On the risk in deviating from Nash equilibrium

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Abstract

The purpose of this work is to offer for any zero-sum game with a unique strictly mixed Nash equilibrium, a measure for the *risk* when deviating from the Nash equilibrium. We present two approaches regarding the nature of deviations; strategic and erroneous. Accordingly, we define two models. In each model we define risk measures for the row-player (PI) and the column player (PII), and prove that the risks of PI and PII coincide. This result holds for any norm we use for the size of deviations. We develop explicit expressions for the risk measures in the $\|\cdot\|_1$ - and $\|\cdot\|_2$ -norms, and compute it for several games. Although the results hold for all norms, we show that only the $\|\cdot\|_1$ -norm is suitable in our context, as it is the only norm which is consistent in the sense that it gives the same size to potentially equivalent deviations. The risk measures defined here enables testing and evaluating predictions on the behavior of players. For example: Do players deviate more in a game with lower risks than in a game with higher risk?

Introduction

One of the foci of experimental game theory in recent years has been the attempts to validate the predictions of game theory through classical tests of statistical hypotheses (e.g., O'Neille, 1986; Brown and Rosenthal, 1990; Nagel, Rohde and Zamir, 2006). This encounters many difficulties. For example, in repeated games, mixed strategies involve independence (Brown and Rosenthal, 1990), but human subjects cannot produce perfect randomization and thus cannot play the Nash equilibrium *exactly* (e.g., Baddeley, 1966) thus often failing the statistical tests. Acknowledging this, Brown and Rosenthal suggest replacing the question: "*Did* the players play the Nash strategy?" with the question: "*To what extent* did the players play the Nash strategy?" This view presents a weaker or wider interpretation of "playing Nash", by concentrating on the *closeness* of the play to the prediction. In line with this approach, a basic prediction regarding the behavior of players would be that players deviate less in games that are "riskier" in some sense, than in games that are less risky. In order to validate this prediction a measure for the riskiness of a game should be defined for each of the players.

In two-person zero-sum games, Nash-equilibrium points coincide with Minmax points, thus in addition to the stability property captured by the equilibrium concept, the solution has also a security property: in playing the Nash equilibrium, each player *guarantees* to receive at least the equilibrium payoff, independently of the behavior of his opponent.

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Hence zero-sum games should provide the most promising domain for finding empirical support for the Nash equilibrium theory. In particular, if the Nash equilibrium is unique then it makes a strong prediction regarding the behavior of players. Deviating from the Nash equilibrium is risky because it may result in a lower payoff if the other player is also deviating from the Nash equilibrium. The degree of risk depends on the specific game matrix. Although deviation from the Nash equilibrium does not seem reasonable in this context, there are two main reasons to still expect such deviations. The first is that if a rational player suspects that his opponent is *not* playing according to the Nash equilibrium then he should try to take advantage of this (playing a best-reply to the other player's strategy) in order to get more than the equilibrium payoff. Another reason to expect deviations, is that a deviation can be accidental, a manifestation of a 'trembling hand'. Accordingly, we define two models for deviations. The first model corresponds to the strategic and conscious deviation thus we name this model "the Strategic-model" (or "S-model"). The second model relates to the erroneous (accidental) deviation and thus is named "the Error-model" (or "E-model"). In each of the two models, we define a "risk-measure" representing, for each player, the potential risk in deviating from the Nash-play in that game. Given a direction of deviation, the resulting difference in payoff is linear in the magnitude of the deviation. Hence, the 'risk' (in any reasonable definition) must also be linear in the amount of deviation. Thus, in order to compare different directions of deviations, we restrict our attention to deviations of 'size' 1 and define a measure R for the risk *per unit deviation*. Thus from now on we assume that the magnitude of the deviations is 1. The choice of an appropriate norm to measure the size of deviations is a matter of importance and will be discussed in the sequel.

The more interesting and challenging model among the two, is the S-model. In the S-model the row player (PI) and the column player (PII) each *chooses* the direction of deviation and then PII pays PI the resulting difference from the Nash payoff. Thus the S-model can be viewed as a new zero-sum game, defined over the original game. We call this new game: "The game on the risk in deviation". Clearly this new game does not have a value in the usual sense, as we removed the *unique* Nash equilibrium from the strategy sets (by "forcing" the players to deviate). *Our main result is that the risk of PI and the risk of PII always coincide.* This result holds for any norm we use for the size of the deviations. We prove the equality between the risks of the two players in both the S-model and the E-model.

We develop an explicit expression for the risk measures for the $\|\cdot\|_1$ - and $\|\cdot\|_2$ -norms,[1] and compute it for several games. Although the results hold for all norms, we show (Section 5) that only the $\|\cdot\|_1$ -norm is suitable in our context, as it is the only norm which identifies between potentially equivalent deviations by giving them the same size.

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[1] Recall that for $x = (x_1, \ldots, x_n)^t \in R^n$, $\|x\|_1 = \sum_i |x_i|$ and $\|x\|_2 = \sum_i x_i^2$

1. Deviation vectors

Consider a two-person zero-sum game with an $n \times n$ payoff matrix A with a unique strictly mixed Nash equilibrium. Denote the unique pair of Nash strategies of PI and PII by x_E, y_E respectively. If PI uses the strategy $(x_E + x)$ and PII uses the strategy $(y_E + y)$ we will say that x and y are the "deviation vectors" of PI and PII respectively.[2]

As probability vectors, x_E, $(x_E + x)$ both satisfy that their entries sum up to 1. It follows that: $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 0.$

A well-known property of *strictly* mixed Nash equilibrium, referred to as the "indifference property" (see Maschler, Solan and Zamir, 2013), states that if *only one* of the players deviates from the Nash equilibrium he does not lose nor does he gain by it, thus:

$$\left(x_E + x\right)^t A y_E = x_E^t A y_E.$$

That is for all deviation vectors x, y:

$$x^t A y_E = 0, \quad x_E^t A y = 0. \qquad (1.1)$$

Therefore in order to measure the risk of a deviation one should look at the case where *both* of the players deviate. Then we have for all deviation vectors x, y:

$$\left(x_E + x\right)^t A\left(y_E + y\right) = x_E^t A y_E + x^t A y.$$

The term $x_E^t A y_E$ is the equilibrium payoff. Therefore from now on we will call $x^t A y$ "the deviation payoff".

