim_{\eta \to 0} \eta \log P^{\eta}_{s,s'} = c_{s,s'}.$$

Proof. See Sawa (2013). □

Lemma 3.2 implies that the amount of payoff losses plays a significant role in determining transitions. An interesting observation is that the dynamic captures the notion of "taking one for team". A transition in which an agent sacrifices a smaller amount of her payoffs to benefit others will be more likely than transitions which lose a greater amount of their payoffs. For instance, if everyone wants to go out but someone has to stay home to watch kids, it is more likely that a person will volunteer than that everyone will stay home without volunteers.

Let $\mathcal{T}(s)$ denote the set of s-trees. The waste of a tree $T \in \mathcal{T}(s)$ is defined as

$$W(T) = \sum_{(s',s'') \in T} c_{s',s''}. \tag{8}$$

Note that Equation (7) shows the main difference from the standard stochastic stability analysis which assumes unilateral deviations. The cost of (s, s') evaluates the payoff disadvantages of coalitional deviation s'_J for $J \in R_{s,s'}$ instead of individual deviations, i.e. evaluating $u_i(s'_J, s_{-J})$ instead of $u_i(s'_i, s_{-i})$. The waste of a tree is the sum of the payoff disadvantages along the tree. The stochastic potential of state s is defined as[12]

$$W(s) = \min_{T \in \mathcal{T}(s)} W(T).$$

As η approaches zero, the stationary distribution converges to a unique limiting stationary distribution. Our main result is the following theorem which offers the characterization of $\mathcal{R}$-stochastically stable states.

[11]See Chapter 12 of Sandholm (2010) for a discussion of defining costs for stochastic dynamics.

[12]Our stochastic potential is a simplified version of that of Alós-Ferrer and Netzer (2010) which defines the stochastic potential minimizing the waste over trees and transition mappings which map every edge of trees to a set of revising agents. Lemma 3.2 embeds the minimization over transition mappings into transition costs of Equation (7) which gives the minimum rate of decay over sets of revising agents. This allows us to directly apply the technique of Freidlin and Wentzell (1988), and makes two technical contributions; (i) enabling us to characterize stochastically stable states with other noisy best responses in Section **??**, and (ii) characterizing an exact expression for the limiting stationary distribution.

Theorem 3.3. *A state is $\mathcal{R}$-stochastically stable if and only if it minimizes $W(s)$ among all states.*

Proof. See Sawa (2013). □

4 Stochastically Stable Allocations

Recall that the dynamic chooses each $J \in \mathcal{R}$ with positive probability in each period. In this sense, our model is closer to the 'random proposer' model of Okada (1996) and Compte and Jehiel (2010) rather than the 'rejecter proposer' model of Chatterjee et al. (1993). We will compare our result with the former papers in Section 5.

The next lemma guarantees that the set of stochastically stable allocations is a subset of the collection of strict core allocations. The stochastic stability approach will allow us to select core allocations that are most robust against perturbations.

Lemma 4.1. *For $s \notin \mathscr{C}_\Delta$, the unperturbed dynamic induced by P^0 reaches some $s \in \mathscr{C}_\Delta$ with positive probability.*

For what follows, we assume that Δ is sufficiently small such that $v(\{i\}) > \Delta$ for all $i \in N$. For $s \in S$, let $s_{(i)}$ denote the i-th largest share in s. Let

$$s_{\min} = \min_{s' \in \mathscr{C}_\Delta} s'_{(1)}, \ s_{\max} = \max_{s' \in \mathscr{C}_\Delta} s'_{(1)}.$$

In words, $s_{\min}$ is the lowest claim of the richest agent among all strict core allocations, and $s_{\max}$ is the highest one.

Let $\mathcal{R}_i = \{J \in \mathcal{R} : i \in J\}$. In words, $\mathcal{R}_i$ is a set of coalitions including agent i. Let $i_\# \in \{i : s_i = s_{(1)}\}$ denote one of the richest agents in $s \in \mathscr{C}_\Delta$. We define a condition: for all $i_\#$, the inequality below holds:

$$\sum_{i \in J} s_i \geq v(J) + 2\Delta \quad \forall J \in \mathcal{R}_{i_\#}. \tag{9}$$

We say that allocation s satisfies Condition (9) if Inequality (9) holds for all $i_\# \in \{i : s_i = s_{(1)}\}$. Any allocation satisfying (9) is a strict core allocation. Furthermore, even if there is a transfer of Δ from someone to another in an allocation satisfying (9), the resulting allocation satisfies Inequality (2), i.e. it is still a strict core allocation.

