Stability of Networks under Limited Farsightedness

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Abstract

We provide a tractable concept that can be used to study the influence of the degree of farsightedness on network stability. A set of networks G_K is a level-K farsightedly stable set if three conditions are satisfied. First, external deviations should be deterred. Second, from any network outside of G_K there is a a sequence of farsighted improving paths of length smaller than or equal to K leading to some network in G_K. Third, there is no proper subset of G_K satisfying the first two conditions.

We show that a level-K farsightedly stable set always exists and we provide a sufficient condition for the uniqueness of a level-K farsightedly stable set. There is a unique level-1 farsightedly stable set G_1 consisting of all networks that belong to closed cycles. Level-K farsighted stability leads to a refinement of G_1 for generic allocation rules. We then provide easy to verify conditions for a set to be level-K farsightedly stable and we consider the relationship between limited farsighted stability and efficiency of networks.

Key words: Limited farsightedness, Stability, Networks.

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1 Introduction

Networks of relationships help determine the careers that people choose, the jobs they obtain, the products they buy, and how they vote. The many aspects of our lives that are governed by social networks make it critical to understand how they impact behavior and which network structures are likely to emerge in a society. A simple way to analyze the networks that one might expect to emerge in the long run is to examine the requirement that individuals do not benefit from altering the structure of the network. A prominent example of such a condition is the pairwise stability notion defined by Jackson and Wolinsky (1996).[1] A network is pairwise stable if no individual benefits from deleting a link and no two individuals benefit from adding a link between them, with at least one benefiting strictly. While pairwise stability is natural, easy to work with and a very important tool in network analysis,[2] it assumes that individuals are myopic, and not farsighted, in the sense that they do not forecast how others might react to their actions. Indeed, the adding or deleting of one link might lead to subsequent addition or deletion of another link. For instance, individuals might not add a link that appears valuable to them given the current network, as this might induce the formation of other links, ultimately leading to lower payoffs for them.

Herings, Mauleon and Vannetelbosch (2009) introduce the notion of pairwise farsighted stability. A set of networks is pairwise farsightedly stable (i) if all possible farsighted pairwise deviations from any network within the set to a network outside the set are deterred by the threat of ending worse off or equally well off, (ii) if there exists a farsighted improving path from any network outside the set leading to some network in the set, and (iii) if there is no proper subset satisfying conditions (i) and (ii).[3] Pairwise farsighted stability makes sense if players have very good information

[1] An alternative way to model network stability is to explicitly model a game by which links form and then to solve that game using the concept of Nash equilibrium or one of its refinements. See Aumann and Myerson (1988), Myerson (1991) and Dutta and Mutuswami (1997) among others.

[2] Krishnan and Sciubba (2009) find that pairwise stability leads to testable predictions for the network architectures generated by labour-sharing groups in village economies of rural Ethiopia. In addition, their empirical results confirm strongly that the architecture of a social network and not just number of links, has an important role to play in understanding network formation, and the role of social networks on economic performance.

[3] Other approaches to farsightedness in network formation are suggested by the work of Chwe (1994), Xue (1998), Herings, Mauleon and Vannetelbosch (2004), Mauleon and Vannetelbosch

about how others might react to changes in the network. But in general, especially when the set of players becomes large, it requires too much foresight on behalf of the players.[4]

Our aim is to provide a tractable concept that can be used to study the influence of the degree of farsightedness on network stability. We define the notion of a level-K farsightedly stable set. A set of networks G_K is a level-K farsightedly stable set if three conditions are satisfied. First, external deviations should be deterred. That is, adding a link ij to a network $g \in G_K$ that leads to a network outside of G_K, is deterred by the threat of ending in g'. Here g' is such that either there is a farsighted improving path of length smaller than or equal to $K-2$ from $g+ij$ to g' and g' belongs to G_K or there is a farsighted improving path of length equal to $K-1$ from $g+ij$ to g' and there is no farsighted improving path from $g+ij$ to g' of smaller length. A similar requirement is imposed for the case where a link is severed. Second, external stability is required or, in other words, the networks within the set should be robust to perturbations. That is, from any network outside of G_K there is a a sequence of farsighted improving paths of length smaller than or equal to K leading to some network in G_K. Third, a minimality condition is required. That is, there is no proper subset of G_K satisfying the first two conditions.

We show that a level-K farsightedly stable set always exists and we provide a sufficient condition for the uniqueness of a level-K farsightedly stable set. We find that there is a unique level-1 farsightedly stable set G_1. It is given by the set consisting of all networks that belong to closed cycles (pairwise stable networks included). Level-K farsighted stability leads to a refinement of myopic stability for generic allocation rules: for any $K \geq 1$, the myopically stable set G_1 contains a level-K farsightedly stable set G_K. Thus, an analysis based on myopic behavior may not rule out some networks that are not stable when players are sufficiently farsighted. At the same time, a myopic analysis is compatible with farsightedness, and for any value of K there is always a level-K farsightedly stable set that consists exclusively of networks that belong to closed cycles. But, some networks that are

(2004), Dutta, Ghosal and Ray (2005), Page, Wooders and Kamat (2005), Page and Wooders (2009), Mauleon, Vannetelbosch and Vergote (2011), and Ray and Vohra (2013).

[4] Kirchsteiger, Mantovani, Mauleon and Vannetelbosch (2013) design a simple network formation experiment to test between pairwise stability and farsighted stability, but find evidence against both of them. Their experimental evidence suggests that subjects are consistent with an intermediate rule of behavior, which can be interpreted as a form of limited farsightedness.

not part of any closed cycle may become stable under limited farsightedness.

We then provide easy to verify conditions for a set to be level-K farsightedly stable and we consider the relationship between limited farsighted stability and efficiency of networks. We show that if there is a network that Pareto dominates all other networks, then that network is the unique prediction of level-K farsighted stability if K is greater than the maximum number of links in a network. In addition, we introduce a property on the allocation rule under which level-K farsighted stability singles out the complete network. Finally, we illustrate the tractability of our new concept by analyzing the criminal network model of Calvo-Armengol and Zenou (2004). We find that in criminal networks with n players, the set consisting of the complete network (where all criminals are linked to each other) is the unique level-$(n-1)$ farsightedly stable set.

Recent experimental and empirical studies suggest that players's initial choices in games often deviate systematically from equilibrium, that structural nonequilibrium level-k (Stahl and Wilson, 1994; Nagel, 1995; Costa-Gomes, Crawford and Broseta, 2001) or cognitive hierarchy (Camerer, Ho and Chong, 2004) models often out-predict equilibrium,[5] and that players only look a finite number of steps ahead when making choices.[6] We assume that players are limited farsighted, but we do not require that players choose best responses to some beliefs on opponents' strategies. In our concept, players cannot even think about a strategy since they are not able to reason about what takes place after a certain horizon.

Recently, Morbitzer, Buskens and Rosenkranz (2011) develop a model of network formation where players look a finite number of steps ahead when anticipating the reaction of other players to their change. The decision to initiate a change to the network is based on some ad hoc rules that weight improving paths that might follow their change, but which are not necessarily improving paths for the players

[5] Level-k theory and the closely-related cognitive hierarchy theory distinguish types of players according to the level at which they reason. Assumptions about level-0 behavior provide an anchor for beliefs and strategies at higher levels. At each higher level, players are assumed to know the probability distributions of strategies at lower levels. Level-1 players choose best responses to level-0 choices, while level-2 players choose best responses to level-1 choices (level-k theory) or to some probability distribution over level-0 and level-1 strategies (cognitive hierarchy theory). See Crawford, Costa-Gomes and Iriberri (2013) for a review of the literature.

[6] Players who are motivated by substantial incentives often violate backward induction even in simple sequential games such as the centipede game. See McKelvey and Palfrey (1992) among others.

who made the initial change. Using computer simulations they show that, in the co-author model of Jackson and Wolinsky (1996), limitedly farsighted players can overcome the tension between stability and efficiency only if the number of players is small. However, their concept is a refinement of the set of pairwise stable networks, and so fails to exist even more often.

The paper is organized as follows. In Section 2 we introduce some notations and basic properties and definitions for networks. In Section 3 we define the notions of improving paths and level-K pairwise stability and we show that level-K pairwise stability often fails to exist. In Section 4 we define the notion of a level-K farsightedly stable set and we characterize it. In Section 5 we study the relationship to pairwise stability. In Section 6 we provide easy to verify sufficient conditions for a set to be level-K farsightedly stable. We look at the relationship between limited farsighted stability and efficiency of networks in Section 7. In Section 8 we analyze Calvo-Armengol and Zenou (2004) model of criminal networks when players have limited farsightedness. Finally, in Section 10 we conclude.

2 Networks

Let $N = \{1, \ldots, n\}$ be the finite set of players who are connected in some network relationship. The network relationships are reciprocal and the network is thus modeled as a non-directed graph. Individuals are the nodes in the graph and links indicate bilateral relationships between individuals. Thus, a network g is simply a list of which pairs of individuals are linked to each other. We write $ij \in g$ to indicate that i and j are linked under the network g. The complete network on the set of players $S \subseteq N$ is denoted g^S and is equal to the set of all subsets of S of size 2.[7] It follows in particular that the empty network is denoted by $g^{\emptyset}$. The set of all possible networks or graphs on N is denoted by $\mathbb{G}$ and consists of all subsets of g^N. The cardinality of $\mathbb{G}$ is denoted by $n' = 2^{n(n-1)/2}$.

The network obtained by adding link ij to an existing network g is denoted by $g + ij$ and the network that results from deleting link ij from an existing network g by $g - ij$. Let

$$g_{|S} = \{ij \in g \mid i, j \in S\}.$$

[7] Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion. Finally, $\#$ will refer to the notion of cardinality.

Thus, $g_{|S}$ is the network found by deleting all links except those that are between players in S. For any network g, let $N(g) = \{i \in N \mid \exists j \in N$ such that $ij \in g\}$ be the set of players who have at least one link in the network g.

A path in a network $g \in \mathbb{G}$ between players i and j of length $K \geq 1$ is a finite sequence of players $i_0, \ldots, i_K$ with $i_0 = i$ and $i_K = j$ such that for any $k \in \{0, \ldots, K-1\}$, $i_k i_{k+1} \in g$, and such that each player in the sequence $i_0, \ldots, i_K$ is distinct. A network g is connected if for each pair of players i and j such that $i \neq j$ there exists a path between i and j in g. A non-empty network $h \subseteq g$ is a component of g if for all $i \in N(h)$ and $j \in N(h) \setminus \{i\}$, there exists a path in h connecting i and j, and for any $i \in N(h)$ and $j \in N(g)$, $ij \in g$ implies $ij \in h$. The set of components of g is denoted by $C(g)$. Using the components of a network, we can partition the players into maximal groups within which players are connected. Let $P(g)$ denote the partition of N induced by the network g. That is, $S \in P(g)$ if and only if either there exists $h \in C(g)$ such that $S = N(h)$ or there exists $i \notin N(g)$ such that $S = \{i\}$.

An allocation rule is a function $Y : \mathbb{G} \to \mathbb{R}^N$ which gives for every player i and network g a payoff $Y_i(g)$.

3 Improving Paths

A farsighted improving path of length $K \geq 0$ from a network g to a network $g' \neq g$ is a finite sequence of networks $g_0, \ldots, g_K$ with $g_0 = g$ and $g_K = g'$ such that for any $k \in \{0, \ldots, K-1\}$ either (i) $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_K) > Y_i(g_k)$ or $Y_j(g_K) > Y_j(g_k)$, or (ii) $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_K) > Y_i(g_k)$ and $Y_j(g_K) \geq Y_j(g_k)$. Since the set $\{0, \ldots, K-1\}$ is empty for $K = 0$, this definition implies that there is a farsighted improving path of length 0 from each network g to itself, but clearly there are no farsighted improving paths of length 0 from g to any other network. If there exists a farsighted improving path of length K from g to g', then we write $g \to_K g'$.

For a given network g and some $K' \geq 0$, let $f_{K'}(g)$ be the set of networks that can be reached from g by a farsighted improving path of length $K \leq K'$. That is,

$$f_{K'}(g) = \{g' \in \mathbb{G} \mid \exists K \leq K' \text{ such that } g \to_K g'\}.$$

This defines $f_{K'}$ as a correspondence on the set $\mathbb{G}$. The set of networks that can be reached from g by some farsighted improving path is denoted by $f_\infty(g)$, so

$$f_\infty(g) = \{g' \in \mathbb{G} \mid \exists K \in \mathbb{N} \text{ such that } g \to_K g'\}.$$

The following lemma follows almost immediately and is presented without proof.