Let $\|\cdot\|$ be an arbitrary norm. Following the above discussion, given an $n \times n$ payoff matrix we define the set of strategies S of both PI and PII, as:

$$S = \left\{x \in R^n \ ; \sum x_i = 0, \ \|x\| = 1\right\}.$$

Note that real deviations are of the form $\varepsilon x, \delta y,$ with $\varepsilon, \delta,$ sufficiently small so that $x_E + \varepsilon x, \quad y_E + \delta y$ are in fact probability vectors. The deviation payoff is then, as mentioned $\varepsilon\delta(x^t A y).$ Note also that since the Nash equilibrium in the original game is strictly mixed, one can deviate in all directions.

Clearly, the set S is not convex, therefore the generalized Nash existence theorem (see Maschler, Solan and Zamir, 2013) does not apply here.

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[2] We denote by x a column vector and by x^t its corresponding row vector.

2. The risk measures

In this section we define for each model (the S- and E-models), and for each player (PI and PII) a measure for the amount that is risked as a result of a deviation**.**

2.3. The S-model

In the *S-model* the players *control* their deviations (of magnitude 1). They each choose a strategy (x and y respectively) from $S,$ and then PII pays PI the resulting deviation payoff $x^t Ay$. Thus the S-model presents a new zero-sum game defined over the original game. This new zero sum game is between the same players and with the same payoff matrix as the original game but the set of mixed strategies is the $(n-2)-$dimensional sphere S (rather than the $(n-1)-\text{dimensional}$ simplex in the original game). If we assume the players use a minmax approach (just like we assume they do in the original game) then the players should deviate in the least risky direction. Accordingly:

PI can guarantee not to get less than: $\max_{x\in S}\min_{y\in S} x^t Ay$

and PII can guarantee not to pay more than: $\min_{y\in S}\max_{x\in S} x^t Ay.$ (2)

In Proposition 2.3 we prove that $\max_{x\in S}\min_{y\in S} x^t Ay$ is negative and $\min_{y\in S}\max_{x\in S} x^t Ay$ is positive. In other words, if deviating, PI can guarantee not to *lose* more than $-\max_{x\in S}\min_{y\in S} x^t Ay,$ and similarly, if deviating, PII can guarantee not to *pay* more than $\min_{y\in S}\max_{x\in S} x^t Ay.$ It is natural to define the risk as the potential damage in deviating. Hence we define the risk measures $R_1(A)$ of PI, and $R_2(A)$ of PII, in the S-model as follows.

Definition 2.1

The risk measure $R_1(A)$ for a strategic deviation of PI in the A matrix game is defined as:

$$R_1(A) = -\max_{x\in S}\min_{y\in S} x^t Ay.$$

Similarly, the risk measure $R_2(A)$ for a strategic deviation of PII in the A matrix game is defined as:

$$R_2(A) = \min_{y\in S}\max_{x\in S} x^t Ay.$$

The main result of the present work is that the risk of PI and the risk of PII always coincide, namely $R_1(A) = R_2(A),$ and that this is true for any norm we choose to use for the size of the deviations. See Theorem 2.4.

2.2 The E-model

Unlike the S-model, in the E-model, in which the players are *not* in control over their deviations one should be interested in the worst possible outcome, namely:

PI will not lose more than: $-\min_{x\in S} \min_{y\in S} x^t Ay$

and PII will not pay more than: $\max_{y\in S} \max_{x\in S} x^t Ay.$

The risks in the E-model are thus defined as follows:

Definition 2.2

The error-risk measure $e_1(A)$ of PI in the A matrix game is:

$$e_1(A) = -\min_{x\in S} \min_{y\in S} x^t Ay \ .$$

Similarly, the error-risk measure $E_2(A)$ of PII in the A matrix game is:

$$e_2(A) = \max_{y\in S} \max_{x\in S} x^t Ay.$$

In the E-model too, we show that the risk measures of the players coincide, namely that: $e_1(A) = e_2(A).$

Proposition 2.3. For any $n \times n$ matrix $A,$ that has a strictly mixed and unique Nash equilibrium, and for $i = 1, 2$:

$$(a)\ R_i(A) > 0.$$
$$(b)\ e_i(A) > 0.$$

Proof.

We will prove the proposition for $i=2$ (i.e., for the column player PII). Applying this to $-A^t$ (in which PI is the column player) establishes it for PI ($i=1$) as well.

Note first that since S is a compact set, then all the expressions appearing in the Proposition are well defined. Note also that:
$e_2(A)=\max_{y\in S}\max_{x\in S}x^tAy\geq\min_{y\in S}\max_{x\in S}x^tAy=R_2(A),$ thus the proof of (b) will follow from the proof of (a).

In order to prove (a), assume on the contrary, that $\min_{y\in S}\max_{x\in S}x^tAy\leq 0.$ Then there exists $y_0\in S,$ such that: $\max_{x}x^tAy_0\leq 0,$ and thus for all $x\in S:\quad x^tAy_0\leq 0.$

Since y_E, the equilibrium strategy of PII in the original game, is strictly mixed then for a small enough $\varepsilon>0,$ the deviation $y_d=\varepsilon y_0$ is a feasible deviation from the equilibrium strategy, that is y_E+y_d is a probability vector and $x^tAy_d\leq 0,$ for all $x\in S.$

Now:

$$\left(x_E+x\right)^t A\left(y_E+y_d\right)=x_E^tA\left(y_E+y_d\right)+x^tAy_E+x^tAy_d.$$

By (1.1): $x^tAy_E=0,$ and since we assumed that $x^tAy_d\leq 0,$ we get that:

$$\left(x_E+x\right)^t A\left(y_E+y_d\right)\leq x_E^tA\left(y_E+y_d\right),\quad \forall x\in S. \qquad (5)$$

On the other hand, from the indifference property of the strictly mixed Nash equilibrium $(x_E,y_E),$ we also have, for all deviation vectors y:

$$x_E^tA\left(y_E+y_d\right)=x_E^tAy_E=x_E^tA\left(y_E+y\right). \qquad (6)$$

Combining (5) and (6), we get that for all deviation vectors x and y in S:

$$\left(x_E+x\right)^t A\left(y_E+y_d\right)\leq \quad x_E^tA\left(y_E+y_d\right)\ =x_E^tA\left(y_E+y\right).$$

Hence the pair of strategies: (x_E,y_E+y_d) is also a Nash equilibrium of the original game $A,$ in contradiction to the assumption that (x_E,y_E) is the unique equilibrium (note that this equilibrium is different from (x_E,y_E) since $y_0\in S$ and $\varepsilon>0$ imply $y_d\neq 0$).

■

Proposition 2.3 implies that it is not possible to deviate without any risk.

2.3 Main Results

The main result of this work is:

Theorem 2.4

If the payoff $n \times n$ matrix A has a strictly mixed and unique Nash equilibrium, then

$$R_1(A) = R_2(A),$$

and this holds for any norm we use for the deviations.