Recall that $R(s)$ in Equation (**??**) denotes the minimum waste for the process to escape from the basin of attraction of s. We call a sequence of transitions from $s \in \mathscr{C}_\Delta$ to some other $s' \in \mathscr{C}_\Delta$ the least-cost escape from s if its waste is $R(s)$. Condition (9) is a key to identify which allocation the process will most likely visit when it departs from a strict core allocation, as shown in the following lemma.

Lemma 4.2. *For all $s \in \mathscr{C}_\Delta$,*

$$R(s) = u(s_{(1)}) - u(s_{(1)} - \Delta). \tag{10}$$

Furthermore, if allocation s satisfies Condition (9), then the least-cost escape from s leads the process to $s' \in \mathscr{C}_\Delta$ where s' is either with the richest agent claiming $s_{(1)} - \Delta$ or with one fewer richest agents claiming $s_{(1)}$. If allocation s violates Condition (9), then the least-cost escape from s leads the process to any $s' \in \mathscr{C}_\Delta$.

Lemma 4.2 shows that the stability of a core allocation depends on the richest player. Moreover, the concavity of u implies that $u(x) - u(x - \Delta) < u(y) - u(y - \Delta)$ for all $x > y$; the stability decreases in the wealth of the richest player. Our main result in this section is the following proposition which characterizes the stochastically stable allocations.

Proposition 4.3. *Allocation s is stochastically stable if and only if $s \in \mathscr{C}_\Delta$ and*

$$s_{(1)} = s_{\min} \tag{11}$$

Its formal proof, which is in Appendix, uses the modified Radius-Coradius theorem (Theorem **??**). We provide a sketch here. For each $h \in \{0, 1, \ldots\}$, we classify the strict core allocations into two sets: those with $s_{(1)} \geq s_{\max} - h\Delta$ and those with $s_{(1)} < s_{\max} - h\Delta$. Starting with $h = 0$, we show that the radius of the latter set is greater than its coradius and so exclude allocations with $s_{(1)} = s_{\max}$ from the stochastically stable allocations. By the inductive step, we show that the radius of the latter set is greater than its coradius for all $h = \{1, 2, \ldots\}$ such that $s_{\min} < s_{\max} - h\Delta$.

5 Discussions

Model	Agents	Resulting allocation (among those in the core)
Okada (1996)	rational	Maximizing per capita (i.e. $\max_{J \subset N} v(J)/\lvert J\rvert$)
Compte and Jehiel (2010)	rational	Maximizing product of payoffs
Agastya (1999)	myopic	Minimizing payoff for the richest
Newton (2012)	myopic	Maximizing payoff for the poorest
This paper	myopic	Minimizing payoff for the richest

Table 1: Coalitional bargaining models and their resulting allocations

Table 1 summarizes related papers in coalitional bargaining, both with rational and myopic agents. We note a few general observations among them, and then we compare our model with the other papers which assume myopic agents. We say $(s, \mathcal{M})$ is an egalitarian allocation if $s_i = v(N)/n$ for all $i \in N$ and $\mathcal{M} = \{N\}$. Observe that despite various differences between the models, all papers favor the egalitarian outcome if it is in the core (or strict core).[13] Even if the egalitarian outcome is not in the core, equity considerations play a significant role in all papers.

[13] The model of Okada (1996) requires one more condition for the egalitarian outcome:

$$\frac{v(S)}{|S|} \leq \frac{v(T)}{|T|} \quad \text{for all } S, T \subseteq N \text{ with } S \subseteq T.$$

In the stochastically stable outcome of present paper, the richest agents transfer to others as much money as possible subject to Constraint (2). As an example for other papers, surplus is equally distributed among coalition members even if it is not a grand coalition in Okada (1996).

