k-price auction revisited

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Abstract

This paper shows the correct derivation of the equilibrium bid function for a k-price auction with n bidders, where $k \geq 3$ and $n \geq k$.

1 Introduction

This paper explains the derivation of the k-price auction bidding equilibrium.[1] We show the complete derivation of equilibrium bid function for a k-price auction with n bidders (with $k \geq 3$ and $n \geq k$) for the case where the valuations of bidders are independently drawn from a continuous distribution that has the property that the first derivative of its density function is a constant. This implies that all second and higher order derivatives of the density function are zero. Apart from the uniform distribution, one specific example of such distributions is: $F(x) = x^2, x \in [0, 1]$. It is shown that whenever the first derivative of the density function is a non-zero constant, then for $k \geq 4$, bid function has additional terms that do not appear in the derivation of [1]-[4].

The mathematical foundations of independent private value (IPV) auctions have been well understood since Vickrey [8]. A key result in the IPV auction theory is the remarkable Revenue Equivalence Theorem (RET) which states that, under some mild conditions, the seller's expected profits are the same on average from all standard (English, Dutch, first-price sealed-bid,and second-price sealed-bid) and non-standard (e.g. k-price sealed-bid, and all-pay) auction formats, and that buyers are also indifferent among them all. The RET was later developed by Riley and Samuelson [7] under more general conditions.

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[1] A derivation of k-price auction had previously appeared in [1], [2], [3] and [4]. Unfortunately, that derivation, despite being widely referenced, is incorrect.

2 k-price auction

Consider a k-price auction with n bidders, where the highest bidder wins, and pays only the k-highest bid. It is assumed that $k \geq 2$ and $n \geq k$. We also assume that the valuations of the bidders are independent and identically distributed with distribution function $F(x)$, $x \in R_+$, which is log concave continuous (cumulative) distribution function with density $f(x)$.

Let $b_k : R_+ \to R_+$ be the equilibrium (Nash Equilibrium) bid function of a k-price auction. Kagel and Levine [5] and later on Wolfstatter [1] have provided an elegant derivation for $b_k(x)$ by using the Revenue Equivalence Theorem (RET). From RET, we have

$$\int_0^v b_k(x) \frac{(n-1)!}{(n-k)!(k-2)!} [F(x)]^{n-k} [F(v) - F(x)]^{k-2} f(x) dx$$
$$\equiv \int_0^v x(n-1) F(x)^{n-2} f(x) dx \tag{1.1}$$

By iteratively differentiating the above $k - 1$ times with respect to v, and dividing by $f(v)$ on each iteration yields the following for $k = 3$:

$$b_3^*(v) = v + \frac{1}{n-2} \frac{F(v)}{f(v)} \tag{1.2}$$

Attempting to generalize to the k-price auction, Wolfstetter [1] and [2], derived the following closed-form function:

$$b_k^*(v) = v + \frac{k-2}{n-k+1} \frac{F(v)}{f(v)} \tag{1.3}$$

This result is not correct. The error with Wolfstetter's derivation pertains in particular to dealing with the term $F(v)/f(v)$, which seems to be completely missing from the derivation.

3 Derivation of correct bid function

Let us denote

$$\phi_0(v) := \int_0^v b_k(x)[F(x)]^{n-k}[F(v) - F(x)]^{k-2} f(x) \mathrm{d}x, \ \psi_0(v) := \int_0^v x[F(x)]^{n-2} f(x) \mathrm{d}x \tag{1}$$

Then from (1.1), we have

$$\binom{n-2}{k-2} \phi_0(v) \equiv \psi_0(v) \tag{2}$$

Iteratively define

$$\phi_{t+1}(v) := \phi_t'(v)/f(v) \text{ and } \psi_{t+1}(v) := \psi_t'(v)/f(v) \text{ for } t = 0, 1, \ldots \tag{3}$$

From (2) and (3), it follows that

$$\binom{n-2}{k-2}\phi_t(v) = \psi_t(v) \text{ for } t = 0, 1, \dots \tag{4}$$

Observe from (1) that

$$\phi_0(v) = \int_0^v b_k(x)[F(x)]^{n-k}\left[\sum_{\ell=0}^{k-2}(-1)^\ell\binom{k-2}{\ell}[F(x)]^\ell[F(v)]^{k-2-\ell}\right]f(x)\mathrm{d}x \tag{5}$$

