On Time Preferences and Bargaining*

Sebastian Kodritsch†

April 15, 2014

Abstract

This paper analyses dynamically inconsistent time preferences in the seminal Rubinstein [1982] model of sequential bilateral bargaining. I consider any continuous time preferences which satisfy a weak impatience property and study multiple-selves equilibrium for sophisticated players. Employing a novel analytical approach to account for the temporal structure of equilibrium outcomes, I characterise (i) the set of equilibrium outcomes for any preference profile and (ii) the set of preference profiles for which equilibrium is unique. Previous findings for (dynamically consistent) exponential discounting carry over to any preferences which satisfy a form of present bias, where, for both players, the most costly period of delay is always the first one from the immediate present, e.g. any hyperbolic or quasi-hyperbolic discounting. In this case, the restriction to stationary equilibrium is without loss of generality for characterising the set of equilibrium surplus divisions. More generally, this is not the case, however: if there is a player who always finds a near-future period of delay sufficiently more costly than the first one, there exist equilibrium divisions and delays which rely on non-stationary threats for all off-path subgames.

Keywords: alternating offers, bargaining, time preference, impatience, discounting, dynamic inconsistency, delay, non-stationary equilibrium

JEL classification: C78, D03, D74

*This work is a generalisation of an earlier paper titled "On Time-Inconsistency in Bargaining", which was also part of my doctoral thesis at the London School of Economics.

†School of Business and Economics, Humboldt-University Berlin and WZB Berlin Social Science Center. Email: sebastian.kodritsch@gmail.com.

1 Introduction

As a mechanism to share economic surplus, bargaining is pervasive in decentralised exchange and accordingly fundamental to the economic analysis of contracts. In the absence of irrevocable commitments, time becomes a significant variable of bargaining agreements; parties may not only agree now or never, but also sooner or later. At the heart of economists' understanding of how the bargaining parties' "time preferences" shape the agreement they will reach lies the seminal work of Rubinstein [1982].[1] Explicitly formalizing the bargaining process as one where parties alternate over time in making and answering proposals, without any deadline, it reaches surprisingly sharp conclusions about how two completely informed and impatient parties share an economic surplus: under seemingly weak assumptions on the players' preferences, there is a unique subgame-perfect Nash equilibrium with the properties that (i) agreement is reached immediately, (ii) a player's "bargaining power" can be measured by her tolerance of a bargaining period's delay, and (iii) the initial proposer enjoys a strategic advantage. Moreover, this equilibrium has a simple—"stationary"—structure: whenever it is her turn in the respective role, a player always makes the very same offer and follows the very same acceptance rule, and in any round the proposer's offer equals the smallest share the respondent accepts, given that upon a rejection, roles reverse and the same property holds true.[2]

In this framework, the fundamental motive for reaching agreement is the parties' impatience as a property of their time preferences. Yet, the basic model is fully understood only for a very particular form of such impatience, namely exponential discounting.[3] In view of the large body of empirical evidence on individual intertemporal choice, which shows that people systematically violate this assumption, and of the success in applied work of alternative discounting models, most prominently the (β, δ)-model of "quasi-hyperbolic" discounting as proposed by Laibson [1997], this currently restricts the theory's applicability and raises the question of how robust the original

[1] Ståhl [1972] had pioneered a similar approach to bargaining theory but appears to have been largely ignored.

[2] The original Rubinstein [1982] model actually permits multiplicity, but only in a knife-edge case of preference profiles, which I ignore in this discussion as did the subsequent literature (indeed, that case is ruled out in all of the model's later reproductions involving the author himself: see Binmore et al. [1986], Rubinstein [1987], Osborne and Rubinstein [1990, Chapter 3], Binmore et al. [1992], Osborne and Rubinstein [1994, Chapter 7] and Rubinstein [1995]).

[3] Rubinstein [1982] works directly with preference relations, but Fishburn and Rubinstein [1982] show that his axioms imply exponential discounting (see also Osborne and Rubinstein [1990, Section 3.3]).

conclusions are to various other forms of impatience that have been documented.[4]

I address these issues here by extending the basic Rubinstein [1982] model to any continuous time preferences satisfying a weak impatience property. More specifically, I study any continuous preferences over bargaining agreements represented by some function $U_i(x_i, t)$ for a player i, where x_i is i's share of the surplus in an agreement and t is its delay, such that a greater share is always preferred and i is impatient in the following sense: (i) for any share, later is not preferred over sooner, (ii) a positive share now is preferred over the same share later, and (iii) in the limit as the delay approaches infinity, if the present utility value does not vanish for all agreements, there is a finite delay after which further delay is costless.

Such preferences of a player i are dynamically consistent if and only if $U_i(x_i, t) = \delta^t \cdot u(x_i)$ for some $\delta \in (0, 1)$ and $u : [0, 1] \to \mathbb{R}$, which is exponential discounting. Hence my innovation in this paper is the analysis of dynamically inconsistent time preferences. What makes this analysis challenging is the need to account for the temporal structure of the set of equlibrium agreements.[5] I achieve this here via a novel analytical approach to the alternating-offers protocol reminiscient of the framework proposed by Abreu [1988] that introduced simple strategies and proved existence of *optimal penal codes* as simple strategies for the theory of repeated games with (exponential) discounting, thus simplifying the history-dependence of (non-stationary) equilibrium strategies to consider.[6]

The central results of this analysis are (i) a characterisation of equilibrium outcomes for any preference profiles and (ii) a characterisation of those preference profiles that yield equilibrium uniqueness.[7] These generate the following main insights. First, the aforementioned celebrated

[4] Frederick et al. [2002] survey a large number of mostly psychological studies from as early as the 1970s and, referring to the exponential-discounting model as the "DU model", conclude that "virtually every assumption underlying the DU model has been tested and found to be descriptively invalid in at least some situations" (p. 352), with hyperbolic discounting as "the best documented DU anomaly" (p. 360). Recently, there has been a surge of interest by experimental economists in the study of time preferences, with the result that previous findings have been somewhat qualified with respect to the domain of choice and the elicitation method used (see for example the discussions in Attema [2012] and Augenblick et al. [2013]).

[5] Existing analytical approaches exploit recursions on *payoff* extrema; see the survey of Binmore et al. [1992] for two alternative approaches to proving the Rubinstein [1982] results, which culminated in the formulation of Merlo and Wilson [1995] in their extension to stochastic environments. When preferences are dynamically inconsistent, the possibility of equilibrium delay through history-dependent behaviour means that payoffs alone do not encode sufficient information for such a recursion, whence those approaches become inadequate.

[6] I am deeply grateful to Can Çeliktemur for pointing out this similarity to me.

[7] When some player's preferences are dynamically inconsistent, the equilibrium concept has to take a stance

conclusions found under exponential discounting are robust to various forms of *present bias*, where, whenever an immediate reward and an indifferent delayed reward are shifted into the future by one period, the decision maker is at least as patient, i.e. the indifference may only be broken in favour of the larger later reward. Because present bias is in particular a property of hyperbolic and quasi-hyperbolic discounting, this result opens the door to the use of non-cooperative bargaining theory in applied economic modelling studying such preferences.

Second, for preferences which—in violation of present bias—exhibit greater impatience about a *near-future* period's delay than the present period, there emerges a novel form of (non-stationary) equilibrium delay in reaching agreement, where to avoid such a particularly costly near-future period of delay to some given agreement as a respondent, such a player is willing to accept shares which are low enough to present a threat which is in turn sufficiently severe to induce the same player to indeed favour that delayed agreement as a proposer (facing one period of delay less). Interestingly, and in contrast to previously proposed delay equilibria, such equilibrium can be constructed without the use of stationary equilibrium in any subgame and in this sense is "purely non-stationary". A qualitative property of time preferences which permits this type of delay has very recently been documented for a large proportion of participants in a number of experimental studies that study monetary trade-offs within distinctively short horizons and may therefore be particularly relevant for bargaining applications; in loose graphic terms for separable time preferences (discounting), it is that the discounting function is initially concave, dropping most sharply over some near-future period rather than the first period of delay.[8]

regarding the conflict that exists within that player's own objectives across time. The concept I employ here assumes that the players always correctly predict their opponents' as well as their own behaviour, which they then take as given in evaluating the different actions that are currently available. Chade et al. [2008] analyze repeated games with (β, δ)-discounting using the very same concept. It is the natural extension to dynamic strategic environments with multiple persons of *Strotz-Pollak equilibrium*, also known as *multiselves equilibrium*, which was pioneered by Strotz [1955-1956] and Pollak [1968] and further developed by Peleg and Yaari [1973] and Goldman [1980] for dynamic single-person decision problems. Technically, this equilibrium's defining property is robustness against one-stage deviations; the one-stage deviation principle (e.g. see Fudenberg and Tirole [1991, Theorem 4.2]) here ensures equivalence with subgame-perfect Nash equilibrium under exponential discounting.

[8]The survey sections of Attema [2012, p. 1390] and Olea and Strzalecki [2014, pp. 23-24] list several references for this finding called "increasing impatience" and "future bias". Besides these, also a number of field studies in development economics, investigating the effects of providing various commitment opportunities on savings, conducted surveys to elicit time preferences where the answers of significant proportions of participants showed patterns of this type (e.g. Ashraf et al. [2006], Brune et al. [2013], Dupas and Robinson [2013] and Giné et al. [2013]). Bleichrodt et al. [2009], Takeuchi [2011] and Pan et al. [2013] propose discounting models that can rationalise such choices.