Proof.

Clearly: $R_1(A) = R_2(-A^t)$.

This is so since one can view PI as being the *column* player (and PII being the row player) in the $-A^t$ matrix game. In fact, it also holds that:

$$R_2(-A^t) = R_2(A^t).$$

This is true since:

$$R_2(-A^t) = \min_{y \in S} \max_{x \in S} x^t(-A^t)y = \min_{y \in S} \max_{x \in S} (-x^t)A^t y,$$

and since $x \in S$ *iff* $-x \in S$, then the above equals: $\min_{y \in S} \max_{x \in S} x^t A^t y = R_2(A^t)$.

So finally, it remains to show that:

$$R_2(A^t) = R_2(A).$$

Namely, that: $\min_{y \in S} \max_{x \in S} x^t \boldsymbol{A} y = \min_{y \in S} \max_{x \in S} x^t \boldsymbol{A^t} y.$

The $(n-1)$-dimensional subspace of deviation vectors (that is, the subspace containing S) is:

$$V = \left\{ x \in R^n \ : \ \sum_{i=1}^{n} x_i = 0 \right\}.$$

Since $S \subseteq V$, we need to consider the restriction of $x^t Ay$ to V, and we identify V with R^{n-1} via an appropriate choice of basis.

The following is straightforward.

Lemma 2.5 The columns of the following $n\times(n-1)$ matrix are an orthonormal basis of V,

$$E=\begin{bmatrix} 1+t & t & \cdots & t \\ t & 1+t & \cdots & t \\ \vdots & \vdots & \ddots & \vdots \\ t & t & \cdots & 1+t \\ 1/\sqrt{n} & 1/\sqrt{n} & \cdots & 1/\sqrt{n} \end{bmatrix}$$

where $t=\dfrac{1}{\sqrt{n}-n}$.

Since we identify V with R^{n-1} via E, we define a norm $\|\cdot\|_E$ on R^{n-1} as follows:

$$\|u\|_E := \|Eu\|.$$

For $x,y\in V$, denote by $v,w\in R^{n-1}$ the vectors that satisfy: $Ev=x,\quad Ew=y.$

So in terms of $v,w\in R^{n-1}$, the deviation payoff function is:

$$(Ev)^t A(Ew)=v^tE^tAEw=v^tBw,$$

where $B=E^tAE.$

Thus: $\min\limits_{\substack{y\in V\\ \|y\|=1}}\max\limits_{\substack{x\in V\\ \|x\|=1}} x^tAy=\min\limits_{\|w\|_E=1}\max\limits_{\|v\|_E=1} v^tBw.$

Similarly: $\min\limits_{\substack{y\in V\\ \|y\|=1}}\max\limits_{\substack{x\in V\\ \|x\|=1}} x^tA^ty=\min\limits_{\|w\|_E=1}\max\limits_{\|v\|_E=1} v^tB^tw.$

So in order to prove Theorem 2.4, we need to prove that:

$$\min_{\|w\|_E=1}\max_{\|v\|_E=1} v^tBw=\min_{\|w\|_E=1}\max_{\|v\|_E=1} v^tB^tw. \qquad (7)$$

For the sake of convenience, from this point until the end of the proof of Theorem 2.4, we drop the subscript E from $\|\cdot\|_E$ and write simply $\|\cdot\|$. This will cause no confusion with the norm on R^n since we will be working only in R^{n-1}.

We need the following definition for the dual-norm (see e.g., Lax, 1997):

Definition 2.6

Given any norm $\|\cdot\|$ on R^{n-1}, define the dual-norm $\|\cdot\|^*$ on R^{n-1} as follows:

$$\|u\|^* = \max_{\substack{v \in R^{n-1} \\ \|v\|=1}} v^t u.$$

Note that $\min_{\|w\|=1} \max_{\|v\|=1} v^t Bw = \min_{\|w\|=1} \|Bw\|^*$, and similarly $\min_{\|w\|=1} \max_{\|v\|=1} v^t B^t w = \min_{\|w\|=1} \|B^t w\|^*$.

Thus we need to prove that:

$$\min_{\|w\|=1} \|Bw\|^* = \min_{\|w\|=1} \|B^t w\|^*.$$

This is proved by the following two Propositions.

Proposition 2.7. If B is invertible then: $\min_{\|w\|=1} \|Bw\|^* = \min_{\|w\|=1} \|B^t w\|^*$.

Proposition 2.8. If the payoff matrix A has a strictly mixed and unique Nash equilibrium, then the matrix $B = E^t AE$ is invertible.

Proof of Proposition 2.7.

$$\min_{\|w\|=1} \|Bw\|^* = \min_{w \neq 0} \frac{\|Bw\|^*}{\|w\|} = \frac{1}{\max_{w \neq 0} \left(\frac{\|w\|}{\|Bw\|^*} \right)}.$$

We will focus on simplifying $\max_{w \neq 0} \left(\frac{\|w\|}{\|Bw\|^*} \right)$.

Substitute: $u = Bw,$ then since B is invertible we may write:

$$\max_{w \neq 0} \left(\frac{\|w\|}{\|Bw\|^*} \right) = \max_{u \neq 0} \left(\frac{\|B^{-1}u\|}{\|u\|^*} \right).$$

Now, since $\|\cdot\|^{**} = \|\cdot\|$ (see e,g., Lax, 1997), then:

$$\max_{u\neq 0}\left(\frac{\|B^{-1}u\|}{\|u\|^{*}}\right) = \max_{u\neq 0}\left(\frac{\|B^{-1}u\|^{**}}{\|u\|^{*}}\right) = \max_{\|u\|^{*}=1}\|B^{-1}u\|^{**} = \max_{\|u\|^{*}=1}\max_{\|s\|^{*}=1} s^{t}B^{-1}u.$$

Hence: $$\min_{\|w\|=1}\|Bw\|^{*} = \frac{1}{\max\limits_{\substack{\|u\|^{*}=1\\ \|s\|^{*}=1}} s^{t}B^{-1}u}.$$

Similarly: $$\min_{\|w\|=1}\|B^{t}w\|^{*} = \frac{1}{\max\limits_{\substack{\|u\|^{*}=1\\ \|s\|^{*}=1}} s^{t}(B^{t})^{-1}u}.$$

Now:

$$\min_{\|w\|=1}\|B^{t}w\|^{*} = \frac{1}{\max\limits_{\substack{\|u\|^{*}=1\\ \|s\|^{*}=1}} s^{t}(B^{t})^{-1}u} = \frac{1}{\max\limits_{\substack{\|u\|^{*}=1\\ \|s\|^{*}=1}} s^{t}(B^{-1})^{t}u} = \frac{1}{\max\limits_{\substack{\|u\|^{*}=1\\ \|s\|^{*}=1}} u^{t}B^{-1}s} = \min_{\|w\|=1}\|Bw\|^{*}.$$

■

To prove Proposition 2.8**,** we need to prove first the following lemma.