It is interesting to note that both Agastya (1999)'s centralized approach and our decentralized approach lead to similar results. Both approaches have a common observation; agents who do better in a coalition are less reluctant to reduce their demands. Then, a transition in which the richest agent reduces her demand is the easiest way to leave a strict core allocation, as we see in Lemma 4.2. This determines the stochastically stable allocations. The difference from Newton (2012) comes from a tailored setting of Newton (2012) that agents evaluate a correlated strategy and switch to a pure strategy profile in its support. We roughly sketch the setting here. Suppose that $N = \{1, 2, 3\}$ and that strict core allocations are $s = \{1, 6, 6\}$, $s' = \{2, 5, 6\}$ and $s'' = \{2, 6, 5\}$. In our model, all are stochastically stable, since the wealth of the richest agent is 6 in all the allocations. Now, suppose that agents form a grand coalition with allocation s. If we allow correlated strategies, then agents may evaluate a correlated strategy in which they play each of s' and s'' with probability $1/2$. In this way, agents 2 and 3 can (probabilistically) share the cost of accommodating the poorest with one unit of money. Newton (2012) showed that, in his setting, the cost of the transition above is lower than the cost of transitions in pure strategies, e.g. evaluating switching from s to s'. It will be interesting to study our model with correlated strategies in future research.[14]

There are two crucial differences between our paper and others assuming myopic agents. The first is that others assume a central authority which collect agents' claims and chooses coalition(s) to be formed.[15] Ours does not assume such an authority, but assumes instead that agents randomly meet and decide whether to form a coalition by themselves. The second is about restrictions on production function v. Agastya (1999) assumes strict convexity (see Equation (3)), and Newton (2012) assumes super-additivity:

$$v(J \cup J') \geq v(J) + v(J') \quad \text{if } J \cap J' = \varnothing.$$

In contrast, our model does not require any restriction on v except the existence of strict core allocations, which other two papers assume as well.

A Appendix

Proofs for Section 4

[14] We conjecture that our result will not change so long as logit choice is assumed. The cost of transition from s to the correlated strategy is given by the sum of differences in agents 2 and 3's expected payoffs:

$$u(6) - \frac{1}{2}(u(6) + u(5)) + u(6) - \frac{1}{2}(u(6) + u(5)) = u(6) - u(5).$$

Observe that the cost above is indifferent from the cost of both transitions (s, s') and (s, s''). However, the result would differ with the probit choice (see Equation (**??**)), and we may obtain a similar result to Newton (2012).

[15] In Newton (2012), each agent submits to the authority her claim and acceptable players to form a coalition together.

Proof of Lemma 4.1. By definition, it is obvious that any strict core allocation is an absorbing state in the unperturbed dynamic. Let s^* denote an arbitrary strict core allocation. We will show that, for any $s \notin \mathscr{C}_\Delta$, the unperturbed dynamic starting from s reaches s^* with positive probability.

First, suppose that allocation $s \notin \mathscr{C}_\Delta$ but in the core. Then, there exist a coalition $J \subset N$ and an allocation s' such that $s'_i = 0$ for all $i \notin J$,

$$s'_i \geq s_i \ \forall i \in J$$

with at least one equality, and

$$\sum_{i \in J} s'_i = v(J).$$

Let agents form coalition J and accept s'.[16] Let s'' be such that $s''_i = s'_i$ if $i \in J$ and $s''_i = v(\{i\})$ otherwise. Note that s'' is feasible for a grand coalition, i.e. $v(J) + \sum_{i \notin J} v(\{i\}) \leq v(N)$.[17] Let agents form a grand coalition and accept s''. Then, let agent $i^* \notin J$ form a singleton coalition and accept $s'''_{i^*} = v(\{i^*\})$. Note that the grand coalition is dissolved due to the deviation by i^*. Let agents form a grand coalition again and accept $s^* \in \mathscr{C}_\Delta$.

