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ASYMMETRIC ALL-PAY AUCTIONS, MONOTONE AND NON-MONOTONE EQUILIBRIUM.

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ABSTRACT. We re-visit the two-bidder, all-pay auction of Amann and Leininger (1996) allowing for interdependent values and correlation à la Lizzeri and Persico (1998) and Siegel (2014). We study both monotone and non-monotone pure strategy equilibria (MPSE and NPSE): First, we show the allocation and bidding strategies of MPSE can be obtained in the same manner as in the independent private values environment. For correlated private values, the allocation is the same regardless of correlation. For common-values, the allocation is determined by the signals' percentiles. Second, we present a local single-crossing condition which is necessary for existence of MPSE. Using the allocation, we provide a standard single crossing, which is sufficient for MPSE. Third, we exhibit common-value, families of examples that violate the local single-crossing and thus lack MPSE. We also construct a correlated private values example, where the slightest amount of correlation breaks down MPSE that exists under independence. And lastly, we explicitly obtain NPSE for quadratic valuations in cases where no MPSE exists.

1. INTRODUCTION

In rent-seeking contests, distinct individuals may entertain different estimates of the prize. Such estimates maybe of varying precision or accuracy; possibly they maybe interdependent and/or correlated.

In this paper we model rent-seeking contests as all-pay auctions. Our aim here is limited to provide a tractable characterization of pure strategy (monotone or not!) equilibria (henceforth MPSE and NPSE) of the (first-price) all-pay auction with two (possibly asymmetric) players with interdependent valuations and correlated, continuous signals.

Our model can be viewed either as an extension of Amann and Leininger (1996) as we add correlation and interdependent values and or as specialization of Lizzeri and Persico (1998) to the all-pay auction. It is also closely related to Siegel (2014) who studies a discrete

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signals model. However, we also study non-monotone equilibria which starkly departs from previous studies of the all-pay auction. Araujo et al. (2008) study non-monotonic equilibria in auctions but their assumptions rule out non-monotonic equilibrium in the all-pay auction.

As Siegel (2014), we do not restrict attention neither to affiliated signals, as in Lizzeri and Persico (1998) nor to independent signals as in, Araujo et al. (2008). We allow for positive or negative correlated signals. Speaking plainly, affiliation is (mostly) useless – in the context of all-pay auctions with interdependent valuations. In Lemma 1, we prove that any equilibrium with correlated signals must also be an equilibrium of some all-pay auction with independent signals (but different valuations).

After Lemma 1 is established, the characterization of MPSE is a straightforward application of the recursive algorithm of Amann and Leininger (1996) for the independent private values case.[1] The algorithm first solves for the allocation rule or tying function and next computes bid functions. It can be implemented with any available differential equation solver, which are available in all computer algebra systems[2]. Siegel (2014)'s algorithm is the analogous version for discrete type spaces.

In any MPSE, the allocation rule (*i.e.* the assignment of the object given the signal of the players) only depends on the players' expected values for the object conditional on their signals. In particular: For the *correlated private values* environment, the allocation rule is the same regardless of the nature of the correlation in the sense it coincides with the allocation of the independent private values case. For the *common-value* environment, the allocation is dictated by the percentiles of the distribution of the agents' signals[3]: when agent 1 gets a signal in the p-percentile, he bids the same as when agent 2 gets a signal in the p-percentile.[4]

In Section 4, we show economically interesting all-pay auctions have MPSE when signals are independent but even "small doses" of correlation "break-down" the MPSE. That motivates us to study non-monotonic equilibria. For the class of quadratic valuations, we provide an explicit equilibrium strategies of NPSE when no MPSE exists. Discrete pooling (*i.e.* two or more types placing the same bid) allows the bidders incentives to be properly aligned without requiring different tie-break rules to assure existence.

[1]See also Parreiras (2006) for an application to the first-price auction, common-value, affiliated case.

[2]See Mathematica, Maple, or for an open-source alternative, see Sage

[3]In the context of discrete signals, Siegel (2014) obtain this result.