Third, the approach of this paper highlights the generalisability of the Abreu [1988] framework to other stochastic games satisfying some form of stationarity beyond repeated games. This may provide a useful starting point in further related theoretical analyses.

The paper proceeds as follows: section 2 presents the model, with a focus on the preferences and the equilibrium concept studied here. Section 3 first explains how the analytical approach used in the related literature on bargaining with exponential discounting fails to be feasible when preferences are dynamically inconsistent and introduces some notation, then informally develops and formally exploits a more general approach based on the framework of Abreu [1988] in the repeated-games context leading to the first main result which is the characterisation of equilibrium outcomes for any preference profile, then presents the second main result which is the characterisation of those preference profiles for which equilibrium is unique together with simpler sufficient conditions on individual preferences in isolation and ends by providing an simple example of time preferences which imply unbounded delay in equilibrium. Section 4 briefly discusses the implications of various more or less recent alternative models of time preferences inspired by empirical evidence, and section 5 concludes the paper by summarising its contribution and pointing out some questions that it raises. Relations to the existing literature are made throughout.

2 Model

I follow Rubinstein [1982] exactly with regards to the bargaining protocol of indefinitely alternating offers and will therefore describe this part of the model only informally, focussing instead on the generalisation of preferences and the equilibrium concept investigated here.

Bargaining Protocol and Strategies There are two players $\{1, 2\} \equiv I$ who bargain over a perfectly divisible surplus of (normalised) size one. In each round $n \in \mathbb{N}$ one of them proposes a surplus division $x \in \{(x_1, x_2) \in \mathbb{R}_+ | x_1 + x_2 = 1\} \equiv X$ to the other who then responds by either accepting or rejecting the proposal. If it is accepted, the game ends in agreement on x; otherwise, one period of time elapses until the subsequent round takes place, with the roles of proposer and respondent reversed. This process of alternating offers begins with player 1's proposal and continues until there is agreement, possibly without ever terminating.

A history of play to the beginning of round $n \in \mathbb{N}$, denoted h^{n-1}, is a sequence of $n-1$ rejected proposals, and a strategy σ_i of a player i assigns to every possible history $h^{n-1} \in X^{n-1}$ an available action, where $h^0 = X^0 = \emptyset$. For instance, if n is odd, player 1's strategy assigns to such a history h^{n-1} a proposal $\sigma_1(h^{n-1}) = x$, and player 2's strategy specifies for every possible such proposal whether she accepts or rejects it; while, in general, $\sigma_2(h^{n-1})$ is therefore an "acceptance rule" which partitions X into a subset of accepted proposals and its complement of rejected proposals, for the purposes of this paper it is without loss of generality to restrict attention to "threshold rules" described by a single number $q \in [0,1]$ where for instance $\sigma_2(h^{n-1}) = q$ means that player 2 accepts a proposal x by player 1 if and only if $x_2 \geq q$. A particularly simple strategy of a given player is one where in any round that player makes the same proposal and follows the same acceptance rule irrespective of the history of play; if σ_i is such a *stationary strategy*, it is then characterised simply by a pair $(x,q) \in X \times [0,1]$, where $x \in X$ is the division that i offers as the proposer and $q \in [0,1]$ specifies the threshold for i's acceptance as the respondent.

Outcomes and (Time) Preferences Any pair of strategies $\sigma = (\sigma_1, \sigma_2)$ induces an outcome in an obvious way. Players will be assumed to care only about the division of the surplus and the delay of an eventual agreement, whence the preference-relevant such outcomes are either of the form $(x,t) \in X \times T \equiv A$, $T \equiv \mathbb{N}_0$, where agreement on division x is reached with a delay of t periods, or (perpetual) disagreement, denoted simply by D. After any (non-terminal) history of play, in terms of relative time (delay), the set of feasible outcomes $A \cup \{D\}$ is therefore identical. The focus of this paper is on the players' time preferences over this domain of feasible bargaining outcomes, which satisfy the following assumption, where I find it notationally convenient to let $A_i \cup \{D\}$, $A_i \equiv [0,1] \times T$, be a player i's set of personal feasible outcomes and A_i is obtained from A as projecting any $x \in X$ to i's own surplus share $q = x_i \in [0,1]$; due to the fact that $x_1 + x_2 = 1$, this is without loss of generality.

Assumption 1. *In any round $n \in \mathbb{N}$, player $i \in I$ has preferences over personal feasible outcomes $A_i \cup \{D\}$ are represented by a utility function $U_i : A_i \cup \{D\} \to \mathbb{R}$ such that the following properties hold for any $(q,q') \in [0,1]^2$, any $(t,t') \in T^2$ and any $k \in \mathbb{R}$:*

(1) Disagreement is worst: $U_i(q,t) \geq U_i(D)$,

(2) Continuity: $\{z \in A_i \cup \{D\} | U_i(z) \geq k\}$ *and* $\{z \in A_i \cup \{D\} | U_i(z) \leq k\}$ *are closed,*

(3) Desirability: $q > q'$ *implies* $U_i(q,t) > U_i(q',t)$,

(4) Impatience:

(i) $t > t'$ *implies* $U_i(q,t) \leq U_i(q,t')$,

(ii) $q > 0$ *implies* $U_i(q,0) > U_i(q,1)$, *and*

(iii) if $\lim_{t\to\infty} U_i(1,t) > U_i(0,0)$, *there exists a* $\hat{t} \in T$ *such that* $\lim_{t\to\infty} U_i(q,t) = U_i(q,\hat{t})$.

This class of preferences covers all main models of time preferences that have been put forward in the literature; in particular, it generalises the most widely studied class of separable time preferences, where $U_i(q,t) = d(t) \cdot u(q)$ and d is a decreasing "discounting" function, as axiomatised by Fishburn and Rubinstein [1982, Theorem 1], to also cover non-separable time preferences such as those put forward by Benhabib et al. [2010] and Noor [2011].[9] While property (1) reflects once more the consequentialist nature of preferences, properties (2) and (3) are elementary preference properties, and property (4) formalises a rather weak notion of impatience toward agreement: for any given division of the surplus, players do not prefer agreeing on it later over sooner (i), if a division yields them a positive share, players prefer immediate agreement over delayed agreement on it (ii), and either they become "infinitely impatient" as the delay approaches infinity—the standard case which guarantees "continuity at infinity"—or they are impatient only about a finite number of periods of delay (iii).

Halevy [2012] shows that, given that the preferences studied here are dynamically consistent if and only if they satisfy the stationarity axiom. The latter requires that the preference over two delayed rewards (q,t) and (q',t') depends only on the relative delay, i.e. $U_i(q,t) \geq U_i(q',t')$ if and only if $U_i(q,t+\tau) \geq U_i(q',t'+\tau)$ for any $\tau \in T$, and imposing this property yields exponential discounting, where $U_i(q,t) = \delta^t \cdot u(q)$ for some $\delta \in (0,1)$ and some continuous increasing function u. Hence, with the exception of exponential discounting preferences, for which Rubinstein

[9] Ok and Masatlioglu [2007] propose a theory of relative discounting which relaxes transitivity for comparisons across different delays and accommodates the procedural models suggested by Read [2001a] and Rubinstein [2003a]. While assumption 1 imposes such transitivity, section 4.3 argues that the characterisation results obtained are more general and also apply to any preferences that fall under their theory. Another procedural model of intertemporal choice is found in Manzini and Mariotti [2007]; their model violates impatience, however, which means it is not covered here.

[1982] developed his bargaining theory, all of the time preferences studied here are dynamically inconsistent.

Equilibrium Concept I assume that the players' preferences are common knowledge. In particular, and employing the recent terminology proposed by O'Donoghue and Rabin [1999], players are sophisticated about their own as well as their opponent's dynamic inconsistency. The equilibrium concept has to incorporate how this intertemporal conflict in a player's preferences is resolved. For single-person decision problems, the standard solution concept for such sophisticated decision makers is that of Strotz-Pollak equilibrium (named after the pioneering contributions in this area of Strotz [1955-1956] and Pollak [1968]), also known as multiple-selves equilibrium (see e.g. Piccione and Rubinstein [1997]), which is the subgame-perfect Nash equilibrium of an auxiliary game in which the decision-maker at any point in time is a distinct non-cooperative player. Technically, therefore, one then looks for strategies which are robust to "one-stage deviations", and this formalises the presumption that a dynamically consistent decision-maker cannot "intrinsically" commit to future behaviour.[10]

The equilibrium notion here is the natural extension of this concept to multiple-player games. In its definition provided below, $z_i^h(\sigma)$ denotes a player i's outcome in $A_i \cup \{D\}$ that obtains if, following history h, players adhere to strategy profile σ; e.g. if player j is the respondent in some round n following a beginning-of-round history h^{n-1} and facing a proposal x, then $h = (h^{n-1}, x)$, and if $\sigma_j(h^{n-1}) = q$ with $q_j \leq x_j$, then $z_j^h(\sigma) = (x_j, 0)$.

