Negotiation across Multiple Issues*

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April 8, 2014

Abstract

A single agreement on the allocation of payments from multiple issues requires unanimous consent of all parties involved. This framework applies to many real-world problems, such as cooperation in R&D and organizational behavior. We present a novel solution concept to the problem termed the multi-core, which is a generalization of the core. It is assumed that an agent knows the aggregate payoffs but is uniformed about their decomposition by issues. An agent consents to participate in the grand coalition if she can envision a decomposition of the proposed allocation for which each coalition to which she belongs derives greater benefit on each issue by cooperating with the grand coalition rather than operating unilaterally. We provide an existence theorem for the multi-core, and show that the multi-core increases cooperation relative to solving issues independently. In addition, the multi-core, where agents can take into account the specifics of the original issues, is a refinement of the core of the summation game, in which such information is ignored.

Keywords: Cooperative games, Issue linkage, Multi-issue bargaining, Multi-core.
JEL Classification: C71, C78, D74.

*We thank Shiri Alon, Geoffroy de Clippel, Eddie Dekel, Ran Eilat, Chaim Fershtman, Itzhak Gilboa, Yoram Halevy, Ehud Lehrer, Mike Peters, Anna Rubinchik, Ariel Rubinstein, David Wettstein and the participants of the seminar in UBC Economics department for fruitful discussion and insightful comments.

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1 Introduction

A common practice for collaborating firms is to form a joint venture, whereby a new firm is established. The joint venture is granted the sole responsibility for the joint activity, in spite of the fact that the participating firms are its actual owners. When interested in collaborating on multiple projects, a firm could form either a separate joint venture for each project or a single joint venture that is responsible for all projects, thereby linking the projects. In this work we show that issue linkage improves cooperation and affects the allocation of profits.

In the present work a group of agents is aspiring to solve several issues simultaneously. An agreement, in our setup is a single contract that divides the aggregate payoffs of all issues. We explore how such linkage of issues affects the set of acceptable aggregate allocations. This framework can accommodate several other multilateral, multi-issue bargaining situations, such as wage bargaining where an employer and workers sign a single contract regulating the performance of several tasks, coalition formation in parliamentary systems where the participating parties reach a single agreement on promulgating various governmental policies, and trade-environment linkage expressing the ongoing need for countries to sign one agreement including issues of international trade and environmental policy, see, for example, WTO (2014)).

We use cooperative games with transferable utility to model the multiple-issue problem. This we term the multi-game,[1] which is a reduced form approach to bargaining. In contrast to the standard approach of non-cooperative game theory that strongly depends on specific protocols,[2] the setup of cooperative game theory

[1]Assa et al. (2013) study a different problem in an environment with multiple games where every player must participate in exactly one game. A set of players, in this setting, may partition itself among the games to attain maximal benefits.

[2]See Section 6.1 for a review of the non-cooperative theory literature on multi-issue bargaining.

allows us to concentrate on the linkage of issues while removing other considerations that are limited to a particular context, providing a general perspective on the problem.

In standard cooperative game theory an issue is represented by a characteristic function that assigns a value to each coalition. This value is typically interpreted as the benefit to the coalition in the event it chooses to resolve the issue without the cooperation of the other agents. The most prominent solution to the single-issue problem is the core, which is the set of all feasible allocations such that each coalition derives greater benefit from cooperating with the grand coalition than from operating unilaterally. A trivial extension of the core to multiple issues is to require that the solution be composed of solutions in the cores of individual issues, so that the agents receive the sum of these individual allocations. This approach solves each issue separately, and thus forgoes the possibility of making use of issue linkage to enhance cooperation. Obviously in this case, a solution exists if only if the cores of all the individual issues are non-empty.

In the present work we propose a generalization of the core to multiple issues that does allow for issue linkage. This we term the multi-core.[3] The solution concept reflects the assumption that while each agent knows the aggregate payoffs of all agents, they are uninformed of the decomposition of the payoffs derived from individual issues. Therefore, in spite of the issues being independent, the lack of information links the issues to each other. Uncertainty regarding the decomposition may be inherent to the problem so that there is no way to track payments back to any specific issue (e.g. the wage of a worker performing various tasks) or else it may be a deliberate choice on the part of the agents desirous to avoid conflict (e.g. a joint venture for the development of several products). An implicit assumption is that the agents cannot recant on their agreement even if at some point in the

[3] The term multi-core appears in Hwang and Liao (2011). However, apart from the name there is no other real connection to the solution concept suggested here.

future the actual decomposition does become known. There are many methods to decompose aggregate payoffs. However, an agent will accept an allocation of wealth generated by the grand coalition only if the agent can be shown that no coalition to which she belongs is better off with regard to the payoff on any specific issue by operating alone. This decomposition of payoffs is referred to as a sufficient justification for this agent. An allocation is in the multi-core, if it can be justified by all agents.

A special instance of this situation is an allocation that can be decomposed into solutions for the individual issues such that all agents use the same justification. This implies that any solution that can be implemented by resolving each issue independently can also be obtained when issues are linked. Generally, however a justification for one agent may not be a justification for another. This is because each agent's only concern is for full compensation for the coalitions to which she belongs, having no interest in the compensation obtained by the other coalitions.[4] In fact, it is precisely when the agents have no common justification for supporting a given allocation that solving problems collectively is beneficial. In this case there may be no possibility to implement this allocation by an overriding agreement that covers all issues when discussed separately, yet, such an agreement may exist when the issues are linked.

Furthermore, there may be allocations in the multi-core even when some or all individual issues have an empty core. We provide a characterization of the non-empty multi-core employing the rubric of Bondareva (1963) and Shapley (1967) and use it to identify cases where at least one of the individual cores is empty while the multi-core is non-empty. In such cases, when resolving issues independently, there is a subset of agents that recognizes that it is under compensated, ruling out the possibility of reaching an agreement. However, when the agents are only informed

[4]This idea is a reminiscent of the Nash equilibrium solution concept, according to which each player only considers his own profitable deviations but not those of the other players.

of aggregate payments, each agent still views the formation of the grand coalition as a win-lose situation aware that some agents are inadequately compensated yet, believing that she is on the winning side.

A different approach to solving the multi-issue problem is the summation game whereby the values of each coalition in the individual games are summed together to form a new single game. It is well-known that the core of the summation game contains the sum of the cores of the individual games, thereby increasing cooperation. Bloch and de Clippel (2010) provide conditions for the core of the summation game to coincide with the sum of the cores of the individual games.[5] Fernández et al. (2002, 2004) introduce new solution concepts for multi-games that assume that the various games are weighted, though the weights are unknown. They show that their dominance core is generally a weaker concept than the core of the summation game.[6]

In the summation game (and its variants), issues are linked together under the presumption that agents are unaware that the problem is composed of multiple issues. Alternatively, one may interpret this as a situation in which the agents do know that there are multiple issues, but must operate with the same subset of agents in all issues.[7] In this paper we show that the multi-core is a refinement of the core of the summation game. While in both solution concepts the payments per issue are unknown, the multi-core can be interpreted as allowing coalitions to cooperate on a subset of issues. Therefore the multi-core is a stricter concept than

[5]A similar question is addressed by Kalai (1977) and Ponsati and Watson (1997) in the context of Nash bargaining. They both characterize solution concepts of the multi-issue Nash bargaining problem that are not affected by the underlying bargaining process being simultaneous or sequential.

[6]A similar approach can be found in the multi-objective cooperative game literature (see, for example, Tanino (2009)).

[7]Nax (2008) studies an environment where there are externalities between the issues. He proposes an extension of the core whereby a deviating coalition will partition itself in the manner that maximizes its wealth while constrained to deviate in all issues at once. Diamantoudi et al. (2013) explore the Shapley value in a similar environment.

that of the summation game.

The rest of the paper is organized as follows: Section 2 describes the basic framework and the solution concept of the multi-core followed by an example; Section 3 characterizes multi-games with non-empty multi-core; Section 4 shows the advantages of using the multi-core over resolving issues independently; Section 5 characterizes the relation between the multi-core and the core of the summation game. Section 6 contains a review the relevant literature on non-cooperative games. It also introduces a computer program that finds the multi-core of a multi-game. Finally, it discusses the relation between the multi-core and decision making under ambiguity. All proofs are relegated to the appendix.

2 The Model

2.1 Preliminaries

The problem under consideration is that of a group of agents $N = \{1, 2, \dots, n\}$ trying to reach unanimous consent on m issues. The aim is to understand when all agents would agree to cooperate on all issues, thereby forming the grand coalition. For convenience, we denote the set of non-empty coalitions by $P(N) \equiv \{S \neq \phi | S \subseteq N\}$, the set of coalitions that include Agent i by $P_i(N) \equiv \{S \cup \{i\} | S \subseteq N \backslash \{i\}\}$ and the set of non-empty coalitions that do not include Agent i by $P_{-i}(N) \equiv P(N) \backslash P_i(N)$.

This setting is explored in the framework of cooperative game theory in which a single game, $G = (N; v)$ is defined by a set of agents N and a single characteristic function v which assigns a real number to every non-empty coalition $S \in P(N)$ and zero to the empty set. Typically $v(S)$ is interpreted as the value attained by coalition S when operating independently. We extend this definition to our setting of multiple issues by defining the multi-game as a set of agents and a set of characteristic functions.

Definition 1. *An m-issue Multi-Game* $\tilde{G}$ *is a pair* $\tilde{G} = (N; V)$ *where* V *is a set of characteristic functions* $V = \{v_1, v_2, \ldots, v_m\}$ *such that for every* $j \in \{1, \ldots, m\}$, $v_j : P(N) \to \mathbb{R}$ *and* $v_j(\emptyset) = 0$.

For convenience, the single cooperative game defined by the j^{th} characteristic function of the multi-game $\tilde{G}$ is denoted by $\tilde{G}_j = (N; v_j)$.

As in standard cooperative game theory an efficient aggregate payoff vector of the multi-game $\tilde{G}$ allocates all available surplus among the agents.

Definition 2. *The allocation* $x \in \mathbb{R}^n$ *is an efficient aggregate payoff vector of* $\tilde{G} = (N; \{v_1, v_2, \ldots, v_m\})$ *if* $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} v_j(N)$.

A payoff vector x is in the core, $C(G)$ of a single game if it is efficient and there is no coalition that is strictly better off operating independently relative to operating within the grand coalition ($\forall S \in P(N)$, $\sum_{i \in S} x_i \geq v(S)$).

We present an extension of this well-known solution to the multi-game setup. The agents consider a payoff vector x, which contains their total payoff on all issues. A key aspect is the unspecified breakdown by issue. The following defines the set of possible breakdowns:

Definition 3. *The set of efficient decomposition matrices of an aggregate payoff vector* x *is*

$$\hat{Y}(\tilde{G}, x) = \{y \in \mathbb{R}^{n \times m} | \forall i \in N : \sum_{j=1}^{m} y_{i,j} = x_i, \forall v_j \in V : \sum_{k=1}^{n} y_{k,j} = v_j(N)\}$$

An aggregate payoff vector is in the multi-core if every agent has an efficient decomposition matrix that justifies her participation in the grand coalition.

Definition 4. *The multi-core of $\tilde{G}$, $MC(\tilde{G})$, is the set of efficient aggregate payoff vectors such that for every Agent i there exists an efficient decomposition matrix $y^i \in \hat{Y}(\tilde{G}, x)$ such that $\forall v_j \in V, \forall i \in N, \forall S \in P_i(N) : \sum_{k \in S} y^i_{k,j} \geq v_j(S)$. We refer to y^i as a justification matrix of Agent i regarding x.*

When applying the solution of the multi-core, it is assumed that the agents know the structure of the multi-game and the proposed aggregate payoff vector, but do not know the decomposition of these payoffs by issue. As a result, they have to decide whether to agree to the proposed vector based on a belief of its decomposition. Definition 3 ensures that the decomposition matrices that the agents envision conform to the available information. It guarantees that the decomposition of payoffs adds up to the proposed vector x and that the resources are exhausted in each issue. The latter restriction is a consequence of agents' knowledge of the resource allocation in each issue so that resources cannot be shifted from one issue to another.[8] Since all agents share the same information, their set of efficient decomposition matrices is the same.

Definition 4 certifies that y^i is a justification for Agent i to consent to x if the coalitions that include Agent i have no reason to block the formation of the grand coalition in any one of the issues. While many efficient decomposition matrices may exist, for each agent only one needs to satisfy the condition in Definition 4. Furthermore, justification matrices may differ among agents since, as mentioned above, each agent considers only coalitions in which she participates and disregards all others. This is demonstrated by the following example.

[8]Technically, this requirement is redundant. It is implied by the efficiency of x together with Definition 4's coalitional rationality (with respect to the grand coalition).