[4]Corollary: if signals are conditionally (on the value) independent, winning probabilities of both agents are identical. Einy et al. (2013) and Warneryd (2013) independently obtain this. They study models where one agent's signals is Blackwell's sufficient for the other's, which implies condition independence.

2. The Model

There are two agents, $i = 1, 2$. Let V_i be the random variable describing the value of the object for player i. Let X_1 and X_2 be the agents' signals. The conditional expected value is $v_i(x, y) = \mathrm{E}[V_i | X_1 = x, X_2 = y]$. The cumulative distribution of X_i is F_i and, $F_{i|j}$ is the conditional cumulative distribution of X_i given X_j. The lower-case f denotes the respective probability density function. Finally, we also define

$$\lambda_i(x, y) \stackrel{\text{def}}{=} E[V_i | X_i = x, X_j = y] \cdot f_{X_j|X_i}(y|x) \text{ where } i, j = 1, 2 \text{ and } j \neq i.$$

We assume:

CONTINUITY: F_{X_i} is absolutely continuous.

FULL SUPPORT: For all $(x, y) \in [0, 1]^2$, $f_{X_1, X_2}(x, y) > 0$.

UNIFORM MARGINALS: Without any loss of generality, $X_i \sim \mathrm{U}[0, 1]$.[5]

As we are interested in non-monotone equilibria of the all-pay auction, unlike the previous literature we do not assume $\lambda_i(x, y)$ increasing in x.[6]

With interdependent valuations, there is no loss of generality in assuming independent signals in the context of the all-pay auction as the reasoning below shows.

THE FICTITIOUS AUCTION: Given an all-pay auction, the corresponding fictitious (or auxiliary) auction is the all-pay auction where signals are independently and uniformly distributed on the unit interval, and expected conditional valuations are $\tilde{v}_i(x, y) \stackrel{\text{def}}{=} \lambda_i(x, y)$.

Lemma 1. *The fictitious auction and the original auction are payoff equivalent.*

Proof of lemma 1. Pick any strategy profile $\mathbf{b} = (b_1, b_2)$ then

$$\tilde{U}_i(b|x) = \int_{\{y: b_j(y) \leq b\}} \tilde{v}_i(x, y) dy - b = \int_{\{y: b_j(y) \leq b\}} v_i(x, y) f_{j|i}(y|x) dy - b = U_i(b|x), \ i = 1, 2. \quad \blacksquare$$

Best reply correspondences in the fictitious and original auctions coincide, so do equilibria sets.

3. Monotone Equilibrium

Let ϕ_1 and ϕ_2 denote the inverse bidding functions of a (increasing) monotone[7]. As in Amann and Leininger (1996) or Parreiras (2006), the *tying function* Q (or allocation rule)

[5] Say S_i is the original signal, re-parametrize signals by taking as the new signal, $X_i = F_{S_i}(S_i)$.

[6] See Amann and Leininger (1996),Krishna and Morgan (1997), Lizzeri and Persico (1998), Araujo et al. (2008), and Siegel (2014).

[7] For brevity, we restrict attention to increasing equilibrium because analogous results hold for decreasing equilibrium.

maps the type of player 1 to the type of player 2 that bids the same in equilibrium, that is $Q(\phi_1(b)) \stackrel{\text{def}}{=} \phi_2(b)$.

Proposition 1. *The tying function solves the differential equation,*

$$Q'(x) = \frac{v_2(x, Q(x))}{v_1(x, Q(x))} \quad \text{and} \quad Q(1) = 1.$$

Also $b_1(x) = \int_0^x v_2(z, Q(z)) f_{1|2}(z|Q(z))\, dz$.