Second, suppose that the process is in allocation s that is not in the core. If no team is formed in s, let agents form a grand coalition and accept s^*. Next, suppose that a set of teams $\mathcal{M}$ exists in s. Let $\hat{s}$ be such that $\hat{s}_i = s_i$ for all $i \in N$. Due to the existence of the core, such $\hat{s}$ must be feasible for a grand coalition, i.e. $\sum_i \hat{s}_i \leq v(N)$.[18] Let agents form a grand coalition and accept $\hat{s}$. If $\hat{s}$ is a core allocation, then there is positive probability the process reaches some $s^* \in \mathscr{C}_\Delta$ as shown above. If $\hat{s}$ is not a core allocation, then there exist J and s' such that J blocks $\hat{s}$. Let agents form J and accept s'. Following the discussion in the previous paragraph, we can show that there is positive probability the process reaches some $s^* \in \mathscr{C}_\Delta$. □

Proof of Lemma 4.2. First, observe that RHS of Equation (10) gives the minimum cost of a mistake over all mistakes in allocation s because of the concavity of $u(\cdot)$.[19] We will prove that this least-cost mistake is enough for the process to switch to another strict core allocation.

Recall that $i_\# \in \{i : s_i = s_{(1)}\}$, i.e. one of the richest agents. If allocation s satisfies Condition (9), then any transfer of Δ from $i_\#$ to another agent will result in a new strict core allocation.

[16] We mean by "let agents form J and accept s" that there is positive probability that coalition J and allocation s are chosen, and agents accept it. We assume that that event is realized in the dynamic.

[17] This comes from non-emptiness of $\mathscr{C}_\Delta$. For $s \in \mathscr{C}_\Delta$,

$$\sum_{i \in J} s_i \geq v(J) + \Delta, \quad \text{and} \quad s_i \geq v(\{i\}) + \Delta \quad \forall i \notin J.$$

Summing up all inequalities, we have

$$v(N) \geq \sum_{i \in N} s_i > v(J) + \sum_{i \notin J} v(\{i\}).$$

[18] Note that $\hat{s}$ is not necessarily a core allocation.

[19] Since s is core allocation, some player's share must decrease by any switch from s. Due to the concavity of $u(\cdot)$, the least cost is Δ decrease in the richest's share.

More specifically, let s' be such that $s'_{i_\#} = s_{(1)} - \Delta$, $s'_h = s_h + \Delta$ for some $h \in N$, and $s'_i = s_i$ otherwise. Note that if $s_{(1)} > s_{\min}$, we can choose h such that $s_h \leq s_{(1)} - 2\Delta$. Observe that s' satisfies Inequalities (2). Thus, the escaping-cost from s to s' is given by (10). And the resulting allocation s' is either with the richest agent claiming $s_{(1)} - \Delta$ or with one fewer richest agents claiming $s_{(1)}$.

Next, suppose that Condition (9) does not hold for allocation s. In other words, there exist at least one richest agent $i_\#$ and one coalition $J \in \mathcal{R}_{i_\#}$ such that

$$\sum_{i \in J} s_i = v(J) + \Delta.$$

Consider allocation s' such that $s'_{i_\#} = s_{i_\#} - \Delta$, $s'_h = s_h + \Delta$ for some $h \notin J$, and $s'_i = s_i$ for $i \notin \{i_\#, h\}$. Note that the cost to switch from s to s' is given by (10). Suppose that the process start with s and that the following events occur sequentially.

(i) Agents form a grand coalition and accept s'. This costs a waste of $u(s_{(1)}) - u(s_{(1)} - \Delta)$.

(ii) Let $s''_J \in S^J$ be such that $s''_j = s'_j$ for all $j \in J$. Agents in J form a coalition and accept s''_J. Note that the grand coalition is dissolved.

(iii) Let $\hat{s} \in \mathscr{S}$ be such that $\hat{s}_i = s''_i$ for all $i \in J$ and $\hat{s}_i = v(\{i\})$ otherwise. Agents forms a grand coalition and accept $\hat{s}$.

(iv) Let $i \notin J$ and $\tilde{s}_i = v(\{i\}) \in S^{\{i\}}$. Agent i forms a singleton team $\{i\}$ and switches from $\hat{s}$ to $\tilde{s}_i$. Note that the grand coalition is dissolved by the agent i's deviation.

(v) Let $s^* \in \mathscr{C}_\Delta$. Agents forms a grand coalition and accept s^*.

