Using a Sequential Game to Distribute Talent in a Professional Sports League

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Abstract In this paper, a professional sports league is modeled as a duopoly. I introduce a sequential game in which in the first stage, the two teams in the league make a bid on the cost of talent. Then, teams formulate their talent demand in a subgame that depends on the cost implemented in the first stage. I find that revenue sharing has no impact on competitive balance and that this model cannot sustain the usual competitive equilibrium as an equilibrium. Also, I find that the supply of talent is not exhausted in equilibrium.

Keywords: Duopoly, Sports, Labour Market, Sequential Game, Competitive Equilibrium, Revenue Sharing

JEL Classification Numbers: D43, L13, J44.

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1 Introduction

The economic theory of professional team sports has yet to fully integrate Game Theory into its field. Although sports economics is a young and growing field of research, the current modelling approach tends to follow the so-called "conjectural variation" approach from the Industrial Organization literature. In this paper, I introduce a model that can be used to describe and analyse the professional sports labour market. I depart from the "conjectural variation" standard and use the the tools of Game Theory to forge my model.

To appreciate the relevance of Game Theory to this field, let's consider the equilibrium concept that is standard in the labour market; the competitive equilibrium. Think of the labour market as a noncooperative game in normal form. There are three players: two professional teams, A and B, and the Market. Assume that there is a fixed supply of workers. Each team wishes to maximize their profits. The only action that a team is allowed to take is the following: given the cost of labour, team i submits a demand to the market representing the quantity of workers i wishes to hire at that price. Teams have interfering desires as more workers hired by one team relatively decreases the power of its opponent to earn revenues. The only desire of the market is to maximize the value of residual demand which is defined by the demand for talent minus the supply. If the demand is greater than the supply the Market increases the cost. If the demand is lower than the supply the Market decreases the cost and if the demand equals the supply, the residual demand is equal to zero and so the Market has no incentive to change the cost. The two teams formulate their demands and the Market sets the cost of labour independently and simultaneously. The Nash equilibrium of the game is precisely what economists may know better as the competitive equilibrium. At the competitive equilibrium, neither the teams nor the Market have an incentive to change their actions since they cannot be strictly better off by doing so.

With the conjectural variation approach, there are two equilibrium concepts in the professional sports labour market. These concepts rely on two distinct assumptions which are known as the "Walras" and the "Nash" assumptions. The former states that prior to formulating a demand to the market, team A must internalize the fact that his quantity demanded will be taken away from team B and thus affecting his revenue directly through the increase of his input *and* indirectly through the decrease of its opponent's input. The latter assumption stipulates that the indirect effect is restricted to be null. Unfortunately, neither the "Walras" assumption nor the "Nash" is coherent with the game-theoretical perspective on competitive equilibrium. The definition of a Nash equilibrium is straightforward: For every individual among the n players, taking as given the actions of the $n-1$ other players, one cannot strictly be better off by changing his action. The Nash equilibrium concept does not allow players to assume subsequent movements in response to their own. This is precisely why the conjectural variation approach is inconsistent with the concept of competitive equilibrium.

In this paper, I present a game-theoretical analysis of the market for playing talent using an alternative to the perfectly competitive model for the labour market in professional team sports. Recall that the usual competitive equilibrium concept relies on the assumption that no team has market power. That is, no team can affect the cost of labour by submitting demands to the market. However, one may argue that in a sports league, teams may have considerate power over the cost of talent. In line with this critique, I propose a sequential game model in which teams have full control over the unit cost of talent.

The main finding is that the implementation of a system of revenue sharing has no impact of competitive balance. The second result is that for the usual contest revenue functions, the standard competitive equilibrium is not sustained as an equilibrium of the model. Consequently, the supply of talent is not fully exhausted in equilibrium.

This paper augments the sports economics literature in two ways. First, I introduce a richer model that offers an alternative way to think about the formation of professional teams in a sports league. Second, the model proposed in this paper is used to analyse a situation where the cost of labour can be affected by the actions of the teams.

2 Review of Literature

Gerard Debreu (Debreu [2]) shows the close link that exists between the concept of competitive equilibrium and that of Nash equilibrium of a normal-form game. Proving the existence of a competitive equilibrium necessitates the same tools that are required to prove the existence of a Nash equilibrium in any finite normal form game.

The use of the term "competitive equilibrium" in Szymanski [6] refers to the so-called Walrasian fixed-supply conjecture model while the "Nash" solution to the noncooperative game of talent choice in a professional sports league is called the "Contest-Nash" solution. This equilibrium concept has been adopted in the subsequent work of Szymanski and Késenne [8]. The conjectural variation hypothesis in the field of sports economics is well-documented in Késenne [4] and in the references therein.

My research follows the work of Madden [5] who initiated a transition towards a more game-theoretically oriented approach to club formation. Madden suggested a new equilibrium concept where, instead of formulating demands to the market in terms of quantity of talent, teams would first decide the total budget dedicated to acquiring playing talent. Then the market decides of the cost of talent such that the whole supply of talent is distributed to teams. Szymanski [7] stated that the work of Madden is "the most significant contribution to this literature since 2004." However, the situation where teams may have full power over both the cost of talent and the quantity of talent has not been looked at yet. My paper focusses on this last issue.

My model shares some resemblance with the model of Jackson and Moulin [3], but in a different context. Jackson and Moulin use a multi-stage mechanism in order to efficiently provide a public good. My model also fits in the Industrial Organization literature that takes game theory as its workhorse. The books of Basu [1] and Vives [9] offer an extensive description of the relevance of Game Theory to the IO literature.

3 The Model

A sports league consists of two teams; team A and team B. They are engaged in a contest against each other. The only input used by teams is called *talent*, which is a positive real number. This input represents what professional sports player are assumed to be endowed with. We assume that talent is a continuous variable and that the total quantity of talent is equal to 1. Throughout the paper, the quantity of talent associated with team A and team B will be denoted t_a and t_b, respectively.

Let the revenue functions for team A and team B be R_a and R_b, respectively. For $i = a, b$,

$$\begin{aligned} R_i \; : \; & \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+ \\ & (t_a, t_b) \longmapsto R_i(t_a, t_b) \end{aligned}$$

The profit functions are for $i = a, b$,

$$\pi_i(t_a, t_b; c) = R_i(t_a, t_b) - c \cdot t_i \tag{1}$$

For $i = a, b$ and $i \neq j$,

Assumption 1. *R_i is continuous on $\mathbb{R}_+^2$.*

Assumption 2. *R_i is concave in t_i.*

In order to distribute the total quantity of talent among the two teams, I introduce a game that runs in two stages.