Lemma 2.9. The matrix E satisfies:

$E^{t}E = I,$ and $EE^{t}\big|_{V} = Id_{V},$

where E^{t} is the transpose of $E,$ I is the $(n-1)\times(n-1)$ unit matrix, $\big|_{V}$ denotes restriction to V , and Id_{V} is the identity on $V.$

Proof of Lemma 2.9. Since V and R^{n-1} are of the same dimension, it is sufficient to prove $E^{t}E = I.$ This follows from the fact that the columns of E are orthonormal.

■

Proof of Proposition 2.8

Assume on the contrary that B is not invertible, then there is a vector $w \in R^{n-1}$, $w \neq 0$, such that $Bw = 0$. Since $Ew \neq 0$, and $Ew \in V$, then $\hat{y} = \frac{Ew}{\|Ew\|} \in S$.

By Lemma. 2.9, and the fact that: $B = E^t AE$, we get that for all $x, y \in V$:

$$x^t EBE^t y = x^t EE^t AEE^t y = x^t Ay.$$

Therefore, for all $x \in S$:

$$x^t A\hat{y} = x^t EBE^t \frac{Ew}{\|Ew\|} = \frac{1}{\|Ew\|} x^t EBw = 0, \qquad \text{since } Bw = 0.$$

The above is true for *all* $x \in S$. In particular it is true for the max, that is:

$\max_{x \in S} x^t A\hat{y} = 0$, and so $\min_{y \in S} \max_{x \in S} x^t Ay \leq 0$, in contradiction to Proposition 2.3(a).

■

The proofs of Propositions 2.7 and 2.8 complete the proof of Theorem 2.4.

■

We now prove the corresponding result for the E-model:

Theorem 2.10

If the payoff $n \times n$ matrix A has a strictly mixed and unique Nash equilibrium, then

$$e_1(A) = e_2(A),$$

and this holds for any norm we use for the deviations.

Proof.

Note first that: $-\min_{x \in S} \min_{y \in S} x^t Ay = \max_{y \in S} \max_{x \in S} x^t A^t y.$

The above is obtained (as before) by viewing PI as the column player in $-A^t$. Now:

$\max_{y \in S} \max_{x \in S} x^t A^t y = \max_{\substack{y \in S \\ x \in S}} y^t Ax = \max_{y \in S} \max_{x \in S} x^t Ay = e_2(A).$ Hence: $e_1(A) = e_2(A).$

■

Theorems 2.4 and 2.10 imply that once a norm for the deviation has been chosen, the risk measure is a property of the payoff matrix A only. Accordingly, instead of the risk measures that were defined earlier for each player separately, we can now define the following risk measures for a given A-matrix game as:

$R(A) := R_1(A) = R_2(A),$ in the S-model, and:

$e(A) := e_1(A) = e_2(A),$ in the E-model.

Denote by $[1]$ the $n \times n$ matrix with $a_{ij} = 1,$ for all $i, j.$

It is straight forward that:

Lemma 2.31 For all $c \in R$:

(1) $R\left(A + c[1]\right) = R(A)$

(2) $R(cA) = |c|\, R(A)$

(3) Adding a constant to any row or column of A does not change $R(A)$.

The same is true for $e(A)$.

Lemma 2.31 states that the risk measures we have defined are invariant under addition, and linear under multiplication. These properties seem desirable for any risk measure, since an addition may be viewed as paying (or charging) the players a fixed payment before the game starts. This should not affect their rational choices during the game that follows. However, multiplying the matrix by a scalar presents a different game and thus *should* affect the rational behavior. One might still wish that the risk measure would be invariant also under positive multiplication, so that the risk would be *relative* to the magnitude of payoffs in the matrix. This would make the risk a property of the *configuration* (i.e., the relations between the elements) of the payoff matrix, and would enable a comparison between games with different configurations, ignoring differences between the order of magnitude of payoffs in each game. However, this together with the invariance under addition, would make the risk measure meaningless, since through addition and multiplication, one can change any payoff matrix A into a matrix that is close as one wishes to the matrix $[1]$, e.g.: $[1] + \frac{1}{M} A,$ where M is a large positive number.

Theorems 2.4 and 2.10 prove that our result holds for *any* norm we choose to use for the size of deviations. It may be of interest to see and compare the risk values obtained using different norms. In the following two sections we develop explicit expressions for the risk measures for the $\|\cdot\|_1$-norm (Section 3), and the $\|\cdot\|_2$-norm (Section 4).

3. The $\|\cdot\|_1$ -norm

In this case, the set S of vector-deviations of PI and PII is:

$$S = \left\{ x \ : \quad \sum_{i=1}^{n} x_i = 0, \quad \sum_{i=1}^{n} | x_i | = 1 \right\}.$$

3.1. The S-model (minmax approach) with the $\|\cdot\|_1$ -norm

3.1.1. Calculating the risk

In the proof of Proposition 2.7, we showed that the risk $R(A)$ satisfies that:

$$R(A) = \frac{1}{\max\limits_{\substack{\|s\|^*=1 \\ \|u\|^*=1}} s^t B^{-1} u},$$

where $s, u \in R^{n-1}$, and $B = E^t AE.$

Recall that the norm $\|\cdot\|$ we use on R^{n-1} is not the $\|\cdot\|_1$ -norm of R^{n-1}, but rather the norm $\|\cdot\|_{1E}$ induced on R^{n-1} via E by the $\|\cdot\|_1$ -norm of R^n, i.e., for $u \in R^{n-1}$:

$$\|u\|_{1E} := \|Eu\|_1 .$$

Since it is easier to calculate $\|\cdot\|_1$ on V than $\|\cdot\|_{1E}$ on R^{n-1}, we return from R^{n-1} to $V \subset R^n$. The corresponding *dual* norm $\|\cdot\|_{1V}^*$ on V is defined as follows.

Extension of Definition 2.6

$$\|y\|_{1V}^* = \max_{\substack{x \in V \\ \|x\|_1 = 1}} x^t y.$$

Let us focus on the denominator: $\max\limits_{\substack{\|s\|_{1E}^{*}=1\\ \|u\|_{1E}^{*}=1}} s^t B^{-1} u,$ where $s,u \in R^{n-1}.$ As E^t is a one-to-one map from V onto $R^{n-1},$ there exist $x, y \in V,$ such that: $u = E^t y,$ $s = E^t x,$ and so:

$$s^t B^{-1} u = x^t E B^{-1} E^t y.$$

Denote $C = EB^{-1}E^t,$ then:

$s^t B^{-1} u = x^t C y,$ and $C(V) = C^t(V) = V,$ i.e. C and C^t map V onto V .