Definition 1. A strategy profile $\sigma = (\sigma_1, \sigma_2)$ is a multiple-player Strotz-Pollak equilibrium ("equilibrium") if, for any round $n \in \mathbb{N}$ and history $h^{n-1} \in X^{n-1}$ to the beginning of this round,

$$\begin{aligned} U_i\left(z_i^{h^{n-1}}(\sigma)\right) &\geq U_i\left(z_i^{\left(h^{n-1},x\right)}(\sigma)\right) \\ U_j\left(z_j^{\left(h^{n-1},x\right)}(\sigma)\right) &\geq U_j\left(z_j^{\left(h^{n-1},x,\sigma_j\left(h^{n-1},x\right)\right)}(\sigma)\right) \end{aligned}$$

for any division $x \in X$ and $\{i, j\} = I$ with $i = 1$ if n is odd and $i = 2$ if n is even.

[10] There exists a relatively small literature that has developed refinements of this concept, see Kocherlakota [1996], Asheim [1997] and Plan [2010].

Observe that this indeed defines the subgame-perfect Nash equilibrium of the auxiliary game where the set of players is taken to be $I \times \mathbb{N}$. The well-known one-stage deviation principle (e.g. Fudenberg and Tirole [1991, Theorem 4.1]) says that such equilibrium coincides with the subgame-perfect Nash equilibrium of the actual game played by I whenever both players' preferences satisfy exponential discounting. In what follows, "equilibrium" will refer to the above definition.

Final Remarks on the Model It is worthwhile to emphasise here the focus of the paper and distinguish it from other bargaining theories. The model's central premise is that parties are impatient about reaching a bargaining agreement and consequentialist in their preferences. While impatience about enjoying the fruits of agreement can plausibly be expected to be prevalent in any bargaining problem, consequentialism means a theoretical abstraction which allows to focus on the role of impatience alone. In particular, the latter means that the players' preferences do not intrinsically respond to observed past bargaining behaviour as in Fershtman and Seidmann [1993], Compte and Jehiel and Li [2007] where the bargaining parties come to prefer impasse over any agreement that yields less than they could have obtained from an earlier rejected proposal (where "less" refers to an agreement's material terms in the two former models and to its discounted utility terms in the latter).[11]

Moreover, I assume common knowledge of preferences, so players are fully aware of their opponent's as well as their own (future) preferences; there is neither private information about preferences which is the most prominent explanation for delay in the literature (see Kennan and Wilson [1993]) nor naiveté about future preferences (as investigated by Akin [2007] for (β, δ)-discounting preferences under a certain stationarity restriction on equilibrium). Given the generality of the model in terms of time preferences and the strategy space considered, this is a natural starting point. While the basic mechanisms of signalling and screening identified in bargaining under incomplete information can be expected to be orthogonal to the interest in impatience per se, naiveté may yield different implications depending on the type of dynamic inconsistency (e.g. contrast present-biased (β, δ)-discounting with the preferences of the example in section 3.4).

[11] Rejecting a somewhat generous offer by the opponent then buys commitment similar to the role of history-dependent outside options as in Compte and Jehiel [2004].

3 Results

3.1 Preliminaries

An important property of the game studied here is its stationarity: all subgames beginning with a given player i's proposal are identical; denote this game by G_i. Consequently, their respective sets of equilibria are identically equal to those of game G_i, in particular the proposing players' infimal and supremal equilibrium payoffs.

With exponential discounting, two rounds of simple backwards-induction recursions on these extremal payoffs results in the very same extreme values and therefore characterises them as fixed points; the underlying temporal structure of equilibrium outcomes is irrelevant.[12] Moreover, in order to characterise the set of equilibrium bargaining divisions, it is without loss of generality to restrict attention to stationary equilibria, which are also used to establish that the extrema of the set of equilibrium payoffs obtained from the fixed point problem are themselves equilibrium payoffs.

In contrast, this procedure is not feasible with dynamically inconsistent preferences. To illustrate, suppose player 1 has (β, δ)-discounting preferences with linear instantaneous utility and her infimal equilibrium payoff as the initial proposer equals some v_i, where the maximal equilibrium delay is one period and the infimal equilibrium payoffs are x_i among immediate-agreement equilibria and $\beta\delta x_i'$ among delay equilibria, respectively, so $v_i = \min\{x_i, \beta\delta x_i'\}$. Then i's infimal rejection equilibrium value—her infimal equilibrium "threat point"—equals $\min\{\beta\delta x_i, \beta\delta^2 x_i'\} = \beta\delta \cdot \min\{x_i, \delta x_i'\}$ which is $\beta\delta v_i$ if $x_i \leq \beta\delta x_i'$, $\beta\delta x_i$ with $\beta\delta v_i < \beta\delta x_i < \delta v_i$ if $\beta\delta x_i' < x_i < \delta x_i'$, and δv_i if $\delta x_i' \leq x_i$. Hence, knowledge of v_i alone is insufficient to even execute the first recursion unless delay is ruled out beforehand.[13] This observation leads to a novel approach which, by necessity, directly deals with the possibility of equilibrium delay.

A first result is immediate, however: perpetual disagreement D is not an equilibrium outcome. Due to the players' impatience, there exist proposals which the respondent must accept even if he

[12] Shaked and Sutton [1984] first demonstrated the effectiveness of this approach; for variations see for instance Binmore et al. [1992] and Merlo and Wilson [1995].

[13] This is in fact the flaw in the uniqueness proof of Lu [2006] for such preferences; his claim of equilibrium uniqueness is true, however, as corollary 1 in section 3.3 shows.

were to otherwise obtain the maximal feasible rejection value and which still leave a positive share for the proposer; this implies that an initial proposer i's infimal equilibrium payoff strictly exceeds $u_i(0)$ and, since D is the worst possible outcome, also $U_i(D)$.

The (short) proof of this claim uses the following definitions for each $i \in I$, which will be prominent also in the subsequent analysis. First, define an instantaneous utility function $u_i : [0,1] \to \mathbb{R}$ such that $u_i(q) \equiv U_i(q,0)$ for any $q \in [0,1]$, which is continuous and increasing by assumption 1, properties (2) and (3), respectively. Second, let $\pi_i : U_i(A_i \cup \{D\}) \to [0,1]$ with

$$\pi_i(U) \equiv \min \{q \in [0,1] \,|u_i(q) \geq U\}$$

be player i's minimal acceptable share for rejection value U, which is non-decreasing and continuous. Finally, in all that follows, for any given player $i \in I$, j will denote the opponent, i.e. $j = 3 - i$

Lemma 1. *All equilibrium outcomes are agreement outcomes.*

Proof. By contradiction. Suppose D were an equilibrium outcome of G_i, $i \in I$, and consider its first round. In any equilibrium, respondent j accepts any offered share $x_j > \pi_j(U_j(1,1))$ because $U_j(1,1)$ the maximal possible rejection value. Proposing division x such that $\pi_j(U_j(1,1)) < x_j < 1$, which exists due to j's impatience, i obtains utility $u_i(x_i) > u_i(0) \geq U_i(D)$. □

The remaining part of this subsection introduces further notation. In view of lemma 1, for each player $i \in I$, let $A_i^* \subseteq A_i$ be the set of i's personal feasible equilibrium outcomes of G_i and define the following extrema on the basis of A_i^*, which are i's infimal equilibrium payoff as the initial respondent in G_i, v_i^*, i's infimal equilibrium threat point as the initial respondent in G_j, w_i^*, and the supremal equilibrium delay in G_i, t_i^*:

$$\begin{aligned} v_i^* &\equiv \inf \{U_i(q,t) \,|\, (q,t) \in A_i^*\} \\ w_i^* &\equiv \inf \{U_i(q,t+1) \,|\, (q,t) \in A_i^*\} \\ t_i^* &\equiv \sup \{t \in T | \exists q \in [0,1],\ (q,t) \in A_i^*\}. \end{aligned}$$

It will also be useful to define a function $\phi_i : u_i([0,1]) \times T \to [0,1]$ such that

$$\phi_i(u,t) \equiv \max\{q \in [0,1] \,|\, u \geq U_i(q,t)\}$$

which gives the maximal share with delay t that is not preferred over instantaneous utility u and is non-decreasing in both u as well as t and continuous in u; note that whenever $\phi_i(u,t) < 1$, it is in fact the delay-t share which is utility-equivalent to u. Finally, I will employ a function $\kappa_i : T \times U_i(A_i) \times U_i(A_i) \times U_j(A_j) \to \mathbb{R}_+$ such that

$$\kappa_i(t, v_i, v_j, w_j) = \begin{cases} \phi_i(v_i,t) + \pi_j(w_j) & t = 0 \\ \phi_i(v_i,t) + \max\{\phi_j(v_j, t-1), \phi_j(u_j(0), t)\} & t > 0 \end{cases}$$

for the surplus cost of delay t in G_i with immediate-agreement value v_k for proposer k, any $k \in I$, and rejection value w_j for (initial) respondent j. The terminology will become clearer when recognising κ_i in the central formal results which follow.

3.2 Equilibrium

The work of Abreu [1988] presented an analytical framework for repeated games with (exponential) discounting, which showed that the history-dependence of "off-path" equilibrium strategies is tractable: in terms of the most extreme credible threats, or "punishment" equilibrium paths, which support the game's entire set of equilibrium paths—this is called an "optimal penal code"—it is sufficient to look for one such path per player, for *any* deviation of that player; there is a "simple penal code" which is optimal. Note that if in some repeated game this simple penal code which is optimal were to consist only of stationary equilibrium paths, this would mean that upon any deviation by a player, it would specify infinite repetition of that player's least preferred Nash equilibrium of the stage game in the ensuing subgame.