Definition 1 (The Talent Allocation Game). *The Talent Allocation Game is $\mathcal{G} = \langle N, (C_i)_{i=A,B}, G \rangle$ where*

- $N = \{A, B\}$
- $C_i = \mathbb{R}_{++}$ *and* $c_i \in C_i$*, for* $i = A, B$
- $G : C_A \times C_B \longrightarrow \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\},\ (c_a, c_b) \mapsto \begin{cases} \mathcal{G}_1 & \text{if } c_a = c_b \\ \mathcal{G}_2 & \text{if } c_a > c_b \\ \mathcal{G}_3 & \text{if } c_a < c_b \end{cases}$
- *For* $k = 1, 2, 3$*,* $\mathcal{G}_k = \langle N, (T_i^k)_{i=A,B}, (\pi_i)_{i=A,B} \rangle$ *is a strategic subgame with perfect information*
- *For* $i = A, B$ *and* $k = 1, 2, 3$ $t_i \in T_i^k$ *and*

$$\begin{aligned} \pi_i : T_A^k \times T_B^k &\longrightarrow \mathbb{R} \\ (t_a, t_b) &\mapsto \begin{cases} R_i(t_a, t_b) - c \cdot t_i & \text{if } t_a + t_b \leq 1 \\ R_i\left(\frac{t_a}{t_a+t_b}, \frac{t_b}{t_a+t_b}\right) - c \cdot \frac{t_i}{t_a+t_b} & \text{otherwise} \end{cases} \end{aligned}$$

- $c = \max\{c_a, c_b\}$
- $T_A^1 = T_B^1 = [0, 1]$
- $T_A^2 = [0, 1],\ T_B^2 = [0, 1 - t_a]$
- $T_A^3 = [0, 1 - t_b],\ T_B^3 = [0, 1]$

The Talent Allocation Game is a sequential game in which the profile (c_a, c_b) determines what subgame is to be played next. In the first stage, teams simultaneously bid on the cost of talent. For $i = A, B$, team i submits a positive real number c_i that represents a price at which team i is willing to pay for a unit of talent. The implemented cost will be equal to the highest bid[1]. Then the first stage ends and players move on to the subgame $G(c_a, c_b)$. If in the first stage the bids are not equal, then the team who submitted the highest bid will be allowed to choose any quantity of talent on the interval $[0, 1]$. In turn, the other team is constrained by the choice of the unconstrained player. The quantities of talent chosen by the teams is to be paid at the implemented cost. Then, profits are realized and the game ends.

If in the first stage $c_a = c_b$ then, in the second stage, both teams are unconstrained. They are asked to simultaneously submit a quantity of talent they wish to hire at the implemented cost. If $t_a + t_b \leq 1$ then teams are allocated exactly the quantity they submitted. Otherwise, both teams receive a quantity of talent proportional to their own bid relative to the sum of the bids. That is, team i receives $t_i = \frac{t_i}{t_a + t_b}$, $i = a, b$. Then, profits are realized and the game ends.

Definition 2 (Pure Strategy)**.** *For $i = A, B$ a pure strategy for team i is a list $s_i = (c_i, t_i(c|\mathcal{G}_1), t_i(c|\mathcal{G}_2), t_i(c|\mathcal{G}_3))$, specifying the action taken in the first stage and, $\forall c \in \mathbb{R}_{++}$, the action taken in the second stage conditional on the subgame that has been reached.*

3.1 Optimal choices of talent

3.1.1 Subgame $\mathcal{G}_1$

It is understood that in $\mathcal{G}_1$, c has already been decided and thus is considered to be fixed. Teams play mutual best responses to each other. This means that each team must maximize an objective function – a profit function, in this case – subject to a demand that is restricted to the interval $[0, 1]$. For

[1] This assumption is based on the existence of arbitration rules in the National Hockey League and the Major League Baseball in North America. [Add the details]

$i = A, B$, define the *Lagrange* function

$$\mathcal{L}_i = \pi_i(t_a, t_b; c) - \lambda_i(t_i - 1)$$

where λ_i is a so-called *Lagrange multiplier*. The Nash equilibrium in this subgame is a solution to the system of constrained maximization problems

$$\left(\max_{t_i} \mathcal{L}_i \text{ subject to } \lambda_i, t_i \geq 0,\ t_i - 1 \leq 0 \text{ and } \lambda_i(t_i - 1) = 0 \right)_{i=A,B}$$

A solution to this system is a list$(t_a, t_b, \lambda_a, \lambda_b)$ satisfying the first-order conditions

$$\frac{\partial}{\partial t_a}\mathcal{L}_a = 0,\ \frac{\partial}{\partial t_b}\mathcal{L}_b = 0$$

along with

$$t_i - 1 \leq 0,\ \frac{\partial}{\partial \lambda_i}\mathcal{L}_i\lambda_i = 0,\text{ and } \lambda_i, t_i \geq 0,\ i = a, b$$

When the constraint $t_i - 1 \leq 0$ is not binding for team i, then the solution is such that $t_i - 1 < 0$ which implies that $\lambda_i = 0$. The Lagrange multiplier is then interpreted as a measure of the extent to wich $t_i - 1 \leq 0$ is restrictive to i. The greater λ_i is, the more restrictive is the constraint.

3.1.2 Subgames $\mathcal{G}_2$ and $\mathcal{G}_3$

Define for $i, j = a, b$ and $i \neq j$,

$$\phi_i(t_j, c) = \arg \max_{t_i \in [0,1]} \{R_i(t_a, t_b, c) - c \cdot t_i\}$$

and

$$\psi_i(t_j, c) = \arg \max_{t_i \in [0,1-t_j]} \{R_i(t_a, t_b, c) - c \cdot t_i\}$$

Without loss of generality, assume that $c_a > c_b$. Taking t_a and c_a as given, team B will choose $\psi_b(t_a, c_a)$. And so, A finds it optimal to choose $\phi_a(\psi_b(t_a, c_a), c_a)$.