2.2 Example 1

Let $\tilde{G} = (\{1,2,3\}; \{v_1, v_2\})$ be a two-issue game with three agents where the characteristic functions are:

$$v_1(S) = \begin{cases} 0 & \text{if} \quad |S| = 1 \\ \frac{3}{4} & \text{if} \quad |S| = 2 \\ 1 & \text{if} \quad |S| = 3 \end{cases} \quad ; \quad v_2(S) = \begin{cases} 0 & \text{if} \quad |S| = 1 \\ 0 & \text{if} \quad |S| = 2 \\ 1 & \text{if} \quad |S| = 3 \end{cases}$$

The core of $\tilde{G}_1 = (N; v_1)$ is empty[9] and therefore can be interpreted as a difficult problem to resolve. The core of $\tilde{G}_2 = (N; v_2)$ includes every non-negative payoff vector whose elements add up to one, and therefore can be interpreted as an problem with an easy solution. While it is impossible to reach unanimous agreement on all issues when they are solved independently, such an agreement can be reached by linking the issues, since the multi-core of $\tilde{G}$ is non-empty. For example, the payoff vector in which every agent gets $\frac{2}{3}$ is in the multi-core. The following decomposition matrices, one for every agent, support such an aggregate payoff vector:

$$\mathbf{y^1} = \begin{pmatrix} \frac{2}{3} & 0 \\ \frac{1}{6} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix} \quad ; \quad \mathbf{y^2} = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} \\ \frac{2}{3} & 0 \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix} \quad ; \quad \mathbf{y^3} = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{2} \\ \frac{2}{3} & 0 \end{pmatrix}$$

Each decomposition matrix y^i allocates a total of one to each issue so that they all are efficient. Moreover, in both issues, according to y^i every coalition S to which agent i belongs achieves at least its value. Notice that, neither Agent 2 nor Agent 3 would be convinced by y^1 to consent to the proposed payoff vector since their payoffs in $\tilde{G}_1$ are too low (agents 2 and 3 receive together $\frac{1}{3}$ according to y^1

[9]Each pair must receive at least $\frac{3}{4}$ and so the total payoff must exceed $\frac{9}{8}$.

while they can obtain $\frac{3}{4}$ by forming independent coalitions).[10]

3 Non Emptiness of the Multi-Core

The celebrated Bondareva-Shapley Theorem (Bondareva (1963) and Shapley (1967)) presents a necessary and sufficient condition for the non-emptiness of the core of a standard cooperative game. In this section we provide a similar characterization of the multi-core.

3.1 Bondareva-Shapley Theorem

For all $S \in P(N)$, let $\chi^S \in \{0,1\}^n$ denote the characteristic vector of S, so that $\chi^S[i] = 1$ if $i \in S$ and $\chi^S[i] = 0$ otherwise.

Definition 5. *A function $\delta : P(N) \to \mathbb{R}_+$ is a system of balancing weights if $\sum_{S \in P(N)} \delta(S)\chi^S = \chi^N$.*

An interpretation of these weights is that each agent is endowed with one unit of time that can be divided among the different coalitions to which she belongs. A system of balancing weights is an allocation of the agents' time among the different coalitions, where $\delta(S)$ is the fraction of time devoted to coalition S by its members. Then $v(S)$ is the amount produced by coalition S when its members devote their entire time to it and $\delta(S)v(S)$ the proportional amount when S's members devote only $\delta(S)$ of their time to it.

Theorem 1 (Bondareva-Shapley Theorem)**.** *The core $C(G)$ is non-empty if and only if every system of balancing weights, $\delta(S)$, satisfies $v(N) \geq \sum_{S \in P(N)} \delta(S)v(S)$.*

[10] $MC(\tilde{G}) = \left\{ x \in [\frac{1}{2}, 1]^3 | x_1 + x_2 + x_3 = 2 \right\}$.

Thus, when the core is non-empty a planner trying to maximize production will instruct all agents to devote their entire time to the grand coalition. However, when the core is empty the planner prefers a different allocation of the agents' time as there is a more beneficial allocation whose payoff is greater than $v(N)$.

3.2 Systems of Balancing Multi-weights

The following definition adapts Definition 5 to the multi-game framework.

Definition 6. *A function* $\tilde{\delta} : P(N) \times N \times V \to \mathbb{R}_+$ *is a system of balancing multi weights if it satisfies the following two requirements,*

1. *Zero to Non-members:* $\forall v_j \in V, \forall i \in N, \forall S \in P_{-i}(N) : \tilde{\delta}(S, i, j) = 0$.
2. *Resource Exhaustion:* $\forall v_j \in V : \sum_{i \in N} \sum_{S \in P(N)} \tilde{\delta}(S, i, j)\chi^S = \chi^N$.

Once again considering the context of production wherein each agent is endowed with one unit of time per issue: in every issue $v_j \in V$, a general manager is in charge of allocating agents' time among junior managers who are the agents themselves. Such allocations, denoted $\{\alpha_{1j}, \ldots, \alpha_{nj}\}$, must satisfy $\sum_{i \in N} \alpha_{ij} = \chi^N$. The vector $\alpha_{ij} \in [0, 1]^n$ contains the fractions of time of all agents operating under the jurisdiction of junior manager (agent) i in issue j. Junior manager i then chooses the amount of time, $\tilde{\delta}(S, i, j)$ to be devoted to the various coalitions S in issue v_j. This is a system of balancing multi-weights if each junior manager's allocation satisfies the condition Zero to Non-members so that time is assigned only to coalitions in which the junior manager participates and $\alpha_{ij} = \sum_{S \in P(N)} \tilde{\delta}(S, i, j)\chi^S$ implying that the time assigned to the various coalitions exhausts the time allocated to her by the general manager.

We define two special types of balancing multi-weights:

Definition 7. *A system of balancing multi weights with Constant Shares is a system that satisfies* $\forall v_j, v_{j'} \in V : \alpha_{ij} = \alpha_{ij'}$. *The set of all such systems of balancing multi weights is denoted by* $\tilde{\Delta}$.

A system of balancing multi-weights with constant shares requires that the general manager's allocation be identical across issues, but it allows for junior manager allocations to differ across issues. The following definition restricts the allocations of junior managers to being the same across all issues.

Definition 8. *A system of balancing multi weights with Constant Allocations is a system that satisfies* $\forall v_j, v_{j'} \in V : \tilde{\delta}(S, i, j) = \tilde{\delta}(S, i, j')$. *The set of all such systems of balancing multi weights is denoted by* $\hat{\Delta}$.

The set of systems of balancing multi-weights with constant allocations is a subset of the set of systems of balancing multi-weights with constant shares.

3.3 Example: Three Systems of Balancing Multi-weights

Suppose the general manager in the first issue ascribes to every agent half of his own time and a quarter of the time of the two other agents, $\alpha_{11} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $\alpha_{21} = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ and $\alpha_{31} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. In the second issue the general manager allocates each agent one unit of her own time and no time to any of the others, $\alpha_{12} = (1, 0, 0)$, $\alpha_{22} = (0, 1, 0)$ and $\alpha_{32} = (0, 0, 1)$.

In this example only the allocation of the first junior manager (Agent 1) is specified. Junior Manager 1 needs to allocate $\alpha_{11} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ in the first issue and $\alpha_{12} = (1, 0, 0)$ in the second. She may exhaust her resources by allocating one quarter to both $\{1, 2\}$ ($\tilde{\delta}(\{1, 2\}, 1, 1) = \frac{1}{4}$) and to $\{1, 3\}$ ($\tilde{\delta}(\{1, 3\}, 1, 1) = \frac{1}{4}$) in the first issue, and one unit to $\alpha_3 = \alpha_{31} = \alpha_{32} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$) in the second issue.[11]

[11] This system of balancing multi-weights should be completed by specifying the allocations of the other two junior managers so that the Zero to Non-members and the Resource Exhaustion requirements are fulfilled.

In a system of balancing multi-weights with constant shares the general manager's allocations should be identical across issues. For example, $\alpha_1 = \alpha_{11} = \alpha_{12} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $\alpha_2 = \alpha_{21} = \alpha_{22} = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ and $\alpha_3 = \alpha_{31} = \alpha_{32} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$). In this case there are no special requirements made upon junior managers besides Zero to Non-members and Resource Exhaustion. Lastly, a system of balancing multi-weight with constant allocations further requires that junior managers make the same allocations across issues. For example, if $\alpha_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ then constant allocations for Junior Manager 1 may be $\tilde{\delta}(\{1,2\}, 1, 1) = \tilde{\delta}(\{1,2\}, 1, 2) = \frac{1}{4}$ and $\tilde{\delta}(\{1,3\}, 1, 1) = \tilde{\delta}(\{1,3\}, 1, 2) = \frac{1}{4}$.

3.4 Non-Emptiness of the Multi-Core

Next, we present a characterization of multi-games with non-empty multi-cores.

Theorem 2. *The multi-core of $\tilde{G}$ is non-empty if and only if every $\tilde{\delta}(S,i,j) \in \tilde{\Delta}$ satisfies*

$$\sum_{j=1}^{m} v_j(N) \geq \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{S \in P(N)} \tilde{\delta}(S,i,j) v_j(S)$$

Theorem 2 affirms that the multi-core is non-empty if and only if whenever the general manager's allocation abides by the constant shares condition (Definition 7), production is maximized when junior managers instruct agents to devote their entire time to the grand coalition in all issues.

In the proof, the problem is translated into a linear program that minimizes the total amount of payoffs subject to two types of constraints: one guaranteeing that each agent decomposes the aggregate payoffs correctly and the other that coalitional rationality holds. Issue efficiency follows from $\sum_{j=1}^{m} v_j(N)$ creating the upper boundary of the total payments and coalitional rationality with respect to the grand coalition creates the lower boundary. The multi-core is non-empty if and only if the minimal amount of payoffs needed to solve this problem is no greater than the total amount of resources available (the sum of values of the grand coalitions across all issues).

The objective function is defined on the aggregate payoffs while the constraints are defined on the elements of the decomposition matrices. The next step in the proof is to use the constraints on the decomposition to obtain an equivalent problem for which both the objective function and the constraints are defined over the decomposition matrices. The dual program is the maximization of the weighted sum of the values of the coalitions subject to a set of equality constraints on these weights. We prove that the set of weights that satisfy these constraints is equivalent to the set of all systems of balancing multi-weights with constant shares. This concludes the proof since the duality theorem implies that the multi-core is non-empty if and only if the value of this dual linear program is no greater than the sum of the values related to the grand coalition across all issues.

4 The Rent for Linkage

A solution to the unlinked multi-game is the sum over the solutions in the cores of the single issues, $C(\tilde{G}) = \{ \sum_{j=1}^{m} x^j \big| \forall v_j \in V : x^j \in C(\tilde{G}_j) \}$.[12] Obviously, when the issues are unlinked, the agents will be aware of how much they would get in each issue. In this section we compare the multi-core to the set of solutions of the unlinked multi-game, showing how this information affects the prospects of cooperation.

4.1 Non-Emptiness of $C(\tilde{G})$

The following proposition presents the necessary and sufficient conditions for the formation of a grand coalition when issues are solved independently.

[12] If $\exists v_j \in V$ such that $C(\tilde{G}_j) = \emptyset$ then $C(\tilde{G}) = \emptyset$.

Proposition 3. *The following statements are equivalent:*
(i) $\forall v_j \in V : C(\tilde{G}_j) \neq \emptyset$.
(ii) $\forall \bar{\delta}(S,i,j) \in \bar{\Delta} : \sum_{j=1}^{m} v_j(N) \geq \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{S \in P(N)} \bar{\delta}(S,i,j) v_j(S)$.
(iii) There exists an aggregate payoff vector x and a justification matrix y such that all agents can justify x using y.

The first part of the proof concerning the equivalence of *(i)* and *(ii)* relying on the standard Bondareva-Shapley Theorem shows that $\bar{\delta}(S,i,j)$ corresponds to $\delta_j(S)$ and vice versa. The second part of the proof, concerning the equivalence of *(i)* and *(iii)* is based on the observation that a matrix whose columns are allocations in the cores of the corresponding issues can serve as a justification matrix for all agents and that a column of a justification matrix that is common to all agents is an allocation in the core of the corresponding single issue.

4.2 Multi-games where $C(\tilde{G}) = \emptyset$

According to statement *(iii)* in Proposition 3 issue-by-issue cooperation is achievable only if there is a common justification matrix for all agents. Because the decomposition of payoffs is unspecified, it allows agents to have subjective justifications for supporting an allocation even when no common rational justification exists.