Lemma 1. *For $K \geq 0$, for every $g \in \mathbb{G}$, it holds that $f_K(g) \subseteq f_{K+1}(g)$. For $K \geq n' - 1$, for every $g \in \mathbb{G}$, it holds that $f_K(g) = f_{K+1}(g) = f_\infty(g)$.*

For $K \geq 0$, we define the relation $\tilde{f}_K$ on $\mathbb{G}$ as $\tilde{f}_K(g) = f_K(g) \setminus \{g\}$, $g \in \mathbb{G}$, so the network g is dropped from $f_K(g)$ and the set $\tilde{f}_K(g)$ corresponds to the networks different from g that can be reached from g by a farsighted improving path of length at least one and at most K. Similarly, we define $\tilde{f}_\infty$ by $\tilde{f}_\infty(g) = f_\infty(g) \setminus \{g\}$ for every $g \in \mathbb{G}$.

An important concept in the analysis of networks is the one of pairwise stability as introduced in Jackson and Wolinsky (1996).

Definition 1. A network $g \in \mathbb{G}$ is pairwise stable if

1. for every $ij \in g$, $Y_i(g) \geq Y_i(g - ij)$ and $Y_j(g) \geq Y_j(g - ij)$,
2. for every $ij \notin g$, if $Y_i(g) < Y_i(g + ij)$, then $Y_j(g) > Y_j(g + ij)$.

We say that a network g' is adjacent to g if $g' = g + ij$ or $g' = g - ij$ for some ij. A network g' defeats g if either $g' = g - ij$ and $Y_i(g') > Y_i(g)$ or $Y_j(g') > Y_j(g)$, or if $g' = g + ij$ with $(Y_i(g'), Y_j(g')) > (Y_i(g), Y_j(g))$.[8] A network is pairwise stable if and only if it is not defeated by another network.[9] It is also easy to see that $g' \in \tilde{f}_1(g)$ if and only if g' defeats g. We can therefore characterize the pairwise stable networks as those $g \in \mathbb{G}$ for which $\tilde{f}_1(g) = \emptyset$, or, alternatively, $f_1(g) = \{g\}$. This formulation readily suggests the following stability notion when players are less myopic.

Definition 2. For $K \geq 1$, a network $g \in \mathbb{G}$ is level-K pairwise stable if $f_K(g) = \{g\}$. The set of level-K pairwise stable networks is denoted by P_K.

[8] We use the notation $(Y_i(g'), Y_j(g')) > (Y_i(g), Y_j(g))$ for $Y_i(g') \geq Y_i(g)$ and $Y_j(g') \geq Y_j(g)$ with at least one inequality holding strictly, $(Y_i(g'), Y_j(g')) \geq (Y_i(g), Y_j(g))$ for $Y_i(g') \geq Y_i(g)$ and $Y_j(g') \geq Y_j(g)$, and $(Y_i(g'), Y_j(g')) \gg (Y_i(g), Y_j(g))$ for $Y_i(g') > Y_i(g)$ and $Y_j(g') > Y_j(g)$.

[9] Dutta and Mutuswami (1997) and Jackson and van den Nouweland (2005) introduce the notion of strong stability, where stability of the network against deviations by arbitrary coalitions is required. In the same spirit, our theory of limited farsightedness can easily be modified to study coalitional moves rather than pairwise moves.

We may replace the condition $f_K(g) = \{g\}$ for level-K pairwise stability by the equivalent condition $\tilde{f}_K(g) = \emptyset$. The set P_K of level-K pairwise stable networks might be worth studying in its own right. However, similar to the case of myopic players, there is no guarantee that this set is non-empty. It follows from Lemma 1 that $P_K \supseteq P_{K+1}$, so emptiness is more likely to become a problem for higher values of K.[10] In the next section we present a stability notion that does not suffer from this emptiness problem.

The set $f_K^2(g) = f_K(f_K(g)) = \{g'' \in \mathbb{G} \mid \exists g' \in f_K(g) \text{ such that } g'' \in f_K(g')\}$ consists of those networks that can be reached by a composition of two farsighted improving paths of length at most K from g. We extend this definition and, for $m \in \mathbb{N}$, we define $f_K^m(g)$ as those networks that can be reached from g by means of m compositions of farsighted improving paths of length at most K. Since there are n' networks in $\mathbb{G}$, it follows that f_K^m is the same for all values of m greater than or equal to $n' - 1$. The resulting correspondence for such values of m is called the transitive closure of f_K and is denoted by f_K^∞.[11]

Lemma 1 extends to compositions of f_K and in particular to the transitive closure f_K^∞ of f_K as is shown in the following lemma, which is presented without proof.

Lemma 2. *For $K \geq 0$, for every $g \in \mathbb{G}$, it holds that $f_K^\infty(g) \subseteq f_{K+1}^\infty(g)$. For $K \geq n' - 1$, for every $g \in \mathbb{G}$, it holds that $f_K^\infty(g) = f_{K+1}^\infty(g) = f_\infty^\infty(g)$.*

Jackson and Watts (2002) have defined the notion of a closed cycle. A set of networks C is a cycle if for any $g' \in C$ and $g \in C \setminus \{g'\}$, there exists a sequence of improving paths of length 1 connecting g to g', i.e. $g' \in f_1^\infty(g)$. A cycle C is a maximal cycle if it is not a proper subset of a cycle. A cycle C is a closed cycle if $f_1^\infty(C) = C$, so there is no sequence of improving paths of length 1 starting at some network in C and leading to a network that is not in C. A closed cycle is necessarily a maximal cycle. For every network $g \in P_1$, the set $\{g\}$ is a closed cycle. The set of networks belonging to a closed cycle is non-empty.

[10] Jackson (2008) defines a network to be farsightedly pairwise stable if there is no farsighted improving path emanating from it. This concept reverts to P_∞ and refines the set of pairwise stable. A drawback of the definition is that it does not require that a farsighted improving path ends at a network that is stable itself. The set P_∞ is similar to the farsighted core when only one link at a time can be deleted or added.

[11] Page and Wooders (2009) use the path dominance relation to define the notion of path dominance core. A network g path dominates g' if $g \in \tilde{f}_\infty^m(g')$. The path dominance core contains all networks that are not path dominated, but it often fails to exist.

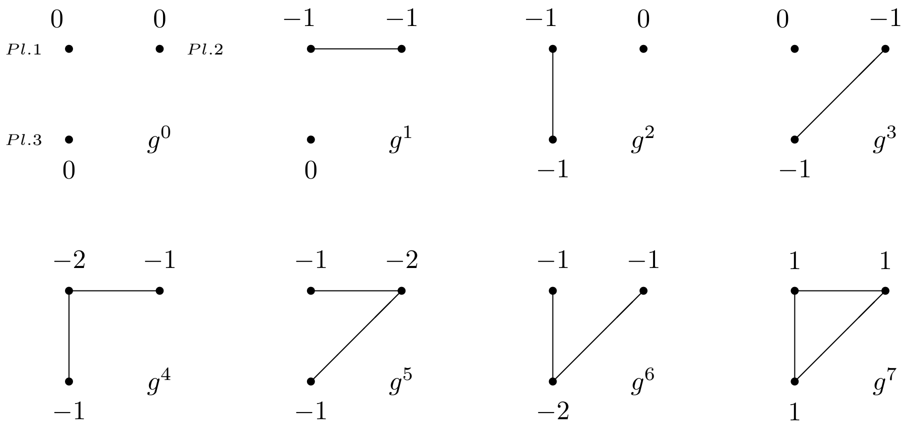

Figure 1: The 3-player investment networks.

We next present an example of an investment game as an example to illustrate the notion of farsighted improving paths and to point out some of its subtleties.

Example 1. The investment game - myopic analysis. Every player can have a link with another player at a cost of 1. Every player receives a benefit of n if all players have formed a link with all other players, but benefits are zero if at least one link is missing. Let $d_i(g)$ denote the number of links player i has in g. Then it holds that $Y_i(g) = -d_i(g)$ if g is not the complete network, and $Y_i(g^N) = n - d_i(g^N) = 1$. Figure 1 presents the resulting payoffs for the case with 3 players.

We compute the farsighted improving paths of length $K = 1$ from a given network g to find the pairwise stable networks. It can easily be verified that

$$\begin{aligned} f_1(g) &= \{g' \in \mathbb{G} \mid g' \subseteq g,\ \#(g \setminus g') \leq 1\}, & \#g \leq n(n-1)/2 - 2, \\ f_1(g) &= \{g' \in \mathbb{G} \mid g' \subseteq g,\ \#(g \setminus g') \leq 1\} \cup \{g^N\}, & \#g = n(n-1)/2 - 1, \\ f_1(g^N) &= \{g^N\}. \end{aligned}$$

In an investment game with three or more players, it holds that both the empty network and the complete network are pairwise stable, whereas there are no other pairwise stable networks.

Next we consider the transitive closure of f_1 to compute the closed cycles in the investment game. It can easily be computed that

$$\begin{aligned} f_1^\infty(g) &= \{g' \in \mathbb{G} \mid g' \subseteq g\}, & \#g \leq n(n-1)/2 - 2, \\ f_1^\infty(g) &= \{g' \in \mathbb{G} \mid g' \subseteq g\} \cup \{g^N\}, & \#g = n(n-1)/2 - 1, \\ f_1^\infty(g^N) &= \{g^N\}. \end{aligned}$$

The empty network $g^\emptyset$ belongs to $f_1^\infty(g)$ for every g that is not complete. Now it is not hard to verify that the closed cycles in the investment example coincide with the pairwise stable sets, so are given by the empty and the complete network. □

It is a priori reasonable that the complete network is stable. However, this is less clear for the empty network and the question when the empty network is stable or not should be intimately linked to the number of players and their degree of farsightedness. We continue the example by studying the farsighted improving paths of length $K \geq 2$.

Example 2. The investment game - farsighted analysis. When we consider farsighted improving paths of length $K = 2$ or 3, the complete network belongs to $f_K(g)$ if and only if $\#(g^N \setminus g) \leq K$, so the network g^N can be obtained from g by adding K links. When the network g is not complete, $f_K(g)$ also includes those networks that are obtained by deleting less than or equal to K links from g, and no other networks.

The picture changes slightly when we consider farsighted paths of length 4 or higher. Although it is generally the case that $g^N \in f_K(g)$ if and only if $\#(g^N \setminus g) \leq K$, and a subset g' of $g \neq g^N$ belongs to $f_K(g)$ if and only if $\#(g \setminus g') \leq K$, new possibilities arise. For instance, in a 4-player investment game it holds that $\{14, 23\} \in f_4(\{12, 13, 23, 24\})$, so the link 14, which did not exist in the starting network, is added. Indeed, starting from the network $g = \{12, 13, 23, 24\}$, first Players 1 and 4 form a link, next Player 1 severs his links with Players 2 and 3, and finally Player 2 cuts his link with Player 4. This constitutes a farsighted improving path since none of the networks involved in the path is complete, the degree of Player 1 in the network $\{12, 13, 23, 24\}$ is one higher than in the network $\{14, 23\}$ and the degree of Player 4 is the same, so the addition of the link 14 in the beginning is feasible. From then on, only links are deleted, which improves the payoffs of the players involved and does not affect the other players. □

Despite the subtleties for higher values of K, it is straightforward to verify that the set P_K of level-K pairwise stable sets consists of $g^\emptyset$ and g^N when $n(n-1)/2 > K$ and is equal to $\{g^N\}$ otherwise. When the level of farsightedness of players is greater than or equal to $n(n-1)/2$, the number of links needed to go from the empty network to the complete network, the complete network emerges as the unique level-K pairwise stable set. In many examples, however, the set P_K will be empty.

4 Limited Farsighted Stability

To analyze the influence of the degree of farsightedness on the stability of networks, we define the notion of a level-K farsightedly stable set. In the next definition, we use the notational convention that $f_{-1}(g) = \emptyset$ for every $g \in \mathbb{G}$.