Proof. First-order conditions for an optimal bid are

$$v_1(x, \phi_2(b)) f_{2|1}(\phi_2(b)|x)\phi_2'(b) - 1 = 0 \quad \text{and} \quad v_2(\phi_1(b), y) f_{1|2}(\phi_1(b)|y)\phi_1'(b) - 1 = 0. \quad (3.1)$$

Combine the first-order conditions with the identity $Q' \cdot \phi_1' = \phi_2'$ and remember that, since wlog. signals are uniformly distributed in the unit interval, the conditional density coincides with the joint density by Baye's rule. ∎

Under some assumptions, the equilibrium described by Proposition 1 is unique:

Proposition 2. *Assume that $v_1(\cdot)$ is bounded away from zero and, $v_1(\cdot)$ and $v_2(\cdot)$ are continuous in the signal of player 1 and continuously differentiable in the signal of player 2. If a continuous, monotone equilibrium exists, it is the unique.*

Proof. Since the space of signals is compact and by assumption, the derivative of $\frac{v_2(x,y)}{v_1(x,y)}$ with respect to y is continuous, it satisfies the Lipschitz condition: exists $K > 0$ such that

$$\left| \frac{v_2(x, y)}{v_1(x, y)} - \frac{v_2(x, \hat{y})}{v_1(x, \hat{y})} \right| \le K \cdot |y - \hat{y}| \text{ for all } (x, y) \in [0, 1]^2.$$

the differential equation characterizing the tying function has a unique maximal solution that satisfies the boundary condition $Q(1) = 1$. By Proposition 1, the uniqueness of the bid functions follows from the uniqueness of Q. ∎

Notice that distributions with unbounded support will typically violate the assumptions of Proposition 2 because valuations are not continuous at the boundary of the signal space.

Proposition 1 says that the interdependence of valuations, as opposed to the correlation between the signals, is the only factor that matters for determining the tying function. We illustrate this remark in a couple of interesting environments:

Corollary 1. CORRELATED PRIVATE VALUES. *In any monotone equilibrium, the tying function is the identical to the tying function of the independent private values environment.*

Corollary 2. COMMON-VALUES. *In any monotone equilibrium, the tying function is the identity*[8]

[8] Siegel (2014) obtains this result for the discrete signals case.

Without re-scaling and with common values, the tying function is $Q(x) = F_2^{-1}(F_1(x))$. In the statistical literature, this Q is also known as quantile-quantile plot, or simply $Q-Q$ plot.

To establish existence (or not) of a monotone equilibrium we define for player 1:
(LOCAL SINGLE CROSSING) For all x, $\lambda_1(\cdot, Q(x))$ is non-decreasing in neighborhood of x.
(INCREASING) For all $x, z \in [0,1]$ $\int_z^x (\lambda_1(x,y) - \lambda_1(y,y))\, dy \geq 0$.
(SINGLE CROSSING) For all $\hat{x} < x < \tilde{x}$: $\lambda_1(\hat{x}, Q(x)) < \lambda_1(x, Q(x)) < \lambda_1(\tilde{x}, Q(x))$. And similarly define the analogous conditions for player 2. We say a condition holds iff it holds for both players.

Proposition 3. LOCAL SINGLE-CROSSING *is necessary,* SINGLE-CROSSING *is sufficient and* INCREASING *is necessary and sufficient for $b_1(x) = \int_0^x v_2(z, Q(z)) f_{1|2}(z|Q(z)dz$ and $b_2(y) = b_1(Q^{-1}(x))$ be an increasing equilibrium.*

Proof. The function $v_1(x, \phi_2(b)) f_{2|1}(\phi_2(b)|x)\phi_1'(b) - 1$ satisfies LOCAL SINGLE-CROSSING and only if $v_1(x, \phi_2(b)) f_{1,2}(z, \phi_2(b))$ is non-decreasing in x at $x = \phi_1(b)$. Remember, marginals are uniform, $f_1 = f_2 = 1$. Differentiating the identity, $v_1(\phi_1(b), \phi_2(b)) f_{2|1}(\phi_2(b)|\phi_1(b))\phi_2'(b) - 1 = 0$, with respect to b, and assuming $\phi_1' > 0$, the local single-crossing in z at $x = \phi_1(b)$ is equivalent to the second-order condition for 1's optimal bid. Clearly, local single crossing is necessary.

The argument to establish SINGLE-CROSSING is sufficient is standard:[9] single-crossing implies that, at $b = b_1(x)$, $\frac{\partial U_1}{\partial b}(b|\hat{x}) < \frac{\partial U_1}{\partial b}(b|x) = 0 < \frac{\partial U_1}{\partial b}(b|\tilde{x})$.