Observe that (ii)–(v) occur with positive probability even in the unperturbed dynamic. The process reaches any strict core allocation without cost (after reaching s'), and the least escaping-cost from s to s^* is again given by (10). □

Proof of Proposition 4.3. First, we show the 'only if' part. Let $h \in \{0, 1, \ldots, \bar{h}\}$ where $s_{\max} - \bar{h}\Delta = s_{\min} - \Delta$. Define

$$U_{s_{\max}-h\Delta} = \left\{ s \in \mathscr{C}_\Delta \,\middle|\, s_{(1)} = s_{\max} - h\Delta \right\}$$
$$U^c_{s_{\max}-h\Delta} = \mathscr{C}_\Delta \setminus U_{s_{\max}} \setminus U_{s_{\max}-\Delta} \setminus U_{s_{\max}-2\Delta} \ldots \setminus U_{s_{\max}-h\Delta}.$$

$U_{s_{\max}-h\Delta}$ is the set of strict core allocations with the richest agent claiming $s_{\max} - h\Delta$, and $U^c_{s_{\max}-h\Delta}$ is the set of strict core allocations in which the richest agent's share is at most $s_{\max} - (h+1)\Delta$. The proof of the 'only if' part is reduced to $\lim_{\eta \to 0} \pi^\eta(U_{s_{\min}}) = 1$. We will prove it by induction. First, we will consider $h = 0$ and show that

$$R(U^c_{s_{\max}}) \geq u(s_{\max} - \Delta) - u(s_{\max} - 2\Delta), \tag{12}$$

$$CR^*(U^c_{s_{\max}}) = u(s_{\max}) - u(s_{\max} - \Delta). \tag{13}$$

Lemma 4.2 shows that the radius of allocation s is given by Equation (10) and the process might fall in a basin of attraction of any strict core allocation. Together with the concavity of $u(\cdot)$, this gives $R(U^c_{s_{\max}})$ above. Recall that CR^* is the maximum of the (modified) least escaping-costs from $U_{s_{\max}}$ to $U^c_{s_{\max}}$ over all $s \in U_{s_{\max}}$. We will show that the least escaping cost from any state in $U_{s_{\max}}$ is given by (13). Choose $s^1 \in U_{s_{\max}}$. Lemma 4.2 implies that the leas-cost escape can cause the process switching to either some $s' \in U^c_{s_{\max}}$ or $s^2 \in U_{s_{\max}}$. In the case of switching to s', the least escaping cost to $U^c_{s_{\max}}$ is given by $R(s^1) = u(s_{\max}) - u(s_{\max} - \Delta)$, which is consistent to Equation (13). Suppose the case of s^2. Lemma 4.2 implies that s^2 has one fewer richest agents than s^1, i.e.

$$\left|\{s_i \in s^2 : s_i = s^1_{(1)}\}\right| = \left|\{s_i \in s^1 : s_i = s^1_{(1)}\}\right| - 1.$$

According to Lemma 4.2 again, the process can further move to either some $s'' \in U^c_{s_{\max}}$ or $s^3 \in U_{s_{\max}}$ by the least-cost mistake. In the case of s'', observe that

$$W(d(s^1, s'')) = R(s^1) + R(s^2),$$
$$OW(d(s^1, s'')) = R(s^2).$$

Thus, $W(d(s^1, s'')) - OW(d(s^1, s'') = R(s^1)$. Again, it is consistent to Equation (13). For s^3, now let us turn to a general discussion. Suppose that a sequence of the least-cost mistakes makes the process move from s^1 to s^2 to ... to s^k and then to $\hat{s}$, where $s^i \in U_{s_{\max}}$ for $1 \le i \le k$ and $\hat{s} \in U^c_{s_{\max}}$. Lemma 4.2 guarantees that such a sequence of mistakes exists. Since the number of the richest agents is finite, k must be finite. Let $d(s^1, \hat{s})$ denote a path induced by this sequence of mistakes. Its waste and offset are given by

$$W(d(s^1, \hat{s})) = \sum_{i=1}^{k} R(s^i), \qquad OW(d(s^1, \hat{s})) = \sum_{i=2}^{k} R(s^i).$$

Observe that

$$W(d(s^1, \hat{s})) - OW(d(s^1, \hat{s})) = R(s^1) = u(s_{\max}) - u(s_{\max} - \Delta). \tag{14}$$

Since the choice of $s^1 \in U_{s_{\max}}$ is arbitrary, Equation (14) implies that the modified Coradius of $U^c_{s_{\max}}$ is given by (13). The concavity of $u(\cdot)$ again implies that $R(U^c_{s_{\max}}) > CR^*(U^c_{s_{\max}})$. According to the modified Radius-Coradius theorem, $\lim_{\eta \to 0} \pi^\eta(U^c_{s_{\max}}) = 1$.