Definition 3. For $K \geq 1$, a set of networks $G_K \subseteq \mathbb{G}$ is a level-K farsightedly stable set with respect to Y if

(i) $\forall\ g \in G_K$,

(ia) $\forall\ ij \notin g$ such that $g + ij \notin G_K$,
$\exists g' \in [f_{K-2}(g+ij) \cap G_K] \cup [f_{K-1}(g+ij) \setminus f_{K-2}(g+ij)]$ such that
$(Y_i(g'), Y_j(g')) = (Y_i(g), Y_j(g))$ or $Y_i(g') < Y_i(g)$ or $Y_j(g') < Y_j(g)$,

(ib) $\forall\ ij \in g$ such that $g - ij \notin G_K$,
$\exists g', g'' \in [f_{K-2}(g-ij) \cap G_K] \cup [f_{K-1}(g-ij) \setminus f_{K-2}(g-ij)]$ such that
$Y_i(g') \leq Y_i(g)$ and $Y_j(g'') \leq Y_j(g)$.

(ii) $\forall g' \in \mathbb{G} \setminus G_K$, $f_K^{\infty}(g') \cap G_K \neq \emptyset$.

(iii) $\forall G'_K \subsetneq G_K$, at least one of the Conditions (ia), (ib), and (ii) is violated by G'_K.

The move from a network g to an adjacent network is called a deviation. Condition (i) in Definition 3 requires the deterrence of external deviations. Condition (ia) captures that adding a link ij to a network $g \in G_K$ that leads to a network outside of G_K, is deterred by the threat of ending in g'. Here g' is such that either there is a farsighted improving path of length smaller than or equal to $K-2$ from $g+ij$ to g' and g' belongs to G_K or there is a farsighted improving path of length equal to $K-1$ from $g+ij$ to g' and there is no farsighted improving path from $g+ij$ to g' of smaller length. Condition (ib) is a similar requirement, but then for the case where a link is severed.[12]

Since level-K farsightedness models a reasoning horizon of the players of length K, we have to distinguish farsighted improving paths of length less than or equal to

[12] Chwe (1994) defines the notion of largest consistent set. A set G is a consistent set if both external and internal deviations with respect to $\widetilde{f}_\infty$ are deterred. The largest consistent is the set that contains any consistent set.

$K-2$ after a deviation from g to $g+ij$ and farsighted improving paths of length equal to $K-1$. In the former case, the reasoning capacity of the players is not yet reached, and the threat of ending in g' is only credible if it belongs to the farsightedly stable set G_K. In the latter case, the only way to reach g' from g requires K steps or even more; one step in the deviation to $g+ij$ and at least $K-1$ additional steps in any farsighted improving path from $g+ij$ to g'. Since this exhausts the reasoning capacity of the players, the threat of ending in g' is credible, irrespective of whether it belongs to G_K or not.

Condition (ii) in Definition 3 requires external stability and implies that the networks within the set are robust to perturbations. From any network outside of G_K there is a a sequence of farsighted improving paths of length smaller than or equal to K leading to some network in G_K.[13] Condition (ii) implies that if a set of networks is level-K farsightedly stable, it is non-empty. Condition (iii) is the minimality condition.

Condition (i) in Definition 3 guarantees that networks inside the set are stable for players whose reasoning horizon is of length K. Hence, f_K is used for deterring deviations from networks inside the set. Condition (ii) in Definition 3 deals with the robustness to perturbations of the networks inside the set. Perturbations may be due to exogenous forces acting on the network, or simply errors on the part of some players. After some perturbation, players make linking decisions inside their horizon and move according to some level-K farsighted improving path without being able to anticipate that other linking decisions might be taken afterwards. Hence, f_K^{∞} is used to capture what could happen after some perturbation occurs. Condition (ii) together with Condition (i) simply imply that if we allow limited farsighted players to successively create or delete links, they will come back to the set G_K without moving away from it.

Theorem 1. *A level-K farsightedly stable set of networks exists.*

Proof. Notice that $\mathbb{G}$ trivially satisfies Conditions (i) and (ii). Let us proceed by contradiction. Assume that there does not exist any set of networks $G_K \subseteq \mathbb{G}$

[13] Chwe (1994) defines the notion of von Neumann-Morgenstern farsightedly stable set. A set G is a von Neumann-Morgenstern farsightedly stable set if both external and internal stability with respect to $\widetilde{f}_{\infty}$ are satisfied. Pages and Wooders (2009) extends this notion by requiring both external and internal stability with respect to $\widetilde{f}^{m}_{\infty}$.

that is level-K farsightedly stable. This means that for any $G_K^0 \subseteq \mathbb{G}$ that satisfies Conditions (i) and (ii) in Definition 3, we can find a proper subset G_K^1 that satisfies Conditions (i) and (ii). Iterating this reasoning we can construct an infinite sequence $\{G_K^k\}_{k \geq 0}$ of subsets of $\mathbb{G}$ satisfying Conditions (i) and (ii) with the property that $G_K^k \subsetneq G_K^{k-1}$. But since $\mathbb{G}$ has finite cardinality n', this is not possible. $\square$

For the special case where K is equal to 1, we can use the fact that $f_{-1}(g) = \emptyset$ and $f_0(g) = \{g\}$, so Definition 3 simplifies as follows.

Theorem 2. *A set of networks $G_1 \subseteq \mathbb{G}$ is a level-1 farsightedly stable set with respect to Y if*

(i) $\forall\, g \in G_1$,

- **(ia)** $\forall\, ij \notin g$ *such that* $g' = g + ij \notin G_1$ *it holds that* $(Y_i(g'), Y_j(g')) = (Y_i(g), Y_j(g))$ *or* $Y_i(g') < Y_i(g)$ *or* $Y_j(g') < Y_j(g)$,
- **(ib)** $\forall\, ij \in g$ *such that* $g' = g - ij \notin G_1$ *it holds that* $Y_i(g') \leq Y_i(g)$ *and* $Y_j(g') \leq Y_j(g)$.

(ii) $\forall g' \in \mathbb{G} \setminus G_1$, $f_1^{\infty}(g') \cap G_1 \neq \emptyset$.

(iii) $\forall G_1' \subsetneq G_1$, *at least one of the Conditions (ia), (ib), and (ii) is violated by* G_1'.

Theorem 2 shows that a level-1 farsightedly stable set is identical to a myopically stable set as defined in Herings, Mauleon and Vannetelbosch (2009). Herings, Mauleon and Vannetelbosch (2009) have shown that there is a unique myopically stable set. It is equal to the set of networks consisting of all networks that belong to a closed cycle. Theorem 3 below follows.

Theorem 3. *There is a unique level-1 farsightedly stable set. It is given by the set consisting of all networks that belong to a closed cycle.*

Since a farsightedly stable set cannot be empty, it follows from Theorem 3 that there is at least one closed cycle. Level-1 farsightedly stable sets are unique. This result does not carry over to higher levels of K.

Also for $K = 2$, the definition of a level-K farsightedly stable set simplifies somewhat, since if a network $g + ij$ belongs to $\mathbb{G} \setminus G_2$ for some set G_2, it holds that $f_0(g + ij) \cap G_2 = \{g + ij\} \cap G_2 = \emptyset$.

Theorem 4. *A set of networks $G_2 \subseteq \mathbb{G}$ is a level-2 farsightedly stable set with respect to Y if*

(i) $\forall\ g \in G_2$,

(ia) $\forall\ ij \notin g$ *such that* $g+ij \notin G_2$, $\exists g' \in \tilde{f}_1(g+ij)$ *such that* $(Y_i(g'), Y_j(g')) = (Y_i(g), Y_j(g))$ *or* $Y_i(g') < Y_i(g)$ *or* $Y_j(g') < Y_j(g)$,

(ib) $\forall\ ij \in g$ *such that* $g-ij \notin G_2$, $\exists g', g'' \in \tilde{f}_1(g-ij)$ *such that* $Y_i(g') \leq Y_i(g)$ *and* $Y_j(g'') \leq Y_j(g)$.

(ii) $\forall g' \in \mathbb{G} \setminus G_2$, $f_2^{\infty}(g') \cap G_2 \neq \emptyset$.

(iii) $\forall G_2' \subsetneq G_2$, *at least one of the Conditions (ia), (ib), and (ii) is violated by* G_2'.

Theorem 4 is useful when computing level-2 farsightedly stable sets in examples. At the other extreme, when K is greater than or equal to $n' + 1$, it follows from Lemma 1 that $f_{K-2}(g) = f_{K-1}(g)$ for every $g \in \mathbb{G}$, and from Lemma 2 that $f_K^{\infty}(g) = f_{n'-1}^{\infty}(g)$ for every $g \in \mathbb{G}$. We therefore have the following result.

Theorem 5. *For $K \geq n' + 1$, a set of networks $G_K \subseteq \mathbb{G}$ is a level-K farsightedly stable set with respect to Y if*

(i) $\forall\ g \in G_K$,

(ia) $\forall\ ij \notin g$ *such that* $g + ij \notin G_K$, $\exists g' \in f_{n'-1}(g + ij) \cap G_K$ *such that* $(Y_i(g'), Y_j(g')) = (Y_i(g), Y_j(g))$ *or* $Y_i(g') < Y_i(g)$ *or* $Y_j(g') < Y_j(g)$,

(ib) $\forall\ ij \in g$ *such that* $g - ij \notin G_K$, $\exists g', g'' \in f_{n'-1}(g - ij) \cap G_K$ *such that* $Y_i(g') \leq Y_i(g)$ *and* $Y_j(g'') \leq Y_j(g)$.

(ii) $\forall g' \in \mathbb{G} \setminus G_K$, $f_{n'-1}^{\infty}(g') \cap G_K \neq \emptyset$.

(iii) $\forall G_K' \subsetneq G_K$, *at least one of the Conditions (ia), (ib), and (ii) is violated by* G_K'.

It follows immediately from Theorem 5 that the collection of K-farsightedly stable sets is independent of K when $K \geq n' + 1$.

Herings, Mauleon and Vannetelbosch (2009) define a farsightedly stable set as a set G_∞ of networks satisfying Conditions (i) and (iii) of Theorem 5, but with Condition (ii) replaced by the requirement that

$$\forall g' \in \mathbb{G} \setminus G_\infty,\ f_\infty(g') \cap G_\infty \neq \emptyset,$$

so the correspondence $f^\infty_{n'-1}$ is replaced by $f_\infty = f_{n'-1}$, and one could interpret the Herings, Mauleon and Vannetelbosch (2009) concept as level-∞ farsighted stability. In many applications, the correspondence f_∞ is transitive, in which case it coincides with $f^\infty_{n'-1}$, and level-∞ farsighted stable sets are identical to level-$(n'+1)$ farsightedly stable sets, but in general it only holds that $f_\infty(g) \subseteq f^\infty_{n'-1}(g)$ for $g \in \mathbb{G}$. We can therefore conclude that for every level-∞ farsightedly stable set G_∞ there is a set $G' \subseteq G_\infty$ such that G' is level-$(n'+1)$ farsightedly stable.

Example 3. Investment game - farsightedly stable sets. We now analyze the concept of a level-K farsightedly stable set for the investment game of Example 1. For level-1 farsightedly stable sets, we can use Theorem 3 and have to identify all the closed cycles. Using the analysis in Example 1, we find that the unique farsightedly stable set consists of the empty and the complete network whenever there are at least three players.

We will argue next that with $n \geq 3$ players, a reasoning horizon of length K equal to $n(n-1)/2$ or higher is needed to obtain the complete network as the unique level-K farsightedly stable set. For $K < n(n-1)/2$, we show that the unique level-K farsightedly stable set consists of the empty and the complete network.

We argue first that $\{g^N\}$ is the unique level-K farsightedly stable set when $K \geq n(n-1)/2 \geq 3$. The analysis in Example 1 reveals that $g^N \in f_1(\bar{g})$ for all networks $\bar{g}$ that are adjacent to g^N, so by Lemma 1 we have $g^N \in f_{K-2}(\bar{g})$ for all networks $\bar{g}$ that are adjacent to g^N, and Condition (i) of Definition 3 is satisfied since a deviation from g^N to an adjacent network $\bar{g}$ is deterred by the return to g.

We have argued in Example 2 that $g^N \in f_K(g')$ if and only if $\#(g^N \setminus g') \leq K$. So $g^N \in f_K(g')$ for every $g' \neq g^N$, since $\#(g^N \setminus g') \leq n(n-1)/2 \leq K$ for every $g' \neq g^N$. Since $f_K(g) \subseteq f^\infty_K(g)$, we have for every $g' \in \mathbb{G} \setminus \{g^N\}$, $f^\infty_K(g') \cap \{g^N\} \neq \emptyset$ and Condition (ii) of Definition 3 is satisfied. Obviously, $\{g^N\}$ satisfies minimality as expressed in Condition (iii) of Definition 3, so g^N is a level-K farsightedly stable set.