Finally to establish INCREASING is necessary and sufficient, we write payoffs of the profile as in a direct mechanism $U_1(z|x) = \int_0^z (\lambda_1(x,y) - \lambda_1(y,y))\, dy$. The incentive compatible condition, for all x and z, $U(z|x) - U(x|x) \geq 0$ is equivalent to INCREASING. Moreover, as $U(0|x) = 0$, the individual rational constrained is satisfied.

■

Often[10] the following monotonicity assumption, or one of its variants, is used:
(M) $\lambda_i(x,y)$ is increasing in x for all y. We have M $\Rightarrow$ SINGLE CROSSING $\Rightarrow$ INCREASING $\Rightarrow$ LOCAL SINGLE CROSSING.

To the best of our knowledge, Araujo et al. (2008) introduced INCREASING in the context of auctions with independent signals, although they use the more stringent assumption M. In section B of the Appendix, we characterize explicitly (in the context of quadratic valuations) the parameter space regions where each of the conditions in proposition **??** holds.

[9] See Krishna and Morgan (1997, p. 351), Lizzeri and Persico (1998, p. 104) or Athey (2001).

[10] See Krishna and Morgan (1997), Lizzeri and Persico (1998), Araujo et al. (2008) or Siegel (2014)

4. Non-Existence of Monotone Equilibrium

When signals are correlated, the all-pay auction may lack monotone equilibria. Even for ”small doses” of correlation, provided the support of signals is sufficiently large.

Example 1. (Correlated Private Values) The signals (X_1, X_2) follow a truncated, symmetric, bivariate normal distribution specified by $(\mu, \sigma^2, \rho)$and truncation points $\mu - M$ and $\mu + M$. Valuations are $v_i(x) = \exp(h(x_i))$ for $i = 1, 2$ where h is a given increasing function.

Proposition 4. *If $h'(x) \geq \frac{2\rho(\mu - x)}{\sigma^2(1+\rho)}$ for some x, there is no increasing equilibrium for example 1.*

Proof. As players are symmetric, by Proposition 1, if a monotone, pure strategy equilibrium exists then it must be symmetric. However, using the fact that $X_j|X_i \sim \mathcal{N}\left((1-\rho)\mu + \rho X_i, (1-\rho^2)\sigma^2\right)$ we obtain $\frac{\partial}{\partial x} v_i(x,y) \cdot f_{X_j|X_i}(y|x)\Big|_{y=x} > 0 \Leftrightarrow h'(x) > \frac{\rho}{\sigma^2(1+\rho)}(\mu - x)$. ■

As a result, the symmetric monotone equilibrium is not robust to the introduction of a small degree of correlation for a family of examples:

Corollary 3. *Assume $\|h'\|_\infty < K$ then for any $\rho > 0$ there is $M > 0$ such that the private values model of example 1 has no monotone equilibrium.*

For common-values we have similar non-existence problems:

Example 2. (Common-Value) There is no monotone equilibrium an all three families given by the table below, where signals and the value are affiliated; and the parameter θ measures the precision of the players' information.

V	$S_1\|V$	$\mathrm{E}[V\|S_1 = x, S_2 = y]$	$f_{S_2\|S_1}(y\|x)$
$\ln \mathcal{N}(\mu, \tau^{-1})$	$\mathcal{N}(V, \theta^{-1})$	$\exp\left(\frac{\tau\mu+\theta x+\theta y+\frac{1}{2}}{\tau+2\theta}\right)$	$\mathcal{N}\left(\frac{\tau\mu+\theta x}{\tau+\theta}, \frac{\tau+2\theta}{\theta(\tau+\theta)}\right)$
$\mathtt{Pareto}(\omega, \alpha)$	$V \cdot \mathrm{B}(\theta, 1)$	$\frac{\alpha+2\theta}{\alpha+2\theta-1}\max(\omega, x, y)$	$\frac{(\alpha+\theta)\theta}{\alpha+2\theta} \frac{\omega^{(\alpha+\theta)\mathbf{1}_{[x<\omega]}} x^{(\alpha+\theta-2)\mathbf{1}_{[x>\omega]}} y^{\theta-1}}{\max(\omega,x,y)^{\alpha+2\theta}}$
$\mathrm{Inv}-\Gamma(\alpha, \beta)$	$\mathrm{Exp}(\theta V^{-1})$	$\frac{\Gamma(\alpha+2\theta-1)}{\Gamma(\alpha+2\theta)}(x+y+\beta)$	$\frac{\Gamma(\alpha+2\theta)}{\Gamma(\alpha+\theta)\Gamma(\theta)} \frac{y^{\theta-1}(x+\beta)^{\alpha+\theta}}{(x+y+\beta)^{\alpha+2\theta}}$