Note that $E: R^{n-1} \to V$ and $E^t\big|_V : V \to R^{n-1}$ preserve the inner product (as they transform an orthonormal basis to an orthonormal basis). Furthermore, by definition of the norm on $R^{n-1},$ they preserve the norm. It follows that they also preserve the dual norm since it is determined by the norm and the inner product, namely: If $y \in V$ and $u = E^t y$ then $\|u\|_{1E}^{*} = \|y\|_{1V}^{*}$.

So $\max\limits_{\substack{\|s\|_{1E}^{*}=1\\ \|u\|_{1E}^{*}=1}} s^t B^{-1} u = \max\limits_{\substack{\|x\|_{1V}^{*}=1\\ \|y\|_{1V}^{*}=1}} x^t C y.$

For any $a \in R,$ denote by $\bar{a}$ the constant vector $\bar{a} = (a, \quad a, \quad \dots \quad a)^t \in R^n.$

Lemma 3.1

For all constant vectors $\bar{a}$ and $\bar{b}$:

$$(x+\bar{a})^t C(y+\bar{b}) = x^t C y.$$

Proof.

$(x+\bar{a})^t C(y+\bar{b}) = x^t C y + x^t C\bar{b} + \bar{a}^t C(y+\bar{b}).$

But

$$x^t C\bar{b} = x^t E B^{-1} E^t \bar{b} = 0.$$

This is so since the rows of E^t belong to V and thus the sum of the entries of each row is $0,$ so $E^t\bar{b} = 0.$ Similarly, since $\bar{a}^t E = 0$:

$\bar{a}^t C(y+\bar{b}) = \bar{a}^t E B^{-1} E^t (y+\bar{b}) = 0.$

■

Denote $K=\left\{x:\forall i,\ x_i=\pm 1\right\}$, and $K^-=K\setminus\left\{(1,\ 1,\ \ldots\ ,1)^t,\ (-1,\ -1,\ \ldots\ ,-1)^t\right\}$.

For the sake of convenience, from this point on we drop the subscript V from $\|\cdot\|_{1V}^*$ and write simply $\|\cdot\|^*$.

Proposition 3.2

$$\max_{\substack{\|x\|^*=1\\ \|y\|^*=1\\ x,y\in V}} x^tCy=\max_{k,r\in K^-} k^tCr.$$

This proposition says that for finding the maximum in the infinite set $\left\{x,y\in V\ ;\quad \|x\|^*=1,\ \|y\|^*=1\right\}$, it is enough to find the maximum in the finite set K^-. Note that K^- is not contained in V.

Proof

If $x\in V$, and $\|x\|^*=1$, then $\max_{\substack{\|z\|_1=1\\ z\in V}} z^tx=1$. This maximum will be attained at a vector $z\in S$ for which the entry multiplying $x_{\max}$ is 0.5, the entry multiplying $x_{\min}$ is -0.5 and the rest of the entries are 0, where $x_{\max}$ is the maximal entry of x, and $x_{\min}$ is the minimal entry of x. Thus we get: $1=\|x\|^*=\max_{\substack{\|z\|_1=1\\ z\in V}} z^tx=0.5x_{\max}-0.5x_{\min}$, from which we get that: $x_{\max}-x_{\min}=2$. That is, we have obtained that $x\in V$ satisfies that $\|x\|^*=1$, *iff* it satisfies that: $x_{\max}-x_{\min}=2$.

Let L be the set $L=\{x\in R^n : x_{\max}=1,\ x_{\min}=-1\}$. Given any $x\in L$ let e_x be the average of the entries of x , and let $\hat{x}=x-\overline{e}_x$. Then $x\mapsto\hat{x}$ is a bijection from L onto the set $\{x\in V:\|x\|^*=1\}$, and by Lemma 3.1 $x^tCy=\hat{x}^tC\hat{y}$. We note that $K^-\subset L$, and so it remains to show that $m=\max_{x,y\in L} x^tCy$ is attained at a pair of points in K^-. Let x_0,y_0 be a pair of points in L where the maximum is attained, i.e. $m=x_0^tCy_0$.

Fixing y_0 it is clear that $\max_{x \in L} x^t C y_0$ is attained at the point $k \in K^-$ defined as follows. For all i:

If $\left(Cy_0\right)_i \geq 0,$ then $k_i := 1,$ and (8)

if $\left(Cy_0\right)_i < 0,$ then $k_i := -1.$

So we must have $m = k^t C y_0$.

Note that indeed $k \in K^-$, namely that $k \neq (1, \quad 1, \quad \ldots \quad ,1)^t, \ (-1, \quad -1, \quad \ldots \quad ,-1)^t$. This is true because $C(V) = V$, and so $0 \neq y_0 \in V$ implies $0 \neq Cy_0 \in V$. Hence the entries of $0 \neq Cy_0$ sum to 0 which implies that it has both positive and negative entries. Thus k, which is determined by (8) must have both 1 and -1 entries.

We may now similarly replace y_0 with $r \in K^-$ defined as follows. For all i:

If $\left(k^t C\right)_i \geq 0,$ then $r_i := 1,$ and (9)

if $\left(k^t C\right)_i < 0,$ then $r_i := -1.$

As before, we must have $m = k^t C r$, so we have established that the maximum is attained at the pair $k, r \in K^-$

(Again $k^t C$ has both positive and negative entries since if $k = \hat{z}$ where $z \in V$ then $\bar{e}_z{}^t C = 0$ so $k^t C = z^t C,$ and $0 \neq z \in V$ so $0 \neq C^t z \in V$, since $C^t(V) = V$).

■

As a conclusion of our results so far, we have obtained:

$$R(A) = \frac{1}{\max_{k,r \in K^-} k^t C r}.$$

In order to find $\max_{k,r \in K^-} k^t C r,$ one needs to consider all vectors $k \in K^-$. Each such vector is matched with a vector r that is determined by (9), in order to calculate $k^t C r,$ and find the pair $\tilde{k}, \tilde{r},$ for which $k^t C r$ is maximal.

For a game matrix of dimension $n,$ one needs to calculate $k^t C r$ for the $2^n - 2$ possibilities of $k,$ (corresponding to the number of vectors in the set K^-). Clearly

$\max_{r\in K^-}(-k)^t Cr = \max_{r\in K^-} k^t Cr,$ therefore one needs to make only $\frac{2^n-2}{2} = 2^{n-1}-1$ calculations.