Since $Y(g^N) \gg Y(g)$ for every $g \in \mathbb{G} \setminus \{g^N\}$, it holds that $f_K(g^N) = \{g^N\}$ for every value of $K \geq 1$, and so $f_K^\infty(g^N) = \{g^N\}$. By Condition (ii) of Definition 3, it follows that $g^N \in G_K$ for every level-K farsightedly stable set G_K. Minimality as expressed by Condition (iii) of Definition 3 now implies that $\{g^N\}$ is the unique level-K farsightedly stable set when $K \geq n(n-1)/2$.

Consider next the case $K < n(n-1)/2$. It holds that $g^N \notin f_K(g^\emptyset)$, since one needs to form $n(n-1)/2$ links to go from the empty to the complete network. Since $Y(g^\emptyset) \geq Y(g)$ for every $g \in \mathbb{G} \setminus \{g^N\}$, it follows that $f_K(g^\emptyset) = \{g^\emptyset\}$. By Condition (ii) of Definition 3, it follows that $g^\emptyset \in G_K$ for every level-K farsightedly stable set G_K. The previous paragraph argued that $g^N \in G_K$. The analysis in Example 2 reveals that $f_K^\infty(g') \cap \{g^\emptyset, g^N\} \neq \emptyset$ for every $g' \in \mathbb{G} \setminus \{g^\emptyset, g^N\}$. Together with Condition (iii) of Definition 3, we now find that $\{g^\emptyset, g^N\}$ is the unique level-K farsightedly stable set when $K < n(n-1)/2$. □

5 The Relation to Pairwise Stability

In this section, we discuss how limited farsightedly stable sets are related to notions based on pairwise stability such as the set of pairwise stable networks P_1, the set of closed cycles G_1, and the set of level-K pairwise stable networks P_K.

Theorem 3 implies that any pairwise stable network belongs to G_1. The following theorem shows that this result carries over to higher values of K.

Theorem 6. *For $K \geq 1$, the set P_K of level-K pairwise stable networks is a subset of any level-K farsightedly stable set G_K.*

Proof. Suppose G_K is level-K farsightedly stable, but does not contain some $g \in P_K$. By Definition 2, we have $f_K(g) = \{g\}$. We find that $f_K^\infty(g) = \{g\}$, so $f_K^\infty(g) \cap G_K = \emptyset$. By Condition (ii) of Definition 3, it holds that $f_K^\infty(g) \cap G_K \neq \emptyset$, a contradiction. □

Theorem 6 shows that any network g from which there are no farsighted improving paths of length smaller than or equal to K to networks different from g belongs to G_K. Level-K pairwise stability is quite demanding for higher levels of K, since even pairwise stable networks may fail to exist. Indeed, since $f_K(g) \subseteq f_{K+1}(g)$, we have that $P_K \supseteq P_{K+1}$.

Theorem 6 yields an easy sufficient condition for the uniqueness of a level-K farsightedly stable set as a corollary, where we make use of the minimality requirement as expressed in Condition (iii) of Definition 3.

Corollary 1. *For $K \geq 1$, if P_K is a level-K farsightedly stable set, then it is uniquely so.*

An allocation rule is said to be generic if for every $g, g' \in \mathbb{G}$ such that g and g' are adjacent it holds that either $g \in f_1(g')$ or $g' \in f_1(g)$. If an allocation rule is not generic, then some arbitrarily small perturbation of it will be, and genericity can therefore be thought of as a weak requirement on allocation rules. The next result shows that level-K farsighted stability leads to a refinement of myopic stability for generic allocation rules.

Theorem 7. *Let the allocation rule be generic. For every $K \geq 1$, the myopically stable set G_1 contains a level-K farsightedly stable set G_K.*

Proof. The statement is trivial for $K = 1$, so we consider $K \geq 2$.

We show first that the set G_1 satisfies Condition (i) of Definition 3. Consider some $g \in G_1$ and a deviation to $g' \in \mathbb{G} \setminus G_1$.

Suppose that $g' \in f_1(g)$. Since G_1 contains all networks in a closed cycle by Theorem 3, it follows that $g' \in G_1$, a contradiction to $g' \in \mathbb{G} \setminus G_1$. Consequently, it holds that $g' \notin f_1(g)$.

Since the allocation rule is generic, we find that $g \in f_1(g')$. We have that $g \in f_1(g') \setminus \{g'\}$, so for $K = 2$ the deviation from g to g' is deterred by g. For $K \geq 3$, we have by Lemma 1 that $g \in f_1(g') \cap G_1 \subseteq f_{K-2}(g') \cap G_1$, so again the deviation from g to g' is deterred by g.

We show next that the set G_1 satisfies Condition (ii) of Definition 3. Since G_1 is level-1 farsightedly stable, it holds for every $g' \in \mathbb{G} \setminus G_1$ that $f_1^\infty(g') \cap G_1 \neq \emptyset$. By Lemma 2 it holds that $f_1^\infty(g') \subseteq f_K^\infty(g')$, so $f_K^\infty(g') \cap G_1 \neq \emptyset$, and it follows that G_1 satisfies Condition (ii).

Either the set G_1 is a minimal set satisfying Conditions (i) and (ii) of Definition 3 and is therefore level-K farsightedly stable, or it has a proper subset G_K which is a minimal set satisfying Conditions (i) and (ii), so G_K is level-K farsightedly stable. In both cases, the statement of the theorem holds. □

Theorem 3 asserts that there is a unique level-1 farsightedly stable set G_1, given by the union of all closed cycles. Theorem 7 shows that higher levels of farsightedness lead to a refinement of the networks that belong to closed cycles. For any value of K, there is always a subset of G_1 that is level-K farsightedly stable. Theorem 7 shows that an analysis based on myopic behavior may not rule out some networks

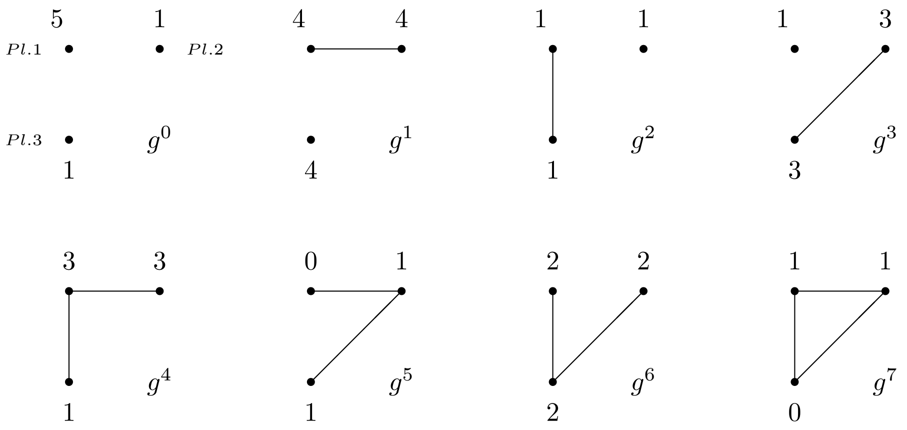

Figure 2: Networks outside closed cycles can be farsightedly stable in Example 4.

that are not stable when players are sufficiently farsighted. At the same time, a myopic analysis is compatible with farsightedness, and for any value of K there is always a farsightedly stable set that consists exclusively of networks that belong to closed cycles.

Theorem 7 does not claim that farsightedly stable sets are always subsets of networks in G_1. The following example shows that networks that are not part of any closed cycle may become stable under limited farsightedness.

Example 4. Consider the situation where three players can form and sever links and where the payoffs are given as in Figure 3. The farsighted improving paths of various lengths are presented in Table 1.

In this example there is a unique pairwise stable network, g^3. By inspecting $\tilde{f}_1$ as presented in Table 1, it is easily verified that there are no other closed cycles in this example. So, $G_1 = \{g^3\}$. By Theorem 7, and the fact that each farsightedly stable set contains at least one element, it holds that $\{g^3\}$ is a level-K farsightedly stable set for any value of K. At the same time, the payoffs resulting from the network g^3 are Pareto dominated by those of g^1. The problem with network g^1 is that Player 1 has myopic incentives to cut his link with Player 2 to obtain a payoff of 5 from the network g^0 instead of 4 from the network g^1. Once at g^0, Players 2 and 3 have myopic incentives to form a link and form the pairwise stable network g^3. The question is whether the network g^1 is stable when the players are less myopic.

We first show that $\{g^1\}$ is a level-2 farsightedly stable set by verifying that $\{g^1\}$ satisfies the three conditions in Theorem 4. There are three possible deviations from

g	$\tilde{f}_1(g)$	$\tilde{f}_2(g)$	$\tilde{f}_3(g)$	$\tilde{f}_K(g),\ K \geq 4$
g^0	g^3	g^3	g^1, g^3	g^1, g^3, g^4
g^1	g^0	g^0	g^0	g^0
g^2	g^0, g^4, g^6	g^0, g^1, g^3, g^4, g^6	g^0, g^1, g^3, g^4, g^6	g^0, g^1, g^3, g^4, g^6
g^3		g^1	g^1, g^4	g^1, g^4
g^4	g^1	g^0, g^1	g^0, g^1, g^3	g^0, g^1, g^3
g^5	g^1, g^3	g^1, g^3, g^4, g^6	g^1, g^3, g^4, g^6	g^0, g^1, g^3, g^4, g^6
g^6	g^3	g^3, g^4	g^1, g^3, g^4	g^1, g^3, g^4
g^7	g^4, g^5, g^6	g^1, g^3, g^4, g^5, g^6	$g^0, g^1, g^3, g^4, g^5, g^6$	$g^0, g^1, g^3, g^4, g^5, g^6$

Table 1: The elements of $f_K(g)$ in Example 4.

g^1. Players 1 and 2 can cut their link and move to g^0, Players 1 and 3 can form a link to arrive at g^4, and Players 2 and 3 can form a link to go to g^5. From Table 1 it follows immediately that $g^3 \in \tilde{f}_1(g^0)$, $g^1 \in \tilde{f}_1(g^4)$, and $g^1 \in \tilde{f}_1(g^5)$. Since Players 1 and 2 both have lower payoffs at g^3 than at g^1, the first deviation is deterred. The other two deviations are deterred by the possible return to g^1. We conclude that Condition (i) of Theorem 4 holds.

One degree of farsightedness is needed to move from g^4 or g^5 to g^1, and two such degrees are needed to move from g^2, g^3, or g^7 to g^1. Since $g^0 \to_1 g^3 \to_2 g^1$ and $g^6 \to_1 g^3 \to_2 g^1$, we have shown that for every $g' \in \mathbb{G} \setminus \{g^1\}$, $g^1 \in f_2^\infty(g')$, so Condition (ii) of Theorem 4 holds. Condition (iii) of Theorem 4 is trivially satisfied by $\{g^1\}$. □

We conclude this section by observing that farsightedly stable sets may depend in a non-monotonic way on the degree of farsightedness by showing that $\{g^1\}$ is not a level-3 farsightedly stable set in Example 4.

Example 5. Consider the same network situation as in Example 4, so $\{g^1\}$ is a level-2 farsightedly stable set. We argue by contradiction, so suppose that $\{g^1\}$ is a level-3 farsightedly stable set and consider a deviation by Player 1, who cuts the link with Player 2 to arrive at the network g^0. When Player 1 has only two degrees of farsightedness, he might fear a further move to g^3, which would deter the deviation. With three degrees of farsightedness, Player 1 realizes that the threat of ending in g^3 is not credible, since g^3 does not belong to the level-3 farsightedly stable set $\{g^1\}$.

Since the set $f_2(g^0) \setminus f_1(g^0) = \emptyset$, the deviation by Player 1 to g^0 is not deterred according to Definition 3.