An equilibrium in this subgame is a pair (t_a^*, t_b^*) such that

$$t_a^* = \phi_a(\psi_b(t_a, c_a), c_a) \text{ and } t_b^* = \psi_b(t_a^*, c_a)$$

3.1.3 Equilibrium of $\mathcal{G}$

Definition 3 (Subgame Perfect Equilibrium). *Let $(\hat{s}_1, \hat{s}_2)$ be a list of pure strategies where for $i = a, b$, $\hat{s}_i = \big(\hat{c}_i, \hat{t}_i(\hat{c}|\mathcal{G}_1), \hat{t}_i(\hat{c}|\mathcal{G}_2), \hat{t}_i(\hat{c}|\mathcal{G}_3)\big)$ and $\hat{c} =$ $\max\{\hat{c}_a, \hat{c}_b\}$. $(\hat{s}_1, \hat{s}_2)$ is a (subgame perfect) equilibrium of $\mathcal{G}$ if for $k = 1, 2, 3$, the pair $(\hat{t}_a(c|\mathcal{G}_k), \hat{t}_b(c|\mathcal{G}_k))$ is Nash equilibrium in the subgame $\mathcal{G}_k$ and if there is no $c \neq \hat{c}_a$ such that*

$$\pi_a\left(\hat{t}_a(\hat{c}|G(\hat{c}_a, \hat{c}_b)), \hat{t}_b(\hat{c}|G(\hat{c}_a, \hat{c}_b)); \hat{c}\right) < \pi_a\left(\hat{t}_a(c|G(c, \hat{c}_b)), \hat{t}_b(c|G(c, \hat{c}_b)); c\right)$$

and there is no $c \neq \hat{c}_b$ such that

$$\pi_b\left(\hat{t}_a(\hat{c}|G(\hat{c}_a, \hat{c}_b)), \hat{t}_b(\hat{c}|G(\hat{c}_a, \hat{c}_b)); \hat{c}\right) < \pi_b\left(\hat{t}_a(c|G(\hat{c}, c)), \hat{t}_b(c|G(\hat{c}, c)); c\right)$$

4 A General Case

Let the revenue functions be

$$R_a(t_a, t_b) = z\frac{t_a}{t_a + t_b} \quad \text{and} \quad R_b(t_a, t_b) = \frac{t_b}{t_a + t_b}$$

with $z \in [1, 2)$.

4.1 Optimal choices of talent

4.1.1 Subgame $\mathcal{G}_1$

Proposition 1. *In $\mathcal{G}_1$, the list $(t_a(c), t_b(c))$ such that*

$$t_a(c) = \begin{cases} \frac{z^2}{c(z+1)^2} & \text{if } c \geq \frac{z}{z+1} \\ 1 & \text{otherwise} \end{cases}$$

and

$$t_b(c) = \begin{cases} \frac{z}{c(z+1)^2} & \text{if } c \geq \frac{z}{z+1} \\ 1 & \text{otherwise} \end{cases}$$

is a Nash equilibrium of $\mathcal{G}_1$.

Proof. See appendix □

The corresponding profits are then

$$\pi_a = \begin{cases} \frac{z^3}{(z+1)^2} & \text{if } c \geq \frac{z}{z+1} \\ \frac{z}{2}(1-c) & \textit{otherwise} \end{cases}$$

and

$$\pi_b = \begin{cases} \frac{1}{(z+1)^2} & \text{if } c \geq \frac{z}{z+1} \\ \frac{1}{2}(1-c) & \textit{otherwise} \end{cases}$$

4.2 Subgame $\mathcal{G}_2$

Proposition 2. *In $\mathcal{G}_2$, , the list $(t_a(c), t_b(c))$ such that*

$$t_a(c) = \begin{cases} 1 & \text{if } c \in \left(0, z - \frac{1}{4}z^2\right] \\ \frac{z^2}{4c} & \text{otherwise} \end{cases}$$

and

$$t_b(c) = \begin{cases} 0 & \text{if } c \in \left(0, z - \frac{1}{4}z^2\right] \\ \frac{z(2-z)}{4c} & \text{otherwise} \end{cases}$$

is a Nash equilibrium of $\mathcal{G}_2$.

Proof. See Appendix □

It leads team A to a profit of

$$\pi_a = \begin{cases} z - c & \text{if } c \in \left(0, z - \frac{1}{4}z^2\right] \\ \frac{1}{4}z^2 & \text{otherwise} \end{cases}$$

and B to a profit of

$$\pi_b = \begin{cases} 0 & \text{if } c \in \left(0, z - \frac{1}{4}z^2\right] \\ \frac{(z-2)^2}{4} & \text{otherwise} \end{cases}$$

4.3 Subgame $\mathcal{G}_3$

Proposition 3. *In $\mathcal{G}_3$, , the list $(t_a(c), t_b(c))$ such that*

$$t_a(c) = \begin{cases} 0 & \textit{if } c \in \left(0, \frac{4z-1}{4z}\right] \\ \frac{2z-1}{4cz} & \textit{otherwise} \end{cases}$$

and

$$t_b(c) = \begin{cases} 1 & \textit{if } c \in \left(0, \frac{4z-1}{4z}\right] \\ \frac{1}{4cz} & \textit{otherwise} \end{cases}$$

is a Nash equilibrium of $\mathcal{G}_3$.

Proof. See Appendix □

It leads team A to a profit of

$$\pi_a = \begin{cases} 0 & \text{if } c \in \left(0, \frac{4z-1}{4z}\right] \\ \frac{(2z-1)^2}{4z} & \text{otherwise} \end{cases}$$

and B to a profit of

$$\pi_b = \begin{cases} 1 - c & \text{if } c \in \left(0, \frac{4z-1}{4z}\right] \\ \frac{1}{4z} & \text{otherwise} \end{cases}$$

4.4 Equilibrium

Proposition 4. *Let*

$$\hat{t}_a(c|\mathcal{G}_1) = \begin{cases} \frac{z^2}{c(z+1)^2} & \text{if } c \geq \frac{z}{z+1} \\ 1 & \text{otherwise} \end{cases}$$

$$\hat{t}_a(c|\mathcal{G}_2) = \begin{cases} 1 & \text{if } c \in \left(0, z - \frac{1}{4}z^2\right] \\ \frac{z^2}{4c} & \text{otherwise} \end{cases}$$

and

$$\hat{t}_a(c|\mathcal{G}_3) = \begin{cases} 0 & \text{if } c \in \left(0, \frac{4z-1}{4z}\right] \\ \frac{2z-1}{4cz} & \text{otherwise} \end{cases}$$

Let

$$\hat{t}_b(c|\mathcal{G}_1) = \begin{cases} \frac{z}{c(z+1)^2} & \text{if } c \geq \frac{z}{z+1} \\ 1 & \text{otherwise} \end{cases}$$

$$\hat{t}_b(c|\mathcal{G}_2) = \begin{cases} 0 & \text{if } c \in \left(0, z - \frac{1}{4}z^2\right] \\ \frac{z(2-z)}{4c} & \text{otherwise} \end{cases}$$

and

$$\hat{t}_b(c|\mathcal{G}_3) = \begin{cases} 1 & \text{if } c \in \left(0, \frac{4z-1}{4z}\right] \\ \frac{1}{4cz} & \text{otherwise} \end{cases}$$

Let $\hat{c}_a \geq \frac{4z-1}{4z}$ *and* $\hat{c}_b > \hat{c}_a$. *The list* $(\hat{s}_1, \hat{s}_2)$ *is a subgame perfect equilibrium of* $\mathcal{G}$.