3.1.2. The optimal strategies

The optimal strategies x^* and y^* of PI and PII respectively are given by:

$$x^* = \frac{EB^{-1}E^t\tilde{k}}{\left\|EB^{-1}E^t\tilde{k}\right\|_1} \qquad y^* = \frac{EB^{-1}E^t\tilde{r}}{\left\|EB^{-1}E^t\tilde{r}\right\|_1} \qquad (10)$$

This is obtained by starting at the end of the process described above, namely at the optimal vectors $\tilde{k}, \tilde{r},$ and following the process in the reverse direction. Note that we can skip the first step of $\tilde{k} \mapsto \tilde{k} - \overline{e}_{\tilde{k}},$ (where $e_{\tilde{k}}$ is the average of the entries of $\tilde{k}$). This is because the next step is multiplication by E^t, and $E^t\overline{e}_{\tilde{k}} = 0.$ The same goes for $\tilde{r}.$

Note also that the normalization appearing in (10) is needed because the first step of the proof of Proposition 2.7 is $\min_{\|w\|=1}\|Bw\|^* = \min_{w\neq 0}\frac{\|Bw\|^*}{\|w\|}$, which relaxes the requirement $\|w\| = 1.$

Remark. Note that the matrix we are finally using is $EB^{-1}E^t$, which can be expressed in terms of the matrix A as: $E(E^tAE)^{-1}E^t.$

That is, we induce A on R^{n-1} via $E,$ take the inverse in R^{n-1} and then induce back to R^n via $E^t.$ If A itself is invertible then one might think that the round trip through R^{n-1} can be saved by simply taking $A^{-1}.$ The action of A^{-1} is however completely different. Note e.g. that in general $A^{-1}(V) \not\subset V$.

In the following section we will demonstrate the procedure described here.

3.1.3. Examples.

We look at four different games:

MP	**O'Neille's game**	**Non-symmetric 3X3 game**	**Non-symmetric 4x4 game**
$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.1 & 0.8 & 0.7 \\ 0.9 & 0.2 & 0.5 \\ 0.3 & 0.1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0.5 & 0.25 \\ 0 & 1 & 0.25 & 0.5 \\ 0.25 & -0.25 & 0.75 & 0 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{pmatrix}$

The first two games are symmetric; the first denoted MP is the well-known game of "Matching Pennies", and the second game was constructed by O'Neille (O'Neille, 1986). The last two games are non-symmetric. We will follow in detail the example of the non-symmetric 3×3 game, and only give the final results for the other 3 games.

In the two symmetric games above, only two different payoffs appear: 1 and -1. This, theoretically, makes irrelevant the subjects' attitude to risk assuming they have von-Neumann Morgenstern linear utility functions (see O'Neille, 1986). We are interested in MP as an example for a very simple game that has a unique mixed Nash equilibrium. Our interest in O'Neille's game derives from the fact that it was carefully chosen to hold several desirable properties, one of which is that it is the simplest nontrivial game according to a definition of simplicity given in O'Neille, 1986.

O'Neille's game was comprehensively discussed and debated upon in several papers (see O'Neille, 1986; Brown and Rosenthal 1990; O'Neille 1991). It is interesting to compare between the risks of MP and O'Neille's game. Nevertheless looking at symmetric games only would not be satisfactory since our main result, that the risks of the players coincide, holds trivially in symmetric games. Hence we look at non-symmetric games as well.

Note that if x is optimal then $-x$ is optimal as well. In our solutions below we will not mention both.

The Non-symmetric 3x3 game

$$A = \begin{pmatrix} 0.1 & 0.8 & 0.7 \\ 0.9 & 0.2 & 0.5 \\ 0.3 & 0.1 & 1 \end{pmatrix}$$

The matrix E appearing in Corollary 2.5 for the case of $n=3,$ is:

$$E=\begin{pmatrix} \frac{3-\sqrt{3}}{6} & \frac{-3-\sqrt{3}}{6} \\ \frac{-3-\sqrt{3}}{6} & \frac{3-\sqrt{3}}{6} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$B=E^tAE=\begin{pmatrix} \frac{3-\sqrt{3}}{6} & \frac{-3-\sqrt{3}}{6} & \frac{1}{\sqrt{3}} \\ \frac{-3-\sqrt{3}}{6} & \frac{3-\sqrt{3}}{6} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} 0.1 & 0.8 & 0.7 \\ 0.9 & 0.2 & 0.5 \\ 0.3 & 0.1 & 1 \end{pmatrix}\begin{pmatrix} \frac{3-\sqrt{3}}{6} & \frac{-3-\sqrt{3}}{6} \\ \frac{-3-\sqrt{3}}{6} & \frac{3-\sqrt{3}}{6} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}=\begin{pmatrix} .028 & .612 \\ .555 & -.261 \end{pmatrix}.$$

$$B^{-1}=\begin{pmatrix} 0.752 & 1.764 \\ 1.600 & -0.081 \end{pmatrix}.$$

$$C=EB^{-1}E^t=\begin{pmatrix} -0.578 & 0.962 & -0.385 \\ 1.057 & -0.096 & -0.960 \\ -0.480 & -0.866 & 1.345 \end{pmatrix}.$$

The possibilities we need to check for the vector k appear in the first column of the following table. For each vector k we define the vector r according to (9). For each pair k and $r,$ we calculate the value of k^tCr in the third column.

k	r	k^tCr
$(1,1,-1)$	$(1,1,-1)$	5.381
$(-1,1,1)$	$(1,-1,1)$	3.849
$(1,-1,1)$	$(-1,1,1)$	4.227

From the table above, we see that the maximum is attained in the first case, namely for:
$\tilde{k}=(1,1,-1), \quad \tilde{r}=(1,1,-1).$

And the value of the game is thus:

$$R(A)=\frac{1}{\max\limits_{x,y\in S} x^tCy}=\frac{1}{5.381}=0.18584.$$

To find the optimal strategies x^*, y^*, we use (10) and get:

$x^* = (0.178, 0.322, -0.5), \qquad y^* = (0.143, 0.357, -0.5).$

The results for the four games are summarized in the following table. Note that MP is riskier than the O'Neill's game.