However, when players are sufficiently farsighted, $\{g^1\}$ reemerges as a level-K farsightedly stable set. We consider some $K \geq n' + 1$ and verify that $\{g^1\}$ satisfies the conditions of Theorem 5. As before, the only deviations from g^1 are to g^0, g^4, and g^5. Since $f_{n'-1}(g^0)$, $f_{n'-1}(g^4)$, and $f_{n'-1}(g^5)$ all contain g^1, such deviations are deterred, and Condition (i) of Theorem 3 is satisfied. Since $g^1 \in f_{n'-1}(g') \subseteq f^{\infty}_{n'-1}(g')$ for all $g' \in \mathbb{G} \setminus \{g^1\}$, we know that Condition (ii) of Theorem 3 is satisfied by $\{g^1\}$. Condition (iii) of Theorem 4 is trivially satisfied. It follows that $\{g^1\}$ is a level-K farsightedly stable set for every $K \geq n' + 1$. □

6 Sufficient Conditions for Limited Farsighted Stability

In this section, we present two sets of sufficient conditions for a set to be level-K farsightedly stable. In many examples, these conditions are easy to verify.

A refinement of pairwise stability is obtained when we require the network g to defeat every other adjacent network, so $g \in f_1(g')$ for every network g' adjacent to g. We call such a network g pairwise dominant. The following definition generalizes this idea and allows for farsighted improving paths of any length K.

Definition 4. For $K \geq 1$, a network g is level-K pairwise dominant if for every g' adjacent to g it holds that $g \in f_K(g')$. The set of level-K pairwise dominant networks is denoted by D_K.

It follows immediately from the definition that $D_1 \subseteq P_1$. For generic allocation rules, the concepts of pairwise stability and pairwise dominance coincide, $D_1 = P_1$. This coincidence does not hold for values of K greater than or equal to 2. By Lemma 1 it follows that $D_K \subseteq D_{K+1}$, whereas $P_K \supseteq P_{K+1}$.

The first set of sufficient conditions applies to the case where $K = 1$.

Theorem 8. *If $g \in P_1$ and for every $g' \in \mathbb{G} \setminus \{g\}$ it holds that $g \in f_1^{\infty}(g')$, then $\{g\}$ is the unique level-1 farsightedly stable set.*

Proof. We show that $\{g\}$ is a level-1 farsightedly stable set by applying Theorem 2. The uniqueness then follows from Theorem 3. Since $g \in P_1$ it holds that $f_1(g) = \{g\}$, so for a deviation from g to $g' = g + ij$ it holds that $(Y_i(g'), Y_j(g')) = (Y_i(g), Y_j(g))$ or $Y_i(g') < Y_i(g)$ or $Y_j(g') < Y_j(g)$ and for a deviation from g to $g' = g - ij$ it holds that $Y_i(g') \leq Y_i(g)$ and $Y_j(g') \leq Y_j(g)$, so Condition (i) of Theorem 2 is satisfied. Conditions (ii) and (iii) of Theorem 2 are trivially satisfied. □

The next result applies when $K \geq 2$.

Theorem 9. *Consider some $K \geq 2$. If $g \in D_J$ for some $J < K$ and for every $g' \in \mathbb{G} \setminus \{g\}$ it holds that $g \in f_K^\infty(g')$, then $\{g\}$ is a level-K farsightedly stable set. If, moreover, $g \in P_K$, then $\{g\}$ is the unique level-K farsightedly stable set.*

Proof. We show first that $\{g\}$ is a level-K farsightedly stable set.

We first consider $K = 2$ and apply Theorem 4. If $K = 2$, then the only possibility is that $J = 1$, so $g \in D_1$, or equivalently $g \in \tilde{f}_1(\bar{g})$ for every $\bar{g}$ adjacent to g. Condition (i) of Theorem 4 is satisfied since a deviation from g to $\bar{g}$ is deterred by the return to $g \in \tilde{f}_1(\bar{g})$. Conditions (ii) and (iii) of Theorem 4 are trivially satisfied.

We next consider $K \geq 3$ and apply Definition 3. Since $g \in D_J$ for some $J < K$, it holds that $g \in f_J(\bar{g}) \subseteq f_{K-1}(\bar{g})$ for every $\bar{g}$ adjacent to g, where the inclusion uses Lemma 1. It holds that either $g \in f_{K-2}(\bar{g})$, so $g \in f_{K-2}(\bar{g}) \cap \{g\}$, or $g \notin f_{K-2}(\bar{g})$, so $g \in f_{K-1}(\bar{g}) \setminus f_{K-2}(\bar{g})$. Condition (i) of Definition 3 is satisfied since a deviation from g to $\bar{g}$ is deterred by the return to $g \in [f_{K-2}(\bar{g}) \cap \{g\}] \cup [f_{K-1}(\bar{g}) \setminus f_{K-2}(\bar{g})]$. Conditions (ii) and (iii) of Definition 3 are trivially satisfied.

We complete the proof by showing that $\{g\}$ is the unique level-K farsightedly stable set if in addition $g \in P_K$. Since $g \in f_K^\infty(g')$ for every $g' \in \mathbb{G} \setminus \{g\}$ and $g \in P_K$, we have that $P_K = \{g\}$, and therefore P_K is a level-K farsightedly stable set. Corollary 1 yields the desired result. □

The conditions of Theorems 8 and 9 are usually easy to verify. To show that $g \in P_1$ requires that $f_1(g)$ does not contain networks different from g. To show that $g \in f_1^\infty(g')$ for all $g' \neq g$, we have to find a sequence of farsighted improving paths of length one that connect g' to g. In Theorem 9 the requirement of Theorem 8 that $g \in P_1$ is replaced by the requirement that $g \in D_J$ for some $J < K$, so we have to show that $g \in f_J(g')$ for all g' adjacent to g. The higher J, the weaker is this requirement, so we could replace the requirement $g \in D_J$ for some $J < K$ by $g \in D_{K-1}$. To show that $g \in f_K^\infty(g')$ for all $g' \neq g$, we have to find a sequence of farsighted improving paths of length at most K that connect g' to g. Very often the analysis of farsighted improving paths of small lengths is already sufficient. The higher K, the easier it is to satisfy the conditions of Theorem 9 and to find a singleton level-K farsightedly stable set. Finally, to show that $g \in P_K$ requires that $f_K(g)$ does not contain networks different from g. This requirement is more difficult to satisfy for increasing values of K.

In Example 4 it holds that $g^3 \in P_1$ and for every $g \in \mathbb{G} \setminus \{g^3\}$, $g^3 \in f_1^{\infty}(g)$. We can then apply Theorem 8 to conclude that $\{g^3\}$ is the unique level-1 farsightedly stable set.

In Example 4 it also holds that $g^3 \in D_1$. Since for every $g \in \mathbb{G} \setminus \{g^3\}$, $g^3 \in f_1^{\infty}(g)$, we have by Lemma 2 that $g^3 \in f_K^{\infty}(g)$ for every $K \geq 2$. We can then apply Theorem 9 to conclude that $\{g_3\}$ is a level-K farsightedly stable set for any value of $K \geq 2$.[14]

We have illustrated in Example 4 that there are other farsightedly stable sets for higher values of K, in particular $\{g^1\}$ can be sustained as a farsightedly stable set for higher values of K. Indeed, for $K \geq 2$, $f_K(g^3)$ contains networks different from g^3, so the condition $g^3 \in P_K$ in Theorem 9, which is sufficient for uniqueness of $\{g^3\}$ as a level-K farsightedly stable set, does not hold.

In Example 4, $\{g^1\}$ has been shown to be a level-2 farsightedly stable set. In Example 5 we have argued that g^1 is not a level-3 farsightedly stable set. We show next that Theorem 9 can be used to show that $\{g_1\}$ is a level-K farsightedly stable set for any $K \geq 4$. The adjacent networks of g^1 are g^0, g^4, and g^5. It follows from Table 1 that $f_3(g^0)$, $f_3(g^4)$, and $f_3(g^5)$ all contain g^1, so $g^1 \in D_3$. We have already argued in Example 4 that for every $g' \in \mathbb{G} \setminus \{g^1\}$ it holds that $g^1 \in f_2^{\infty}(g')$ so, by Lemma 2, we have that $g^1 \in f_K^{\infty}(g')$ for all $K \geq 2$. Combining the conclusions in the previous two sentences and applying Theorem 9 proves that $\{g_1\}$ is a level-K farsightedly stable set for any $K \geq 4$.

7 Efficiency and Stability

We now turn to the question of the relationship between limited farsighted stability and efficiency of networks. A network g is strongly efficient if $\sum_{i \in N} Y_i(g) > \sum_{i \in N} Y_i(g')$ for all $g' \neq g$. Assume that there is a network $\widetilde{g}$ that strictly Pareto dominates all other networks. That is, $Y_i(\widetilde{g}) > Y_i(g)$ for all $i \in N$ and for all $g \in \mathbb{G} \setminus \{\widetilde{g}\}$. Hence, $\widetilde{g}$ is both Pareto efficient and strongly efficient.

[14] Alternatively, since the allocation rule in Example 4 is generic and $G_1 = \{g^3\}$, applying Theorem 7, we have that $\{g^3\}$ is a level-K farsightedly stable set for any $K \geq 2$.

Theorem 10. *Suppose that there is some network $\widetilde{g}$ strictly Pareto dominating all other networks $g \in \mathbb{G} \setminus \{\widetilde{g}\}$. Then, $\{\widetilde{g}\}$ is the unique level-K farsightedly stable set for all $K \geq n(n-1)/2$.*

Proof. First, we show that $\widetilde{g} \in D_1$. Since $\widetilde{g}$ is such that $Y_i(\widetilde{g}) > Y_i(g)$ for all $i \in N$ and for all $g \in \mathbb{G} \setminus \{\widetilde{g}\}$, we have that $\widetilde{g} \in f_1(g)$ for any network g adjacent to $\widetilde{g}$, and then we have $\widetilde{g} \in D_1$. Moreover, we have that $\widetilde{g} \in f_K(g)$ for all $g \in \mathbb{G} \setminus \{\widetilde{g}\}$ for $K \geq n(n-1)/2$. Indeed, all players like to move from ant network g to $\widetilde{g}$ given that $Y_i(\widetilde{g}) > Y_i(g)$ for all $i \in N$ and for all $g \in \mathbb{G} \setminus \{\widetilde{g}\}$, and the maximum number of links that one needs to cut and/or to form from any other network g in order to form $\widetilde{g}$ is equal to the number of links in the complete network, $n(n-1)/2$. Hence, $\widetilde{g} \in f_K^{\infty}(g)$ for all $g \in \mathbb{G} \setminus \{\widetilde{g}\}$. Finally, since $\widetilde{g}$ strictly Pareto dominates all other networks, we have that $f_K(\widetilde{g}) = \{\widetilde{g}\}$ for all $K \geq 1$. Thus, $\widetilde{g} \in P_K$ for all $K \geq 1$. Thus, by Theorem 9 we have that $\{\widetilde{g}\}$ is the unique level-K farsightedly stable set for all $K \geq n(n-1)/2$. □

In the investment game of Example 1, the complete network g^N strictly Pareto dominates all other networks. Hence, from Theorem 10 we have that $\{g^N\}$ is the unique level-K farsightedly stable set for all $K \geq n(n-1)/2$. In fact, the most demanding case in terms of level of farsightedness is when $\widetilde{g}$ is either the complete network or the empty network in Theorem 10. So, Theorem 10 holds for levels of farsightedness relatively small compared to the number of possible networks.

There are many situations where a Pareto dominating network does not exist. Two properties imposed on allocation rules will play a role in selecting the complete network for players having some sufficient level of farsightedness. An allocation rule Y displays no externalities across components (NEC) if for any $g \in \mathbb{G}$ and $h \in C(g)$, we have $Y_i(g) = Y_i(h)$ for all $i \in N(h)$. That is, an allocation rule satisfies NEC if the allocation of every player belonging to a given component of a network does not depend on the structure of other components.

Let $C^+(g) = \{h \in C(g)$ such that $\sum_{i \in N(h)} Y_i(h) \geq 0\}$. An allocation rule Y satisfies increasing returns to link creation[15] (IRL) if:

[15] Dutta, Ghosal and Ray (2005) defines the property of increasing returns to link creation for a value function. A value function satisfies this property if there is a threshold network for which the value is nonnegative, and each time a new link is added to this treshold network, both aggregate payoffs and payoffs of players who are adding a link to the network increase. Here, we translate the main idea behind this property for an allocation rule.