Theorem 2 and statement *(ii)* in Proposition 3 show that if there is no solution in the multi-core there is no solution to the unlinked multi-game as well, since $\tilde{\Delta} \subseteq \bar{\Delta}$.[13] The expression of this inclusion in the production interpretation is that the general manager in the linked problem is restricted to allocating the same amount of resources across issues to the junior managers, whereas the general manager of the unlinked problem is free to choose any set of allocations that

[13] Emptiness of the multi-core does not rule out the possibility of cooperation being achieved on a subset of issues when they are discussed one at a time. But then there is also a solution in the multi-core when the multi-game problem is defined over this subset of games.

she wishes. This suggests that agents' lack of knowledge of the decomposition of aggregate payoffs corresponds to the inability of the general manager to distinguish between issues.

The next example demonstrates that the multi-core can bring about cooperation even if cooperation cannot be achieved in any single issue considered independently (as opposed to Example 1 where one of the issues had a non-empty core). Consider the 4-agent 2-issue multi-game $\tilde{G}(\{1,2,3,4\},\{v_1,v_2\})$

$$v_1(S) = \begin{cases} 9 & \text{if} \quad S \in \{S \subset N | \{1,2\} \subset S\} \\ 10 & \text{if} \quad |S| = N \\ 1 & \text{if} \quad otherwise \end{cases} ; v_2(S) = \begin{cases} 9 & \text{if} \quad S \in \{S \subset N | \{3,4\} \subset S\} \\ 10 & \text{if} \quad |S| = N \\ 1 & \text{if} \quad otherwise \end{cases}$$

The cores of both issues are empty.[14] Therefore, there is no issue-by-issue solution to this problem.

The same conclusion could be attained by realizing that the following system of balancing multi-weights violates *(ii)* of Proposition 3:

$$\begin{aligned}
&\delta(\{1,2\},1,1) = \delta(\{1,2,3\},1,1) = \delta(\{1,2,4\},1,1) = \frac{1}{3} \\
&\delta(\{3\},3,1) = \delta(\{4\},4,1) = \frac{2}{3} \\
&\delta(\{3,4\},3,2) = \delta(\{1,3,4\},3,2) = \delta(\{2,3,4\},3,2) = \frac{1}{3} \\
&\delta(\{1\},1,2) = \delta(\{2\},2,2) = \frac{2}{3} \\
&\text{Otherwise} \qquad \delta(S,i,j) = 0
\end{aligned}$$

[14]The core of Issue v_1 is empty as agents 3 and 4 must get at least 1 each, andagents 1 and 2 must get at least 9 together, adding up to more than the value of the grand coalition which is 10. For similar reasons the core of Issue v_2 is also empty.

Then,

$$\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{S\in P(N)}\delta(S,i,j)v_j(S) = 2\times(3\times\frac{1}{3}\times 9+2\times\frac{2}{3}\times 1) = \frac{62}{3} > 20 = \sum_{j=1}^{m} v_j(N)$$

Nevertheless, the allocation $x = (5,5,5,5)'$ is in the multi-core since it is supported by the following justifications matrices

$$\mathbf{y^1} = \mathbf{y^3} = \begin{pmatrix} 4 & 1 \\ 5 & 0 \\ 1 & 4 \\ 0 & 5 \end{pmatrix} \quad ; \quad \mathbf{y^2} = \mathbf{y^4} = \begin{pmatrix} 5 & 0 \\ 4 & 1 \\ 0 & 5 \\ 1 & 4 \end{pmatrix}.$$

Theorem 2 implies that since the multi-core is non empty, there is no system of balancing multi-weights that both violates condition *(ii)* of Proposition 3 and satisfies the constant shares property. Indeed, δ above does not satisfy the constant shares property.[15]

4.3 Multi-games where $C(\tilde{G}) \neq \emptyset$

Similar to statement *(iii)* in Proposition 3, the next proposition uses the observation that a matrix whose columns are all allocations in the cores of the corresponding issues can serve as a justification matrix for all agents.

Proposition 4. $C(\tilde{G}) \subseteq MC(\tilde{G})$.

Proposition 4 ascertains that the set of allocations that can be agreed upon when issues are linked is larger than when issues are considered independently. We say that the multi-core is effective when the former set is strictly larger, and ineffective when the sets are the same.

[15]The corresponding α_{ij}s are $\alpha_{11} = [1,1,\frac{1}{3},\frac{1}{3}]'$, $\alpha_{21} = [0,0,0,0]'$, $\alpha_{31} = [0,0,\frac{2}{3},0]'$, $\alpha_{41} = [0,0,0,\frac{2}{3}]'$, $\alpha_{12} = [0,0,\frac{2}{3},0]'$, $\alpha_{22} = [0,0,\frac{2}{3},0]'$, $\alpha_{32} = [1,1,\frac{1}{3},\frac{1}{3}]'$, $\alpha_{42} = [0,0,0,0]'$.

4.3.1 Convex Games

A class of games whose cores are non-empty is the class of convex games. These games satisfy $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for every pair of coalitions $S, T \subseteq N$. The next proposition shows that in cases where the multi-issue problem is composed of issues that are all convex the multi-core is ineffective.

Proposition 5. *Let $\tilde{G}$ be a multi-game where for every $v_j \in V$, $\tilde{G}_j$ is convex. The multi-core of $\tilde{G}$ is ineffective.*

When a game is convex, an agent has a higher incentive to join a coalition the larger it is,[16] making it relatively easy to support formation of a grand coalition. Proposition 5 determines that when all issues are easy to solve, there is no rent for linkage.

4.3.2 Superadditive Games

A game is superadditive if for every $S \cap T = \phi$, $v(S) + v(T) \leq v(S \cup T)$. Supperadditivty will certainty apply when two coalitions that merge, still have the option to behave as they did when they were separate, so their total payoff does not decrease. We use the decomposition Lemma from Gayer and Persitz (2014) to show that if three agents encounter any number of superadditive games with non-empty cores any solution in the multi-core can be obtained by solving the issues separately.[17]

Proposition 6. *Let $\tilde{G} = (\{1, 2, 3\}; V)$ be an m-issue cooperative game with three agents. If $\forall v_j \in V$, $\tilde{G}_j$ is superadditive with a non-empty core then the multi-core is ineffective.*

[16] It can be shown that a game is convex if and only if it satisfies the condition that $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ $\forall i$ and $S \subseteq T \subseteq N$.

[17] Example 1 demonstrates that this result does not extend to the case where the core of one of the games is empty.

This result does not extend to the case of four agents as demonstrated in the following example of two supperadditive games with non-empty core:

$$v_1(S) = \begin{cases} 0 & \text{if} \quad |S| \leq 2, S \notin \{\{2,4\},\{3,4\}\} \\ \frac{1}{2} & \text{if} \quad S \in \{\{2,4\},\{3,4\}\} \\ \frac{1}{2} & \text{if} \quad |S| = 3, S \neq \{1,2,3\} \\ 1 & \text{if} \quad S \in \{\{1,2,3\},\{1,2,3,4\}\} \end{cases} \quad ; \quad v_2(S) = \begin{cases} 0 & \text{if} \quad S \notin \{\{2,3,4\},\{1,2,3,4\}\} \\ \frac{3}{4} & \text{if} \quad S = \{2,3,4\} \\ 1 & \text{if} \quad |S| = 4 \end{cases}$$

The core of the first issue includes only the vector $(0, \frac{1}{2}, \frac{1}{2}, 0)'$.[18] The core of the second issue includes many elements, all restricting Agent 1's payoff to be no more than $\frac{1}{4}$ since agents 2, 3 and 4 must get at least $\frac{3}{4}$ together (by $v_2(\{2,3,4\})$). Thus, the set of payoff vectors that can be represented as a sum of members of the cores of the first and the second issues are characterized by allocating no more than $\frac{1}{4}$ to Agent 1. Nevertheless, the vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})'$ which divides the payoffs equally among the agents is in the multi-core.[19] This example determines that the multi-core may offer additional desirable solutions even when all games are superadditive. In fact both games in the example are totally balanced[20] so that a slightly stronger result that the multi-core may be effective even if the games are totally balanced can be established (provided that the issues involve more than 3 agents).

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[18]The value of x_4 must be zero since $v_1(\{1,2,3,4\}) = v_1(\{1,2,3\}) = 1$. Moreover, the values of x_2 and x_3 must be $\frac{1}{2}$ each since $v_1(\{2,4\}) = v_1(\{3,4\}) = \frac{1}{2}$ showing that $(0, \frac{1}{2}, \frac{1}{2}, 0)'$ is unique.

[19]This is confirmed by the following justification matrices:

$$y^1 = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix} \quad ; \quad y^2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix} \quad ; \quad y^3 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad ; \quad y^4 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} \\ \frac{1}{8} & \frac{3}{8} \\ \frac{1}{2} & 0 \end{pmatrix}$$

[20]A subgame of $G = (N; v)$ is a game $G_T(T; v^T)$ where $\emptyset \neq T \subseteq N$ and $v^T(S) = v(S)$ for all $S \subseteq T$. A coalitional game G is totally balanced if every subgame of G has a non-empty core.

5 The Core of the Summation Game

The summation game induced by a multi-game $\tilde{G}$ is a single coalitional game whose characteristic function is the sum of all characteristic functions in $\tilde{G}$.

Definition 9. *The Summation Game $S\tilde{G} = (N; v_{\tilde{G}})$ induced by $\tilde{G} = (N; V)$, is a cooperative game where $v_{\tilde{G}}(S) = \sum_{v_j \in V} v_j(S)$ for every coalition $S \subseteq N$.*

The core of the summation game is the core of the single game induced by the multi-game. When considering the grand coalition in the summation game, the agents disregard the original issues that compose the problem, and act as if they face a single issue.

This is demonstrated in Example 1, in which the summation game is $(\{1,2,3\}, v_{\tilde{G}})$

$$v_{\tilde{G}}(S) = \begin{cases} 0 & |S| = 1 \\ \frac{3}{4} & |S| = 2 \\ 2 & |S| = 3 \end{cases}$$

whose core consists of all the non-negative payoff vectors in which all the elements are at most $\frac{5}{4}$ that add up to 2.[21] Thus, the multi-core of $\tilde{G}$ is a strict subset of the core of $S\tilde{G}$ (see footnotes 10 and 21). Consider the aggregate payoff whereby agents 1 and 2 get 1 each, while Agent 3 gets 0. In this case according to the multi-core solution concept, Agent 3 realizes that one of the coalitions consisting of another agent and herself is not rewarded enough in the first issue, and therefore she can not justify this aggregate payoffs.[22] This type of consideration about the original structure of issues is totally absent when solving the core of the

[21]Every pair must get at least $\frac{3}{4}$ and the total surplus is 2. Therefore, no individual can get more than $\frac{5}{4}$. Individual rationality accounts for the lower bound, so that $C(S\tilde{G}) = \Big\{x \in [0, \frac{5}{4}]^3 | x_1 + x_2 + x_3 = 2\Big\}$.

[22]Coalition rationality for Agent 3 in the first issue entails that both $y^3_{1,1} + y^3_{3,1} \geq \frac{3}{4}$ and $y^3_{2,1} + y^3_{3,1} \geq \frac{3}{4}$, and together with issue efficiency this implies that $y^3_{3,1} \geq \frac{1}{2}$.

summation game.

In spite of these differences, there is a clear connection between these the multi-core and the core of the summation game, which is established in the next proposition.

Proposition 7. *$MC(\tilde{G}) \subseteq C(S\tilde{G})$.*

The proof sums over the payments of the agents in S according to Agent i's decomposition matrix, for some $i \in S$, to show that their total payment is weakly higher than $v_{\tilde{G}}(S)$.[23] This corroborates that the core of the summation game contains the multi-core (and as shown above this inclusion can be strict).

Propositions 4 and 7 together establish that the multi-core falls between the sum of the cores of the individual games and the core of the summation game. This is a reflection of the underlying information the agents are assumed to have according to each solution concept. When considering the sum of the core of the individual games, the agents are aware of both the structure and the payoff vector of each individual issue. In the summation game, the agents are assumed to know only the aggregate structure and the aggregate payoffs, but have no knowledge of their breakdowns by issues. The multi-core represents a hybrid information structure in which the agents are aware of the characteristic function of the individual games but have only a subjective assessment of the payoff vectors that are attached to each game.

Proposition 7 presents a sufficient condition for the emptiness of the multi-core

[23]A solution in the multi-core assumes that there are decomposition matrices that for every coalition, all members are convinced that the coalition is not better off on its own in each issue. For Proposition 7 to hold, this requirement may be weakened so that for each coalition there exists a member that is convinced that in each issue this coalition is not better off on its own.

is emptiness of the core of the summation game.[24]

Further insight about the gap between these two solution concepts can be gained by the following proposition.