The game	R	x^* = optimal strategy of PI	y^* = optimal strategy of PII
MP	1	$(-.5, .5)$	$(-.5, .5)$
O'Neille's game	0.625	$(.25, -.5, .125, .125)$	$(.25, .125, .125, .-.5)$
3X3 non-symmetric game	0.18584	$(0.178, 0.322, -0.5)$	$(0.143, 0.357, -0.5).$
4X4 non-symmetric game	0.10227	$(.318, .091, -.409, .182)$	$(.316, .-.227, -.273, .364)$

3.2. The E-model (maxmax approach) with the $\|\cdot\|_1$ -norm

As explained earlier, in the E-model each player is concerned with the worst possible outcome. In particular, PII is concerned with $\max_{x,y\in S} x^t Ay.$

Denote: $z^t = x^t A,$ and denote z_i and z_j as the minimal and maximal entries of z respectively. Then given $z^t = x^t A,$ the strategy y^x of PII, which will maximize $x^t Ay,$ satisfies that: $y_i^x = -0.5, \quad y_j^x = 0.5$ and for all $k \neq i, j: \ y_k^x = 0.$ This is so since: $y \in S,$ and so: $\sum y_i^x = 0, \quad$ and $\quad \sum |y_i^x| = 1.$ Hence the vectors $x^*, y^*,$ are both of the form: $0.5 \times (0 \ldots 0, 1, 0 \ldots 0, -1, 0 \ldots 0).$ Define the set X of such vectors as follows:

$$X = \left\{ x: \quad \exists i \ s.t: \ x_i = 0.5, \quad \exists j \ s.t: \ x_j = -0.5, \quad \forall k \neq i, j: \ x_k = 0 \right\}.$$

Then:

$$\max_{x,y\in S} x^t Ay = \max_{x\in X} x^t Ay^x,$$

where y^x was defined above.

Thus in order to find the risk one needs to make only $\binom{n}{2}$ calculations and not $2\binom{n}{2}$ calculations since $y^{-x} = -y^x,$ and so: $(-x)^t Ay^{-x} = x^t Ay^x.$

3.2.3. Examples

We will continue with our four examples above.

For the non-symmetric 3x3 game

$$A = \begin{pmatrix} 0.1 & 0.8 & 0.7 \\ 0.9 & 0.2 & 0.5 \\ 0.3 & 0.1 & 1 \end{pmatrix}$$

$2x$	$2y^x$	$4x^t A y^x$
$(1,-1,0)$	$(-1,1,0)$	1.4
$(-1,0,1)$	$(0,-1,1)$	1
$(0,-1,1)$	$(-1,0,1)$	1.1

We see from the table above that the maximal value is attained in the first row and that it is: $1.4/4 = 0.35$. Hence: $e(A) = 0.35$,

and the strategies x^*, y^*, are:

$$x^* = (0.5, -0.5, 0)$$
$$y^* = (-0.5, 0.5, 0).$$

The results for the four games are summarized in the following table.

The game	e	x^*	y^*
MP	1	$(0.5 \quad -0.5)$	$(0.5 \quad -0.5)$
O'Neille's game	1	$(0 \quad 0 \quad 0.5 \quad -0.5)$	$(0 \quad 0 \quad 0.5 \quad -0.5)$
3X3 non-symmetric game	0.35	$(0.5, -0.5, 0)$	$(-0.5, 0.5, 0)$
4X4 non-symmetric game	0.5	$(-0.5 \quad 0.5 \quad 0 \quad 0)$	$(-0.5 \quad 0.5 \quad 0 \quad 0)$

4. The $\|\cdot\|_2$ -norm

As before, for any deviation vectors x and $y,$ denote by $v, w \in R^{n-1}$ the vectors s.t. $Ev = x,\ Ew = y,$ and denote: $B = E^t AE.$ Then, like before, the deviation payoff is:

$$x^t Ay = (Ev)^t A(Ew) = v^t (E^t AE)w = v^t Bw.$$

Now, given $w,$ then by the Cauchy-Swartz inequality $\max_{\|v\|_2=1} v^t Bw$ is obtained when v is in the direction of $Bw.$ That is:

$$\max_{\|v\|_2=1} v^t Bw = \frac{(Bw)^t Bw}{\|Bw\|_2} = \|Bw\|_2 = \sqrt{w^t B^t Bw}.$$

Denote: $D = B^t B,$ then D is a symmetric and non-negative $(n-1)\times(n-1)$ matrix, thus it can be diagonalized, namely there exists an orthonormal basis $\{u_1, \ldots, u_{n-1}\}$ such that if $w = \alpha_1 u_1 + \cdots \alpha_{n-1} u_{n-1},$ then:

$$\sqrt{w^t Dw} = \sqrt{\sum_{i=1}^{n-1} d_i \alpha_i^2} \qquad (11)$$

where $\forall i,\ d_i \geq 0,$ are the eigenvalues of $D.$ Note that $\|w\|_2 = 1$ so $\sum_{i=1}^{n-1} \alpha_i^2 = 1.$

Hence: $R(A) = \min \sqrt{\sum_{i=1}^{n-1} d_i \alpha_i^2},$ and $e(A) = \max \sqrt{\sum_{i=1}^{n-1} d_i \alpha_i^2},$ so if we denote by $d_{\min}$ and $d_{\max}$ the minimal and maximal eigenvalues of D respectively, then:

$$R(A) = \sqrt{d_{\min}}, \quad \text{and} \quad e(A) = \sqrt{d_{\max}}.$$

Accordingly, in the S-model, an optimal strategy of PII is $Ew,$ where w is an eigenvector of $D,$ corresponding to $d_{\min}$, and in the E-model, an optimal strategy of PII is $Ew,$ where w is an eigenvector of $D,$ corresponding to $d_{\max}$.

Similarly, for PI with $\hat{D} = BB^t$ instead of $D = B^t B.$

4.3. Examples

For the non-symmetric 3x3 game

$$A = \begin{pmatrix} 0.1 & 0.8 & 0.7 \\ 0.9 & 0.2 & 0.5 \\ 0.3 & 0.1 & 1 \end{pmatrix}$$

The matrix B for this case was already found in section 3.1.3:

$$B = \begin{pmatrix} .028 & .612 \\ .555 & -.261 \end{pmatrix}.$$

$$D = B^t B = \begin{pmatrix} 0.309 & -0.128 \\ -0.128 & 0.443 \end{pmatrix}.$$

The eigenvalues of D and their eigenvectors are:

$$\begin{aligned} &0.231, \quad (-0.855, 0.518). \\ &0.52, \quad (-0.518, -0.855). \end{aligned}$$

Hence:

$$\begin{aligned} R(A) &= \sqrt{0.231} = 0.4806, \\ e(A) &= \sqrt{0.52} = 0.7211. \end{aligned}$$