Proof. See Appendix □

5 Revenue Sharing

The rule for profit sharing is as follows: it is decided by the league that a team can only keep a fraction α of the revenue generated. The rest, $1-\alpha$, goes to the other team. We thus have the profit functions: for $i, j = a, b$,

$i \neq j$

$$\begin{aligned}\pi_a(t_a, t_b; c, \alpha) &= z\frac{\alpha t_a + (1-\alpha)t_b}{t_a + t_b} - c \cdot t_a \\ \pi_b(t_a, t_b; c, \alpha) &= \frac{\alpha t_b + (1-\alpha)t_a}{t_a + t_b} - c \cdot t_b\end{aligned}$$

with $\alpha \in (0.5, 1)$.

5.1 Optimal Choices

5.1.1 Subgame $\mathcal{G}_1$

Proposition 5. *In $\mathcal{G}_1$, the optimal actions are*

$$t_a^*(c) = \begin{cases} \frac{z^2(2\alpha-1)}{c(z+1)^2} & \textit{if } c \geq \frac{z(2\alpha-1)}{z+1} \\ 1 & \textit{otherwise} \end{cases}$$

and

$$t_b^*(c) = \begin{cases} \frac{z(2\alpha-1)}{c(z+1)^2} & \textit{if } c \geq \frac{z(2\alpha-1)}{z+1} \\ 1 & \textit{otherwise} \end{cases}$$

Proof. See Appendix □

5.1.2 Subgame $\mathcal{G}_2$

Proposition 6. *In $\mathcal{G}_2$,*

$$t_a^*(c) = \phi_a(\psi_b(t_a, c), c) = \begin{cases} 1 & \textit{if } c \in \left(0, \frac{1}{4}\frac{z(2\,\alpha-1)(4\,\alpha+2\,z\,\alpha-3\,z)}{\alpha+z\,\alpha-z}\right] \\ \frac{1}{4}\frac{z^2(2\,\alpha-1)^2}{c(\alpha+z\,\alpha-z)} & \textit{otherwise} \end{cases}$$

and

$$t_b^*(c) = \psi_b(t_a^*(c), c) = \begin{cases} 0 & \textit{if } c \in \left(0, \frac{1}{4}\frac{z(2\,\alpha-1)(4\,\alpha+2\,z\,\alpha-3\,z)}{\alpha+z\,\alpha-z}\right] \\ \frac{1}{4}\frac{z\,(2\,\alpha-1)(-z+2\,\alpha)}{c(\alpha+z\,\alpha-z)} & \textit{otherwise} \end{cases}$$

Proof. See Appendix □

It leads team A to a profit of

$$\pi_a = \begin{cases} \alpha z - c & \text{if } c \in \left(0, \frac{1}{4}\frac{z(2\,\alpha-1)(4\,\alpha+2\,z\,\alpha-3\,z)}{\alpha+z\,\alpha-z}\right] \\ -\frac{1}{4}\frac{z\left(4\,\alpha^2+3\,z-4\,\alpha-4\,z\,\alpha\right)}{\alpha+z\,\alpha-z} & \text{otherwise} \end{cases}$$

and B to a profit of

$$\pi_b = \begin{cases} 0 & \text{if } c \in \left(0, \frac{1}{4}\frac{z(2\,\alpha-1)(4\,\alpha+2\,z\,\alpha-3\,z)}{\alpha+z\,\alpha-z}\right] \\ -\frac{1}{4}\frac{-6\,z\,\alpha-4\,\alpha^2+8\,z\,\alpha^2+2\,z-2\,z^2\alpha+z^2}{\alpha+z\,\alpha-z} & \text{otherwise} \end{cases}$$

5.2 Subgame $\mathcal{G}_3$

Proposition 7. *In* $\mathcal{G}_3$*,*

$$t_b^*(c) = \phi_b(\psi_a(t_b, c), c) = \begin{cases} 1 & \textit{if } c \in \left(0, \frac{1}{4}\frac{(4\,z-1)(2\,\alpha-1)}{z}\right] \\ \frac{2\alpha-1}{4cz} & \textit{otherwise} \end{cases}$$

and

$$t_a^*(c) = \psi_a(t_b^*(c), c) = \begin{cases} 0 & \textit{if } c \in \left(0, \frac{1}{4}\frac{(4\,z-1)(2\,\alpha-1)}{z}\right] \\ \frac{(2z-1)(2\alpha-1)}{4cz} & \textit{otherwise} \end{cases}$$

It leads team A to a profit of

$$\pi_a = \begin{cases} 0 & \text{if } c \in \left(0, \frac{1}{4}\frac{(4\,z-1)(2\,\alpha-1)}{z}\right] \\ \frac{1}{4}\frac{2\,\alpha+4\,\alpha\,z^2-8\,\alpha\,z+4\,z-1}{z} & \text{otherwise} \end{cases}$$

and B to a profit of

$$\pi_b = \begin{cases} \alpha - c & \text{if } c \in \left(0, \frac{1}{4}\frac{(4\,z-1)(2\,\alpha-1)}{z}\right] \\ -\frac{1}{4}\frac{-2\,\alpha+4\,\alpha\,z+1-4\,z}{z} & \text{otherwise} \end{cases}$$

5.3 Equilibrium of $\mathcal{G}$

Define

$$c_2 = \frac{1}{4}\frac{z\,(2\,\alpha - 1)\,(4\,\alpha + 2\,z\,\alpha - 3\,z)}{\alpha + z\,\alpha - z} \text{ and } c_3 = \frac{1}{4}\frac{(4\,z - 1)\,(2\,\alpha - 1)}{z}$$

We have that for $z \in \left[1, \frac{\alpha}{1-\alpha}\right)$ and $\alpha \in (0.5, 1)$, if $\alpha < \frac{z(3\,z-1)}{2\,z^2+2\,z-1}$ then $c_2 < c_3$.