Since all terms of (5) are bounded, the order of summation and integration can be switched. Denoting

$$g_\ell(v) := \int_0^v b_k(x)[F(x)]^{n-k+\ell}f(x)\mathrm{d}x \tag{6}$$

from (5) we have

$$\phi_0(v) = \sum_{\ell=0}^{k-2}(-1)^\ell\binom{k-2}{\ell}[F(v)]^{k-2-\ell}g_\ell(v) \tag{7}$$

Lemma 1

$$\phi_{k-1}(v) = (k-2)!b_k(v)[F(v)]^{n-k} \tag{8}$$

Proof See the Appendix. ■

In contrast to $\phi_t(v)$, determination of $\psi_t(v)$ is generally complex. Here we derive $\psi_t(v)$ for a class of density functions whose second or higher order derivatives vanish. Let $f_t(v)$ denote the t-th order derivative of the density function f. Suppose f be such that $f_1(x) = c$, where c is a constant. Consequently all derivatives of second and higher orders of $f(x)$ are zero, i.e., $f_t(x) = 0$ for all $t = 2, 3, \dots$.

To determine $\psi_t(v)$, note from (1) and (3) that

$$\psi_1(v) = v[F(v)]^{n-2} \tag{9}$$

Since $F'(v) = f(v)$, from (3) and (9),

$$\psi_2(v) = v(n-2)[F(v)]^{n-3} + F(v)^{n-2}/f(v) \tag{10}$$

Denote

$$\rho^n(\ell) :=^{(n-2)} P_\ell = (n-2)!/(n-2-\ell)! \tag{11}$$

and observe that

$$(n-2-\ell)\rho^n(\ell) = \rho^n(\ell+1) \tag{12}$$

To avoid notational clutter, we shall suppress the superscript n and denote ρ^n by simply ρ.

For $t = 3, 4, \dots$ and $\ell = 0, 1, \dots$, let θ_ℓ^t be determined as follows:

$$\theta_\ell^t = 0 \text{ for } \ell = t-2, t-1, \dots. \tag{13}$$

$$\theta_0^3 = 1 \text{ and } \theta_0^{t+1} = t - 1 + \theta_0^t \text{ for } t = 3, 4, \ldots. \tag{14}$$

$$\theta_\ell^{t+1} = \theta_\ell^t + (2\ell + 1)\theta_{\ell-1}^t \text{ for } \ell = 1, \ldots, t-2 \text{ and } t = 3, 4, \ldots. \tag{15}$$

Note from (14) that

$$\theta_0^t = (t-1)(t-2)/2 > 0 \text{ for } t = 3, 4, \ldots \tag{16}$$

Using (15) and the solution (16), it follows that

$$\theta_\ell^{t+1} = \left[\prod_{j=0}^{\ell-1}\{2(\ell - j) + 1\}\right] \sum_{i_\ell=\ell+2}^{t} \sum_{i_{\ell-1}=\ell+1}^{i_\ell - 1} \cdots \sum_{i_2=4}^{i_3-1} \sum_{i_1=3}^{i_2-1} \frac{(i_1 - 1)(i_1 - 2)}{2}$$

$$\text{for } \ell = 1, \ldots, t-2 \text{ and } t = 3, 4, \ldots. \tag{17}$$

Lemma 2 *Suppose* $f_1(x) = c$, *where* c *is a constant. Then for* $t \geq 3$,

$$\psi_t(v) = \rho(t-1)v[F(v)]^{n-t-1} + (t-1)\rho(t-2)[F(v)]^{n-t}/f(v)$$

$$+ \sum_{\ell=0}^{t-3}(-1)^{\ell+1}\rho(t-3-\ell)c^{\ell+1}\theta_\ell^t[F(v)]^{n-t+1+\ell}/[f(v)]^{3+2\ell} \tag{18}$$

where θ_ℓ^t *are constants determined by* (13)-(15).

Proof See the Appendix. ∎

Taking $t = k - 1$ in (4), we have

$$\binom{n-2}{k-2}\phi_{k-1}(v) = \psi_{k-1}(v)$$

Using the expression of $\phi_{k-1}(v)$ from (8) of Lemma 1, we have

$$\binom{n-2}{k-2}(k-2)!b_k(v)[F(v)]^{n-k} = \psi_{k-1}(v) \tag{19}$$

Using the permutation function ρ from (11) in (19), we have

$$b_k(v) = \psi_{k-1}(v)/\rho(k-2)[F(v)]^{n-k} \tag{20}$$

Let us first dispose off the cases of $k = 2$ and 3.