Table 1. Common-value models without MPSE.

5. Non-Monotone Equilibria

Consider a pure strategy equilibrium profile in which every bid strategy, $b_i(\cdot)$, is piecewise monotone, that is, b_i' (provided it is well defined) changes sign a finite number of times[11].

[11]This is related to the 'limited complexity strategies' of Athey's 1997 working-paper version of Athey (2001)

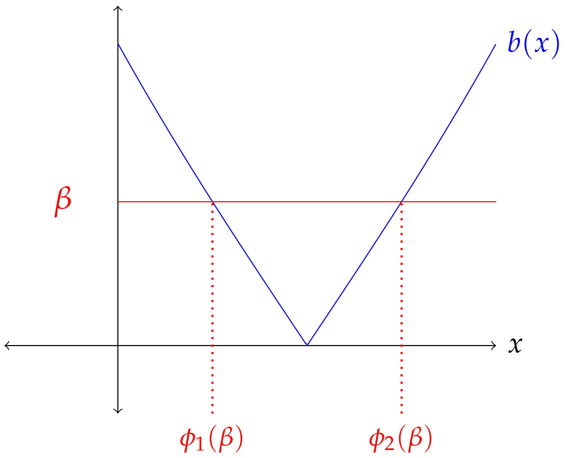

FIG. 1. A piecewise monotone strategy and its local inverse bids.

$$v(x,y) = x + y(3 - 4x + 2x^2)$$

$$b(x) = \begin{cases} -\frac{2}{3}x^3 + x^2 - 3x + \frac{4}{3} & \text{if } x \leq \frac{1}{2} \\ \frac{2}{3}x^3 - x^2 + 3x - \frac{4}{3} & \text{otherwise.} \end{cases}$$

Valuations and bid of the eq. in figure 1.

Next partition i's type space into finite intervals $[0,1] = \bigcup_{k=1}^{n_i} I_k^i$ such these intervals are maximal with respect the property each restriction $b_i|_{I_k^i}(\cdot)$ is monotone. For exposition purposes, let's focus on the case where $b_i(\cdot)$ is increasing (decreasing) in odd (even) intervals. The cases where one or both of the $b_i(\cdot)$ is increasing in even intervals are analogous.

Now define the k_{th} local inverse bid function of player i: $\phi_k^i : b^{-1}(I_k^i) \to I_k^i$. Using inverse bids, the payoff of a type x of player i who bids b is:

$$U_i(b|x) = \left(\sum_{k=1}^{n_j} \int_{\phi_{2k-2}^j(b)}^{\phi_{2k-1}^j(b)} \lambda_i(x,y)dy \right) - b,$$

where by convention, $\phi_0^j(b) = 0$, and for odd n, $\phi_{2n_j-1}^j(b) = 1$.

Later we shall use the $n_i \times n_j$ – matrices $\Lambda^i(b)$ with entries given by $\Lambda_{k,l}^i(b) = \lambda_i\left(\phi_k^i(b), \phi_l^j(b)\right)$. We abuse notation and write $n_i(b)$ for the number of types of player i that bid b.

Definition 1. A piecewise-monotone equilibrium **b** is *regular* if, $n_i(b) = n_j(b) < +\infty$ is constant in a neighborhood[12] of b for almost all b and, $\Lambda^i(b)$ is full-rank for $i = 1, 2$.