Proposition 8. *The core of the summation game $C(S\tilde{G})$ is non-empty if and only if every $\hat{\delta}(S,i,j) \in \hat{\Delta}$ satisfies*

$$\sum_{j=1}^{m} v_j(N) \geq \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{S \in P(N)} \hat{\delta}(S,i,j) v_j(S)$$

As in the proof of Proposition 3, this proof also relies directly on the standard Bondareva-Shapley Theorem. The importance of this proposition (as well as that of Proposition 3) is that it provides necessary and sufficient conditions for non-emptiness using the terms of balancing multi-weights, making it comparable to Theorem 2. Specifically, Definitions 7 and 8 entail that $\hat{\Delta} \subseteq \tilde{\Delta}$. Therefore, by Theorem 2 and Proposition 8, if the multi-core is empty while the core of the summation game is not, then all balanced multi-weights with constant allocations satisfy the condition in Proposition 8 but at least one system of balanced multi-

[24] The following example shows that this is not a necessary condition. Let $\tilde{G} = (\{1,2,3,4\}; \{v_1, v_2\})$ where

$$v_1(S) = \begin{cases} 0 & \text{if} \quad |S| = 1 \text{ or } S \in \{\{1,2\},\{1,3\},\{2,3\}\} \\ \frac{2}{3} & \text{if} \quad S \in \{\{1,4\},\{2,4\},\{3,4\}\} \text{ or } |S| = 3 \\ 1 & \text{if} \quad |S| = 4 \end{cases} \quad ; \quad v_2(S) = \begin{cases} 0 & \text{if} \quad |S| = 1, |S| = 2 \\ \frac{5}{6} & \text{if} \quad |S| = 3 \\ 1 & \text{if} \quad |S| = 4 \end{cases}$$

and therefore,

$$S\tilde{G} = \begin{cases} 0 & \text{if} \quad |S| = 1 \text{ or } S \in \{\{1,2\},\{1,3\},\{2,3\}\} \\ \frac{2}{3} & \text{if} \quad S \in \{\{1,4\},\{2,4\},\{3,4\}\} \\ \frac{3}{2} & \text{if} \quad |S| = 3 \\ 2 & \text{if} \quad |S| = 4 \end{cases}$$

The core of the $S\tilde{G}$ contains the single payoff vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})'$, which is not in the multi-core (since Agent 4 requires at least half in each issue). Therefore by Proposition 7, $MC(\tilde{G}) = \emptyset$.

weights with constant shares violates it.[25]

In the production context, it is possible to map the available information in the problem to the restrictions on the general and junior managers. Ignorance regarding the structure of the game (ignorance of the characteristic functions that represent the issues) corresponds to restricting the junior managers to allocate the same amount of time to the same coalitions in the various issues, whereas ignorance regarding the decomposition of payoffs corresponds to restricting the general manager to allocate the same shares of the agent time to the junior managers across issues. In the core of the summation game, both the decomposition of payoffs and the structure of the multi-game are unknown so that both the general manager and the junior managers are constrained when choosing their optimal allocation of the agents time among coalitions. In the multi-core the agents are ignorant about the decomposition of payoffs, but are aware of the structure of the multi game, so that only the general manager is constrained. Finally, when solving the issues separately, all information is available so that both constraints are removed.[26] In the case discussed above, allocating all resources to production by the grand coalition is the best course of action when both general and junior managers are constrained but no longer optimal once removing the restriction on the junior managers.

[25]The example in Footnote 24 is such an instance where all systems of balanced multi-weights with constant allocations satisfy the required condition but there is a system of balanced multi-weights with constant shares that violates it. One such system is $\tilde{\delta}(\{1,2,3\},1,1) = \tilde{\delta}(\{1,3,4\},1,1) = \tilde{\delta}(\{1,2,3\},1,2) = \tilde{\delta}(\{1,3,4\},1,2) = \tilde{\delta}(\{1,2,3,4\},4,1) = \tilde{\delta}(\{2,4\},4,1) = \tilde{\delta}(\{1,2,4\},4,2) = \tilde{\delta}(\{2,4,3\},4,2) = \frac{1}{3}$ and all other values of $\tilde{\delta}(S,i,j)$ are zero.

[26]Note that the restrictions on the junior managers imply that on the general manager. This is also true for the information interpretation since knowing the decomposition of payoffs is meaningless without knowledge of the structure of the game.

6 Short Discussions

6.1 Issue Linkage in the Non-Cooperative Literature

Modelling bargaining using non-cooperative game theory requires an explicit specification of the bargaining procedure. Most of the literature on non-cooperative bargaining over multiple issues modifies the alternating offers procedure introduced by Rubinstein (1982) to account for multiple issues. In this framework, Fershtman (1990, 2000) and Busch and Horstmann (1997, 1999a) show that agents may have opposite preferences on the order in which issues are discussed since it influences the outcomes. In a slightly different bargaining procedure Winter (1997) shows that the important issues should be discussed first to guarantee efficiency. In these works the agenda is set before the bargaining process begins. In contrast, Inderst (2000), In and Serrano (2003, 2004) and In (2006) study a model where the agenda is determined during the bargaining process, whereby each agent in his turn proposes an allocation of one or more pies. The responder can then accept or reject the offer with the risk of negotiations breaking down. The main issues in these literature are the multiplicity of equilibria and the existence of equilibira where agendas are admitted with considerable delay. In a similar setup with incomplete information about time preferences Bac and Raff (1996), and Busch and Horstmann (1999b) show that delay may occur as a by-product of patient types separating themselves from impatient ones.

Although not directly related to bargaining the literature of mechanism design in the context of the private-values buyer-seller problem is nevertheless relevant. When trade involves multiple objects it is possible to exploit the structure to increase the probability of efficient trade. The idea is that linking problems allows the designer to "punish" agents that do not trade enough, thereby reducing their inherent tendency to exaggerate their valuation for the objects (e.g. McAfee et al.

(1989), Avery and Hendershott (2000), Eilat and Pauzner (2011)) and Fang and Norman (2010, 2011)).

The applied literature on multiple issue negotiations is usually concerned with international relations. Blonski and Spagnolo (2003) and Spagnolo (2001) consider multiple international issues, each modeled as a repeatedly played prisoner's dilemma game. It is shown that cooperation is easier to sustain when problems are linked, since linkage allows for a broader set of punishments of deviators. Using a different framework, Conconi and Perroni (2002) show that the number of linked issues has an ambiguous effect on the stability of agreements. While more issues bring about less objections, these objections are harder to dismiss.

6.2 Relation to Uncertainty Aversion

The cooperative game theory literature typically ignored the aspect of ambiguity (as opposed to the non-cooperative game theory literature, see for example, Dow and Werlang (1994), Lo (1996) and Marinacci (2000)). An exception is Fernández et al. (2002, 2004) that considers a problem with multiple criteria that should be weighted according to specific weights that are unknown to the agents.

In our context, ambiguity is instigated by the agents not knowing the breakdown of the aggregate payoffs to payoffs from individual games, allowing them to envision any decomposition in the set of efficient decomposition matrices (see Definition 3). If an efficient decomposition matrix y^i is not a sufficient justification for Agent i, there is an issue v_j and a coalition S ($i \in S$) such that $\sum_{k \in S} y^i_{k,j} < v_j(S)$. Namely, y^i cannot serve as a justification for Agent i if it facilitates an action that is more profitable than x_i for Agent i (holding $y^i_{k,l}$ fixed for all $v_l \neq v_j$ and $k \in N$). However, when y^i can serve as a justification for Agent i no such action exists. Therefore a justification matrix is the worst possible implementations of x from Agent i's point of view. In the multi-core solution it is assumed that the agents maintain that x was implemented according to their justification matrix, which is consistent with them being pessimistic or uncertainty averse.

6.3 The Code

The present work is supplemented with a Matlab package that implements the main solution concepts that were discussed above. For a given multi-game this software can verify whether $C(\tilde{G})$, $MC(\tilde{G})$ and $C(S\tilde{G})$ are empty, and for a given aggregate payoff vector x, it can confirm if $x \in C(\tilde{G})$, $x \in MC(\tilde{G})$ and $x \in C(S\tilde{G})$.

The package is user friendly. The user first needs to set the Matlab path to the place that stores the code folder (using the Set Path option with Add Folder with Subfolders). Writing multi_core in the Matlab command window will initiate the program. The user will then we prompted to specify the multi-game. Once the multi-game is inserted, the user we need choose between the option of existence analysis or analysis of a specific aggregate vector. If the first option is selected, the calculation takes place and the answer appears on the Matlab command window. Otherwise, the user is asked to specify the aggregate vector and then the answer is provided.

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Appendix

Theorem 2

Proof. Consider the following linear program that minimizes the sum of aggregate payoffs subject to each player having a decomposition matrix by which all coalitions to which she belongs have no incentive to deviate in any one of the issues,

$$\min_{x\in\mathbb{R}^n} \sum_{i=1}^{n} x_i$$

subject to:

$$\forall i,l \in N : \sum_{j=1}^{m} y_{l,j}^{i} = x_l$$

$$\forall i \in N, \forall v_j \in V, \forall S \in P_i(N) : \sum_{l\in S} y_{l,j}^{i} \geq v_j(S)$$

The constraints include n^2 equalities and $n \times m \times 2^{n-1}$ inequalities. There exists $x \in \mathbb{R}^n$ that satisfies the constraints and since the objective function is linear and bounded from below, there exists a solution to the problem, which we denote by $\bar{x}$. Most importantly, due to the efficiency requirement, the multi-core is non-empty if and only if $\sum_{i=1}^{n} \bar{x}_i \leq \sum_{j=1}^{m} v_j(N)$.

The n equalities of the justification matrix of Agent 1 are substituted into the objective function. The other $n^2 - n$ equalities of the other agents are used to isolate the values ascribed by their justification matrices to the payoff vector in the m^{th} issue. These values are then substituted into the corresponding inequalities, leading to the following linear problem,

$$\min_{y^1\in\mathbb{R}^{n\times m}} \sum_{l=1}^{n}\sum_{j=1}^{m} y_{l,j}^{1}$$

subject to:

$$\forall i \in N, \forall v_j \in V \backslash \{v_m\}, \forall S \in P_i(N) : \sum_{l\in S} y_{l,j}^{i} \geq v_j(S)$$

$$\forall S \in P_1(N) : \sum_{l\in S} y_{l,m}^{1} \geq v_m(S)$$

$$\forall i \in N \backslash \{1\}, \forall S \in P_i(N) : \sum_{l\in S}\sum_{j=1}^{m} y_{l,j}^{1} - \sum_{l\in S}\sum_{j=1}^{m-1} y_{l,j}^{i} \geq v_m(S)$$

This problem in matrix form becomes,

$$\min_{y \in \mathbb{R}^p} c'y$$
$$\text{subject to:} \quad Ay \geq b$$

where y and c are column vectors of length $p = nm + (n-1)[n(m-1)]$. The first nm elements of y are obtained by converting y^1 into a vector by stacking its m columns (issues) one on top of the other, an operation called vectorization. The next $n(m-1)$ elements of y are obtained by vectorizing the first $m-1$ columns of y^2 followed by the vectorization of the first $m-1$ columns of y^3 and so on. To preserve the previous objective function, c is defined such that the first nm cells are ones while the other $n-1[n(m-1)]$ cells are zeros. Therefore, $c'y = \sum_{l=1}^{n} \sum_{j=1}^{m} y^1_{l,j}$.

Let L^i be an $2^{n-1} \times n$ matrix where the rows are the characteristic vectors corresponding to the coalitions that include Agent i. Let $\mu_i(S)$ be an ordering on these characteristic vectors. Thus, the $\mu_i(S)$ row of L^i consists of the characteristic vector of coalition S ($\mu_i(S)$ equals zero for all S such that $i \notin S$).[27] We also use the function $\mu_i^{-1}(l)$ ($l \in \{1, \ldots, 2^{n-1}\}$) which is the coalition in the l^{th} place in the ordering for Agent i. Note that for every $l \in \{1, \ldots, 2^{n-1}\}$, $\mu_i(\mu_i^{-1}(l)) = l$. Let BL^i be a block matrix of size $(m-1)2^{n-1} \times (m-1)n$ where there are $(m-1) \times (m-1)$ blocks, each of size $2^{n-1} \times n$, such that $m-1$ blocks of L^i occupy the diagonal and there are zeros elsewhere.[28] For Agent 1, FL is a block matrix of size $m2^{n-1} \times p$,

[27] The choice of the specific ordering is inconsequential to the rest of the proof. For example, we can order the row vectors by their binary values. Hence, if $N = \{1, 2, 3\}$ then

$$L^1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad ; \quad L^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad ; \quad L^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

For instance, $\mu_1(\{1,3\}) = 2$, $\mu_2(\{1,3\}) = 0$ and $\mu_3(\{1,3\}) = 3$.