The approach taken here is an application of this idea to alternating-offers bargaining with the preferences described by assumption 1. It is worthwhile pointing out an important difference between the bargaining game investigated here and repeated games: not all deviations can be punished, since acceptance of a proposal terminates the game.[14] Consequently, there are only types of deviations from a specified path which may be punished: (1) a proposer's making a deviating offer that is rejected and (2) a respondent's rejection of an offer which she is supposed to accept. Moreover, also the fact that payoffs are realised only upon agreement limits the "richness" of available punishments. As it turns out, however, how rich the space of available punishments—hence the space of equilibrium paths—actually is, depends on the bargaining parties' time preferences. Before the formal statements, which summarise the results of my approach, I briefly sketch the reasoning that underlies it.

To begin with, it is intuitively rather clear that one can identify paths with outcomes in this game as follows: an outcome $(\hat{x}, t)$ is an equilibrium outcome of G_i, $i \in I$, if and only if the path $(1_i^t, \hat{x})$ is an equilibrium path of G_i, where $1_i^t \in X^t$ is the history to the beginning of round $t+1$ in which each previous proposer $k \in I$ proposed division $1_k \equiv x$ such that $x_k = 1$, starting with $k = i$. This reflects the basic property of this theory of bargaining that how parties fail to agree is irrelevant in equilibrium: offering the opponent a positive share when it is understood to be rejected essentially means not to offer anything. Hence, it is without loss of generality to consider only equilibrium paths of this form.

The next observations concern the proposer's strategic advantage, which is typically invoked in intuitions about sequential bargaining equilibria.[15] Its essence is that any equilibrium payoff a player may obtain as the initial proposer is her payoff in an immediate-agreement equilibrium, in particular the player's infimal equilibrium payoff v_i^* can be obtained from the set of immediate-agreement-equilibrium payoffs. This provides an important clue in relating the extreme credible punishments across a given player's roles as proposer and respondent trough impatience: in partic-

[14] Busch and Wen [1995, p. 547], in making a similar comparison and introducing their intuition for equilibrium uniqueness in Rubinstein [1982], observe that "(...) history-dependent strategies do not have the same power in Rubinstein bargaining (...). The reason is that a player will always accept any offer which yields at least as much as rejecting it, since acceptance cannot be punished." Before multiple equilibria and delay are ruled out, this is misleading, however, because a responding player's equilibrium strategy may reject off-path offers which are more worth more than the (delayed) equilibrium outcome. The reason is that the respondent would be rewarded with a rejection value sufficiently above the equilibrium outcome's value in order to constitute a punishment to the proposer that deters the latter from making such an offer in the first place despite the greater overall surplus of this immediate agreement relative to the equilibrium outcome.

[15] For instance, Kreps [1990, p. 564]—referring to equilibrium uniqueness result of Rubinstein [1982] under exponential discounting—states: "The uniqueness result is not quite intuitive or obvious, but you should be convinced that what drives these equilibria is that each party when it is making an offer is able to put the onus of waiting entirely on the other side."

ular, a proposing player i's most extreme punishment in the subsequent round's game G_j, following i's deviation, is an equilibrium of G_j in which i immediately accepts y such that $x_i = \pi_i(w_i^*)$. A rather straightforward argument then establishes that

$$v_i^* = u_i\left(1 - \pi_j\left(U_j\left(1 - \pi_i\left(w_i^*\right), 1\right)\right)\right). \tag{1}$$

Moreover, and related to the proposer's strategic advantage, for equilibrium delay to arise, say agreement on division x after t periods in G_i (in the $t+1$-th round), each player as a proposer along the path before the agreement round must be deterred from making an immediately accepted offer to the respondent. Given the players' impatience and the stationarity of the game, a threat that achieves this the first time around that player i proposes is necessary and sufficient for successful deterrence—and thus delay—in on-path stages where i proposes, and similarly for j. Therefore (x,t) must satisfy

$$\begin{aligned} x_i &\geq \phi_i\left(v_i^*, t\right) \\ x_j &\geq \begin{cases} \pi_j\left(w_j^*\right) & t = 0 \\ \phi_j\left(v_j^*, t-1\right) & t > 0, \end{cases} \end{aligned}$$

and, upon replacement of $\phi_j\left(v_j^*, t-1\right)$ by $\max\left\{\phi_j\left(v_j^*, t-1\right), \phi_j\left(u_j(0), t\right)\right\}$ in the second inequality for $t > 0$, this turns out to suffice for (x,t) to be an equilibrium outcome of G_i: the proposer's strategic advantage is "overwhelming" in the sense that once the players' incentives to delay are put in place at their respective proposer stages, they do not have a profitable deviation as a respondent either.

This yields two further insights and as many equations (per player): first, the initial proposer i is indifferent across all delayed outcomes that are each the worst for the given delay; each of them is worth v_i^*, and, by mere impatience, therefore

$$w_i^* = \inf\left\{U_i\left(\phi_i\left(v_i^*, t\right), t+1\right) \middle| t \in T,\ t \leq t_i^*\right\}. \tag{2}$$

Second, given a delay t, in order for a division x such that (x,t) is an equilibrium outcome of G_i to exist, it is necessary and sufficient that

$$\kappa_i\left(t, v_i^*, v_j^*, w_j^*\right) \leq 1;$$

otherwise at least one of the players who get to propose on the path to (x,t) before the agreement stage would prefer the immediate agreement based on the most extreme threat, worth v_k^* to a player k, over waiting the remaining number of periods for agreement on division x. In other words, the surplus of size one is too small to be able to meet incentive constraints at the proposal stages. But this means that the supremal delay in G_i is determined by

$$t_i^* = \sup\left\{t \in T | \kappa_i\left(t, v_i^*, v_j^*, w_j^*\right) \leq 1\right\}. \tag{3}$$

The set of equations 1 through 3, each for both $i \in I$, characterises equilibrium, where in particular impatience property (iii) of preference assumption 1 ensures that each w_i^* is in fact a minimum based on an actual continuation equilibrium outcome.

The next lemma formalises these insights into this paper's central tool which is a system of equations that implicitly provides the structure of a "simple penal code" in this bargaining game in analogy to what Abreu [1988] proposed for repeated games.

Lemma 2. *Suppose the values* $(v_k, w_k, t_k)_{k\in I}$ *solve*

$$v_i = u_i\left(1-\pi_j\left(U_j\left(1-\pi_i\left(w_i\right),1\right)\right)\right) \tag{4}$$

$$w_i = \min\left\{U_i\left(\phi_i\left(v_i,t\right),t+1\right) | t \in T,\ t \leq t_i\right\} \tag{5}$$

$$t_i = \sup\left\{t \in T | \kappa_i\left(t, v_i, v_j, w_j\right) \leq 1\right\} \tag{6}$$

for both $i \in I$. *Then every outcome* $(x,t) \in A$ *such that*

$$\phi_i\left(v_i,t\right) \leq x_i \leq \begin{cases} 1-\pi_j\left(w_j\right) & t=0 \\ 1-\max\left\{\phi_j\left(v_j,t-1\right),\phi_j\left(u_j\left(0\right),t\right)\right\} & t>0 \end{cases}$$

for $i \in I$ is an equilibrium outcome of G_i.

Proof. Take such a solution $(v_k, w_k, t_k)_{k \in I}$ and note that due to impatience, for $\{i,j\} = I$, $u_j(0) < v_j$ from equation 4, implying $\phi_j(v_j, t) > 0$ for all $t \in T$, so equation 6 yields that whenever $0 < t \leq t_i$, it holds that $\phi_i(v_i, t) < 1$ and hence $U_i(\phi_i(v_i, t), t) = v_i$. Also observe that $w_i \leq U_i(\phi_i(v_i, 0), 1) < U_i(\phi_i(v_i, 0), 0) = v_i$, where the first (weak) inequality follows from equation 5 and the second (strict) inequality follows from $\phi_i(v_i, 0) > 0$ and impatience.

Tthe following main step of the proof shows that, for each $i \in I$, any outcome $\left(x^{(i)}, t^{(i)}\right)$ where $t^{(i)}$ solves equation 5 and $x_i^{(i)} = \phi_i\left(v_i, t^{(i)}\right)$ is an equilibrium outcome of G_i, and it does so by means of a recursive equilibrium construction. To simplify the exposition, I first assume existence of a strategy profile in G_i which supports outcome $\left(x^{(i)}, t^{(i)}\right)$ as an equilibrium outcome of G_i for each $i \in I$ to then recursively construct it, thus actually verifying existence. Suppose then that, for each $i \in I$, $\alpha^{(i)}$ is such a strategy profile in G_i and consider strategy profile $\beta^{(i)}$ in G_j, $j = 3 - i$, such that $\beta_j^{(i)}(\emptyset) = y^{(i)}$, $\beta_i^{(i)}(\emptyset) = y_i^{(i)}$ and, for any $x \in X$, $\beta^{(i)}(\cdot|x) = \alpha^{(i)}$, where $y^{(i)}$ is defined through $y_i^{(i)} = \pi_i(w_i)$; clearly, $\beta^{(i)}$ supports $\left(y^{(i)}, 0\right)$ as an equilibrium outcome of G_j if $\alpha^{(i)}$ is an equilibrium of G_i.