Case 1 $k = 2$: For this case $k - 1 = 1$. By (20) and using the expression of $\psi_1(v)$ from (9) we have,

$$b_2(v) = \psi_1(v)/\rho(0)[F(v)]^{n-2} = \psi_1(v)/[F(v)]^{n-2} = v \tag{21}$$

Case 2 $k = 3$: For this case $k - 1 = 2$. By (20) and using the expression of $\psi_2(v)$ from (10) we have,

$$b_3(v) = \psi_2(v)/\rho(1)[F(v)]^{n-3} = \psi_2(v)/(n-2)[F(v)]^{n-3} = v + F(v)/(n-2)f(v) \tag{22}$$

3.1 The main result

Theorem 1 *Consider a k-price auction where $k \geq 4$. Suppose the valuations are independently drawn from a continuous distribution function F that has density f with the property $f_1(x) = c$, where c is a constant. Then*

$$b_k(v) = v + \frac{(k-2)F(v)}{(n-k+1)f(v)}$$

$$+\frac{1}{\rho(k-2)}\sum_{\ell=0}^{k-4}(-1)^{\ell+1}\rho(k-4-\ell)c^{\ell+1}\theta_\ell^{k-1}[F(v)]^{2+\ell}/[f(v)]^{3+2\ell} \tag{23}$$

where θ_ℓ^t are constants determined by (13)-(15).

Proof Let $k \geq 4$ so that $k-1 \geq 3$. Taking $t = k-1$ in Lemma 2 (since $t = k-1 \geq 3$, Lemma 2 applies), by (20),

$$\psi_{k-1}(v) = \rho(k-2)v[F(v)]^{n-k} + (k-2)\rho(k-3)[F(v)]^{n-k+1}/f(v)$$

$$+\sum_{\ell=0}^{k-4}(-1)^{\ell+1}\rho(k-4-\ell)c^{\ell+1}\theta_\ell^t[F(v)]^{n-k+2+\ell}/[f(v)]^{3+2\ell} \tag{24}$$

Then (23) follows from (20) and (24). ■

Remark The last term of (24) does not appear in the derivation of [1]-[4]. This shows that the derivation of [1]-[4] is incorrect.

3.2 Conclusion

We conclude by providing the expressions of correct bid functions for the class of distributions specified in Theorem 1 when $k = 4, \ldots, 7$. Computer programs can determine the bid function for $k \geq 7$ from (24). So far as more general distributions are concerned, in the Appendix we derive the bid function for a general continuous distribution F when $k = 4$.

Examples Observe from (16), (13) and (15) that

$$\theta_0^3 = 1, \theta_0^4 = 3, \theta_0^5 = 6, \theta_0^6 = 10, \theta_1^3 = 0, \theta_1^4 = 3, \theta_1^5 = 12, \theta_1^6 = 30,$$

$$\theta_2^3 = 0, \theta_2^4 = 0, \theta_2^5 = 15, \theta_2^6 = 75, \theta_3^3 = 0, \theta_3^4 = 0, \theta_3^5 = 0, \theta_3^6 = 105 \tag{25}$$

Let $n \geq 7$. We find $b_k(v)$ for $k = 4, \ldots, 7$.

$k = 4$: Taking $k = 4$ in (23) and noting that $\theta_0^3 = 1$,

$$b_4(v) = v + \frac{2F(v)}{(n-3)f(v)} - \frac{\rho(0)c\theta_0^3[F(v)]^2}{\rho(2)[f(v)]^3} = v + \frac{2F(v)}{(n-3)f(v)} - \frac{c[F(v)]^2}{(n-2)(n-3)[f(v)]^3} \tag{26}$$

$k = 5$: Taking $k = 5$ in (23) and by (25),

$$b_5(v) = v + \frac{3F(v)}{(n-4)f(v)} - \frac{\rho(1)c\theta_0^4[F(v)]^2}{\rho(3)[f(v)]^3} + \frac{\rho(0)c^2\theta_1^4[F(v)]^3}{\rho(3)[f(v)]^5}$$

$$= v + \frac{3F(v)}{(n-4)f(v)} - \frac{3c[F(v)]^2}{(n-3)(n-4)[f(v)]^3} + \frac{3c^2[F(v)]^3}{(n-2)(n-3)(n-4)[f(v)]^5} \tag{27}$$

$k = 6$: Taking $k = 6$ in (23) and by (25),

$$b_6(v) = v + \frac{4F(v)}{(n-5)f(v)} - \frac{\rho(2)c\theta_0^5[F(v)]^2}{\rho(4)[f(v)]^3} + \frac{\rho(1)c^2\theta_1^5[F(v)]^3}{\rho(4)[f(v)]^5} - \frac{\rho(0)c^3\theta_2^5[F(v)]^4}{\rho(4)[f(v)]^7}$$

$$= v + \frac{4F(v)}{(n-5)f(v)} - \frac{6c[F(v)]^2}{(n-4)(n-5)[f(v)]^3} + \frac{12c^2[F(v)]^3}{(n-3)(n-4)(n-5)[f(v)]^5}$$
$$- \frac{15c^3[F(v)]^4}{(n-2)(n-3)(n-4)(n-5)[f(v)]^7} \tag{28}$$