Proposition 8.

$$\hat{t}_a(c;\alpha|\mathcal{G}_1) = \begin{cases} \frac{z^2(2\alpha-1)}{c(z+1)^2} & \text{if } c \geq \frac{z(2\alpha-1)}{z+1} \\ 1 & \text{otherwise} \end{cases}$$

$$\hat{t}_a(c;\alpha|\mathcal{G}_2) = \begin{cases} 1 & \text{if } c \in \left(0, \frac{1}{4}\frac{z(2\alpha-1)(4\alpha+2z\alpha-3z)}{\alpha+z\alpha-z}\right] \\ \frac{z^2(2\alpha-1)^2}{4c(\alpha+z\alpha-z)} & \text{otherwise} \end{cases}$$

and

$$\hat{t}_a(c;\alpha|\mathcal{G}_3) = \begin{cases} 0 & \text{if } c \in \left(0, \frac{(4z-1)(2\alpha-1)}{4z}\right] \\ \frac{(2z-1)(2\alpha-1)}{4cz} & \text{otherwise} \end{cases}$$

Let

$$\hat{t}_b(c;\alpha|\mathcal{G}_1) = \begin{cases} \frac{z(2\alpha-1)}{c(z+1)^2} & \text{if } c \geq \frac{z(2\alpha-1)}{z+1} \\ 1 & \text{otherwise} \end{cases}$$

$$\hat{t}_b(c;\alpha|\mathcal{G}_2) = \begin{cases} 0 & \text{if } c \in \left(0, \frac{1}{4}\frac{z(2\alpha-1)(4\alpha+2z\alpha-3z)}{\alpha+z\alpha-z}\right] \\ \frac{z(2\alpha-1)(-z+2\alpha)}{4c(\alpha+z\alpha-z)} & \text{otherwise} \end{cases}$$

and

$$\hat{t}_b(c;\alpha|\mathcal{G}_3) = \begin{cases} 1 & \text{if } c \in \left(0, \frac{(4z-1)(2\alpha-1)}{4z}\right] \\ \frac{2\alpha-1}{4cz} & \text{otherwise} \end{cases}$$

Let $\hat{c}_a \geq c_3$ and $\hat{c}_b > \hat{c}_a$. The list $(\hat{s}_1, \hat{s}_2)$ is a subgame perfect equilibrium of $\mathcal{G}$.

Proof. See Appendix ∎

6 Discussion

The *competitive balance* is a relative measure of the inequality in talent dispersion across teams. We say that competitive balance reaches its maximal value when all teams own the same quantity of talent. Competitive balance *increases* when the dispersion of talent changes from a relatively unequal state to a relatively *less* unequal state. Inversely, competitive balance decreases when the dispersion of talent changes from a relatively equal state to a relatively less equal state.

From proposition 8, we have that in equilibrium, team A owns a quantity of talent equal to $\frac{(2z-1)(2\alpha-1)}{4cz}$ while team B owns $\frac{2\alpha-1}{4cz}$. We see that α is positively correlated with both quantities of talent in equilibrium.

The competitive balance is

$$\frac{t_a}{t_b} = 2z - 1 \tag{2}$$

We can see that competitive balance is independent of α. This result contrasts with some recent results such as Szymanski and Késenne [8], where revenue sharing was shown to have a negative impact on competitive balance. I explain the result in this paper by the fact that the Talent Allocation Game allows team to add an extra layer of strategy in the choice of the cost of talent. It permits weaker teams, such as team B in this case, to internalize the parameter α in such a way that they are not disadvantaged in equilibrium.

We can also see from (2) that if z is greater than 1, team A owns more talent in equilibrium, which means a negative impact of z on competitive balance. This result is easily explained since a richer team is expected to have an advantage when it comes to acquiring playing talent.

Revenue sharing has a mitigated impact on the profits of team A and a positive impact on the profits of team B. We have that

$$\frac{\partial}{\partial \alpha}\pi_a = \frac{1 + 2\,z^2 - 4\,z}{2z}$$

and

$$\frac{\partial}{\partial \alpha}\pi_b = \frac{1-2\,z}{2z}$$

For $z > 1.71$, $\frac{\partial}{\partial \alpha}\pi_a > 0$. However, the possible positive impact of revenue sharing on the profits of A should not be over emphasized as it happens only for large values of z. Although revenue sharing is successful at increasing the profits of the poor team, it comes at a cost of reducing the quantity of playing talents in both teams. This last concern is absent from other models of league formation such as Szymanski and Késenne [8] or Madden [5]. The reason being that the equilibrium concept of these two models is the competitive equilibrium. At this competitive equilibrium, the cost is computed by forcing teams to jointly hire the total supply of talent. In my model, the supply of talent is not completely exhausted. Consequently, the total quantity of talent in the league varies with α. Moreover, the usual competitive equilibrium which is defined by $(t_a^{ce}, t_b^{ce}, c^{ce})$ such that

$$\begin{aligned}
t_a^{ce} &= \arg\max_{t_a}\ \pi_a(t_a, t_b^{ce}; c^{ce}) \\
t_b^{ce} &= \arg\max_{t_b}\ \pi_b(t_a^{ce}, t_b; c^{ce}) \\
t_a^{ce} &+ t_b^{ce} = 1 \\
t_a^{ce}&, t_b^{ce} \geq 0,\ \ c^{ce} > 0
\end{aligned}$$

cannot be sustained as an equilibrium of the Talent Allocation Game. For simplicity of exposition, assume that $\alpha = 1$. The result holds also for $\alpha < 1$. For all $z \in [1, 2)$ we have that if A deviates by increasing c to $\frac{z}{z+1} + \frac{1}{4}$, A will choose $t_a = 1$ which leads to a profit of $z - \left(\frac{z}{z+1} + \frac{1}{4}\right)$. We have that

$$z - \left(\frac{z}{z+1} + \frac{1}{4}\right) - \frac{z^3}{(z+1)^2} = \frac{(3\,z+1)\,(z-1)}{4\,(z+1)^2}$$

which is positive on $z \in [1, 2)$.

7 Conclusion

In this paper, a formal game-theoretical perspective on professional league formation was considered. It was argued that revenue sharing has no impact on competitive balance and it also leads to a decrease in the quantity of talent hired in equilibrium. It has been shown that the usual competitive equilibrium is subject to beneficial deviations if teams have the power to affect the unit-cost of talent. This brings to light the possible negative effect that revenue sharing may have on the labour market conditions. Future work should concentrate on determining whether this last issue can be confirmed by the data.

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Appendix

Proof of Proposition 1. Assume for the moment that c is high enough for an equilibrium to locate in $\Delta = \{(t_a, t_b) \in [0,1]^2 \mid t_a + t_b \leq 1\}$. Later, I will compute a lower bound on such a c. Since teams are playing best-responses to each other, in an interior solution, it must be the case that

$$\frac{\partial}{\partial t_a} R_a(t_a, t_b; c) = \frac{\partial}{\partial t_b} R_b(t_a, t_b; c) = c \tag{3}$$

The solution to (2) is

$$t_a = \frac{z^2}{c(z+1)^2} \;\Rightarrow\; t_b = \frac{z}{c(z+1)^2}$$

The lower bound on c is given by

$$\frac{z^2}{c(z+1)^2} + \frac{z}{c(z+1)^2} \leq 1 \;\Rightarrow\; c \geq \frac{z}{z+1}$$