Construct now a strategy profile $\sigma^{(i)}$ for G_i as follows: first, take any history 1_i^{n-1} of round $n \in \mathbb{N}$ with $n < t^{(i)} + 1$ where k and l denote the proposer and respondent, respectively of this round; then $\sigma_k^{(i)}\left(1_i^{n-1}\right) = 1_k$, $\sigma_l^{(i)}\left(1_i^{n-1}\right) = \pi_l\left(U_l\left(y_l^{(k)}, 1\right)\right)$ and $\sigma^{(i)}\left(\cdot|1_i^{n-1}, x\right) = \beta^{(k)}$ for any $x \in X$ with $x \neq 1_k$. Second, take history $1_i^{t^{(i)}}$ (of round $n = t^{(i)} + 1$), again denoting by k and l the proposer and respondent, respectively; then let $\sigma_k^{(i)}\left(1_i^{t^{(i)}}\right) = x^{(i)}$, $\sigma_l^{(i)}\left(1_i^{t^{(i)}}\right) = x_l^{(i)}$ and, for any $x \in X$, $\sigma^{(i)}\left(\cdot|1_i^{t^{(i)}}, x\right)$ equals $\beta^{(k)}$ if $x_l < x_l^{(i)}$ and $\alpha^{(l)}$ otherwise. To complete the description by recursion, set $\alpha^{(i)} = \sigma^{(i)}$ for each $i \in I$.

The path resulting under $\sigma^{(i)}$ is indeed $\left(1_i^{t^{(i)}}, x^{(i)}\right)$: while this is clear whenever $\pi_k\left(U_k\left(y_k^{(i)}, 1\right)\right) > 0$ for both $k \in I$, if $\pi_j\left(U_j\left(y_j^{(i)}, 1\right)\right) = 0$, $j = 3 - i$, then $v_i = u_i(1)$ from equation 4, whence $\phi_i(v_i, 0) = 1$ and $t_i = 0$ because of equation 6, whereas in the remaining case of $\pi_j\left(U_j\left(y_j^{(i)}, 1\right)\right) > 0$ but $\pi_i\left(U_i\left(y_i^{(j)}, 1\right)\right) = 0$, it follows similarly that $t_j = 0$ and, moreover, $t_i \leq 1$ from equation 6; therefore also in both of the these cases it is indeed path $\left(1_i^{t^{(i)}}, x^{(i)}\right)$ that results under $\sigma^{(i)}$. To complete the main step, it remains to be shown that $\sigma^{(i)}$ constitutes equilibrium in G_i; due to its recursive construction, it suffices to establish best responses for histories 1_i^{n-1}, any $n \in \mathbb{N}$ such that $n \leq t^{(i)} + 1$.

First, let $n < t^{(i)}+1$, which exists only if $t^{(i)} > 0$, and consider the respective round's respondent, denoted by $l \in I$: given any proposal $x \neq 1_k$, $k = 3 - l$, rejection is worth $U_l\left(y_l^{(k)}, 1\right)$, whereas rejecting 1_k is worth $U_l\left(x_l^{(i)}, t^{(i)} + 1 - n\right) \geq u_l(0)$, where the last inequality follows from impatience and equation 6, which altogether makes $\sigma_l^{(i)}\left(1_i^{n-1}\right) = \pi_l\left(U_l\left(y_l^{(k)}, 1\right)\right)$ a best response for any possible x; on the other hand, proposer k obtains (i) at most $u_k\left(1 - \pi_l\left(U_l\left(y_l^{(k)}, 1\right)\right)\right) = v_k$ when making an immediately accepted proposal, (ii) $U_k\left(y_k^{(k)}, 1\right) \leq u_k\left(y_k^{(k)}\right) = \max\{u_k(0), w_k\} < v_k$ from any rejected proposal not equal to 1_k, and (iii) $U_k\left(x_k^{(i)}, t^{(i)} + 1 - n\right)$ from proposing 1_k, which is no less than v_k both if $k = i$—then because $U_i\left(x_i^{(i)}, t^{(i)} + 1 - n\right) \geq U_i\left(x_i^{(i)}, t^{(i)}\right) = v_i$—and if $k = j$—since this can only arise if $n > 1$, and then, using equation 6, $U_j\left(x_j^{(i)}, t^{(i)} + 1 - n\right) \geq U_j\left(x_j^{(i)}, t^{(i)} - 1\right) \geq U_j\left(\phi_j\left(v_j, t^{(i)} - 1\right), t^{(i)} - 1\right) = v_j$, whence $\sigma_k^{(i)}\left(1_i^{n-1}\right) = 1_k$ is a best response.

Second, take $n = t^{(i)} + 1$ and again begin by considering respondent l: rejecting proposer k's offer $x_l \geq x_l^{(i)}$ is worth w_l, and this is neither greater than $u_l\left(x_l^{(i)}\right)$ if $l = i$, because then $u_i\left(x_i^{(i)}\right) = u_i\left(\phi_i\left(v_i, t^{(i)}\right)\right) \geq u_i\left(\phi_i\left(v_i, 0\right)\right) = v_i > w_i$, nor if $l = j$, since in this case, either $t^{(i)} > 0$ and equation 6 implies $x_j^{(i)} = 1 - \phi_i\left(v_i, t^{(i)}\right) \geq \phi_j\left(v_j, t^{(i)} - 1\right) \geq \phi_j\left(v_j, 0\right)$, whence $u_j\left(x_j^{(i)}\right) \geq u_j\left(\phi_j\left(v_j, 0\right)\right) = v_j > w_j$, or $t^{(i)} = 0$ and equation 6 implies $x_j^{(i)} = 1 - \phi_i\left(v_i, 0\right) \geq \pi_j\left(w_j\right)$, whence $u_j\left(x_j^{(i)}\right) \geq w_j$.

In contrast, rejecting proposer k's offer of $x \in X$ such that $x_l < x_l^{(i)}$ is worth $U_l\left(y_l^{(k)}, 1\right) = U_l\left(1 - \pi_k\left(w_k\right), 1\right)$. Now note that equation 4 implies that $\phi_k\left(v_k, 0\right) = 1 - \pi_l\left(U_l\left(1 - \pi_k\left(w_k\right), 1\right)\right)$, whence

$$\pi_l\left(U_l\left(1 - \pi_k\left(w_k\right), 1\right)\right) \geq 1 - \phi_k\left(v_k, t\right) \tag{7}$$

for any $t \in T$ due to the non-decreasingness of ϕ_k in t. For the case where $l = j$ this says in particular that $\pi_j\left(U_j\left(1 - \pi_i\left(w_i\right), 1\right)\right) \geq 1 - \phi_i\left(v_i, t^{(i)}\right) = x_j^{(i)}$, whereas for $l = i$ it must be that $t^{(i)} > 0$ and inequality 7 together with equation 6 (in this order) can be used to obtain $\pi_i\left(U_i\left(1 - \pi_j\left(w_j\right), 1\right)\right) \geq 1 - \phi_j\left(v_j, t^{(i)} - 1\right) \geq \phi_i\left(v_i, t^{(i)}\right) = x_i^{(i)}$; in any case, $\pi_l\left(U_l\left(1 - \pi_k\left(w_k\right), 1\right)\right) \geq x_l^{(i)}$, so either $U_l\left(1 - \pi_k\left(w_k\right), 1\right) \geq u_l(0)$ and therefore $U_l\left(1 - \pi_k\left(w_k\right), 1\right) \geq u_l\left(x_l^{(i)}\right) > u_l\left(x_l\right)$, or $U_l\left(1 - \pi_k\left(w_k\right), 1\right) < u_l(0)$ implying that $x_l^{(i)} = 0$ so there is no $x \in X$ with $x_j < x_l^{(i)}$.

Now consider proposer k: given l's acceptance rule, the best possible immediate agreement for k is that on $x^{(i)}$, and any proposal that is rejected yields respondent l a share at least as large with positive delay, whence it is worse for k than immediate agreement on $x^{(i)}$.

Finally, the proof that every $(x,t) \in A$ as stated is an equilibrium outcome of G_i is achieved simply by replacing $\left(x^{(i)}, t^{(i)}\right)$ with (x,t) in the above construction "on the path", and verifying that there are no profitable one-stage deviations along that path $(1_i^t, x)$ given the off-path continuation strategy profiles which have been proven to constitute equilibrium in the previous main step. □

Recall the two kinds of deviations that can be punished in the bargaining game studied here: (1) a proposer's making a deviating offer that is rejected and (2) a respondent's rejection of an offer which she is supposed to accept. Simple penal codes specify a punishment outcome which is an equilibrium for each of these, and lemma 2 provides these from the solutions to the system of equations 4-6 for any $i \in I$: if i deviates in the sense of (2) as the respondent by rejecting a proposal, then equilibrium outcome $\left(x^{(i)}, t^{(i)}\right)$ is played in the subsequently ensuing game G_i, where $t^{(i)}$ solves equation 5 and $x_i^{(i)} = \phi_i\left(v_i, t^{(i)}\right)$, and if i deviates in the sense of (1) as the proposer by making a deviating proposal that is rejected, then equilibrium outcome $\left(y^{(i)}, 0\right)$ is played in the subsequently ensuing game G_j, where $y_i^{(i)} = \pi_i\left(U_i\left(\phi_i\left(v_i, t^{(i)}\right)\right), t^{(i)}+1\right)$.