[28] For example, if there are three issues,

$$BL^1 = \begin{pmatrix} L^1 & 0 \\ 0 & L^1 \end{pmatrix} \quad ; \quad BL^2 = \begin{pmatrix} L^2 & 0 \\ 0 & L^2 \end{pmatrix} \quad ; \quad BL^3 = \begin{pmatrix} L^3 & 0 \\ 0 & L^3 \end{pmatrix}$$

obtained from the concatenation of two matrices. On the left, $m \times m$ blocks of $2^{n-1} \times n$ where m blocks of L^1 occupy the diagonal and the other $m^2 - m$ blocks are zeros. On the right, an $m2^{n-1} \times (n-1)[n(m-1)]$ matrix of zeros.[29] For the other agents ($i \in \{2, \dots, n\}$), let ZBL^i be an $(m-1)2^{n-1} \times p$ block matrix that has BL^i starting at the $nm + (i-2)n(m-1) + 1$ column and zeros elsewhere.[30] Let ML^i be an $2^{n-1} \times p$ block matrix, of blocks of size $2^{n-1} \times n$, such that it has m blocks of L^i in the first m blocks, $m-1$ blocks of $-L^i$ starting from the $m + (i-2)(m-1) + 1^{th}$ block and zeros elsewhere.[31] Finally, let A be an $nm2^{n-1} \times p$ block matrix where FL occupies the first $m2^{n-1}$ rows, followed by ZBL^2 and ML^2 and so on to ZBL^n and ML^n. Then, Ay is the left hand side of the inequality constraints of the linear programming problem.

Let b be an $nm2^{n-1}$ length vector where the first 2^{n-1} are the values of the coalitions that include Agent 1 in Issue 1 ordered by $\mu_1(S)$, the next 2^{n-1} are the

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[29] For example, if there are three agents and three issues, FL is the following 12×21 matrix,

$$FL = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

[30] For example, if there are three agents and three issues, ZBL^2 is the following 8×21 matrix,

$$ZBL^2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

[31] For example, if there are three agents and three issues, then

$$ML^2 = \begin{pmatrix} L^2 & L^2 & L^2 & -L^2 & -L^2 & 0 & 0 \end{pmatrix}$$

$$ML^3 = \begin{pmatrix} L^3 & L^3 & L^3 & 0 & 0 & -L^3 & -L^3 \end{pmatrix}$$

values of the coalitions that include Agent 1 in Issue 2 ordered by $\mu_1(S)$ and so on such that the 2^{n-1} elements starting from place $[(i-1)m+(j-1)]\times 2^{n-1}+1$ are the values of the coalitions that include Agent i in Issue j ordered by $\mu_i(S)$. Formally, $b[k]=v_j(\mu_i^{-1}(l))$ where $i=\lceil\frac{k}{m2^{n-1}}\rceil$, $j=\lceil\frac{k-(i-1)m2^{n-1}}{2^{n-1}}\rceil$ and $l=k-(i-1)m2^{n-1}-(j-1)2^{n-1}$. This completes the matrix notation of the linear program. The multi-core is non-empty if and only if $c'\bar{y}\leq\sum_{j=1}^{m}v_j(N)$ where $\bar{y}$ is the solution to the linear program.

The asymmetric dual problem is,

$$\max_{z\in\mathbb{R}^{nm2^{n-1}}} b'z$$
$$\text{subject to:}\quad A'z=c \quad , \quad z\geq 0$$

The Strong Duality theorem states that in a primal-dual pair of linear programs, if either the primal or the dual problem has an optimal feasible solution, then so does the other and the two optimal objective values are equal. Since the primal problem, in this case, has a solution, so does its asymmetric dual problem, denoted by $\bar{z}$. Moreover, $b'\bar{z}=c'\bar{y}$. Thus, the multi-core is non-empty if and only if $b'\bar{z}\leq\sum_{j=1}^{m}v_j(N)$. Equivalently, the multi-core is non-empty if and only if every $z\in\mathbb{R}_+^{nm2^{n-1}}$ such that $A'z=c$ satisfies $b'z\leq\sum_{j=1}^{m}v_j(N)$.

Next we characterize the set $Z=\{z\in\mathbb{R}_+^{nm2^{n-1}}|A'z=c\}$. The first step is to add, for every issue j (except the last one), the corresponding rows of all agents (except Agent 1) to those of Agent 1. Recall that A' is a $p\times nm2^{n-1}$ matrix where $p=nm+(n-1)[n(m-1)]$. For every $k\in\{0,\ldots,m+(n-1)(m-1)-1\}$ we denote the block consisting of the n rows from $nk+1$ to $n(k+1)$ in A' by CC_k. For every $j\in\{1,\ldots,m-1\}$ let $D(j)=\sum\limits_{k\geq m|k \mod m-1=j} CC_k$ be the sum of the blocks corresponding to issue v_j over all agents $i\in N\backslash\{1\}$.[32] Now, let

[32]For simplicity let $km \mod m=m$ instead of $km \mod m=0$.

$\tilde{A}'$ be the matrix that results from replacing, for every $v_j \in V \backslash \{v_m\}$, CC_{j-1} by $CC_{j-1} + D(j)$. Since this is an elementary row operation on A', and since for every $i > nm$, $c[i] = 0$, the solutions set for the linear equations system continues to be $Z = \{z \in \mathbb{R}_+^{nm2^{n-1}} | \tilde{A}'z = c\}$.[33]

We denote the k^{th} element of z by $z[k]$. Let us define the function $\tilde{\delta}(S,i,j)$ in the following manner: if $i \notin S$ then $\tilde{\delta}(S,i,j) = 0$ and if $i \in S$ then $\tilde{\delta}(S,i,j) = z\big[(i-1)m2^{n-1} + (j-1)2^{n-1} + \mu_i(S)\big]$.[34] By Lemma 9 below, for every $z \in Z$, $\tilde{\delta}(S,i,j)$ is a system of balancing multi weights with constant shares as defined in definitions 6 and 7. Moreover, together with Lemma 10, this construction facilitates a one-to-one and onto correspondence between Z and $\tilde{\Delta}$.

Recall that we have shown that the multi-core is non-empty if and only if every $z \in Z$ satisfies $b'z \le \sum_{j=1}^{m} v_j(N)$, or, explicitly, the multi-core is non-empty if and only if every $z \in Z$ satisfies

$$\sum_{j=1}^{m} v_j(N) \ge \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{S \in P_i(N)} z\big[(i-1)m2^{n-1} + (j-1)2^{n-1} + \mu_i(S)\big] v_j(S)$$

[33]To illustrate, if there are three agents and three issues, then

$$A' = \begin{pmatrix} L^{1'} & 0 & 0 & 0 & 0 & L^{2'} & 0 & 0 & L^{3'} \\ 0 & L^{1'} & 0 & 0 & 0 & L^{2'} & 0 & 0 & L^{3'} \\ 0 & 0 & L^{1'} & 0 & 0 & L^{2'} & 0 & 0 & L^{3'} \\ 0 & 0 & 0 & L^{2'} & 0 & -L^{2'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L^{2'} & -L^{2'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L^{3'} & 0 & -L^{3'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L^{3'} & -L^{3'} \end{pmatrix}$$

$$(CC_0 \to CC_0 + CC_3 + CC_5) \qquad \Downarrow \qquad (CC_1 \to CC_1 + CC_4 + CC_6)$$

$$\tilde{A}' = \begin{pmatrix} L^{1'} & 0 & 0 & L^{2'} & 0 & 0 & L^{3'} & 0 & 0 \\ 0 & L^{1'} & 0 & 0 & L^{2'} & 0 & 0 & L^{3'} & 0 \\ 0 & 0 & L^{1'} & 0 & 0 & L^{2'} & 0 & 0 & L^{3'} \\ 0 & 0 & 0 & L^{2'} & 0 & -L^{2'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L^{2'} & -L^{2'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L^{3'} & 0 & -L^{3'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L^{3'} & -L^{3'} \end{pmatrix}$$

[34]The function $\tilde{\delta}(S,i,j)$ is well defined as it is defined for every combination of $S \in P(N)$, $v_j \in V$ and $i \in N$ and the index of z does not exceed its length.

Therefore, the multi-core is non-empty if and only if every system of balancing multi weights with constant shares satisfies

$$\sum_{j=1}^{m} v_j(N) \geq \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{S\in P_i(N)} \tilde{\delta}(S,i,j)v_j(S)$$

and since $\tilde{\delta}(S,i,j) = 0$ if $i \notin S$ then,

$$\sum_{j=1}^{m} v_j(N) \geq \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{S\in P(N)} \tilde{\delta}(S,i,j)v_j(S)$$

□

Lemma 9. *Let* $z \in Z$ *and set* $\tilde{\delta}(S,i,j)$ *to be zero if* $i \notin S$ *and* $\tilde{\delta}(S,i,j) = z\big[(i-1)m2^{n-1} + (j-1)2^{n-1} + \mu_i(S)\big]$ *otherwise. Then,* $\tilde{\delta}(S,i,j) \in \tilde{\Delta}$.

Proof. Since $i \notin S$ implies $\tilde{\delta}(S,i,j) = 0$, "zero to non members" is satisfied.

Consider a typical equation in the first nm rows of $\tilde{A}'z = c$. Given an Agent i and Issue v_j,

$$\sum_{l=1}^{n}\sum_{q=1}^{m}\sum_{S\in P_l(N)} \tilde{A}'\Big[(j-1)n+i,(l-1)m2^{n-1}+(q-1)2^{n-1}+\mu_l(S)\Big] \times z\Big[(l-1)m2^{n-1}+(q-1)2^{n-1}+\mu_l(S)\Big] = 1$$

Note that for every $q \neq j$ the values of $\tilde{A}'$ in row $(j-1)n+i$ are zeros. Therefore,

$$\sum_{l=1}^{n}\sum_{S\in P_l(N)} \tilde{A}'\Big[(j-1)n+i,(l-1)m2^{n-1}+(j-1)2^{n-1}+\mu_l(S)\Big] \times z\Big[(l-1)m2^{n-1}+(j-1)2^{n-1}+\mu_l(S)\Big] = 1$$

By the definition above,

$$\sum_{l=1}^{n} \sum_{S \in P_l(N)} \tilde{A}' \Big[(j-1)n+i, (l-1)m2^{n-1} + (j-1)2^{n-1} + \mu_l(S) \Big] \times \tilde{\delta}(S,l,j) = 1$$

Extending the summation to the entire collection of coalitions,

$$\sum_{l=1}^{n} \sum_{S \in P(N)} \tilde{A}' \Big[(j-1)n+i, (l-1)m2^{n-1} + (j-1)2^{n-1} + \mu_l(S) \Big] \times \tilde{\delta}(S,l,j) = 1$$

Given an Agent l, by the definition of L^l,

$$\tilde{A}' \Big[(j-1)n+i, (l-1)m2^{n-1} + (j-1)2^{n-1} + \mu_l(S) \Big] = 1 \qquad \text{iff} \qquad i \in S$$

and therefore can be substituted by $\chi^S[i]$. Moreover, if $l \notin S$ then $\tilde{\delta}(S,l,j) = 0$. Hence, for a given Agent i and Issue v_j,

$$\sum_{l=1}^{n} \sum_{S \in P(N)} \chi^S[i] \times \tilde{\delta}(S,l,j) = 1$$

By Definition 6, $\sum_{l=1}^{n} \alpha_{lj}[i] = 1$. Since this is true for every Agent i and Issue v_j, $\tilde{\delta}(S,i,j)$ satisfies "resources exhaustion".