When c is below the lower bound, teams are constrained by the requirement that $t_a + t_b \leq 1$. The Nash equilibrium in a subgame where $c < \frac{z}{z+1}$ is characterized by the solution to the system

$$\begin{aligned} \max_{t_a} &\left\{ \frac{t_a}{t_a+t_b}(z-c) - \lambda_a(t_a-1) \right\} \\ \max_{t_b} &\left\{ \frac{t_b}{t_a+t_b}(1-c) - \lambda_b(t_b-1) \right\} \\ &\lambda_i(t_i-1) = 0,\; i = a,b \\ &\lambda_i \geq 0,\; i = a,b \end{aligned}$$

The solution to this system of constrained maximization problems is

$$(t_a, t_b) = (1,1),\; \lambda_a = \frac{1}{4}(z-c) \text{ and } \lambda_b = \frac{1}{4}(1-c)$$

This solution is well-defined since $z \geq 1$ implies that $c < \frac{1}{2}$. The corresponding profits are then

$$\pi_a = \frac{1}{2}(z-c) \text{ and } \pi_b = \frac{1}{2}(1-c)$$

Thus, in $\mathcal{G}_1$, the optimal actions are

$$t_a^*(c) = \begin{cases} \frac{z^2}{c(z+1)^2} & \text{if } c \geq \frac{z}{z+1} \\ 1 & \textit{otherwise} \end{cases}$$

and

$$t_b^*(c) = \begin{cases} \frac{z}{c(z+1)^2} & \text{if } c \geq \frac{z}{z+1} \\ 1 & \textit{otherwise} \end{cases}$$

□

Proof of Proposition 2. In an interior solution, we have that

$$\frac{\partial}{\partial t_b}\pi_b(t_a, t_b; c) = \frac{t_a}{(t_a + t_b)^2} - c = 0$$

The positive root of the quadratic equation is

$$t_b^* = \frac{-ct_a + \sqrt{ct_a}}{c}$$

However, t_b must be contained in $[0, 1 - t_a]$. Consequently,

$$\psi_b(t_a, c) = \begin{cases} 0 & \text{if } c \geq \frac{1}{t_a} \\ 1 - t_a & \text{if } c \leq t_a \\ t_b^* & \textit{otherwise} \end{cases}$$

and

$$\phi_a(\psi_b(t_a, c), c) = \arg\max_{t_a \in [0,1]} \begin{cases} z - c \cdot t_a & \text{if } t_a \geq \frac{1}{c} \\ t_a(z - c) & \text{if } t_a \geq c \\ \frac{zt_a}{t_a + t_b^*} - c \cdot t_a & \textit{otherwise} \end{cases}$$

For $c \geq 1$, choosing $t_a \geq \frac{1}{c}$ gives a maximal profit of $z - 1$. Otherwise, $t_a^* = \frac{z^2}{4c}$ and $t_b^* = \frac{z(2-z)}{4c}$ which gives team A a profit of $\frac{1}{4}z^2$ wich is strictly greater than $z - 1$ on $z \in [1, 2)$. If $c \leq 1$, choosing $t_a \geq c$ gives a maximal profit of $z - c$. Team A will be better off choosing $t_a = 1$ when

$$z - c \geq \frac{1}{4}z^2$$

which is equivalent to

$$c \in \left(0, z - \frac{1}{4}z^2\right]$$

Thus in $\mathcal{G}_2$,

$$t_a^*(c) = \phi_a(\psi_b(t_a, c), c) = \begin{cases} 1 & \text{if } c \in \left(0, z - \frac{1}{4}z^2\right] \\ \frac{z^2}{4c} & \text{otherwise} \end{cases}$$

and

$$t_b^*(c) = \psi_b(t_a^*(c), c) = \begin{cases} 0 & \text{if } c \in \left(0, z - \frac{1}{4}z^2\right] \\ \frac{z(2-z)}{4c} & \text{otherwise} \end{cases}$$

□

Proof of Proposition 3. In an interior solution, We have that

$$\frac{\partial}{\partial t_a}\pi_a(t_a, t_b; c) = \frac{z t_b}{(t_a + t_b)^2} - c = 0$$

The positive root of the quadratic equation is

$$t_a^* = \frac{-c t_b + \sqrt{c z t_b}}{c}$$

t_a must belong to $[0, 1 - t_b]$, hence

$$\psi_a(t_b, c) = \begin{cases} 0 & \text{if } c \geq \frac{z}{t_b} \\ 1 - t_b & \text{if } c \leq z t_b \\ t_a^* & \textit{otherwise} \end{cases}$$

In the second stage, team B chooses

$$\phi_b(t_a, c) = \arg\max_{t_b \in [0,1]} \begin{cases} 1 - c \cdot t_b & \text{if } t_b \geq \frac{z}{c} \\ t_b(1 - c) & \text{if } t_b \geq \frac{c}{z} \\ \frac{t_b}{t_a^* + t_b} - c \cdot t_b & \textit{otherwise} \end{cases}$$

For $c \geq z$, choosing $t_b \geq \frac{z}{c}$ gives team B a maximal profit of $1 - z \leq 0$ since it is assumed that $z \geq 1$. Otherwise, B will choose $t_b^* = \frac{1}{4cz}$ which implies that $t_a^* = \frac{2z-1}{4cz}$ with a profit for B of $\frac{1}{4z}$. If $1 \leq c \leq z$, choosing $t_b \geq \frac{c}{z}$ gives rise to a profit of at most 0. If $c < 1 \leq z$, choosing $t_b \geq \frac{c}{z}$ gives a maximal profit of $1-c$. The interval in which $1-c \geq \frac{1}{4z}$ is $\left(0, \frac{4z-1}{4z}\right]$.

Thus, in $\mathcal{G}_3$,

$$t_b^*(c) = \phi_b(\psi_a(t_b, c), c) = \begin{cases} 1 & \text{if } c \in \left(0, \frac{4z-1}{4z}\right] \\ \frac{1}{4cz} & \text{otherwise} \end{cases}$$

and

$$t_a^*(c) = \psi_a(t_b^*(c), c) = \begin{cases} 0 & \text{if } c \in \left(0, \frac{4z-1}{4z}\right] \\ \frac{2z-1}{4cz} & \text{otherwise} \end{cases}$$