(i) Y displays NEC, $Y_i(g^\emptyset) = 0$ for all $i \in N$ and $\sum_{i \in N} Y_i(g^N) \geq 0$;

(ii) If $h \in C^+(g)$, then $\sum_{i \in N(h')} Y_i(h') \geq 0$ for every $h' \supsetneq h$;

(iii) If $\#h = n(n-1)/2 - 1$ or $h \in C^+(g)$, $i \in N(h)$, $ij \notin g$, then $Y_l(g + ij) \geq Y_l(g)$ for $l = i, j$ with at least one inequality holding strictly;

(iv) There exists a critical network $\overline{g} \neq g^\emptyset$ such that for all $g \subsetneq \overline{g}$, for all $i \in N(\overline{g})$, we have $Y_i(g) < Y_i(\overline{g})$.

An allocation rule with NEC satisfies increasing returns to link creation (IRL) if along every nested chain of increasingly connected networks, there is a threshold network $\overline{g}$ for which the payoff of all players having at least one link turns to be positive and greater than the payoffs they could obtain in any network $g \subsetneq \overline{g}$, and both aggregate payoffs as well as the payoffs of the players who form extra links increase as the network becomes even larger. Notice that even for networks in between the threshold and the complete network, some players may have negative payoffs. The investment game in Example 1 satisfies IRL as well as a generalization of the investment game where

$$Y_i(g) = \begin{cases} -d_i(g)c & \text{if } d_j(g) < \overline{d} \text{ for some } j \text{ who is connected to } i \text{ in } g \\ (1 + d_i(g)) - d_i(g)c & \text{if } d_j(g) \geq \overline{d} \text{ for all } j \text{ who are connected to } i \text{ in } g \end{cases},$$

with each link ij resulting in a cost c to both i and j. The case $c = 1$ and $\overline{d} = n - 1$ corresponds to the investment game in Example 1. Another model that satisfies IRL is the symmetric connections model of Jackson and Wolinsky (1996) when the cost for maintaining a link is small, $c < \delta(1 - \delta)$.

We now show that if the allocation rule satisfies IRL, then there exists a value of K' such that, for all $K \geq K'$, $\{g^N\}$ is a level-K farsightedly stable set. Given any allocation rule Y satisfying IRL, let $\widetilde{K} = \{\#g \mid \#g \geq \#g'$ for any two critical networks $g, g'\}$ be the number of links in the critical network with the highest number of links, and let $\overline{K}$ be the number of links in the network that has the highest number of links among the networks that do not contain any critical network.

Theorem 11. *Suppose the allocation rule satisfies IRL. Then, $\{g^N\}$ is a level-K farsightedly stable set for all $K \geq \max\{\widetilde{K}, \overline{K}\}$.*

Proof. First, we show that $g^N \in D_1$. Since the allocation rule satisfies IRL, we have that $Y_l(g^N) \geq Y_l(g^N - ij)$ for $l = i, j$ with at least one inequality holding strictly.

So, $g^N \in f_1(g)$ for any network g such that $\#g = n(n-1)/2 - 1$, and then we have $g^N \in D_1$. Hence, $g^N \in f_K(g^N - ij)$ for $K \geq 1$ and the deviation from g^N to $g^N - ij$ is deterred.

To apply Theorem 9, we need to show that $g^N \in f_K^{\infty}(g)$ for every $g \neq g^N$. That is, from any network $g \neq g^N$ there is a sequence of farsighted improving paths of length smaller than or equal to K leading to the complete network g^N (external stability). Notice that NEC implies that in any critical network all players having a link are connected to each other.

(a) First, consider any network $g' \nsubseteq \overline{g}$ such that $\sum_{i \in N} Y_i(g') < 0$. Since Y satisfies IRL, for any $g \subseteq g'$ ($g \neq g^{\emptyset}$), there is some player $i \in N(g)$ who has a negative payoff, $Y_i(g) < 0$, and so, i has incentives to cut a link foreseeing the empty network where $Y_i(g^{\emptyset}) = 0$. Then, either we are at $g^{\emptyset}$ or we are at $g'' \subsetneq g'$ with $\sum_{i \in N} Y_i(g'') < 0$ and we can repeat the process until we reach the empty network $g^{\emptyset}$. So, $g^{\emptyset} \in f_K(g')$ for $K \geq \overline{K}$, where $\overline{K}$ is the number of links in the network that has the highest number of links among the networks that do not contain any critical network. From Condition (iv), we have $Y_i(\overline{g}) > Y_i(g^{\emptyset})$ for all $i \in N(\overline{g})$ and players in $N(\overline{g})$ have incentives to form sequentially the missing links in $g^{\emptyset}$ foreseeing $\overline{g}$. The number of links to be added is equal to $\#\overline{g}$. Hence, $\overline{g} \in f_{\#\overline{g}}(g^{\emptyset})$. From $\overline{g}$, Condition (iii) implies that there is a sequence of improving paths of length 1 from $\overline{g}$ to g^N where missing links are sequentially added until g^N is formed.

(b) Second, consider any network $g' \neq g^{\emptyset}$ such that $\sum_{i \in N} Y_i(g') \geq 0$. From g', condition (iii) implies that there is a sequence of improving paths of length 1 from g' to g^N where missing links are sequentially added until g^N is formed.

(c) Third, consider any network $g' \subsetneq \overline{g}$. From Condition (iv), we have $Y_i(\overline{g}) > Y_i(g')$ for all $i \in N(\overline{g})$ and players in $N(\overline{g})$ have incentives to form the missing links in g' foreseeing $\overline{g}$. The number of links to be added is equal to $\#\overline{g} - \#g' \leq \widetilde{K} - \#g^{\emptyset} = \widetilde{K}$, where $\widetilde{K} = \{\#g \mid \#g \geq \#g'$ for any two critical networks $g, g'\}$ is the number of links in the critical network with the highest number of links. As in (a), there is a sequence of improving paths of length 1 from $\overline{g}$ to g^N where missing links are sequentially added until g^N is formed. So, $g^N \in f_K^{\infty}(g')$ for $K \geq \widetilde{K}$.

From (a), (b) and (c), we have that, for any $g \neq g^N$, either $g^N \in f_K^{\infty}(g)$ for $K \geq \overline{K} > \widetilde{K}$ or $g^N \in f_K^{\infty}(g)$ for $K \geq \widetilde{K} > \overline{K}$. Theorem 9 implies that $\{g^N\}$ is a level-K farsightedly stable set for all $K \geq \max\{\widetilde{K}, \overline{K}\}$. □

In the generalized investment game, the set $\{g^N\}$ is the unique level-K farsightedly stable set for all $K \geq n\bar{d}/2$ for N even, and $K \geq n\bar{d}/2 + 1/2$ for N odd. However, in the symmetric connections model, there is a sequence of improving paths of length 1 from any $g \neq g^N$ to g^N when $c < \delta(1-\delta)$. Hence, if $c < \delta(1-\delta)$, then $g^N \in f_K^{\infty}(g)$ for any $g \neq g^N$ and for any $K \geq 1$, and $\{g^N\}$ is the unique level-K farsightedly stable set for all $K \geq 1$. In both models, the complete network is strongly efficient. However, notice that IRL does not prevent the complete network of being strongly inefficient.

8 Criminal Networks

There is empirical evidence suggesting that peer effects and the structure of social interactions matter strongly in explaining an individual's own criminal or delinquent behavior.[16] Calvo-Armengol and Zenou (2004) provide a network analysis of criminal behavior. They develop a model where criminals compete with each other in criminal activities but benefit from being friends with other criminals by improving their knowledge of the crime business. Individuals decide first to work or to become a criminal and then they choose the crime effort to exert if criminals.[17] Here, we present a simplified version of their model, which puts emphasis on the formation of links and keeps the level of criminal activities of the players fixed.[18]

Consider some criminal network g with $n \geq 3$ players. The players in the network are referred to as criminals and a maximally connected set of players as a criminal group.

Each criminal group S has a positive probability $p_S(g)$ of winning the loot $B > 0$. It is assumed that the bigger the criminal group, the higher its probability of getting the loot. This assumption captures the idea that delinquents learn from other criminals belonging to the same group how to commit crime in a more efficient way by sharing the know-how about the technology of crime. We assume that the probability of winning the loot is given by $p_S(g) = \#S/n$.

[16] See Patacchini and Zenou (2008) among others.

[17] Calvo-Armengol and Zenou (2004) mostly focus on the case where the network is exogenously given. They show that multiple equilibria with different members of active criminals and levels of involvement in crime business may coexist.

[18] For simplicity, we also keep the wage on the labour market small enough in Calvo-Armengol and Zenou's model so that all individuals prefer to become a criminal whatever the social network connecting the criminals.

The network architecture determines how the loot is shared among the criminals in the group. Consider some Player $i \in N$ and let $S \in P(g)$ be the criminal group i belongs to. We define $c_i(g) = \max_{j \in S} d_j(g)$ as the maximum degree in this criminal group. A criminal i who is part of a group $S \in P(g)$ expects a share $\alpha_i(g)$ of the loot given by

$$\alpha_i(g) = \begin{cases} \frac{1}{\#\{j \in S | d_j(g) = c_j(g)\}}, & \text{if } d_i(g) = c_i(g), \\ 0, & \text{otherwise.} \end{cases}$$

That is, within each criminal group, the criminal that has the highest number of links gets the loot. If two or more criminals have the highest number of links, then they share the loot equally among them.

Criminal i has a probability $q_i(g)$ of being caught, in which case his rewards are punished at a rate $\phi > 0$. It is assumed that the higher the number of links a criminal has, the lower his individual probability of being caught. We assume that the probability of being caught is simply given by

$$q_i(g) = \frac{n - 1 - d_i(g)}{n}.$$

The total payoffs of criminal i belonging to criminal group $S \in P(g)$ are therefore equal to

$$\begin{aligned} Y_i(g) &= p_S(g)\alpha_i(g)(1 - q_i(g)\phi)B \\ &= \begin{cases} \frac{\#S}{n} \frac{1}{\#\{j \in S | d_j(g) = c_i(g)\}}(1 - \frac{n-1-d_i(g)}{n}\phi)B, & \text{if } d_i(g) = c_i(g), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We require $\phi < n/(n-1)$ to guarantee that payoffs are non-negative and positive for a player with the highest degree in his group.

Figure 2 presents the payoffs for 3-player criminal networks with $B = 9$ and $\phi = 1$. Table 2 shows the farsighted improving paths for the different possible values of K. It can be verified that the farsighted improving paths for the 3-player case do not depend on the specific choices for B and ϕ.

For the three player case, we use Theorem 3 to compute the closed cycles and conclude that $G_1 = P_1 = \{g^1, g^2, g^3, g^7\}$ is the myopically stable set, so G_1 consists of all pairwise stable networks. There are many networks that are stable when players are myopic.

For $K \geq 2$, we apply Theorem 9 to show that $G_K = \{g^7\}$ is the unique level-K farsightedly stable set. It holds that $g^7 \in D_1$ and $g^7 \in f_2^{\infty}(g)$ for every $g \neq g^7$, so

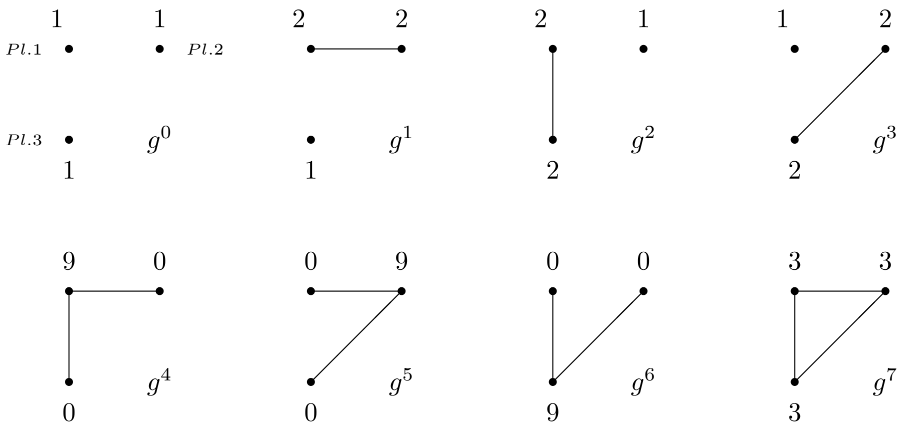

Figure 3: The 3-player criminal networks.

g	$\tilde{f}_1(g)$	$\tilde{f}_2(g)$	$\tilde{f}_K(g),\ K \geq 3$
g^0	g^1, g^2, g^3	g^1, g^2, g^3	g^1, g^2, g^3, g^7
g^1, g^2, g^3		g^7	g^7
g^4	g^1, g^2, g^7	g^1, g^2, g^7	g^1, g^2, g^3, g^7
g^5	g^1, g^3, g^7	g^1, g^3, g^7	g^1, g^2, g^3, g^7
g^6	g^2, g^3, g^7	g^2, g^3, g^7	g^1, g^2, g^3, g^7
g^7			

Table 2: The elements of $\tilde{f}_K(g)$ for 3-player criminal networks with $B = 9$ and $\phi = 1$.