The first main theorem establishes that there is an optimal penal code which takes this form and thus, in conjunction with lemma 2, characterises the sets of equilibrium outcomes and payoffs.

Theorem 1. *The values $(v_k^*, w_k^*, t_k^*)_{k \in I}$ solve the system of equations 4-6 for both $i \in I$ such that if $(v_k, w_k, t_k)_{k \in I}$ is any (other) solution, then $v_k^* \leq v_k$, $w_k^* \leq w_k$ and $t_k^* \geq t_k$ for both $k \in I$.*

Proof. Take any $i \in I$, let $\tilde{w}_i = \inf\left\{U_i(x_i, t+1) \,|\, (x,t) \in A_i^*\right\}$ and consider G_j, $j = 3-i$: since for any $(x,t) \in A_i^*$, there exists an equilibrium of G_j where there is immediate agreement on y such that $y_i = \pi_i(U_i(x_i, t+1))$, and because of continuity as well as impatience of preferences, $\sup\left\{U_j(x_j, 0) \,|\, (x,0) \in A_j^*\right\} = u_j(1 - \pi_i(\tilde{w}_i)) = \sup\left\{U_j(x_j, t) \,|\, (x,t) \in A_j^*\right\}$, which implies also that $\sup\left\{U_j(x_j, t+1) \,|\, (x,t) \in A_j^*\right\} = U_j(1 - \pi_i(\tilde{w}_i), 1)$ because by i's impatience $1 - \pi_i(\tilde{w}_i) > 0$ must be j's supremal equilibrium share in G_j and j is impatient. This means that, for any $\epsilon > 0$, there exists an immediate-agreement outcome $(x', 0)$ which is an equilibrium outcome of G_j such that $U_j\left(x'_j, 1\right) \geq u_j(\tilde{x}_j - \epsilon)$, where $\tilde{x}_j \equiv \pi_j(U_j(1 - \pi_i(\tilde{w}_i), 1))$, and due to j's impatience, for $\epsilon > 0$ sufficiently small, $x'_j \geq \tilde{x}_j$, implying that $U_i(x'_i, 1) < u_i(\tilde{x}_i)$ by i's impatience and $u_i(\tilde{x}_i) > u_i(0)$. Hence there exists an assignment of continuation equilibria such that j's accepting a proposal $x \in X$ if and only if $x_j \geq \tilde{x}_j$ and i's offering j a share $\tilde{x}_j$ are best responses, and therefore immediate agreement on division $\tilde{x}$ is an equilibrium outcome of G_i. Because j must accept any share greater than $\tilde{x}_j$, $v_i^* \geq u_i(\tilde{x}_i)$, and therefore $v_i^* = u_i(\tilde{x}_i)$, proving that $(v_k^*, w_k^*, t_k^*)_{k \in I}$ must solve equation 4.

Next, observe that $\kappa_i\left(t_i^*, v_i^*, v_j^*, w_j^*\right) \leq 1$, $i \in I$, since otherwise, from the definition of κ_i, if $t_i^* = 0$, for any $(x, 0) \in A$, either $u_i(x_i) < u_i(\phi_i(v_i^*, 0)) = v_i^*$ or $u_j(x_j) < u_j\left(\pi_j\left(w_j^*\right)\right) = \max\left\{u_j(0), w_j^*\right\} = w_j^*$, and thus $u_j(x_j) < w_j^*$ since $x_j \geq 0$, whence $(x, 0)$ cannot be an equilibrium outcome, and if $t_i^* > 0$, for any $(x, t_i^*) \in A$, either $U_i(x_i, t_i^*) < U_i(\phi_i(v_i^*, t_i^*), t_i^*) = v_i^*$ or at least one of $U_j(x_j, t_i^* - 1) < U_j\left(\phi_j\left(v_j^*, t_i^* - 1\right), t_i^* - 1\right) = v_j^*$ and $U_j(x_j, t_i^*) < u_j(0)$, whence there cannot be an equilibrium outcome with delay t_i^* either.

Now let $(v_k^0, w_k^0, t_k^0)_{k \in I} \equiv (v_k^*, w_k^*, t_k^*)_{k \in I}$ and define a sequence $\left((v_k^n, w_k^n, t_k^n)_{k \in I}\right)_{n \in \mathbb{N}}$ such that, for each $i \in I$, $j = 3 - i$ and $n \in \mathbb{N}$, $t_i^n \equiv \sup\left\{t \in T | \kappa_i\left(t, v_i^{n-1}, v_j^{n-1}, w_j^{n-1}\right) \leq 1\right\}$, $w_i^n \equiv \inf\left\{U_i\left(\phi_i\left(v_i^{n-1}, t\right), t+1\right) | t \in T,\ t \leq t_i^n\right\}$ and $v_i^n \equiv u_i(1 - \pi_j(U_j(1 - \pi_i(w_i^n), 1)))$. The previous paragraph proved that $t_i^1 \geq t_i^0$, implying that $w_i^1 \leq w_i^0$ and thus also $v_i^1 \leq v_i^0$, whence, by construction, the sequence satisfies generally that $t_i^n \geq t_i^{n-1}$, $w_i^n \leq w_i^{n-1}$ and $v_i^n \leq v_i^{n-1}$ for each $i \in I$ and any $n \in \mathbb{N}$. Moreover, for any $n \in \mathbb{N}$, it is true that $|t_1^n - t_2^n| \leq 1$: whenever $(t_1^n, t_2^n) \neq (t_1^1, t_2^1)$, $t_i^n > 0$ for some $i \in I$ and

$$\begin{aligned}
1 &\geq \phi_i(v_i^n, t_i^n) + \max\left\{\phi_j\left(v_j^n, t_i^n - 1\right), \phi_j(u_j(0), t_i^n)\right\} \\
&\geq \phi_i(v_i^n, t_i^n) + \phi_j\left(v_j^n, t_i^n - 1\right) \\
&= \phi_j\left(v_j^n, t_i^n - 1\right) + \max\left\{\phi_i(v_i^n, t_i^n), \phi_i(u_i(0), t_i^n)\right\} \\
&\geq \kappa_j\left(t_i^n - 1, v_j^n, v_i^n, w_i^n\right),
\end{aligned}$$

for $j = 3 - i$, which implies $t_j^n \geq t_i^n - 1$; in particular, either both sequences t_i^n have a finite limit, or both become infinitely large.

The final step of the proof will establish that there exists $N \in \mathbb{N}$ such that $\left(t_1^{N+1}, t_2^{N+1}\right) =$

$\left(t_1^N, t_2^N\right)$, because then $\left(v_k^n, w_k^n, t_k^n\right)_{k\in I} = \left(v_k^{N+1}, w_k^{N+1}, t_k^{N+1}\right)_{k\in I}$ for all $n \in \mathbb{N}$ with $n \geq N+1$, and these values solve the system of equations 4-6 for any $i \in I$ and $j = 3-i$; since $t_i^{N+1} \geq t_i^*$, $w_i^{N+1} \leq w_i^*$ and $v_i^{N+1} \leq v_i^*$ for each $i \in I$, application of lemma 2 implies that $\left(v_k^*, w_k^*, t_k^*\right)_{k\in I} = \left(v_k^{N+1}, w_k^{N+1}, t_k^{N+1}\right)_{k\in I}$ and thus also that there cannot be another solution which does not satisfy all of the inequalities stated.

Suppose first that $\lim_{t\to\infty} U_i(1,t) \leq u_i(0)$ for $i \in I$. For any u with $u > u_i(0)$, there then exists $\hat{t} \in \mathbb{N}$ such that, for any $t \geq \hat{t}$, $U_i(1,t) < u$ and, consequently, $\phi_i(u,t) = 1$ hold true. In particular, this is the case for $u = u_i(1-\pi_j(U_j(1,1)))$, since $1-\pi_j(U_j(1,1)) > 0$ due to j's impatience, where $j = 3-i$; for any $n \in \mathbb{N}$, since $v_i^n \geq u_i(1-\pi_j(U_j(1,1)))$, there exists therefore $\hat{t} \in \mathbb{N}$ such that $\phi_i(v_i^n, t) = 1$ for any $t \geq \hat{t}$. Because also $\phi_j\left(v_j^n, t\right) > 0$ for any $t \in T$, it follows that $t_i^n \leq \hat{t}$ and $t_j^n \leq \hat{t}+1$ for some $\hat{t} < \infty$, $j = 3-i$, whence each sequence t_i^n has a finite limit, so there indeed exists $N \in \mathbb{N}$ such that $\left(t_1^{N+1}, t_2^{N+1}\right) = \left(t_1^N, t_2^N\right)$.