Any monotone equilibrium is regular. Without regularity, there is little hope to pin-down the equilibrium using the first-order approach. Without regularity, the differential system corresponding to FOCs is undetermined.

For a regular equilibrium we can construct tying functions in the same way we did in the monotone case. See section A in the Appendix. But notice a tying function, unlike in the monotone case, maps a type of a player to another type of the same player!

[12]In a regular equilibrium, $n_i(b)$ may vary with b but it can take at most a countable number of values.

To make it more concrete, let's restrict attention to the symmetric case and also assume (at most) only two types are pooling. The tying function is a solution of the differential equation:

$$Q'(x) = -\frac{\lambda(x,x) - \lambda(Q(x),x)}{\lambda(Q(x),Q(x)) - \lambda(x,Q(x))}.$$

Once we solve for Q we can reduce the first-order conditions to a single ODE. In general, it is hard to obtain a closed form solution for Q. Also the boundary condition is not obvious unlike in the monotone case. One also has to characterize the region where pooling occurs. Despite these difficulties, we show below, that for the case of quadratic preferences one can solve for the equilibrium.

5.1. **Quadratic Preferences.** In this section we construct non-monotone equilibrium displaying mirror-symmetry. Tthat is, the tying function is a translation composed with a reflection: $Q(x) = c - x$ for some suitable constant c. We assume:

QUADRATIC MODEL Players are symmetric, wlog. signals are independent and, player i's value is $v(x,y) = Ax^2 + By^2 + Cxy + Dx + Ey + F$ where $x, y \in [0,1]$, x is the signal of player i and y is the signal of $-i$.

In the section B of the Appendix, we characterize explicitly the parameter space regions where M, single-crossing, increasing, and local single crossing hold.

Some preliminaries and notation: Let $c \stackrel{\text{def}}{=} \frac{-2D}{2A+C}$ and notice it is the unique solution of:

$$v(x,x) + v(x, c-x) = v(c-x, x) + v(c-x, c-x).$$

We also need the auxiliary function, $\hat{\lambda}(x,y) \stackrel{\text{def}}{=} v(x,y) + v(x, c-y)$.

Remark 1. The function $\hat{\lambda}(x,y)$ satisfies $\hat{\lambda}(x,y) = \hat{\lambda}(c-x,y)$ for all x and y.

Remark 2. The derivative $\hat{\lambda}_x(x,y)$ is linear in x with $\hat{\lambda}_x(c/2,y) = 0$ for all y, and $\hat{\lambda}_{xx} = 4A$.

There are four cases to consider:

Proposition 5. *Assume $C < 2A < 0 < 2D < -2A - C$. The equilibrium bidding function is bell-shaped for types in $[0,c]$ and decreasing for types in $[c,1]$. We have $b(0) = \int_c^1 v(x,x)dx$ and,*

$$b(x) = \begin{cases} b(0) + \int_0^x \hat{\lambda}(y,y)dy & \text{if } 0 \le x \le c/2, \\ b(0) + \int_0^{c-x} \hat{\lambda}(y,y)dy & \text{if } c/2 \le x \le c, \text{ and} \\ b(0) - \int_c^x v(y,y)dy & \text{if } c \le x \le 1. \end{cases}$$

Proof. Let's verify that $b(\cdot)$ is indeed a NPSE by a direct mechanism approach. Let $U(z|x)$ be the payoff of a player with type x who bids as if his type were z.

$$U(z|x) = \begin{cases} U(c|x) + \int_0^z \left(\hat{\lambda}(x,y) - \hat{\lambda}(y,y)\right) dy & \text{if } 0 \le z \le c/2, \\ U(c-z|x) & \text{if } c/2 < z < c, \text{ and} \\ \int_z^1 \left(v(x,y) - v(y,y)\right) dy & \text{if } c \le z \le 1. \end{cases}$$