$\{g^7\}$ is a level-K farsightedly stable set. Since $g^7 \in P_K$, it follows from Theorem 9 that $\{g^7\}$ is the unique farsightedly stable set. If criminals behave myopically, they may not go beyond forming a single link in the three player case. But with a degree of farsightedness of at least 2, the complete criminal network emerges as the unique prediction.

The remainder of this section is devoted to the analysis of the n-criminal case. As in the 3-criminal case, there are many networks that are pairwise stable in the n-person case. The complete network is easily verified to be pairwise stable. The generalization of the networks g^1, g^2, and g^3 for the 3-criminal case to the n-criminal case would be any network consisting of complete components, where no two components have the same degree. But also any network with a single component where all players have a degree at least equal to two and one player has a degree that is at least two times higher than the degree of any other player is pairwise stable.

We will argue next that $\{g^N\}$ is a level-K farsightedly stable set whenever $K \geq n-1$.

We show first that the complete network is pairwise dominant.

Lemma 3. *For criminal networks it holds that $g^N \in D_1$.*

Proof. Consider the network $g^N - ij$ for some ij. It holds that

$$d_i(g^N - ij) = d_j(g^N - ij) < c_i(g^N - ij) = c_j(g^N - ij),$$

so

$$Y_i(g^N - ij) = Y_j(g^N - ij) = 0 < Y_i(g^N) = Y_j(g^N),$$

and $g^N \in f_1(g^N - ij)$. We have shown that $g^N \in D_1$. □

We show next that the complete network can be reached from any starting network by repeated application of at most $n-1$ degrees of farsightedness.

Lemma 4. *For criminal networks, it holds for every $g \in \mathbb{G}$ that $g^N \in f^{\infty}_{n-1}(g)$.*

Proof. Step 1. If g has a component which is not complete, then there is $g' \in f_{n-1}(g)$ such that $g \subsetneq g'$.
Let $S \in P(g)$ be a criminal group such that some internal links are missing, $g_{|S} \neq g^S$.

If for every $i \in S$ it holds that $d_i(g) = c_i(g)$, so all players in S have the same degree, then any two players i and j in S create a link to form the network $g+ij$ and improve their payoffs since the increase in their degree increases the share in the loot and lowers the probability of being caught for both players, $\alpha_i(g+ij) > \alpha_i(g)$, $\alpha_j(g+ij) > \alpha_j(g)$, $q_i(g+ij) < q_i(g)$, and $q_j(g+ij) < q_j(g)$, so $Y_i(g+ij) > Y_i(g)$ and $Y_j(g+ij) > Y_j(g)$. We have that $g \to_1 g+ij$, so clearly $g+ij \in f_{n-1}(g)$.

If the players in S do not all have the same degree, let $i \in S$ be a player with $d_i(g) < c_i(g)$, so $\alpha_i(g) = 0$ and therefore $Y_i(g) = 0$.

If $c_i(g) = \#S-1$, then Player i consecutively links to all players $j \in S$ such that $ij \notin g$, thereby forming a network g' where he has degree $\#S-1$. The payoffs of Player i are equal to $Y_i(g) = 0$ in every step until the final one, where his payoffs increase to $Y_i(g') > 0$. Every player j that i links to has degree below $\#S-1$ and therefore payoffs equal to $0 \leq Y_j(g')$. We have that $g' \in f_{\#S-2}(g) \subseteq f_{n-1}(g)$.

If $c_i(g) < \#S - 1$, then let $j \in S$ be a player with $d_j(g) = c_i(g)$. Player i links with Player j to form the network $g + ij$. It holds that $Y_i(g + ij) = Y_i(g) = 0$ and $Y_j(g + ij) > Y_j(g) > 0$, since $\alpha_j(g + ij) \geq \alpha_j(g)$ and $q_j(g + ij) < q_j(g)$. In this case we have that $g \rightarrow_1 g + ij$, so clearly $g + ij \in f_{n-1}(g)$.

Step 2. If all components of g are complete and $g \neq g^N$, then there is $g' \in f_{n-1}(g)$ such that $g \subsetneq g'$.

Let S^1 and S^2 be two criminal groups in $P(g)$.

If $\#S^1 = \#S^2$, then form a link between a Player $i \in S^1$ and a Player $j \in S^2$. Since $q_i(g) > q_i(g + ij)$, we have that

$$Y_i(g) = \tfrac{1}{n}(1 - q_i(g)\phi)B < \tfrac{\#S_1}{n}(1 - q_i(g + ij)\phi)B = Y_i(g + ij).$$

By the same calculation, it follows that $Y_j(g) < Y_j(g + ij)$, so $g \rightarrow_1 g + ij$, and therefore $g + ij \in f_{n-1}(g)$.

Otherwise, it holds without loss of generality that $\#S^1 < \#S^2$. Select some player $i \in S^1$ and a set J consisting of $\#S^2 + 1 - \#S^1$ players in S^2, who are linked consecutively to Player i to form network g'. The resulting finite sequence of networks is denoted $g_0, \ldots, g_K$ with $g_0 = g$ and $g_K = g'$. Notice that $K \leq n - 1$. We show next that for every $k \in \{0, \ldots, K - 1\}$, $(Y_i(g_k), Y_{j_k}(g_k)) < (Y_i(g_K), Y_{j_k}(g_K))$, where $j_k \in J$ is such that $g_{k+1} = g_k + ij_k$, thereby proving that $(g_0, \ldots, g_K)$ is a farsighted improving path and completing the proof of Step 2.

For every player $j \in J$ we have

$$d_j(g_K) = d_i(g_K) = c_i(g_K),$$

and for all other players the degree is strictly less than $c_i(g_K)$, so

$$Y_j(g_K) = Y_i(g_K) = \frac{\#S^1 + \#S^2}{n} \frac{1}{\#S^2 + 2 - \#S^1}(1 - q_i(g_K)\phi)B.$$

For $k = 0$, we have

$$\begin{aligned} Y_i(g_0) &= \frac{1}{n}(1 - q_i(g)\phi)B < Y_i(g_K), \\ Y_{j_0}(g_0) &= \frac{1}{n}(1 - q_{j_0}(g)\phi)B < Y_{j_0}(g_K), \end{aligned}$$

where we use $q_i(g_0) > q_i(g_K)$ and $q_{j_0}(g_0) > q_{j_0}(g_K)$ to get the strict inequalities.

For $k = 1, \ldots, K - 1$, it holds that Player i is connected to Player j_0, so $d_i(g_k) < d_{j_0}(g_k) = c_i(g_k)$, so $\alpha_i(g_k) = 0$ and $0 = Y_i(g_k) < Y_i(g_K)$. Similarly, it holds that Player j_k is connected to Player j_0, so $d_{j_k}(g_k) < d_{j_0}(g_k) = c_{j_k}(g_k)$, so $\alpha_{j_k}(g_k) = 0$

and $0 = Y_{j_k}(g_k) < Y_{j_k}(g_K)$.

Step 3. For every $g \in \mathbb{G}$, it holds that $g^N \in f^{\infty}_{n-1}(g)$.

It is obviously true that $g^N \in f^{\infty}_{n-1}(g^N)$. By combining the results of Step 1 and Step 2, we have that for every $g \in \mathbb{G} \setminus \{g^N\}$, there is $g' \in f_{n-1}(g)$ with strictly more links than g. Since the complete network g^N has $n(n-1)/2$ links, we find that $g^N \in f^{n(n-1)/2}_{n-1}(g) \subseteq f^{\infty}_{n-1}(g)$. □

Using Theorem 9 we prove now that, the complete network $\{g^N\}$ is level-K farsightedly stable set for every $K \geq n-1$.[19] Notice that the level of farsightedness needed to sustain the complete network $\{g^N\}$ is relatively small compared to the number of potential networks and the maximum lenghth of paths.

Theorem 12. *For criminal networks it holds that $\{g^N\}$ is a level-K farsightedly stable set for every $K \geq n-1$.*

Proof. By Lemma 3 we have that $g^N \in D_1$. By Lemma 4 we have that for every $g' \in \mathbb{G} \setminus \{g^N\}$ it holds that $g^N \in f^{\infty}_{n-1}(g') \subseteq f^{\infty}_{K}(g')$, where the inclusion follows from Lemma 2. We are now in a position to apply Theorem 9 and conclude that $\{g^N\}$ is a level-K farsightedly stable set. □

How about the uniqueness of $\{g^N\}$ as a level-K farsightedly stable set? It is tempting to use the approach of Theorem 9 and show such a result by proving that $g^N \in P_K$. However, consider the case with 6 players and let $g' = g^N - 16 - 26 - 35 - 45$. For any value of B and ϕ,[20] we claim that $g' \in f_{12}(g^N)$, so $g^N \notin P_{12}$. Since the network g' is connected, $d_1(g') = d_2(g') = d_3(g') = d_4(g') = 4$, and $d_5(g') = d_6(g') = 3$, it holds for any $i \in \{1, 2, 3, 4\}$ that $Y_i(g') = (1/4 - \phi/24)B > B/6 = Y_i(g^N)$ and for any $j \in \{5, 6\}$ that $Y_j(g') = 0 < B/6 = Y_j(g^N)$. The construction of the farsighted improving path is, however, more subtle than simply deleting the links 16, 26, 35, and 45 in some order. Indeed, after the deletion of three such links, there are exactly two players with the maximum degree and they would get strictly lower payoffs by cutting their link, and would be unwilling to do so. The way to avoid this problem requires more farsightedness and involves players in $\{1, 2, 3, 4\}$

[19] Once the network connecting delinquents is endogenous, Calvo-Armengol and Zenou (2004) find that all complete networks (where.each player in the pool of criminals are linked to each other) are pairwise stable. Notice that the size of the pool of criminals depends on the wage on the labour market.

[20] We maintain the assumption that $\phi < n/(n-1)$.

first cutting two of their mutual links, before severing the links with players 5 and 6, and finally restoring their mutual links. One explicit farsighted improving path results from $g^N - 12 - 23 - 34 - 41 - 16 - 26 - 35 - 45 + 12 + 23 + 34 + 41$ and takes 12 steps. We have denoted the player with an incentive to cut a link first, so -16 for instance means that Player 1 cuts his link with Player 6, whereas -61 would mean that Player 6 cuts his link with Player 1. It can be verified that each step in this farsighted improving path is feasible indeed.

We conclude this section by showing that if players are not too farsighted, then $g^N \in P_K$, so $\{g^N\}$ is then the unique level-K farsightedly stable set. More precisely, we will from now on consider $K = n-1$. We show first that any network in $f_{n-1}(g^N)$ has a single component.

Lemma 5. *For criminal networks it holds that every $g' \in f_{n-1}(g^N)$ has a single component.*

Proof. Consider the criminal group S of Player 1 in g'. We show that it contains all players. Suppose it contains only $s \leq n-1$ players. Then those s players have to cut all their links with all other players in $N \setminus S$. This involves at least $s(n-s)$ steps. For fixed n, the concavity of $s(n-s)$ in s implies that it is minimized at $s = 1$ or $s = n-1$. Substitution of these values of s shows the minimum to be equal to $n-1$ at both $s = 1$ and $s = n-1$. When the s players cut all their links with all other players in $N \setminus S$, all the players in N are strictly worse off, since the probability of being caught has strictly increased and the probability of winning the loot has decreased, contradicting $g' \in f_{n-1}(g^N)$. □

We show next that the complete network g^N is level-$(n-1)$ pairwise stable.