Take the alternative case where $\lim_{t\to\infty} U_i(1,t) > u_i(0)$ for both $i \in I$ and assumption 1, property (4) (iii), says that there exists a $\hat{t} \in \mathbb{N}$ such that, for any $q \in [0,1]$, $\lim_{t\to\infty} U_i(q,t) = U_i\left(q,\hat{t}\right)$. Note that each v_i^n is a non-increasing sequence bounded below by $u_i(1-\pi_j(U_j(1,1)))$, $j = 3-i$, whence it has a limiting value $\bar{v}_i \geq u_i(1-\pi_j(U_j(1,1)))$. There exists then a $\hat{t} \in \mathbb{N}$ such that

$$
\begin{aligned}
w_i^n &\geq \inf\left\{U_i\left(\phi_i\left(v_i^{n-1}, t\right), t+1\right) | t \in T\right\} \\
&\geq \inf\left\{U_i\left(\phi_i\left(\bar{v}_i, t\right), t+1\right) | t \in T\right\} \\
&= \min\left\{U_i\left(\phi_i\left(\bar{v}_i, t\right), t+1\right) | t \in T,\ t \leq \hat{t}-1\right\}
\end{aligned}
$$

for any $n \in \mathbb{N}$, whence w_i^n converges to some $\bar{w}_i \geq \min\left\{U_i\left(\phi_i\left(\bar{v}_i, t\right), t+1\right) | t \in T,\ t \leq \hat{t}-1\right\}$, and this is true for each $i \in I$. This means, however, that if the sequences t_i^n do not possess a finite limit, there exists $N \in \mathbb{N}$ such that $t_i^N \geq \hat{t}$ for both $i \in I$, and then $\left(t_1^{N+1}, t_2^{N+1}\right) = \left(t_1^N, t_2^N\right)$ must hold. $\square$

3.3 Uniqueness

Arguably, the main reason for the success of the Rubinstein [1982] model of sequential bargaining was the uniqueness of its prediction for what then was the standard model of time preferences: exponential discounting. Impatience alone, in the general class of time preferences which fall under assumption 1, does, however, neither guarantee a unique (immediate-agreement) prediction nor stationary equilibrium outcomes as optimal simple penal codes; the example of section 3.4 augments theorem 1 to prove this point. Although multiplicity of equilibrium outcomes is disconcerting in principle, it may only arise for time preferences which are empirically—in a broad sense—implausible. This section therefore investigates the question which preference profiles out of the class considered imply a unique equilibrium. While the main theoretical result is a characterisation of these preference profiles, a corollary to it presents more readily testable—both empirically and in applied work with given preferences—sufficient conditions than which are necessary for equilibrium uniqueness at the level of each player's time preferences in isolation.

Rather unsurprisingly, if there is a unique equilibrium, it is the familiar stationary equilibrium with immediate agreement in any round which Rubinstein [1982] discovered. The assumptions on preferences together with the alternating offers protocol mean that the limit of any sequence of equilibria of the truncated games G_1 with finite horizon n converges to an equilibrium of the infinite-horizon game. If this limit is independent of whether the initial n is odd or even, i.e. who makes the last offer, it is indeed unique. Since, for any finite horizon, the game can be solved via backwards induction, specifically due to impatience property (ii), this will result in immediate agreement in any round; moreover, the stationarity of the infinite-horizon game means that the limiting equilibrium will be stationary.

Define now for each player $i \in I$ a function $f_i : [0,1] \to [0,1]$ as follows:

$$f_i(q) = 1 - \pi_j \left(U_j \left(1 - \pi_i \left(U_i(q,1) \right), 1 \right) \right).$$

The significance of this function lies in its relation to stationary equilibrium: if the division with share q for player i is the outcome in G_i in round 3 in case there is no agreement in the first two rounds, independent of what happens in these rounds, then, from two rounds of backwards induction, $f_i(q)$ is the maximal share which player i can obtain as the initial proposer in round 1. Fixed points of f_i are therefore limiting shares of initial proposer i in G_i of sequences of finite-horizon equilibrium shares and thus stationary-equilibrium shares for each $i \in I$. The following lemma therefore proves existence of stationary equilibrium.

Lemma 3. *For each $k \in I$, the function f_k possesses a fixed point $q = f_k(q)$, and if $q \in [0,1]$ is a fixed point of f_i, $i \in I$, then $q' = 1 - \pi_i(U_i(q,1))$ is a fixed point of f_j; moreover, f_1 has a unique fixed point if and only if f_2 has a unique fixed point.*

Proof. Take any $k \in I$. The function f_k is continuous by the continuity of preferences and satisfies $0 < f_k(0) \leq f_k(1) \leq 1$ by impatience, whence it has a fixed point by the intermediate-value theorem. Note, moreover, that f_k is non-decreasing by desirability.

Now suppose q is a fixed point of f_i, $i \in I$, and let $q' = 1 - \pi_i(U_i(q,1))$; then, for $j = 3 - i$,

$$\begin{aligned} f_j(q') &= 1 - \pi_i\left(U_i\left(1 - \pi_j\left(U_j\left(1 - \pi_i\left(U_i(q,1)\right),1\right)\right),1\right)\right) \\ &= 1 - \pi_i\left(U_i\left(f_i(q),1\right)\right) \\ &= 1 - \pi_i\left(U_i(q,1)\right) \\ &= q'. \end{aligned}$$

For the last claim, suppose q is in fact the unique fixed point of f_i; it remains to show that then q' is the unique fixed point of f_j. Suppose not, and f_j had another fixed point $q'' \neq q'$. Then also $q''' = 1 - \pi_j(U_j(q'',1))$ would be a fixed point of f_i; because q is the unique fixed point of f_i, $q''' = q$ has to hold. Moreover, since q' is a fixed point of f_j, it must be that also $1 - \pi_j(U_j(q',1)) = q$ from repeating the above argument. But this leads to a contradiction as follows:

$$1 - q = \pi_j\left(U_j(q',1)\right) = \pi_j\left(U_j(q'',1)\right) \Rightarrow f_j(q') = f_j(q'') \Rightarrow q' = q''.$$

□

Since stationary equilibrium always exists, a necessary condition for uniqueness of equilibrium is uniqueness of stationary equilibrium, i.e. uniqueness of the fixed point of f_1. In terms of lemma 2, this would say that there is a unique simple penal code in stationary strategy profiles; however, there may then still be simple penal codes in non-stationary strategy profiles and such have to be ruled out for sufficiency.

Theorem 2. *There exists a unique equilibrium if and only if the system of equations 4 through 6,$i \in I$, has a unique solution. This equilibrium is then given by the stationary strategy profile σ such that at any stage when a player $i \in I$ is the proposer, i proposes a division x to responding player $j = 3 - i$ with $x_j = 1 - \phi_i\left(v_i^*, 0\right) = \pi_j\left(U_j\left(\phi_j\left(v_j^*, 0\right), 1\right)\right)$ and this is the minimal share that j accepts.*

Proof. Necessity of a unique solution for a unique equilibrium, whence a unique equilibrium outcome, follows from lemma 2. For sufficiency, first establish that there always exists a solution with $t_1 = t_2 = 0$; to see this, recall lemma 3, take any $q^* = f_1\left(q^*\right)$ and set $v_1 = u_1\left(q^*\right)$ as well as $w_1 = U_1\left(q^*, 1\right)$, which clearly solve equations 4 and 5 for $i = 1$ given $t_1 = 0$. Next, let $v_2 = u_2\left(1 - \pi_1\left(U_1\left(q^*, 1\right)\right)\right)$ as well as $w_2 = U_2\left(1 - \pi_1\left(U_1\left(q^*, 1\right)\right), 1\right)$ and verify that these values solve equations 4 and 5 for $i = 2$ given $t_2 = 0$. For the values $\left(v_k, w_k, 0\right)_{k \in I}$ to indeed solve the system of equations 4 through 6 for any $i \in I$ and $j = 3 - i$, only equation 6 needs to be checked for each $i \in I$ and $j = 3 - i$:

$$\begin{aligned}
\kappa_1\left(0, v_1, v_2, w_2\right) &= q^* + \pi_2\left(U_2\left(1 - \pi_1\left(U_1\left(q^*, 1\right)\right), 1\right)\right) \\
&= f_1\left(q^*\right) + \pi_2\left(U_2\left(1 - \pi_1\left(U_1\left(q^*, 1\right)\right), 1\right)\right) \\
&= 1;
\end{aligned}$$

similarly,

$$\begin{aligned}
\kappa_2\left(0, v_2, v_1, w_1\right) &= \phi_2\left(v_2, 0\right) + \pi_1\left(U_1\left(q^*, 1\right)\right) \\
&= 1 - \pi_1\left(U_1\left(q^*, 1\right)\right) + \pi_1\left(U_1\left(q^*, 1\right)\right) \\
&= 1,
\end{aligned}$$

and noting that $\phi_i(v_i, 0) < \phi_i(v_i, 1)$, $(v_k, w_k, 0)_{k\in I}$ is proven to be a solution. Hence, if there is a unique solution, it is this solution, and by theorem 1, $(v_k, w_k, 0)_{k\in I} = (v_k^*, w_k^*, t_k^*)_{k\in I}$.

Using lemma 2, observe next that this solution has a unique outcome associated with it in each G_i which is $\left(x^{(i)}, 0\right)$, $i \in I$, where $x_1^{(1)} = q^*$ and $x_2^{(2)} = 1 - \pi_1(U_1(q^*, 1))$. There is then a unique equilibrium that yields exactly these outcomes in the two subgames, and it is the stationary one described in the theorem. □

Given a particular preference profile, checking for equilibrium uniqueness on the basis of this characterisation is rather cumbersome, as it would require proving uniqueness of the solution to a system of equations which relates the two bargaining parties' preferences. The following corollary presents a simpler, though somewhat weaker, test by providing sufficient conditions for uniqueness that concern each player's preferences in isolation.