Consider a typical equation in rows $nm+1$ to p of $\tilde{A}'z = c$. Given Player $i > 1$, Player l and Issue $v_j \neq v_m$,

$$\sum_{S \in P_i(N)} \tilde{A}' \big[nm + (i-2)(m-1)n + (j-1)n + l, (i-1)m2^{n-1} + (j-1)2^{n-1} + \mu_i(S) \big] \times z \big[(i-1)m2^{n-1} + (j-1)2^{n-1} + \mu_i(S) \big] =$$
$$\sum_{S \in P_i(N)} \tilde{A}' \big[nm + (i-2)(m-1)n + (j-1)n + l, (i-1)m2^{n-1} + (m-1)2^{n-1} + \mu_i(S) \big] \times z \big[(i-1)m2^{n-1} + (m-1)2^{n-1} + \mu_i(S) \big]$$

By the definition of $\tilde{\delta}(S, i, j)$,

$$\sum_{S \in P_i(N)} \tilde{A}'\big[nm + (i-2)(m-1)n + (j-1)n + l, (i-1)m2^{n-1} + (j-1)2^{n-1} + \mu_i(S)\big] \times \tilde{\delta}(S, i, j) =$$
$$\sum_{S \in P_i(N)} \tilde{A}'\big[nm + (i-2)(m-1)n + (j-1)n + l, (i-1)m2^{n-1} + (m-1)2^{n-1} + \mu_i(S)\big] \times \tilde{\delta}(S, i, m)$$

By the definition of L^i,

$$\tilde{A}'\big[nm + (i-2)(m-1)n + (j-1)n + l, (i-1)m2^{n-1} + (j-1)2^{n-1} + \mu_i(S)\big] = 1 \quad \text{iff} \quad l \in S$$

and therefore can be substituted by $\chi^S[l]$, then

$$\sum_{S \in P_i(N)} \chi^S[l] \times \tilde{\delta}(S, i, j) = \sum_{S \in P_i(N)} \chi^S[l] \times \tilde{\delta}(S, i, m)$$

Since $\tilde{\delta}(S, i, j) = 0$ if $i \notin S$ then,

$$\sum_{S \in P(N)} \chi^S[l] \times \tilde{\delta}(S, i, j) = \sum_{S \in P(N)} \chi^S[l] \times \tilde{\delta}(S, i, m)$$

establishing that for every Agent $i > 1$, Agent l and Issue $v_j \neq v_m$, $\alpha_{ij}[l] = \alpha_{im}[l]$. Hence, for every Agent $i > 1$, and Issue v_j, $\alpha_{ij} = \alpha_{im}$. Moreover, for every player $i > 1$ and for every two issues v_j and $v_{j'}$, $\alpha_{ij} = \alpha_{ij'}$. Finally, since $\tilde{\delta}(S, i, j)$ satisfies "resources exhaustion", for every two issues v_j and $v_{j'}$, $\alpha_{1j} = \alpha_{1j'}$. Hence, $\tilde{\delta}(S, i, j)$ satisfies "constant shares". □

Lemma 10. *Let $\tilde{\delta} \in \tilde{\Delta}$ and set $z\big[k\big] = \tilde{\delta}(\bar{S}(k), \bar{\imath}(k), \bar{\jmath}(k))$ where $\bar{\imath}(k) = \lceil \frac{k}{m2^{n-1}} \rceil$, $\bar{\jmath}(k) = \lceil \frac{k-(\bar{\imath}(k)-1)m2^{n-1}}{2^{n-1}} \rceil$ and $\bar{S}(k) = \mu^{-1}_{\bar{\imath}(k)}(k - (\bar{\imath}(k)-1)m2^{n-1} - (\bar{\jmath}(k)-1)2^{n-1})$. Then, $\tilde{A}'z = c$.*

Proof. By "resource exhaustion", for every Agent l and every Issue $v_j \in V$, we have $\sum_{i\in N} \alpha_{ij}[l] = 1$. By Definition 6, $\sum_{i\in N}\sum_{S\in P(N)} \chi^S[l] \times \tilde{\delta}(S,i,j) = 1$. By the "zero for non members", $\sum_{i\in N}\sum_{S\in P_i(N)} \chi^S[l] \times \tilde{\delta}(S,i,j) = 1$. For every pair of agents i and l and for every Issue v_j, by the definition of L^i, $\chi^S[l]$ can be substituted by $\tilde{A}'\big[(j-1)n+l,(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)\big]$. Hence,

$$\sum_{i\in N}\sum_{S\in P_i(N)} \tilde{A}'\big[(j-1)n+l,(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)\big] \times \tilde{\delta}(S,i,j) = 1.$$

Let $k = (i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)$. By the construction of z, $\bar{\imath}(k) = i$, $\bar{\jmath}(k) = j$ and $\bar{S}(k) = S$ and $z[k] = \tilde{\delta}(S,i,j)$.[35] Therefore, for every Agent i, every Issue $v_j \in V$ and every coalition S such that $i \in S$, $\tilde{\delta}(S,i,j)$ can be replaced by $z[(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)]$.

$$\sum_{i\in N}\sum_{S\in P_i(N)} \tilde{A}'\big[(j-1)n+l,(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)\big]$$
$$\times z[(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)] = 1.$$

Note that every entry of the type $\tilde{A}'\big[(j-1)n+l,(i-1)m2^{n-1}+(h-1)2^{n-1}+\mu_i(S)\big]$ where $j \neq h$ equals zero. Also, for every row $r \leq nm$, $c[r] = 1$. Therefore, for every row $r \leq nm$, the constructed z satisfies $\tilde{A}'z = c$.

Next, since $\tilde{\delta} \in \tilde{\Delta}$, for every two agents i and l and Issue $v_j \in V$, $\alpha_{ij}[l] = \alpha_{im}[l]$. By Definition 6, $\sum_{S\in P(N)} \chi^S[l] \times \tilde{\delta}(S,i,j) = \sum_{S\in P(N)} \chi^S[l] \times \tilde{\delta}(S,i,m)$. By the "zero for non members" condition, for $i \notin S$, $\tilde{\delta}(S,i,j) = 0$, and therefore,

[35]It can be easily seen that $k = (i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)$ is a one-to-one and onto correspondence between $\{1,\ldots,nm2^{n-1}\}$ and $N \times V \times P_i(N)$.

$\sum_{S\in P_i(N)} \chi^S[l] \times \tilde{\delta}(S,i,j) = \sum_{S\in P_i(N)} \chi^S[l] \times \tilde{\delta}(S,i,m)$. For every pair of agents $i > 1$ and l and for every Issue $v_j \neq v_m$, by the definition of L^i, $\chi^S[l]$ can be substituted by $\tilde{A}'\big[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)\big]$ and for issue m, $\chi^S[l]$ can be substituted by $-\tilde{A}'\big[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1}+(m-1)2^{n-1}+\mu_i(S)\big]$. Then,

$$\sum_{S\in P_i(N)} \Bigg\{ \tilde{A}'\Bigg[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1} + (j-1)2^{n-1}+\mu_i(S)\Bigg] \times \tilde{\delta}(S,i,j) + \tilde{A}'\Bigg[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1} + (m-1)2^{n-1}+\mu_i(S)\Bigg] \times \tilde{\delta}(S,i,m)\Bigg\} = 0$$

As we showed earlier, $z[(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)] = \tilde{\delta}(S,i,j)$ and $z[(i-1)m2^{n-1}+(m-1)2^{n-1}+\mu_i(S)] = \tilde{\delta}(S,i,m)$. Then,

$$\sum_{S\in P_i(N)} \Bigg\{ \tilde{A}'\Bigg[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1} + (j-1)2^{n-1}+\mu_i(S)\Bigg] \times z[(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)] + \tilde{A}'\Bigg[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1} + (m-1)2^{n-1}+\mu_i(S)\Bigg] \times z[(i-1)m2^{n-1}+(m-1)2^{n-1}+\mu_i(S)]\Bigg\} = 0$$

Note that every entry of the type $\tilde{A}'\big[nm+(i-2)(m-1)n+(j-1)n+l,x\big]$ where $x \neq (i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)$ and $x \neq (i-1)m2^{n-1}+(m-1)2^{n-1}+\mu_i(S)$ equals zero. Also, for every row $r > nm$, $c[r] = 0$. Therefore, for every row $r > nm$, the constructed z satisfies $\tilde{A}'z = c$, which concludes the proof. □

Proposition 3

if *(i)* then *(ii)*

Proof. For every system of balancing multi weights, $\bar{\delta}(S,i,j)$, define $\delta_j(S) = \sum_{i=1}^{n} \bar{\delta}(S,i,j)$. This is a system of balancing weights since by resource exhaustion, $\sum_{S\in P(N)} \delta_j(S)\chi^S = \chi^N$.

Suppose there exists $\bar{\delta}(S,i,j)$, a system of balancing multi weights such that

$$\sum_{j=1}^{m} v_j(N) < \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{S\in P(N)} \bar{\delta}(S,i,j)v_j(S)$$

Then, there exists $j \in \{1,\ldots,m\}$ such that

$$v_j(N) < \sum_{i=1}^{n}\sum_{S\in P(N)} \bar{\delta}(S,i,j)v_j(S)$$

Or,

$$v_j(N) < \sum_{S\in P(N)} \delta_j(S)v_j(S)$$

By the Bondareva-Shapley Theorem, $C(G_j) = \emptyset$. Thus, if $\forall j \in \{1,\ldots,m\}$: $C(G_j) \neq \emptyset$ then every $\bar{\delta}(S,i,j) \in \bar{\Delta}$ satisfies

$$\sum_{j=1}^{m} v_j(N) \geq \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{S\in P(N)} \bar{\delta}(S,i,j)v_j(S)$$

□

if *(i)* then *(iii)*

Proof. Suppose $\forall v_j \in V : C(\tilde{G}_j) \neq \emptyset$. Then there exist $\{x^1,\ldots,x^m\}$ where $x^j \in C\tilde{G}_j$. Consider the aggregate payoff vector $x = \sum_{j=1}^{m} x^j$. The matrix $Y = [x^1;\ldots;x^m]$ such that $x^j \in C\tilde{G}_j$ is an efficient decomposition matrix that justifies x for all agents, since each of its columns is in the core of the respective issue. Thus, if $\forall j \in \{1, \ldots, m\} : C(\tilde{G}_j) \neq \emptyset$ there exist an aggregate payoff vector x and a justification matrix Y such that all agents may justify x using Y. $\square$

if *(ii)* then *(i)*

Proof. For every system of balancing weights $\delta_j(S)$, define $\bar{\delta}(S, i, l)$ as follows,

1. If $v_l \neq v_j$ and $S \neq N$ then $\bar{\delta}(S, i, l) = 0$.
2. If $v_l \neq v_j$ and $S = N$ then $\bar{\delta}(N, i, l) = \frac{1}{n}$.
3. If $v_l = v_j$ then $\bar{\delta}(S, i, j) = \frac{\delta_j(S)}{|S|}$ if $i \in S$ and 0 otherwise.

Note that $\bar{\delta}(S, i, l)$ satisfies the zero to non members condition.

Also, for $v_l \neq v_j$,

$$\sum_{i \in N} \alpha_{il} = \sum_{i \in N} \sum_{S \in P(N)} \bar{\delta}(S, i, l) \chi^S = \sum_{i \in N} \bar{\delta}(N, i, l) \chi^N = \sum_{i \in N} \frac{1}{n} \chi^N = \chi^N$$

and for $v_l = v_j$

$$\sum_{i \in N} \alpha_{ij} = \sum_{i \in N} \sum_{S \in P(N)} \bar{\delta}(S, i, j) \chi^S = \sum_{S \in P(N)} \sum_{i \in S} \frac{\delta_j(S)}{|S|} \chi^S = \sum_{S \in P(N)} \delta_j(S) \chi^S = \chi^N$$

Therefore, $\bar{\delta}(S, i, l)$ also satisfies the resources exhaustion condition and $\bar{\delta}(S, i, l)$ is a system of balancing multi weights.

Suppose, on the contrary, that every $\bar{\delta}(S, i, j) \in \bar{\Delta}$ satisfies

$$\sum_{j=1}^{m} v_j(N) \geq \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{S \in P(N)} \bar{\delta}(S, i, j) v_j(S)$$

and that there exists an issue v_j such that $C(G_j) = \emptyset$. Then, by the Bondareva-Shapley Theorem, there exists a system of balancing weights, $\delta_j(S)$, such that

$v_j(N) < \sum_{S\in P(N)} \delta_j(S)v_j(S)$.

$$\sum_{l=1}^{m}\sum_{i=1}^{n}\sum_{S\in P(N)} \bar{\delta}(S,i,l)v_l(S) =$$

$$\sum_{l=1,l\neq j}^{m}\sum_{i=1}^{n}\sum_{S\in P(N)} \bar{\delta}(S,i,l)v_l(S) + \sum_{i=1}^{n}\sum_{S\in P(N)} \bar{\delta}(S,i,j)v_j(S) =$$

$$\sum_{l=1,l\neq j}^{m}\sum_{i=1}^{n}\frac{1}{n}v_l(N) + \sum_{i=1}^{n}\sum_{S\in P(N)} \bar{\delta}(S,i,j)v_j(S) =$$

$$\sum_{l=1,l\neq j}^{m} v_l(N) + \sum_{i=1}^{n}\sum_{S\in P(N)} \bar{\delta}(S,i,j)v_j(S) = \sum_{l=1,l\neq j}^{m} v_l(N) + \sum_{S\in P(N)}\sum_{i=1}^{n} \bar{\delta}(S,i,j)v_j(S) =$$

$$\sum_{l=1,l\neq j}^{m} v_l(N) + \sum_{S\in P(N)}\sum_{i\in S} \bar{\delta}(S,i,j)v_j(S) = \sum_{l=1,l\neq j}^{m} v_l(N) + \sum_{S\in P(N)}\sum_{i\in S} \frac{\delta_j(S)}{|S|}v_j(S) =$$

$$\sum_{l=1,l\neq j}^{m} v_l(N) + \sum_{S\in P(N)} \delta_j(S)v_j(S) > \sum_{l=1,l\neq j}^{m} v_l(N) + v_j(N) = \sum_{l=1}^{m} v_l(N)$$

contradiction. Therefore, if every $\bar{\delta}(S,i,j) \in \bar{\Delta}$ satisfies

$$\sum_{j=1}^{m} v_j(N) \geq \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{S\in P(N)} \bar{\delta}(S,i,j)v_j(S)$$

it must be that $\forall j \in \{1,\dots,m\} : C(G_j) \neq \emptyset$. □

if *(iii)* then *(i)*

Proof. Suppose that $x \in MC(\tilde{G})$ and that Y is a matrix that justify x for all agents. Denote the j^{th} column of Y by $Y_{\cdot j}$. Since Y is an efficient decomposition matrix, $\forall v_j \in V : \sum_{i=1}^{n} Y_{ij} = v_j(N)$, meaning that for every $v_j \in V$, $Y_{\cdot j}$ is an efficient payoff vector. Moreover, $\forall i \in N$, $\forall v_j \in V$ and $\forall S \in P_i(N)$, $\sum_{l\in S} Y_{lj} \geq v_j(S)$. Since this is true for every Agent i, the vector $Y_{\cdot j}$ satisfies coalitional rationality for issue v_j. Thus, $(Y_{\cdot 1}, \dots, Y_{\cdot m})$ is a collection of vectors that belong to the cores of the corresponding games ($Y_{\cdot j} \in C(\tilde{G}_j)$). $\forall v_j \in V : C(\tilde{G}_j) \neq \emptyset$. $\square$