Now, we begin the main part of the induction discussion. We assume that the condition below is satisfied for $h \le \bar{h}$ and show that the same condition is satisfied for $h + 1$:[20]

For all $s \in \bigcup_{h' \le h} U_{s_{\max} - h'\Delta}$,

[20] In a subsequent dicussion, we also show that similar equations to (12) and (13) hold if (15) is satisfied.

$$\exists \hat{s} \in U^{c}_{s_{\max}-h\Delta} \text{ and } d(s,\hat{s}) \text{ such that } W(d(s,\hat{s})) - OW(d(s,\hat{s})) = R(s). \tag{15}$$

Note that we have shown that Condition (15) is satisfied for $h = 0$. Lemma 4.2 implies that from any $s^1 \in U_{s_{\max}-(h+1)\Delta}$ there exists path $d = \{(s^1, s^2), \ldots, (s^{k-1}, s^k), (s^k, \hat{s})\}$ with the following properties:

(I) $s^i \in U_{s_{\max}-(h+1)\Delta}$ for $1 \le i \le k$ and $\hat{s} \in U^{c}_{s_{\max}-(h+1)\Delta}$.

(II) $W((s^i, s^{i+1})) = R(s^i) = u(s_{\max} - (h+1)\Delta) - u(s_{\max} - (h+2)\Delta)$ for $1 \le i \le k-1$.

Observe that

$$W(d) = \sum_{i=1}^{k} R(s^i), \qquad OW(d) = \sum_{i=2}^{k} R(s^i).$$

This implies that, for all $s^1 \in U_{s_{\max}-(h+1)\Delta}$, there exist $\hat{s} \in U^{c}_{s_{\max}-h\Delta}$ and $d(s,\hat{s})$ such that

$$W(d(s,\hat{s})) - OW(d(s,\hat{s})) = R(s^1). \tag{16}$$

Together with Condition (15), the above observation implies that, for all $s^1 \in \bigcup_{h' \le h} U_{s_{\max}-h'\Delta}$, there exists path $d(s^1,\hat{s}) = d_1 \cup d_2$ where $d_1 = \{(s^1, s^2), \ldots, (s^l, s^{l+1})\}$ is defined in Assumption (15) and $d_2 = \{(s^{l+1}, s^{l+2}), \ldots, (s^{l+k}, \hat{s})\}$ satisfies properties (I) and (II) above. Observe that[21]

$$\begin{aligned} W(d(s^1,\hat{s})) - OW(d(s^1,\hat{s})) &= W(d_1) - OW(d_1) - R(s^{l+1}) + W(d_2) - OW(d_2) \\ &= R(s^1). \end{aligned} \tag{17}$$

Equations (16) and (17) imply that Condition (15) is satisfied for $h+1$. Furthermore, these equations imply that $\lim_{\eta \to 0} \pi^{\eta}(U^{c}_{s_{\max}-(h+1)\Delta}) = 1$. To see this, observe that

$$\begin{aligned} R(U^{c}_{s_{\max}-(h+1)\Delta}) &\ge u(s_{\max} - (h+2)\Delta) - u(s_{\max} - (h+3)\Delta) \\ &> u(s_{\max} - (h+1)\Delta) - u(s_{\max} - (h+2)\Delta) \ge CR^*(U^{c}_{s_{\max}-(h+1)\Delta}). \end{aligned}$$

We continue this induction discussion until $h = \bar{h}$, and it leads us to conclude that

$$\lim_{\eta \to 0} \pi^{\eta}(U^{c}_{s_{\max}-\bar{h}\Delta}) = 1.$$

The proof of the 'only if' part is complete by observing that $U^{c}_{s_{\max}-\bar{h}\Delta} = U_{s_{\min}}$.