$k = 7$: Taking $k = 7$ in (23) and by (25),

$$b_6(v) = v + \frac{5F(v)}{(n-6)f(v)} - \frac{\rho(3)c\theta_0^6[F(v)]^2}{\rho(5)[f(v)]^3} + \frac{\rho(2)c^2\theta_1^6[F(v)]^3}{\rho(5)[f(v)]^5} - \frac{\rho(1)c^3\theta_2^6[F(v)]^4}{\rho(5)[f(v)]^7} + \frac{\rho(0)c^4\theta_3^6[F(v)]^5}{\rho(5)[f(v)]^9}$$

$$= v + \frac{5F(v)}{(n-6)f(v)} - \frac{10c[F(v)]^2}{(n-5)(n-6)[f(v)]^3} + \frac{30c^2[F(v)]^3}{(n-4)(n-5)(n-6)[f(v)]^5}$$
$$- \frac{75c^3[F(v)]^4}{(n-3)(n-4)(n-5)(n-6)[f(v)]^7}$$
$$+ \frac{105c^4[F(v)]^5}{(n-2)(n-3)(n-4)(n-5)(n-6)[f(v)]^9} \tag{29}$$

Appendix

Proof of Lemma 1 Note that $g'_\ell(v) = b_k(v)[F(v)]^{n-k+\ell}f(v)$. Differentiating (7) with respect to v, by (3) we have

$$\phi_1(v) = \sum_{\ell=0}^{k-2}(-1)^\ell \binom{k-2}{\ell} \Big[(k-2-\ell)[F(v)]^{k-2-\ell-1}g_\ell(v) + [F(v)]^{k-2-\ell}b_k(v)[F(v)]^{n-k+\ell}\Big]$$

$$+ \sum_{\ell=0}^{k-2}(-1)^\ell \binom{k-2}{\ell} [F(v)]^{k-2-\ell}b_k(v)[F(v)]^{n-k+\ell}$$

$$= (k-2)\sum_{\ell=0}^{k-3}(-1)^\ell \binom{k-3}{\ell}[F(v)]^{k-3-\ell}g_\ell(v) + b_k(v)[F(v)]^{n-2}\sum_{\ell=0}^{k-2}(-1)^\ell\binom{k-2}{\ell}$$

Since the last term of the expression above is zero, we have

$$\phi_1(v) = (k-2)\sum_{\ell=0}^{k-3}(-1)^\ell \binom{k-3}{\ell}[F(v)]^{k-3-\ell}g_\ell(v) \tag{30}$$

From (30), using (3) repeatedly for $t = 1, \ldots, k-2$ we have

$$\phi_t(v) = {}^{(k-2)}P_t \sum_{\ell=0}^{k-2-t} (-1)^\ell \binom{k-2-t}{\ell} [F(v)]^{k-2-t-\ell} g_\ell(v) \tag{31}$$

Taking $t = k-2$ in (31) we have

$$\phi_{k-2}(v) = (k-2)! g_0(v) \tag{32}$$

Using (5) in (32), $\phi_{k-1}(v) = (k-2)! g_0'(v)/f(v)$. By (6), $g_0'(v) = b_k(v)[F(v)]^{n-k} f(v)$, which proves (8). ■

Proof of Lemma 2 We prove the lemma by induction. First let $t = 3$. From (10) and (3), and using the definition of ρ from (11), we have

$$\psi_3(v) = v\rho(2)[F(v)]^{n-4} + 2\rho(1)[F(v)]^{n-3}/f(v) - c[F(v)]^{n-2}/[f(v)]^3 \tag{33}$$

As $\rho(0) = 1$ and $\theta^0(3) = 1$, (33) shows that (18) holds for $t = 3$.