□

Proof of Proposition 4. From proposition 1,2 and 3 we know that for $k = 1, 2, 3$, $(\hat{t}_a(c|\mathcal{G}_k), \hat{t}_b(c|\mathcal{G}_k))$ is a Nash equilibrium of $\mathcal{G}_k$. We have that $G(\hat{c}_a, \hat{c}_b) = \mathcal{G}_3$. The profits corresponding to $(\hat{t}_a(\hat{c}|\mathcal{G}_3), \hat{t}_b(\hat{c}|\mathcal{G}_3), \hat{c})$ are

$$\frac{(2z-1)^2}{4z} \text{ and } \frac{1}{4z}$$

for team A and team B, respectively. Note that $\frac{4z-1}{4z} \leq z - \frac{1}{4}z^2$ and $\frac{4z-1}{4z} \geq \frac{z}{z+1}$. If A deviates by setting $c > \hat{c}_b$, $G(c, \hat{c}_b) = \mathcal{G}_2$ and A's maximal profit is

$$\frac{(2z-1)^2}{4z}$$

If A deviates by setting $c = \hat{c}_b$, $G(c, \hat{c}_b) = \mathcal{G}_1$ and A's maximal profit is

$$\frac{z^3}{(z+1)^2} \leq \frac{(2z-1)^2}{4z}$$

Team B has no incentive to deviate by setting $c \in (\hat{c}_a, \hat{c}_b)$ because the induced subgame would still be $\mathcal{G}_3$ and its maximal profit does not depend on c. If B deviates by setting $c = \hat{c}_a$ then $G(\hat{c}_a, c) = \mathcal{G}_1$ and B's maximal profit is

$$\frac{1}{(z+1)^2} \leq \frac{1}{4z}$$

. If B deviates by setting $c < \hat{c}_a$ then $G(\hat{c}_a, c) = \mathcal{G}_2$ and B's maximal profit is

$$\frac{(z-2)^2}{4} \leq \frac{1}{4z}$$

. Thus there is no beneficial deviation from $(\hat{s}_1, \hat{s}_2)$, which satisfies the equilibrium requirement. $\square$

Proof of Proposition 5. Assume for the moment that c is such that the solution to both first-order conditions belongs to Δ. Thus we have that the solution to the system

$$\frac{\partial}{\partial t_a}\pi_a(t_a, t_b; c, \alpha) = \frac{\partial}{\partial t_b}\pi_b(t_a, t_b; c, \alpha) = 0$$

is

$$t_a^* = \frac{z^2(2\alpha - 1)}{c(z+1)^2} \text{ and } t_b^* = \frac{z(2\alpha - 1)}{c(z+1)^2}$$

In order for $t_a^* + t_b^* \leq 1$, it must be the case that $c \geq \frac{z(2\alpha-1)}{z+1}$. Otherwise, both teams are constrained and the solution to the system

$$\begin{aligned} \max_{t_a} \ \{\pi_a(t_a, t_b; c, \alpha) - \lambda_a(t_a - 1)\} \\ \max_{t_b} \ \{\pi_b(t_a, t_b; c, \alpha) - \lambda_b(t_b - 1)\} \\ \lambda_i(t_i - 1) = 0, \ i = a, b \\ \lambda_i \geq 0, \ i = a, b \end{aligned}$$

is

$$(t_a, t_b) = (1, 1), \ \lambda_a = \frac{1}{2}\alpha z - \frac{1}{4}z - c \text{ and } \lambda_b = \frac{1}{2}\alpha - \frac{1}{4} - c$$

Since $c < \frac{z(2\alpha-1)}{z+1}$ and $\alpha > \frac{1}{2}$ we can verify that $\lambda_a, \lambda_b > 0$.

Thus, in $\mathcal{G}_1$, the optimal actions are

$$t_a^*(c) = \begin{cases} \frac{z^2(2\alpha-1)}{c(z+1)^2} & \text{if } c \geq \frac{z(2\alpha-1)}{z+1} \\ 1 & \textit{otherwise} \end{cases}$$

and

$$t_b^*(c) = \begin{cases} \frac{z(2\alpha-1)}{c(z+1)^2} & \text{if } c \geq \frac{z(2\alpha-1)}{z+1} \\ 1 & \textit{otherwise} \end{cases}$$

□

Proof of Proposition 5. In an interior solution,

$$t_b^* = \frac{-ct_a + \sqrt{ct_a\ (\alpha - z\,(1-\alpha))}}{c}$$

t_b^* is well-defined if $z < \frac{\alpha}{1-\alpha}$. Constraining $t_b \in [0, 1-t_a]$, we have that

$$\psi_b(t_a, c) = \begin{cases} 0 & \text{if } c \geq \frac{\alpha - z(1-\alpha)}{t_a} \\ 1 - t_a & \text{if } c \leq t_a(\alpha - z(1-\alpha)) \\ t_b^* & \textit{otherwise} \end{cases}$$

and

$$\phi_a(\psi_b(t_a, c), c) = \arg\max_{t_a \in [0,1]} \begin{cases} z\alpha - c \cdot t_a & \text{if } t_a \geq \frac{\alpha - z(1-\alpha)}{c} \\ t_a(\alpha(z+1) - (c+1)) + (1-\alpha) & \text{if } t_a \geq \frac{c}{\alpha - z(1-\alpha)} \\ z\frac{\alpha t_a + (1-\alpha)t_b^*}{t_a + t_b^*} - c \cdot t_a & \textit{otherwise} \end{cases}$$

For $c \geq \frac{\alpha - z(1-\alpha)}{t_a}$, by choosing $t_a \geq \frac{\alpha - z(1-\alpha)}{c}$, A will get a maximal profit of $z - \alpha \geq 0$. When $c < \alpha - z(1-\alpha)$, it implies that $\alpha(z+1) \geq (c+1)$, thus choosing $t_a \geq \frac{c}{2\alpha - 1}$ gives A a maximal profit of $\alpha z - c > 0$. Otherwise, A chooses $t_a^* = \frac{1}{4}\frac{z^2(2\,\alpha-1)^2}{c(\alpha + z\,\alpha - z)}$ which implies that $t_b^* = \frac{1}{4}\frac{z\,(2\,\alpha-1)(-z+2\,\alpha)}{c(\alpha+z\,\alpha-z)}$. The pair (t_a^*, t_b^*) induces a profit for A of

$$\pi_a = -\frac{1}{4}\frac{z\ (4\,\alpha^2 + 3\,z - 4\,\alpha - 4\,z\,\alpha)}{\alpha + z\,\alpha - z} > 0$$

Note that

$$\pi_a - (z - \alpha) = \frac{1}{4}\frac{(z - 2\,\alpha)^2}{\alpha + z\,\alpha - z} > 0$$

which implies that when $c \geq \alpha - z(1-\alpha)$, A would prefer choosing $t_a = t_a^*$

over choosing $t_a = \frac{\alpha - z(1-\alpha)}{c}$. When $c < \alpha - z(1-\alpha)$ A will choose $t_a = 1$ if

$$\begin{aligned} & \alpha z - c - \pi_a \geq 0 \\ \Leftrightarrow \qquad & c \leq \frac{1}{4} \frac{z\,(2\,\alpha - 1)\,(4\,\alpha + 2\,z\,\alpha - 3\,z)}{\alpha + z\,\alpha - z} \end{aligned}$$

and will choose t_a^* otherwise.