Lemma 6. *For criminal networks it holds that $g^N \in P_{n-1}$.*

Proof. Suppose g' is an element of $\tilde{f}_{n-1}(g^N)$. Let $g_0, \ldots, g_K$ with $g_0 = g^N$ and $g_K = g'$ be a farsighted improving path of length $K \leq n-1$. Since g' consists of a single component, $c_i(g')$ is independent from i and is simply denoted by c. Let $M \subseteq N$ be such that $i \in M$ if and only if $d_i(g') = c$ and denote the cardinality of M by m. It cannot be that $m = n$, since then all players have lower payoffs in g' than in g^N (because the probability of being caught is higher in g' than in g^N). Since g' is connected, it follows that $Y_i(g') = 0$ for all $i \in N \setminus M$. A player $j \in N \setminus M$ will therefore never sever a link starting at g^N. It follows that

$$\sum_{i \in M} d_i(g') \leq \sum_{j \in N \setminus M} d_j(g').$$

Since $d_i(g') > d_j(g')$ whenever $i \in M$ and $j \in N \setminus M$, we have that $m > n/2$.

Since at least one link ij with $i \in M$ and $j \in N$ is missing in g', it follows that the maximum degree in g' satisfies $c \leq n - 2$.

A lower bound on K is provided by the number of times a link ij is severed with $i \in M$ and $j \in N \setminus M$ plus the number of times a link ij is cut with $i, j \in M$. Since all players in $N \setminus M$ experienced the severance of at least two links, and any such link is cut by a player in M, a lower bound for the first number is $2(n - m)$.

For $k = 0, \ldots, K$, let $L(g_k) = \{i \in N \mid d_i(g_k) = n - 1\}$ be the set of players with degree $n - 1$ and let $\ell(g_k) = \#L(g_k)$ be its cardinality. Clearly, it holds that $\ell(g^N) = n$ and $\ell(g') = 0$. Let k' be the lowest value of k such that $\ell(g_k) \leq m$ for all $k \geq k'$. Since $\ell(g_k) - \ell(g_{k+1}) \leq 2$, we find that $\ell(g_{k'}) = m$ or $\ell(g_{k'}) = m - 1$. The sum of the cardinality of $L(g_{k'})$ and the cardinality of M is at least $2m - 1$. Since there are only n players, it follows that $\#(L(g_{k'}) \cap M)$, the cardinality of the set of players in $L(g_{k'})$ that belong to M, is at least $2m - n - 1$.

For all $k \geq k'$, for all $i \in L(g_k)$, it holds that $Y_i(g_k) > Y_i(g')$, since the loot has to be shared with less or the same number of criminals and the probability of being caught is strictly less when comparing g_k to g'. Such a player i will therefore never choose to sever a link himself, so whenever a link involving player $i \in L(g_k)$ is severed when going from g_k to g_{k+1}, it must be by a player in $M \setminus L(g_k)$. It follows that $\ell(g_k) - \ell(g_{k+1}) \leq 1$. Since $\#(L(g_{k'}) \cap M) \geq 2m - n - 1$, we find that going from $g_{k'}$ to g' involves the deletion of at least $2m - n - 1$ links ij with $i, j \in M$.

We have proved that $K \geq 2(n - m) + 2m - n - 1 = n - 1$. □

Using Theorem 9 we prove now that, the complete network $\{g^N\}$ is the unique level-$(n - 1)$ farsightedly stable set.[21]

Theorem 13. *For criminal networks it holds that* $\{g^N\}$ *is the unique level-*$(n - 1)$ *farsightedly stable set.*

Proof. By Lemma 3 we have that $g^N \in D_1$. By Lemma 4 we have that for every $g' \in \mathbb{G} \setminus \{g^N\}$ it holds that $g^N \in f^{\infty}_{n-1}(g')$. By Lemma 6 it holds that $g^N \in P_{n-1}$.

[21] Herings, Mauleon and Vannetelbosch (2009) show that in the example of criminal networks with n players, the complete network $\{g^N\}$ is a pairwise farsightedly stable set.

We are now in a position to apply Theorem 9 and conclude that $\{g^N\}$ is the unique level-$(n-1)$ farsightedly stable set. $\square$

Structural properties of criminal networks must be taken into account to better understand the impact of peer influence on delinquent behavior and to address adequate and novel delinquency-reducing policies. Hence, it is important to acquire knowledge about the level of farsightedness of criminals to determine which criminal networks are likely to emerge in the long run.[22]

9 Conclusion

We study the stability of social and economic networks when players are limitedly farsighted. Pairwise stability is a very important tool in network analysis. One shortcoming of pairwise stability is the lack of farsightedness. Players do not anticipate that other players may react to their changes. However, farsighted stability requires too much foresight on behalf of the players. Hence, we propose a tractable concept, namely level-K farsighted stability, that can be used to study the influence of the degree of farsightedness on network stability.

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[22] Ballester, Calvo-Armengol and Zenou (2010) develop a criminal network game where each delinquent decides how much delinquency effort to exert. The network is determined endogenously by allowing players to join the labour market instead of committing criminal activities. They find that the optimal enforcement policy consists of removing some key player or some key group and is complex since it depends both on the wage and on the network. Indeed, the removal of some players may induce further voluntary moves of other players who now find profitable to leave their criminal activities for joining the labour market.

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10 Material that might be useful.

Lemma 7. *For criminal networks it holds that every $g' \in f_{n-2}(g^{\{1,\ldots,n-1\}})$ has at least two components.*

Proof. Consider the networks g' in $f_{n-2}(g^{\{1,\ldots,n-1\}})$. Assume g' has one component. Then Player n has at least one link, $d_n(g') \geq 1$. Since Player n should have at least the same payoff in g' as in $g^{\{1,\ldots,n-1\}}$, sine otherwise Player n does not join in making a link, Player n should have the highest number of links in g', $d_n(g') = c_n(g')$. We denote this degree by k. To move from $g^{\{1,\ldots,n-1\}}$ to g', the following holds true. Player n makes at least k links to k distinct players, taking k steps. Since ceteris paribus this gives those k players a degree equal to $n-1$, they should all reduce their degree by at least $n-k-1$ to ensure Player n has the highest degree, which in total takes at least $k(n-k-1)/2$ steps. The other $n-k-1$ players should all reduce their degree by at least $n-k-2$ to ensure Player n has the highest degree, which in total takes at least $(n-k-1)(n-k-2)/2$ steps. The total number of steps involved in moving from $g^{\{1,\ldots,n-1\}}$ to g' is therefore greater than or equal to

$$k + \frac{k(n-k-1)}{2} + \frac{(n-k-1)(n-k-2)}{2} = \frac{n^2-(k+2)n+4k+1}{2}.$$

In case $n = 2$, it holds that $k = 1$, so the expression above is equal to 1. In case $n = 3$, it holds that $k = 2$, so the expression above is equal to 2. In both cases, it holds that it takes at least $n-1$ steps to go from network $g^{\{1,\ldots,n-1\}}$ to network g'. Consider the case $n \geq 4$. The expression above is minimized by taking $k = n-1$, the largest value possible for k. Substituting $k = n-1$ and simplifying shows that the expression above is equal to $n-1$. So, in all cases, at least $n-1$ steps are needed to go from $g^{\{1,\ldots,n-1\}}$ to g', showing that there is no network g' in $f_{n-2}(g^{\{1,\ldots,n-1\}})$ with a single component. □

Lemma 8. *For criminal networks it holds for every $g' \in f_{n-2}(g^{\{1,\ldots,n-1\}})$ that $P(g') = \{\{1,\ldots,n-1\},\{n\}\}$.*

Proof. Consider the criminal group of Player 1 in g'. We show that it contains all players, with the exception of Player n. Suppose it contains only $k \leq n-2$ players from the set $\{1,\ldots,n-1\}$. Then those k players have to cut their all their links with all other players in $\{1,\ldots,n-1\}$. This involves at least $k(n-1-k)$ steps. For fixed n, the concavity of $k(n-1-k)$ in k implies that it is minimized at $k = 1$ or $k = n-2$

at both $k = 1$ and $k = n - 2$. Substitution of these values of k shows the minimum to be equal to $n - 2$. When the k players cut all their links with all other players in $\{1, \ldots, n-1\}$, all the players in $\{1, \ldots, n-1\}$ are strictly worse off, since the probability of being caught has strictly increased, contradicting $g' \in f_{n-2}(g^{\{1,\ldots,n-1\}})$. Consequently, the criminal group of Player 1 in any $g' \in f_{n-2}(g^{\{1,\ldots,n-1\}})$ contains all players in $\{1, \ldots, n-1\}$. Lemma 7 now implies it does not contain Player n. □

Lemma 9. *For criminal networks it holds for every* $g' \in f_{n-2}(g^{\{1,\ldots,n-1\}})$ *that* $(n-1)/2 < \#\{i \in \{1, \ldots, n-1\} \mid d_i(g') = c_i(g')\} \leq n - 2$.

Proof. Consider some $g' \in f_{n-2}(g^{\{1,\ldots,n-1\}})$. We know that the players in $\{1, \ldots, n-1\}$ are connected and Player n is a singleton. Let k be the number of players in g' with $d_i(g') = c_i(g')$. It holds that $k \leq n-2$, since otherwise the $Y_i(g') \leq Y_i(g^{\{1,\ldots,n-1\}})$ for all $i \in N$ due to the increased probability of being caught. So there are players in $\{1, \ldots, n-1\}$ who get payoff 0 in g'. These players are never willing to destroy a link. They should be outnumbered by the players with maximum degree, to ensure that they all have degree strictly below the maximum degree. It follows that $k > (n-1)/2$. □

Surprisingly, for n sufficiently high, $\tilde{f}_{n-2}(g^{\{1,\ldots,n-1\}}) \neq \emptyset$. For instance, let $n = 300$ and $\phi < 14850/199 \approx 74.62$. Notice that $d_1(g^{\{1,\ldots,n-1\}}) = \cdots = d_{299}(g^{\{1,\ldots,n-1\}}) = 298$ and $d_{300}(g^{\{1,\ldots,n-1\}}) = 0$. The network g' will be such that $d_1(g') = \cdots = d_{200}(g') = 297$, $d_{201}(g') = \cdots = d_{299}(g') = 296$, and $d_{300}(g') = 0$. Compared to $g^{\{1,\ldots,n-1\}}$ it is such that the link 100,200 is missing, as well as all the links $i, i+200$ and $i+100, i+200$ where $i = 1, \ldots, 99$. For Players $1, \ldots, 200$, it holds that

$$Y_i(g') = \frac{299}{300}\frac{1}{200}(1 - \frac{2}{300}\phi)B > Y_i(g^{\{1,\ldots,n-1\}}) = \frac{299}{300}\frac{1}{299}(1 - \frac{1}{300}\phi)B,$$

whereas $Y_i(g') = 0$ for Players $i = 201, \ldots, 299$. It holds that $Y_{300}(g') = Y_{300}(g^{\{1,\ldots,n-1\}}) > 0$. The farsighted improving path is constructed by first adding the link $100, 300$ to $g^{\{1,\ldots,n-1\}}$, resulting in $g_1 = g^{\{1,\ldots,n-1\}} + 100, 300$. Since $Y_{100}(g') > Y_{100}(g^{\{1,\ldots,n-1\}})$ and $Y_{300}(g') = Y_{300}(g^{\{1,\ldots,n-1\}})$, this step is feasible. Since $d_{100}(g_1) = 299 > d_i(g_1)$ for all $i \neq 100$, it holds that $Y_i(g_1) = 0$ for all $i \neq 100$. Players $1, \ldots, 99$ and Players $101, 199$ get strictly higher payoff in g' than in g_1, so are willing to delete any link. Form networks $g_2, \ldots, g_{199}$ by having Player i delete his link with Player $i+200$, and Player $i + 100$ delete his link with Player $i + 200$, where $i = 1, \ldots, 99$. Notice that the payoffs of all players, except Player 100, in the networks $g_2, \ldots, g_{199}$ are equal to 0. It holds that $d_1(g_{199}) = \cdots = d_{99}(g_{199}) = d_{101}(g_{199}) = \cdots = d_{199}(g_{199}) = 297$, $d_{100}(g_{199}) = 299$, $d_{200}(g_{199}) = 298$, and $d_{300}(g_{199}) = 1$. Next, player 200 cuts his link with Player 100, resulting in $g_{200} = g_{199} - 100, 200$. Finally, Player 300 cuts his link with Player 100, resulting in $g_{201} = g_{200} - 100, 300 = g'$. We have that $g' \in f_{201}(g^{\{1,\ldots,n-1\}})$, so 201 degrees of farsightedness are needed, much less than $n - 2 = 298$.

It also follows that for low values of the fine, less than $n - 1$ degrees of farsightedness may suffice to form the complete network.