Now suppose (18) holds for an integer $t \geq 3$. Differentiating (18) with respect to v,

$$\begin{aligned}
\psi_t'(v) &= v\rho(t-1)(n-t-1)F^{n-t-2} f(v) + \rho(t-1)F^{n-t-1} \\
&+(t-1)\rho(t-2)(n-t)[F(v)]^{n-t-1} - (t-1)\rho(t-2)[F(v)]^{n-t} c/[f(v)]^2 \\
&+\sum_{\ell=0}^{t-3} (-1)^{\ell+1} c^{\ell+1} \theta_\ell^t \Big[\rho(t-3-\ell)(n-t+1+\ell)[F(v)]^{n-t+\ell}/[f(v)]^{2+2\ell} \\
&-\rho(t-3-\ell)[F(v)]^{n-t+1+\ell}(3+2\ell)c/[f(v)]^{4+2\ell}\Big]
\end{aligned} \tag{34}$$

Using (3) and the property of ρ from (12) in (34), we have

$$\begin{aligned}
\psi_{t+1}(v) &= v\rho(t)F^{n-t-2} + t\rho(t-1)F^{n-t-1}/f(v) - (t-1)c\rho(t-2)[F(v)]^{n-t}/[f(v)]^3 \\
&+\sum_{\ell=0}^{t-3} (-1)^{\ell+1} c^{\ell+1} \theta_\ell^t \rho(t-2-\ell)[F(v)]^{n-t+\ell}/[f(v)]^{3+2\ell} \\
&+\sum_{\ell=0}^{t-3} (-1)^{\ell+1+1} c^{\ell+1+1} [2(\ell+1)+1] \theta_\ell^t \rho(t-2-(\ell+1))[F(v)]^{n-t+(\ell+1)}/[f(v)]^{3+2(\ell+1)}
\end{aligned} \tag{35}$$

Denote the last term of (35) by A. Denoting $j = \ell+1$, observe that

$$A = \sum_{j=1}^{t-2} (-1)^{j+1} c^{j+1} (2j+1) \theta_{j-1}^t \rho(t-2-j)[F(v)]^{n-t+j)}/[f(v)]^{3+2j} \tag{36}$$

From (35) and (36)

$$\psi_{t+1}(v) = v\rho(t)F^{n-t-2} + t\rho(t-1)F^{n-t-1}/f(v) - [(t-1)+\theta_0^t]c\rho(t-2)[F(v)]^{n-t}/[f(v)]^3$$

$$+\sum_{\ell=1}^{t-3}(-1)^{\ell+1}c^{\ell+1}[\theta^t_\ell+(2\ell+1)\theta^t_{\ell-1}]\rho(t-2-\ell)[F(v)]^{n-t+\ell}/[f(v)]^{3+2\ell}$$

$$+(-1)^{t-1}c^{t-1}[2(t-2)+1]\theta^t_{t-3}\rho(0)[F(v)]^{n-2)}/[f(v)]^{3+2(t-2)} \tag{37}$$

Using (14) and (15) in (37) and noting that $\theta^t_{t-2}=0$ (by (13)), we have

$$\psi_{t+1}(v)=v\rho(t)F^{n-t-2}+t\rho(t-1)F^{n-t-1}/f(v)$$

$$+\sum_{\ell=0}^{t-2}(-1)^{\ell+1}c^{\ell+1}\theta^{t+1}_\ell\rho(t-2-\ell)[F(v)]^{n-t+\ell}/[f(v)]^{3+2\ell} \tag{38}$$

This shows that if the result holds for an $t\geq 3$, it also holds for $t+1$. Since the result holds for $t=3$, it holds for all $t\geq 3$. ■

Theorem 2 *Consider the fourth-price auction, where the highest bidder wins, and pays only the 4-th highest bid, and assume F is log concave. Then, the equilibrium bid function is*

$$b_4(v)=v+\frac{2}{(n-3)}\frac{F(v)}{f(v)}-\frac{1}{(n-2)(n-3)}\frac{F(v)^2}{f(v)^3}f(v)'$$

Proof. Taking $k=4$ in Lemma 1, we have

$$\phi_3(v)=2b_4(v)[F(v)]^{n-4} \tag{39}$$

Note from (10) that

$$\psi_2(v)=v(n-2)[F(v)]^{n-3}+[F(v)]^{n-2}/f(v) \tag{40}$$

From (39), we have

$$\psi_3(v):=\psi_2'(v)/f(v)=$$

$$=v(n-2)(n-3)[F(v)]^{n-4}+2(n-2)[F(v)]^{n-3}/f(v)-f'(v)[F(v)]^{n-2}/[f(v)]^3 \tag{41}$$

By (4), we have

$$\binom{n-2}{2}\phi_3(v)=\psi_3(v) \tag{42}$$

From (39), (41) and (42), it follows that

$$b_4(v)=v+\frac{2F(v)}{(n-3)f(v)}-\frac{f'(v)[F(v)]^2}{(n-2)(n-3)[f(v)]^3}.$$ ■

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