We want to show that $x = \underset{z}{\operatorname{argmax}}\, U(z|x)$ for all x. Of course, the first-order condition is always satisfied since $U_z(x|x) = 0$. Because since the equilibrium of the direct mechanism (i.e. truth telling) is always monotone, we only to check whether the single crossing condition, $U_{xz}(z|x) > 0$, holds:

$$U_{xz}(z|x) = \begin{cases} \lambda_x(x,z) & \text{if } 0 \le z \le c/2, \\ -\lambda_x(x,z) & \text{if } c/2 < z < c, \text{ and} \\ -v_x(x,z) & \text{if } c \le z \le 1. \end{cases}$$

Due to remark 2 and the fact $A < 0$, we have:

$$\hat{z} < \frac{c}{2} < \tilde{z} \Rightarrow \lambda_x(x,\hat{z}) > \lambda_x(x,c/2) = 0 > \lambda_x(x,\tilde{z}) \text{ for all } x,$$

which proves the single-crossing holds in the pooling region. It remains to prove $U_{xz}(z|x) > 0$ in the monotonic region, $z \in [c,1)$:

$$-v_x(x,z) = -2Ax - D - Cz \ge \min(-D - C\cdot c, -D - C) = \min(D\frac{C-2A}{C+2A}, -D - C) > 0,$$

where the first inequality follows from $A < 0$ and, the last inequality is implied by $C < 2A < 0 < D < -A - C/2$. ∎

We have a dual proposition:

Proposition 6. *If $-2A - C < 2D < 0 < 2A < C$ then the bidding function is U-shaped for types in $[0,c]$ and increasing for types in $[c,1]$. We have $b(0) = \int_0^{c/2} \hat{\lambda}(x,x)dx$ and*

$$b(x) = \begin{cases} b(0) - \int_0^x \hat{\lambda}(z,z)dz & \text{if } 0 \le x \le c/2, \\ \int_{c/2}^x \hat{\lambda}(z,z)dz & \text{if } c/2 \le x \le c, \text{ and} \\ b(0) + \int_c^x v(z,z)dz & \text{if } c \le x \le 1. \end{cases}.$$

Proof. The proof is analogous to the previous case and therefore omitted. ∎

Proposition 7. *Assume $C < 2A < 0 < -2A < D < -2A - C$. The equilibrium bidding function is increasing for types in $[0, c-1]$ and bell shaped for types in $[c-1, 1]$. We have*

$$b(x) = \begin{cases} \int_0^x v(y,y)dy & \text{if } 0 \le x \le c-1, \\ \int_0^{c-1} v(y,y)dy + \int_{c-1}^x \hat{\lambda}(y,y)dy & \text{if } c-1 \le x \le c/2, \text{ and} \\ \int_0^{c-1} v(y,y)dy + \int_{1-\frac{c}{2}}^x \hat{\lambda}(y,y)dy & \text{if } c/2 \le x \le 1. \end{cases}.$$

Proof. The proof is similar to the previous cases.

$$U(z|x) = \begin{cases} \int_0^z \left(v(x,y) - v(y,y)\right) dy & \text{if } 0 \le z \le c-1, \\ U(c-1|x) + \int_{c-1}^z \left(\hat{\lambda}(x,y) - \hat{\lambda}(y,y)\right) dy & \text{if } c-1 \le z \le c/2, \\ U(c-z|x) & \text{if } c/2 \le z \le 1, \end{cases} \quad \text{and so:}$$

$$U_{xz}(z|x) = \begin{cases} v_x(x,z) & \text{if } 0 \le z \le c-1, \\ \hat{\lambda}_x(x,z) & \text{if } 1-c \le z \le c/2, \\ -\hat{\lambda}_x(x,z) & \text{if } c/2 \le z \le 1, \end{cases}$$

Once more, remark 2 implies $U_{xz} > 0$ for the pooling region. As for the monotonic region, we have $v_x(x,z) = 2Ax + D + Cz \ge 2A + D + Cz \ge 2A + D + \min(0, C(c-1)) \ge 2A + D\min(0, -C\frac{2D+2A+C}{2A+C}) > 0$. ■

Proposition 7's dual is:

Proposition 8. *Assume $C > 2A > 0 > -2A > D < -2A - C$. The equilibrium bidding function is decreasing for types in $[0, c-1]$ and U-shaped for types in $[c-1, 1]$. We have*

$$b(x) = \begin{cases} \int_x^{c-1} v(y,y)dy + \int_{c-1}^{c/2} \hat{\lambda}(y,y)dy & \text{if } 0 \le x \le c-1, \\ \int_x^{c/2} \hat{\lambda}(y,y)dy & \text{if } c-1 \le x \le c/2, \text{ and} \\ \int_{\frac{c}{2}}^x \hat{\lambda}(y,y)dy & \text{if } c/2 \le x \le 1. \end{cases}.$$

Proof. The proof is analogous to the previous prop. and so it's omitted here. ■

6. Conclusions

We characterized monotone equilibrium of all-pay auctions in the continuous signals case, allowing for correlation and interdependent values.

We showed the monotone equilibrium may fail to be robust to small degrees of correlation. Motivated by that finding, we characterized non-monotone equilibrium for the class of quadratic preferences.

The existence of non-monotone pure strategy equilibria, however, remains an open question. Rentschler and Turocy (2012)'s results suggest that some models may lack pure strategy equilibria. In this paper, we do not discuss mixed strategy equilibrium.

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Appendix A. Tying functions for Non-Monotonic Equilibrium

Lemma 2. *Consider a regular equilibrium and let $b \in b_i([0,1])$ with $\phi^i_k(b)$ for $k = 1,...,n_i$ as the corresponding local inverse bids. Define the tying functions by $Q^i_k(\phi^i_1(b)) \stackrel{\text{def}}{=} \phi^i_k(b)$. Let $Q^i(x) = (x, Q^i_1(x), \dots, Q^i_{n_i}(x))$. The tying functions satisfy the system of differential equations:*

$$\frac{\partial}{\partial x} Q^i_k(x) = (-1)^{k+1} \frac{\left|L^j_k(x)\right|}{\left|L^j_1(x)\right|}$$

where $(L^j_k)_{r,c}(x) = \lambda^j(Q^j_a(x), Q^j_b(x))$ if $c \neq k$ and $(L_k)_{r,k} = 1$ for all r.

Proof. Now notice the first-order condition for type x of player i who bids b is:

$$\sum_{l=1}^{n_j} (-1)^{l+1} \lambda_i \left(x, \phi^j_l(b)\right) \frac{\partial \phi^j_l}{\partial b}(b) = 1 \tag{FOC}$$

As the FOC must be satisfied for $x = \phi^i_k(b)$ for all k, we obtain an ODE system, which in matrix form reads as, $\Lambda^i(b) \cdot \Phi^j = \mathbf{1}$, where Φ^j is the n_j–column vector $\left((-1)^{l+1} \frac{\partial \phi^j_l}{\partial b}(b)\right)$. Consider the matrices that are Λ^i but with the k_{th} column replaces by a vector of ones: $\Lambda^i_k(b) = \left(\Lambda^i_{1,\dots,k-1}(b), \mathbf{1}, \Lambda^i_{k+1,\dots,n_i}(b)\right)$. Now, applying Cramer's rule yields the lemma. ■

Appendix B. Quadratic Model

Lemma 3. *Single crossing holds true, if and only if,*

$$D \notin (-\max(A + C, 0), -\min(A, 0) - \min(A + C, 0)).$$

Condition M *holds, if and only if,*

$$D \notin (-\max(2A, 0) - \max(C, 0), -\min(2A, 0) - \min(C, 0)).$$

The condition increasing (or decreasing) holds, if and only if, single crossing holds or
$[C \in (-\max(A, 4A), -\min(A, 4A))$ *and*
$D \in (-\max(\frac{2}{3}(A + C), A + C), -\min(\frac{2}{3}(A + C), A + C))$
$D \in (-\max(\frac{1}{3}(4A + C), A), -\min(\frac{1}{3}(4A + C), A))\Big]$.

Proof. ... ■

Proposition 9. *The single crossing condition does not hold under proposition 5, 6, 7, or 8.*

Proof. ... ■