Thus, in $\mathcal{G}_2$,

$$t_a^*(c) = \phi_a(\psi_b(t_a, c), c) = \begin{cases} 1 & \text{if } c \in \left(0, \frac{1}{4} \frac{z(2\,\alpha-1)(4\,\alpha+2\,z\,\alpha-3\,z)}{\alpha+z\,\alpha-z}\right] \\ \frac{1}{4} \frac{z^2(2\,\alpha-1)^2}{c(\alpha+z\,\alpha-z)} & \text{otherwise} \end{cases}$$

and

$$t_b^*(c) = \psi_b(t_a^*(c), c) = \begin{cases} 0 & \text{if } c \in \left(0, \frac{1}{4} \frac{z(2\,\alpha-1)(4\,\alpha+2\,z\,\alpha-3\,z)}{\alpha+z\,\alpha-z}\right] \\ \frac{1}{4} \frac{z\,(2\,\alpha-1)(-z+2\,\alpha)}{c(\alpha+z\,\alpha-z)} & \text{otherwise} \end{cases}$$

□

Proof of Proposition 6. In an interior solution,

$$t_a^* = \frac{-ct_b + \sqrt{czt_b(2\alpha - 1)}}{c}$$

t_a must belong to $[0, 1 - t_b]$, hence

$$\psi_a(t_b, c) = \begin{cases} 0 & \text{if } c \geq \frac{z(2\alpha-1)}{t_b} \\ 1 - t_b & \text{if } c \leq zt_b(2\alpha - 1) \\ t_a^* & \textit{otherwise} \end{cases}$$

and

$$\phi_b(t_a, c) = \arg\max_{t_b\in[0,1]} \begin{cases} \alpha - c \cdot t_b & \text{if } t_b \geq \frac{z(2\alpha-1)}{c} \\ t_b(2\alpha - 1 - c) + 1 - \alpha & \text{if } t_b \geq \frac{c}{z(2\alpha-1)} \\ \frac{\alpha t_b + (1-\alpha)t_a^*}{t_a^* + t_b} - c \cdot t_b & \textit{otherwise} \end{cases}$$

For $c > z(2\alpha - 1)$, choosing $t_b \geq \frac{z(2\alpha-1)}{c}$ gives team B a maximal profit of $\alpha - z(2\alpha - 1) < 0$. Otherwise, B will choose $t_b^* = \frac{2\alpha-1}{4cz}$ which implies that $t_a^* = \frac{(2z-1)(2\alpha-1)}{4cz}$ with a profit for B of

$$\pi_b = -\frac{1}{4}\frac{-2\,\alpha + 4\,\alpha\,z + 1 - 4\,z}{z}$$

If $c \leq z(2\alpha - 1)$, choosing $t_b \geq \frac{c}{z(2\alpha-1)}$ gives a maximal profit of $\alpha - c$. Team B will choose $t_b = 1$ if

$$\begin{aligned} & \alpha - c - \pi_b \geq 0 \\ \Leftrightarrow \quad & c \leq \frac{1}{4}\frac{(4\,z-1)\,(2\,\alpha-1)}{z} \end{aligned}$$

and will choose $t_b = t_b^*$ otherwise.

Thus, in $\mathcal{G}_3$,

$$t_b^*(c) = \phi_b(\psi_a(t_b, c), c) = \begin{cases} 1 & \text{if } c \in \left(0, \frac{1}{4}\frac{(4\,z-1)(2\,\alpha-1)}{z}\right] \\ \frac{2\alpha-1}{4cz} & \text{otherwise} \end{cases}$$

and

$$t_a^*(c) = \psi_a(t_b^*(c), c) = \begin{cases} 0 & \text{if } c \in \left(0, \frac{1}{4}\frac{(4\,z-1)(2\,\alpha-1)}{z}\right] \\ \frac{(2z-1)(2\alpha-1)}{4cz} & \text{otherwise} \end{cases}$$

□

Proof of Proposition 8. From proposition ?,? and ? we know that for $k = 1, 2, 3$, $(\hat{t}_a(c; \alpha|\mathcal{G}_k), \hat{t}_b(c; \alpha|\mathcal{G}_k))$ is a Nash equilibrium of $\mathcal{G}_k$. We have that

$G(\hat{c}_a, \hat{c}_b) = \mathcal{G}_3$. The profits corresponding to $(\hat{t}_a(\hat{c}; \alpha|\mathcal{G}_3), \hat{t}_b(\hat{c}; \alpha|\mathcal{G}_3), \hat{c})$ are

$$\frac{2\alpha + 4\alpha z^2 - 8\alpha z + 4z - 1}{4z} \text{ and } -\frac{-2\alpha + 4\alpha z + 1 - 4z}{4z}$$

for team A and team B, respectively. Note that in $\mathcal{G}_2$, if $c = c_3$ and that $c_3 < c_2$ then A's maxmimal profit is precisely

$$\frac{2\alpha + 4\alpha z^2 - 8\alpha z + 4z - 1}{4z}$$

and so if A deviates to $c > \hat{c}_b$, $G(c, \hat{c}_b) = \mathcal{G}_2$ and A's profits will decrease. If A deviates by setting $c = \hat{c}_b$, $G(c, \hat{c}_b) = \mathcal{G}_1$ and A's maximal profit is

$$-\frac{(-z^3 - 5\,z^2 - 4\,z - 1 + 5\,\alpha\,z^2 + 4\,\alpha\,z + \alpha)\,z}{(2\,z + 1)\,(z + 1)^2} \leq \frac{2\alpha + 4\alpha z^2 - 8\alpha z + 4z - 1}{4z}$$

Team B has no incentive to deviate by setting $c \in (\hat{c}_a, \hat{c}_b)$ because the induced subgame would still be $\mathcal{G}_3$ and its maximal profit does not depend on c. If B deviates by setting $c = \hat{c}_a$ then $G(\hat{c}_a, c) = \mathcal{G}_1$ and B's maximal profit is

$$-\frac{2\,\alpha\,z^2 - \alpha\,z - \alpha - z^2}{(2\,z + 1)\,(z + 1)} \leq -\frac{-2\alpha + 4\alpha z + 1 - 4z}{4z}$$

. If B deviates by setting $c < \hat{c}_a$ then $G(\hat{c}_a, c) = \mathcal{G}_2$ and B's maximal profit is

$$-\frac{1}{4}\frac{-6z\alpha - 4\alpha^2 + 8z\alpha^2 + 2z - 2z^2\alpha - z^2}{\alpha + z\alpha - z} \leq -\frac{-2\alpha + 4\alpha z + 1 - 4z}{4z}$$

. Thus there is no beneficial deviation from $(\hat{s}_1, \hat{s}_2)$, which satisfies the equilibrium requirement. □