Proposition 4

Proof. First, if $C(\tilde{G}) = \emptyset$ the proposition is vacuously true. Otherwise, let $x \in C(\tilde{G})$. Then, by definition, $x = \sum_{j=1}^{m} x^j$ where $\forall v_j \in V : x^j \in C(\tilde{G}_j)$. Consider the matrix $Y = [x^1, x^2, \ldots, x^m]$. By definition, Y is a decomposition matrix. Since $\forall v_j \in V : x^j \in C(\tilde{G}_j)$, Y is an efficient decomposition matrix. For the same reason, $\forall S \subseteq N, \forall v_j \in V : \sum_{i \in S} x_i^j \geq v_j(S)$. Since this condition is satisfied for all coalitions, the coalitional rationality condition is satisfied for all agents. Hence, Y justifies x for every agent $i \in N$ and therefore $x \in MC(\tilde{G})$. $\square$

Proposition 5

Proof. By Propositions 4 and 7, $C(\tilde{G}) \subseteq MC(\tilde{G}) \subseteq SC(\tilde{G})$. Since all the issues are convex by Lemma 11, $C(\tilde{G}) = SC(\tilde{G})$. Therefore, $C(\tilde{G}) = MC(\tilde{G})$, and the multi-core is ineffective. $\square$

Lemma 11. *Let $\tilde{G}$ be a multi-game for which every $v_j \in V$, $\tilde{G}_j$ is convex. Then, $C(\tilde{G}) = SC(\tilde{G})$.*

Proof. The lemma is proved by induction on the number of issues. Dragan et al. (1989) and Bloch and de Clippel (2010) show that if V is a set of two convex issues, $C(\tilde{G}) = SC(\tilde{G})$. Suppose that $C(\tilde{G}) = SC(\tilde{G})$ when V is a set of k convex issues and consider $\tilde{G}$ where V is a set of $k+1$ convex issues. Denote the multi-game that includes the first k issues of $\tilde{G}$ by $\tilde{G}^k$ and the summation game composed of these k issues by $S_k\tilde{G}$. Note that $S_k\tilde{G}$ is convex since the sum of convex games is convex. Therefore, the summation game induced by $\tilde{G}$ is a sum of two convex issues $S_k\tilde{G}$ and $\tilde{G}_{k+1}$. Thus, by Dragan et al. (1989) and Bloch and de Clippel (2010), $SC(\tilde{G}) = \{x + y | x \in C(S_k\tilde{G}), y \in C(\tilde{G}_{k+1})\}$. By the induction assumption, $C(\tilde{G}^k) = SC(\tilde{G}^k) = C(S_k\tilde{G})$ and therefore $SC(\tilde{G}) = \{x + y | x \in C(\tilde{G}^k), y \in C(\tilde{G}_{k+1})\} = C(\tilde{G})$, completing the proof. $\square$

Definition 10 (taken from Gayer and Persitz (2014))**.** *Let $F : P(N) \to \mathbb{R}_+$ be a system of weights. Let $W^F = \sum_{S \in P(N)} F(S)\chi^S$ denote the vector of weights induced by F. We say that F_1 and F_2 are W-equivalent if $W^{F_1} = W^{F_2}$. Denote the set of all W-equivalence classes by Γ. For every class $\gamma \in \Gamma$, denote the agents' weights by W^γ. For every characteristic function v and $\gamma \in \Gamma$ denote $T_v^\gamma \equiv \max_{F \in \gamma} \sum_{S \in P(N)} F(S)v(S)$.*

Proposition 6

Proof. Consider, with no loss of generality, an equivalence set γ such that $W^\gamma[1] \geq W^\gamma[2] \geq W^\gamma[3]$. By Lemma 14, for every characteristic function $v_j \in V$, there exists $F_j \in \gamma$ such that $\sum_{s \in P(N)} F_j(s)v_j(s) = T_{v_j}^\gamma$ and $F_j(\{2\}) = F_j(\{3\}) = F_j(\{2,3\}) = 0$. Alternatively, for every characteristic function $v_j \in V$, there exists $F_j \in \gamma$ such that $\sum_{s \in P_1(N)} F_j(s)v_j(s) = T_{v_j}^\gamma$. Let $x \in MC(\tilde{G})$ and let y^1 be the justification matrix of Agent 1. Therefore, for every $v_j \in V$ and for every coalition $s \in P_1(N)$, $\sum_{i \in s} y_{i,j}^1 \geq v_j(s)$. Then multiplying both sides of each inequality by the corresponding $F_j(s)$ and aggregating over all $s \in P_1(N)$ yields for every $v_j \in V$, $\sum_{s \in P_1(N)} F_j(s) \sum_{i \in s} y_{i,j}^1 \geq \sum_{s \in P_1(N)} F_j(s)v_j(s)$, or equivalently, for every $v_j \in V$, $\sum_{i \in N} y_{i,j}^1 \sum_{s \in P_1(N) \cap P_i(N)} F_j(s) \geq \sum_{s \in P(N)} F_j(s)v_j(s)$.

Since $\sum_{s \in P_1(N) \cap P_i(N)} F_j(s) = W^\gamma[i]$ for every Agent i and $\sum_{s \in P(N)} F_j(s)v_j(s) = T_{v_j}^\gamma$ the inequality becomes $\sum_{i \in N} y_{i,j}^1 W^\gamma[i] \geq T_{v_j}^\gamma$ for every $v_j \in V$. Aggregating over all the issues, $\sum_{v_j \in V} \sum_{i \in N} y_{i,j}^1 W^\gamma[i] \geq \sum_{v_j \in V} T_{v_j}^\gamma$, and changing the order of summation obtains, $\sum_{i \in N} W^{\gamma}[i] \sum_{v_j \in V} y^1_{i,j} \geq \sum_{v_j \in V} T^{\gamma}_{v_j}$. The justification matrix y^1 decomposes the aggregate payoff vector x, therefore $\sum_{i \in N} W^{\gamma}[i] x_i \geq \sum_{v_j \in V} T^{\gamma}_{v_j}$. Then by the decomposition Lemma in Gayer and Persitz (2014), the aggregate payoff vector x can be decomposed into m vectors $\{x^1, \dots, x^m\}$ such that for every $v_j \in V$, $x^j \in C(\tilde{G}_j)$ and $\sum_{v_j \in V} x^j = x$. Hence, $x \in MC(\tilde{G})$ implies $x \in C(\tilde{G})$ and the multi-core in ineffective. □

Lemma 12. *Let $G = (N; v)$ be a superadditive cooperative game. Let $F \in \gamma$ such that there exist two disjoint coalitions s and s' ($s \cap s' = \emptyset$). Define for all $t \in P(N)$,*

$$F_{s,s'}(t) = \begin{cases} F(t) - \min\{F(s), F(s')\} & \text{if} \quad t \in \{s, s'\} \\ F(t) + \min\{F(s), F(s')\} & \text{if} \quad t = s \cup s' \\ F(t) & \text{if} \quad \text{otherwise} \end{cases}$$

Then, $F_{s,s'} \in \gamma$ and $\sum_{t \in P(N)} F_{s,s'}(t) v(t) \geq \sum_{t \in P(N)} F(t) v(t)$.

Proof. First we show that $F_{s,s'} \in \gamma$,

$$\begin{aligned} \sum_{t \in P(N)} F_{s,s'}(t)\chi^t = & \sum_{t \in P(N) \setminus \{s,s',s \cup s'\}} F_{s,s'}(t)\chi^t + F_{s,s'}(s)\chi^s + F_{s,s'}(s')\chi^{s'} + F_{s,s'}(s \cup s')\chi^{s \cup s'} = \\ & \sum_{t \in P(N) \setminus \{s,s',s \cup s'\}} F(t)\chi^t + [F(s) - \min\{F(s), F(s')\}]\chi^s + \\ & [F(s') - \min\{F(s), F(s')\}]\chi^{s'} + [F(s \cup s') + \min\{F(s), F(s')\}]\chi^{s \cup s'} = \\ & \sum_{t \in P(N)} F(t)\chi^t + \min\{F(s), F(s')\}[\chi^{s \cup s'} - \chi^s - \chi^{s'}] = \sum_{t \in P(N)} F(t)\chi^t = W^{\gamma} \end{aligned}$$

Next it is shown that $\sum_{t \in P(N)} F_{s,s'}(t) v(t) \geq \sum_{t \in P(N)} F(t) v(t)$ for every supperadditive characteristic function v,

$$\sum_{t \in P(N)} F_{s,s'}(t)v(t) =$$

$$\sum_{t \in P(N)\setminus\{s,s',s\cup s'\}} F_{s,s'}(t)v(t) + F_{s,s'}(s)v(s) + F_{s,s'}(s')v(s') + F_{s,s'}(s \cup s')v(s \cup s') =$$

$$\sum_{t \in P(N)\setminus\{s,s',s\cup s'\}} F(t)v(t) + [F(s) - \min\{F(s), F(s')\}]v(s) +$$

$$[F(s') - \min\{F(s), F(s')\}]v(s') + [F(s \cup s') + \min\{F(s), F(s')\}]v(s \cup s') =$$

$$\sum_{t \in P(N)} F(t)v(t) + \min\{F(s), F(s')\}[v(s \cup s') - v(s) - v(s')] \geq \sum_{t \in P(N)} F(t)v(t)$$

□

Lemma 13. *Let $G = (\{1,2,3\}; v)$ be a three agents superadditive cooperative game such that $C(G) \neq \emptyset$, let γ be an equivalence class such that $W^\gamma[1] \geq W^\gamma[2] \geq W^\gamma[3]$ and let $F \in \gamma$ be such that $F(\{1\}) = 0$. Define for all $t \in P(N)$,*

$$F_{-23}(t) = \begin{cases} F(t) & \text{if} \quad |t| = 1 \\ F(t) - F(\{2,3\}) & \text{if} \quad |t| = 2 \\ F(t) + 2F(\{2,3\}) & \text{if} \quad |t| = 3 \end{cases}$$

Then, $F_{-23} \in \gamma$ and $\sum_{t \in P(N)} F_{-23}(t)v(t) \geq \sum_{t \in P(N)} F(t)v(t)$.

Proof. First we show that $F_{-23}(t) \geq 0$ for every coalition t, as required by Definition 10. By definition, $W^\gamma[1] \geq W^\gamma[2]$ if and only if $F(1) + F(1,2) + F(1,3) + F(1,2,3) \geq F(2) + F(1,2) + F(2,3) + F(1,2,3)$. Meaning that $W^\gamma[1] \geq W^\gamma[2]$ if and only if $F(1,3) \geq F(2) + F(2,3)$. Therefore, if $W^\gamma[1] \geq W^\gamma[2]$ then $F(1,3) \geq F(2,3)$. Similarly, if $W^\gamma[1] \geq W^\gamma[3]$ then $F(1,2) \geq F(2,3)$.