The eigenvalues of BB^t are the same as for $B^t B,$ but their eigenvectors are different. The eigenvalues and their eigenvectors are:

$$\begin{aligned} &0.231, \quad (-.708, .706). \\ &0.52, \quad (-.706, -.708). \end{aligned}$$

In the S-model:

$$x^* = E\begin{pmatrix} -.708 \\ .706 \end{pmatrix} = \begin{pmatrix} \frac{3-\sqrt{3}}{6} & \frac{-3-\sqrt{3}}{6} \\ \frac{-3-\sqrt{3}}{6} & \frac{3-\sqrt{3}}{6} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -.708 \\ .706 \end{pmatrix} = \begin{pmatrix} -.706 \\ .708 \\ -.001 \end{pmatrix}.$$

And in the E-model:

$$x^* = E\begin{pmatrix} -.706 \\ -.708 \end{pmatrix} = \begin{pmatrix} \frac{3-\sqrt{3}}{6} & \frac{-3-\sqrt{3}}{6} \\ \frac{-3-\sqrt{3}}{6} & \frac{3-\sqrt{3}}{6} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} -.706 \\ -.708 \end{pmatrix} = \begin{pmatrix} .409 \\ .407 \\ -.816 \end{pmatrix}.$$

and similarly, the optimal strategies of PII are:

In the S-model:

$$y^* = (-.589, .784, -.195).$$

And in the E-model:

$$y^* = (.565, .228, -.793).$$

The results for the four games are summarized in the following two tables.

The S-model

The game	R	x^*	y^*
MP	2	$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
O'Neille's game	2	$(0.000\ 0.408\ 0.408\ -0.816)$ $(0.001\ -0.567\ 0.793\ -0.227)$	$(0.000\ 0.408\ 0.408\ -0.816)$ $(0.001\ -0.567\ 0.793\ -0.227)$
3X3 non-symmetric game	0.4806	$(-.706\quad .708\quad -.001)$	$(-.589, .784, -.195)$
4X4 non-symmetric game	0.34	$(.079, -.34, -.52, .781)$	$(-.219, .31, .607, -.698)$

The E-model

The game	e	x^*	y^*
MP	2	$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
O'Neille's game	2.5	$(-0.577,\ 0.577, 0.577)$	$(-0.577,\ 0.577, 0.577)$
3X3 non-symmetric game	0.7211	$(.409\quad .407\quad -.816)$	$(.565, .228, -.793)$
4X4 non-symmetric game	1.122	$(.333, -.795, .506, -.044)$	$(-.703, .608, -.212, .307)$

5. The appropriate norm to use is $\|\cdot\|_1$.

Although Theorem 2.4 proves that our result holds for any norm, we argue that in fact the only norm that should be used is the norm $\|\cdot\|_1$ on V, (or a multiple of it, i.e. $c\|\cdot\|_1$).

This is because the norm we use should give the same norm to deviations which are equivalent in the sense, which is demonstrated in the following example. Consider a payoff matrix A in which the first two rows are identical (or almost identical, in order for A to have a *unique* Nash equilibrium). Then the deviation vector $x=(\varepsilon,0,x_3,\dots,x_n)$ and the deviation vector $x'=(0,\varepsilon,x_3,\dots,x_n)$ satisfy that: $x^tA=x'^tA,$ and so for all $y\in S:$ $x^tAy=x'^tAy.$ Hence x and x' are equivalent deviations with respect to $A,$ and so we would like them to have the same norm.

For this we need the following definition.

Definition 5.1. Given a norm $\|\cdot\|$, and deviation vectors $x,x'\in V$, we say that x and x' are *similar* if there exists a subset of indices $I\subseteq\{1,2\dots n\}$, s.t:

$$\begin{aligned}&\forall i\in I:\\ &x_i\geq 0,\ \ x_i'\geq 0,\ \text{ and}: \sum_{i\in I}x_i=\sum_{i\in I}x_i',\\ &\text{ and } \forall i\notin I:\\ &x_i=x_i'.\end{aligned}$$

Theorem 5.2.

Given a norm $\|\cdot\|,$ if for any similar deviation vectors $x,x'\in V:$ $\|x\|=\|x'\|,$ then there exists a constant $c,$ s.t:

$$\|\cdot\|=c\|\cdot\|_1 \quad \text{on } \ V.$$

Proof.

For $i\neq j$ denote by v^{ij} the vector that satisfies: $v_i=1,\ v_j=-1,\quad$ and $\forall r\neq i,j: v_r=0$ and let $\|v^{12}\|=k.$ Then we claim that $\|v^{ij}\|=k$ for all $i\neq j.$ This is because exchanging between the places of two entries, one which is zero and another which is 1, creates a vector that is similar to the original one and since $\|-v\|=\|v\|.$

Given $0\neq x\in V,$ replace one of its positive entries with the sum of all positive entries, and replace all other positive entries with $0.$ Denote this new vector by $x',$ then

x and x' are similar. By passing to $-x'$ and repeating the procedure we obtain a vector x'' such that $x'' = av^{ij}$ for some $i \neq j$ and some $a > 0$. Now:

$$\|x\| = \|x'\| = \|-x'\| = \|x''\| = \|av^{ij}\| = ak.$$

Clearly $\|\cdot\|_1$ also satisfies the requirement of the Proposition. Namely, that for any similar vectors x, x': $\|x\|_1 = \|x'\|_1$. Hence $\|x\|_1 = a\|v^{ij}\|_1 = 2a.$

Therefore: $\|x\| = ka = \frac{k}{2}\|x\|_1$.

Since this is true for *all* $x \in V$, we have $\|\cdot\| = \frac{k}{2}\|\cdot\|_1$ on V.

■

Bibliography

Baddeley, A. D. (1966) `The capacity for generating information by randomization', *Quarterly Journal of Experimental Psychology,* 18, 119-128.

Brown, James N. and Robert W. Rosenthal (1990), `Testing the Minimax Hypothesis: A Re-Examination of O'Neill's Game Experiment', *Econometrica* 58, No. 5, pp. 1065-1081.

Maschler Michael, Solan Eilon and Zamir Shmuel (2013), `Game Theory'. Cambridge University Press.

Nagel Rosemarie, Ingrid Rohde and Zamir Shmuel (2006), `Playing the field makes mixed strategies observable'. mimeo.

O'Neill, Barry (1986), `Nonmetric test of the Minimax theory of two-person zero-sum games', *Proceedings of the National Academy of Sciences USA*, 84, pp. 2106-2109.

O'Neill, Barry (1991), `Comments on Brown and Rosenthal's Reexamination', *Econometrica*, 59, No. 2. pp. 503-507.

Lax, Peter D. (1997), `Linear Algebra', John Wiley & Sons.

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