[21] To see the first equality, suppose path d along which the process moves from $s^1 \in U_{s_{\max}}$ to $s^2 \in U_{s_{\max}-\Delta}$ and then to $s^3 \in U^{c}_{s_{\max}-\Delta}$. Let $d_1 = (s^1, s^2)$, $d_2 = (s^2, s^3)$ and $d = d_1 \cup d_2$. Observe that

$$\begin{aligned} &W(d) = R(s^1) + R(s^2), && W(d_1) = R(s^1), && W(d_2) = R(s^2), \\ &OW(d) = R(s^2), && OW(d_1) = 0, && OW(d_2) = 0. \end{aligned}$$

$W(d) - OW(d)$ can be rewritten as $W(d) - OW(d) = W(d_1) - OW(d_1) - R(s^2) + W(d_2) - OW(d_2)$.

Next, we show the ’if’ part, i.e. strict core allocations satisfying (11) are stochastically stable. By the way of contradiction, suppose that $s \in U_{s_{\min}}$ satisfying (11) is not stochastically stable. By the existence and the ’only if’ part which we proved, there exists some $s' \in U_{s_{\min}}$ that is stochastically stable. Note that the least costs to escape from s and from s' are identical and given by $R(s) = R(s') = u(s_{\min}) - u(s_{\min} - \Delta)$. Now, consider s'-tree $T^*(s')$ minimizing the stochastic potential of s'. The edge emanating from s in $T^*(s')$ must cost at least $R(s)$. Lemma 4.2 implies that there exists path $d(s', s)$ such that $W(d(s', s)) - OW(d) = R(s')$. Construct a new tree $T(s)$ by removing the edge from s in $T^*(s')$ and adding edges of $d(s', s)$ to $T^*(s')$. Then, observe that

$$\begin{aligned} W(s) \leq W(T(s)) &\leq W(T^*(s')) - R(s) + R(s') \\ &= W(T^*(s')) \\ &= W(s') \end{aligned}$$

This contradicts that s' is stochastically stable, but s is not. □

References

Agastya, M., 1999, “Perturbed Adaptive Dynamics in Coalition Form Games,” *Journal of Economic Theory* 89, 207–233.

Alós-Ferrer, C. and N. Netzer, 2010, “The logit-response dynamics,” *Games and Economic Behavior* 68, 413–427.

Aumann, R. J., 1959, “Acceptable points in general cooperative n-person games,” in A. W. Tucker and R. D. Luce eds. *Contributions to the Theory of Games, Volume IV*: Princeton University Press, 287–324.

Blume, L., 1993, “The statistical mechanics of strategic interaction,” *Games and Economic Behavior* 5, 387–424.

Chatterjee, K., B. Dutta, D. Ray, and K. Sengupta, 1993, “A Noncooperative Theory of Coalitional Bargaining,” *Review of Economic Studies* 60, 463–477.

Compte, O. and P. Jehiel, 2010, “The Coalitional Nash Bargaining Solution,” *Econometrica* 78, 1593–1623.

Foster, D. P. and H. P. Young, 1990, “Stochastic evolutionary game dynamics,” *Theoretical Population Biology* 38, 219–232.

Freidlin, M. and A. Wentzell, 1988, *Random perturbations of dynamical systems*: Springer Verlag, New York, 2nd edition.

Kandori, M., G. J. Mailath, and R. Rob, 1993, “Learning, Mutation, and Long Run Equilibria in Games,” *Econometrica* 61, 1003–1037.

Newton, J., 2012, “Recontracting and stochastic stability in cooperative games,” *Journal of Economic Theory* 147, 364–381.

Okada, A., 1996, “A Noncooperative Coalitional Bargaining Game with Random Proposers,” *Games and Economic Behavior* 16, 97–108.

——— 2011, “Coalitional bargaining games with random proposers: Theory and application,” *Games and Economic Behavior* 73, 227–235.

Sandholm, W. H., 2010, *Population Games and Evolutionary Dynamics*: MIT Press, 1st edition.

Sawa, R., 2013, “Coalitional stochastic stability in games, networks and markets.” Unpublished Manuscript.

Young, H. P., 1993, “The Evolution of Conventions,” *Econometrica* 61, 57–84.