Now we show that $F_{-23} \in \gamma$.

$$\sum_{t \in P(N)} F_{-23}(t)\chi^t =$$

$$F_{-23}(\{1\})\chi^{\{1\}} + F_{-23}(\{2\})\chi^{\{2\}} + F_{-23}(\{3\})\chi^{\{3\}} + F_{-23}(\{1,2\})\chi^{\{1,2\}} +$$

$$F_{-23}(\{1,3\})\chi^{\{1,3\}} + F_{-23}(\{2,3\})\chi^{\{2,3\}} + F_{-23}(\{1,2,3\})\chi^{\{1,2,3\}} =$$

$$F(\{1\})\chi^{\{1\}} + F(\{2\})\chi^{\{2\}} + F(\{3\})\chi^{\{3\}} + [F(\{1,2\}) - F(\{2,3\})]\chi^{\{1,2\}} +$$

$$[F(\{1,3\}) - F(\{2,3\})]\chi^{\{1,3\}} + [F(\{2,3\}) - F(\{2,3\})]\chi^{\{2,3\}} +$$

$$[F(\{1,2,3\}) + 2F(\{2,3\})]\chi^{\{1,2,3\}} =$$

$$\sum_{t \in P(N)} F(t)\chi^t + F(\{2,3\})[2\chi^{\{1,2,3\}} - \chi^{\{1,2\}} - \chi^{\{1,3\}} - \chi^{\{2,3\}}] = \sum_{t \in P(N)} F(t)\chi^t = W^\gamma$$

Next, we show that $\sum_{t \in P(N)} F_{-23}(t)v(t) \geq \sum_{t \in P(N)} F(t)v(t)$,

$$\sum_{t \in P(N)} F_{-23}(t)v(t) =$$

$$F_{-23}(\{1\})v(\{1\}) + F_{-23}(\{2\})v(\{2\}) + F_{-23}(\{3\})v(\{3\}) + F_{-23}(\{1,2\})v(\{1,2\}) +$$

$$F_{-23}(\{1,3\})v(\{1,3\}) + F_{-23}(\{2,3\})v(\{2,3\}) + F_{-23}(\{1,2,3\})v(\{1,2,3\}) =$$

$$F(\{1\})v(\{1\}) + F(\{2\})v(\{2\}) + F(\{3\})v(\{3\}) + [F(\{1,2\}) - F(\{2,3\})]v(\{1,2\}) +$$

$$[F(\{1,3\}) - F(\{2,3\})]v(\{1,3\}) + [F(\{2,3\}) - F(\{2,3\})]v(\{2,3\}) +$$

$$[F(\{1,2,3\}) + 2F(\{2,3\})]v(\{1,2,3\}) =$$

$$\sum_{t \in P(N)} F(t)v(t) + F(\{2,3\})[2v(\{1,2,3\}) - v(\{1,2\}) - v(\{1,3\}) - v(\{2,3\})]$$

The inequality $2v(\{1,2,3\}) \geq v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})$ follows from the

core of $G = (N; v)$ being non-empty.[36] Thus, $\sum_{t \in P(N)} F_{-23}(t)v(t) \geq \sum_{t \in P(N)} F(t)v(t)$.
□

Lemma 14. *Let $G = (\{1, 2, 3\}; v)$ be a three agnets superadditive cooperative game such that $C(G) \neq \emptyset$ and let γ be an equivalence class such that $W^{\gamma}[1] \geq W^{\gamma}[2] \geq W^{\gamma}[3]$. There exists $F \in \gamma$ such that $\sum_{s \in P(N)} F(s)v(s) = T_v^{\gamma}$ and $F(\{2\}) = F(\{3\}) = F(\{2, 3\}) = 0$.*

Proof. The set $A \in \{F \in \gamma | \sum_{s \in P(N)} F(s)v(s) = T_v^{\gamma}\}$ is non-empty since every equivalence class γ is closed and $\sum_{s \in P(N)} F(s)v(s)$ is a linear function on γ and therefore continuous.

The proof is constructive. We take some $F \in A$ and use it to construct, in three steps, $\bar{F} \in A$ such that $\bar{F}(\{2\}) = \bar{F}(\{3\}) = \bar{F}(\{2, 3\}) = 0$.

The first step is to use F to construct $\tilde{F} \in A$ such that $\tilde{F}(\{2\}) = 0$. There are four cases. First, if $F(\{2\}) = 0$ we are done. Second, if $F(\{2\}) \neq 0$ and $F(\{1\}) = 0$, since $W^{\gamma}[1] \geq W^{\gamma}[2]$ then $F(1) + F(1, 2) + F(1, 3) + F(1, 2, 3) \geq F(2) + F(1, 2) + F(2, 3) + F(1, 2, 3)$ or $F(1, 3) \geq F(2) + F(2, 3)$. Thus, $F(1, 3) \geq F(2)$. By Lemma 12 we set $\tilde{F} = F_{\{1,3\},\{2\}}$ and get $\tilde{F} \in A$ and $\tilde{F}(\{2\}) = 0$. Third, if $F(\{2\}) \geq F(\{1\}) > 0$ then by Lemma 12 we set $\tilde{F} = F_{\{1\},\{2\}}$ and get $\tilde{F} \in A$ and $\tilde{F}(\{1\}) = 0$. Then, we start this step from the beginning. Finally, if $F(\{2\}) \geq F(\{1\}) > 0$ then by Lemma 12 we set $\tilde{F} = F_{\{1\},\{2\}}$ and get $\tilde{F} \in A$ and $\tilde{F}(\{2\}) = 0$.

The second step is to construct $\hat{F} \in A$ from $\tilde{F}$ such that $\hat{F}(\{3\}) = 0$, which is similar to the construction of the first step.

The third and last step is to use $\hat{F}$ ($\hat{F} \in A$, $\hat{F}(\{2\}) = 0$ and $\hat{F}(\{3\}) = 0$) to construct $\bar{F} \in A$ such that $\bar{F}(\{2\}) = \bar{F}(\{3\}) = \bar{F}(\{2, 3\}) = 0$. Again, there

[36] Suppose that $x \in C(G)$ and denote by x_i the payoff of Agent i. Then, since $x_1 + x_2 \geq v\{1, 2\}$, $x_1 + x_3 \geq v\{1, 3\}$ and $x_2 + x_3 \geq v\{2, 3\}$, we can assert that $x_1 + x_2 + x_3 \geq \frac{1}{2} \times [v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\})]$. By the efficiency of x, $2v(\{1, 2, 3\}) \geq v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\})$.

are four cases. First, if $\hat{F}(\{2,3\}) = 0$ we are done. Second, if $F(\{2,3\}) \neq 0$ and $F(\{1\}) = 0$, by Lemma 13 we set $\bar{F} = F_{-23}$ and get $\bar{F} \in A$ and $\bar{F}(\{2,3\}) = 0$. Third, if $F(\{2,3\}) \geq F(\{1\}) > 0$ then by Lemma 12 we set $\bar{F} = F_{\{1\},\{2,3\}}$ and get $\bar{F} \in A$ and $\bar{F}(\{1\}) = 0$. Then, we start this step from the beginning. Last, if $0 < F(\{2,3\}) \leq F(\{1\})$ then by Lemma 12 we set $\bar{F} = F_{\{1\},\{2,3\}}$ and get $\bar{F} \in A$ and $\bar{F}(\{2,3\}) = 0$. □

Proposition 7

Proof. Let $x \in MC(\tilde{G})$. Then, $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} v_j(N)$. Therefore, $\sum_{i=1}^{n} x_i = v_{\tilde{G}}(N)$, so x is an efficient payoff vector. Denote the justification matrix of Player i by y^i (such a matrix exists since $x \in MC(\tilde{G})$). Then, for every non-empty coalition $S \subseteq N$, every $i \in S$ satisfies (see Footnote 23),

$$\sum_{k \in S} x_k = \sum_{k \in S} \sum_{v_j \in V} y^i_{k,j} = \sum_{v_j \in V} \sum_{k \in S} y^i_{k,j} \geq \sum_{j=1}^{m} v_j(S) = v_{\tilde{G}}(S)$$

The first equality is due to y^i being a decomposition matrix, the inequality holds since y^i satisfies the multi-core's coalitional rationality condition and the last equality is due to the definition of summation game. Hence, $x \in C(S\tilde{G})$. □

Proposition 8

Proof. Suppose $C(S\tilde{G}) \neq \emptyset$. Assume by negation that there exists $\hat{\delta}(S,i,j) \in \hat{\Delta}$ such that

$$\sum_{v_j \in V} v_j(N) < \sum_{v_j \in V} \sum_{i=1}^{n} \sum_{S \in P(N)} \hat{\delta}(S,i,j) v_j(S)$$

Or,

$$\sum_{v_j \in V} v_j(N) < \sum_{S \in P(N)} \sum_{i=1}^{n} \sum_{v_j \in V} \hat{\delta}(S,i,j) v_j(S)$$

Since $\hat{\delta}(S,i,j)$ is a system of balancing multi weights with constant allocation, for every agent i, coalition S and two issues v_j and $v_{j'}$:

$$\hat{\delta}(S,i,j) = \hat{\delta}(S,i,j') \equiv \hat{\delta}(S,i)$$

and therefore,

$$\sum_{v_j \in V} v_j(N) < \sum_{S \in P(N)} \sum_{i=1}^{n} \hat{\delta}(S,i) \sum_{v_j \in V} v_j(S).$$

By the definition of the summation game, $v_{\tilde{G}}(S) = \sum_{v_j \in V} v_j(S)$ so that,

$$v_{\tilde{G}}(N) < \sum_{S \in P(N)} \left[\sum_{i=1}^{n} \hat{\delta}(S,i) \right] v_{\tilde{G}}(S).$$

Define $\delta(S) = \sum_{i=1}^{n} \hat{\delta}(S,i)$. Then, $\delta(S)$ is a system of balancing weights since $\displaystyle\sum_{S \in P(N)} \delta(S)\chi^S = \sum_{S \in P(N)} \left[\sum_{i \in N} \hat{\delta}(S,i) \right] \chi^S = \sum_{i \in N} \sum_{S \in P(N)} \hat{\delta}(S,i)\chi^S = \chi^N$. Therefore. the inequality above becomes,

$$v_{\tilde{G}}(N) < \sum_{S \in P(N)} \delta(S) v_{\tilde{G}}(S)$$

which by the Bondareva-Shapley Theorem implies that $C(S\tilde{G}) = \emptyset$, which is a contradiction. Thus, if $C(S\tilde{G}) \neq \emptyset$ then every $\hat{\delta}(S,i,j) \in \hat{\Delta}$ satisfies

$$\sum_{j=1}^{m} v_j(N) \geq \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{S \in P(N)} \hat{\delta}(S,i,j) v_j(S)$$

For the other direction, suppose $C(S\tilde{G}) = \emptyset$. Then, by the Bondareva-Shapley Theorem, there exists a system of balancing weights, $\delta(S)$, whereby $\sum_{S \in P(N)} \delta(S)\chi^S =$

χ^N such that

$$v_{\tilde{G}}(N) < \sum_{S \in P(N)} \delta(S) v_{\tilde{G}}(S)$$

Define $\hat{\delta}(S, i, j) = \frac{\delta(S)}{|S|}$ if $i \in S$ and 0 otherwise. Obviously, $\hat{\delta}(S, i, j)$ satisfies the zero to non members condition. Also, for every $v_j \in V$,

$$\sum_{i \in N} \sum_{S \in P(N)} \hat{\delta}(S, i, j) \chi^S = \sum_{S \in P(N)} \sum_{i \in S} \frac{\delta(S)}{|S|} \chi^S = \sum_{S \in P(N)} \delta(S) \chi^S = \chi^N$$

Therefore, $\hat{\delta}(S, i, j)$ also satisfies the resources exhaustion condition. In addition, $\hat{\delta}(S, i, j)$ does not depend on any specific issue and thus it is a system of balancing multi-weights with constant allocations.

$$\sum_{v_j \in V} \sum_{i=1}^{n} \sum_{S \in P(N)} \hat{\delta}(S, i, j) v_j(S) = \sum_{S \in P(N)} \sum_{v_j \in V} \sum_{i=1}^{n} \hat{\delta}(S, i, j) v_j(S) =$$
$$\sum_{S \in P(N)} \sum_{v_j \in V} \sum_{i \in S} \hat{\delta}(S, i, j) v_j(S) = \sum_{S \in P(N)} \sum_{v_j \in V} \sum_{i \in S} \frac{\delta(S)}{|S|} v_j(S) =$$
$$\sum_{S \in P(N)} \delta(S) \sum_{v_j \in V} v_j(S) = \sum_{S \in P(N)} \delta(S) v_{\tilde{G}}(S) > v_{\tilde{G}}(N) = \sum_{v_j \in V} v_j(N)$$

Thus, if $C(S\tilde{G}) = \emptyset$ there exists $\hat{\delta}(S, i, j) \in \hat{\Delta}$ such that

$$\sum_{v_j \in V} v_j(N) < \sum_{v_j \in V} \sum_{i=1}^{n} \sum_{S \in P(N)} \hat{\delta}(S, i, j) v_j(S)$$

$\square$