Corollary 1. *If, for each $i \in I$ and any $(q, t) \in [0, 1] \times T$,*

(i) $q - \pi_i(U_i(q, 1))$ is increasing and

(ii) $u_i(q) = U_i(\phi_i(u_i(q), t), t)$ implies $U_i(\phi_i(u_i(q), 0), 1) \leq U_i(\phi_i(u_i(q), t), t + 1)$,

then equilibrium is unique.

Proof. First, observe that condition (i) ensures that f_1 as well as f_2 (recall lemma 3) have a unique fixed point; to see this, write

$$\begin{aligned} q - f_1(q) &= q - 1 + \pi_2(U_2(1 - \pi_1(U_1(q, 1)), 1)) \\ &= [q - \pi_1(U_1(q, 1))] - [(1 - \pi_1(U_1(q, 1))) - \pi_2(U_2(1 - \pi_1(U_1(q, 1)), 1))] \end{aligned}$$

and note that $q - f_1(q)$ is increasing in q: by condition (i), the first term in square brackets is increasing in q and the second term is increasing in $1 - \pi_1(U_1(q, 1))$, and since $1 - \pi_1(U_1(q, 1))$ is non-increasing in q, overall their difference is increasing. This means that $q - f_1(q)$ has at most one root, and application of lemma 3 yields the result that f_1 and f_2 have a unique fixed point.

Next, recall the proof of lemma 2, at the outset of which it was argued that any solution $(v_k, w_k, t_k)_{k\in I}$ to the system of equations 4 through 6, $\{i, j\} = I$, must satisfy that $v_i = U_i(\phi_i(v_i, t), t)$ for any $t \leq t_i$, any $i \in I$. Condition (ii) then implies that $U_i(\phi_i(v_i, 0), 1) \leq$

$U_i(\phi_i(v_i,t), t+1)$, whence it follows that $w_i = U_i(\phi_i(v_i,0), 1)$ and by consequence

$$\begin{aligned} v_i &= u_i\left(1-\pi_j\left(U_j\left(1-\pi_i\left(U_i\left(\phi_i\left(v_i,0\right),1\right)\right),1\right)\right)\right) \\ &\Leftrightarrow \\ \phi_i\left(v_i,0\right) &= 1-\pi_j\left(U_j\left(1-\pi_i\left(U_i\left(\phi_i\left(v_i,0\right),1\right)\right),1\right)\right) \end{aligned}$$

for each $i \in I$.

Letting $q^* = f_1(q^*)$, $\phi_1(v_1, 0) = q^*$ and $\phi_2(v_2, 0) = 1 - \pi_1(U_1(q^*, 1))$ must hold, so there are unique values (v_1, v_2) in any solution $(v_k, w_k, t_k)_{k \in I}$ to the system of equations 4 through 6, $\{i, j\} = I$. These also pin down (w_1, w_2) uniquely, as above, and any such solution must be of the form studied in the sufficiency part of the proof of theorem 2 satisfying $t_1 = t_2 = 0$. Thus there is a unique solution, and equilibrium uniqueness follows from theorem 2. □

Condition (i) is merely a translation of the "increasing loss to delay" axiom in the context of the analysis of exponential discounting (see Osborne and Rubinstein [1990, chapter 3]). Condition (ii) is novel, however, and—in conjunction with (i)—it extends earlier sufficiency statements to a much larger set of time preferences than exponential discounting, which itself clearly satisfies it. In fact, it can be interpreted as a form of present "bias" in a weak sense: noting that $u_i(q) = U_i(\phi_i(u_i(q), 0), 0)$, it says that whenever individual i is indifferent between receiving a reward q immediately and receiving a reward $q' = \phi_i(u_i(q), t)$ with t periods of delay, as one period of delay is added and both are moved into the future, such indifference may only be resolved in favour of the larger later share.

It is relatively easy to check that any hyperbolic discounting or any quasi-hyperbolic discounting satisfies condition (ii) (in addition to assumption 1, of course). Similarly, the non-separable time preferences proposed by Benhabib et al. [2010] and Noor [2011] do so, although the interpretation of condition (i) is less clear when there is no separability of instantaneous utility and time-discounting. Section 4 discusses the empirical evidence on time preferences and sketches the implications of these alternative models of time preferences in the present bargaining game.

In any case, condition (ii) on its own implies that the optimal simple penal codes of theorem 1 are stationary equilibrium outcomes. The reason is that while the proposer's strategic advantage means that the minimal equilibrium payoff of a player i as the initial proposer for when agreement takes place with a given delay cannot fall below the minimal immediate-agreement equilibrium payoff, such present bias makes the same player in the role of respondent of G_j at least as patient in comparing immediate agreement and delayed agreement continuation outcomes of G_i; hence the rejection value is minimal for the least preferred immediate-agreement equilibrium of G_i.

3.4 An Example of Unbounded Equilibrium Delay

To demonstrate the novel equilibria, which support optimal simple penal codes in non-stationary

Consider the following symmetric separable preferences:

$$U_i\left(x_i, t\right) = d\left(t\right) \cdot x_i,\ d\left(t\right) = \begin{cases} 1 & t = 0 \\ \alpha & t = 1 \\ \alpha\beta & t > 1 \end{cases},$$

where $0 < \beta < \alpha < 1$. It is insightful to decompose $d\left(t\right)$ into the product of discount factors for each period of delay as $d\left(\tau\right) = \prod_{t=1}^{\tau} \delta\left(t\right)$, where $\delta\left(t\right) \equiv \frac{d(t)}{d(t-1)}$ is the discount factor—and measure of patience—for the t-th period of delay, $t \in \mathbb{N}$.[16] Then

$$\delta\left(t\right) = \begin{cases} \alpha & t = 1 \\ \beta & t = 2 \\ 1 & t > 2, \end{cases}$$

and present bias is violated because $\beta < \alpha$; moreover, note that

$$U_i\left(0, 0\right) = 0 < \lim_{t \to \infty} d\left(t\right) \cdot x_i = \alpha\beta x_i = U_i\left(x_i, t\right)$$

for any $x_i > 0$ and $t > 1$, verifying impatience property (iii) of assumption 1 (the remaining

[16] I follow the convention here that the empty product $d\left(0\right)$ equals 1.

properties are obvious).

Let v_i^t denote the minimal equilibrium share for a given delay of t periods of player i in G_i; by symmetry, $v_i^t = v^t$. Recall that $v^0 = v^*$ is the minimal equilibrium payoff of i as evaluated in initial round of G_i. Indifference across delays means that for any $t > 1$:

$$v^* = \alpha v^1 = \alpha\beta v^t \Rightarrow v^1 = \frac{v^*}{\alpha} \wedge v^t = \frac{v^*}{\alpha\beta}.$$

Hence we get the minimal rejection equilibrium value:

$$w^* = \min\left\{\alpha v^*, \alpha\beta\frac{v^*}{\alpha}, \alpha\beta\frac{v^*}{\alpha\beta}\right\} = \beta v^*.$$

This pins down v^* as follows:

$$v^* = 1 - \alpha\left(1 - \beta v^*\right) \Leftrightarrow v^* = \frac{1-\alpha}{1-\alpha\beta}.$$

The condition for equilibrium delay of more than two, in fact arbitrarily many, periods is:

$$\frac{v^*}{\alpha\beta} + \frac{v^*}{\alpha\beta} \leq 1 \Leftrightarrow 2 \leq \frac{\alpha\beta\left(1-\alpha\beta\right)}{1-\alpha}. \tag{8}$$

Now note that the denominator only depends on α which can be arbitrarily close to 1, and the numerator can be maximised to $\frac{1}{4}$ by setting $\alpha\beta = \frac{1}{2}$, pinning down $\beta = \frac{1}{2\alpha}$. Thus we can solve for the minimal value of α so the above holds:

$$2 \leq \frac{\frac{1}{4}}{1-\alpha} \Leftrightarrow \alpha \geq \frac{7}{8}.$$

Note that equation 8 guarantees that v, v^1 and v^2 are actually no larger than one because:

$$v^* < v^1 = \frac{v^*}{\alpha} < v^2 = \frac{v^*}{\alpha\beta} < 2\frac{v^*}{\alpha\beta} \leq 1.$$

For a convenient numerical example, one may take $\alpha = \frac{15}{16} = \frac{30}{32} > \frac{28}{32} = \frac{7}{8}$ which implies

$\beta = \frac{16}{30} = \frac{8}{15}$ and therefore $v = \frac{1}{8}$ and $w = \frac{1}{15}$. For these numerical values, the proof of the following proposition explicitly constructs an optimal penal code which is "purely" non-stationary (there is no subgame in which its restriction is a stationary equilibrium), and this penal code supports agreement with any delay $t \in T$.

Proposition. *Suppose* $z^{(1)} = \left(\left(\frac{2}{15}, \frac{13}{15}\right), 1\right)$ *is a continuation equilibrium outcome of* G_1 *and* $z^{(2)} = \left(\left(\frac{13}{15}, \frac{2}{15}\right), 1\right)$ *is a continuation equilibrium outcome of* G_2*. Then each* $z^{(i)}$*,* $i \in I$*, is an equilibrium outcome of the respective* G_i*. Hence* $z^{(i)}$ *is indeed an equilibrium outcome of* G_i*.*

Proof. Consider G_1 and construct an equilibrium σ as follows (the conclusion for G_2 follows by symmetry), where the respondent's acceptance rule in any given round as a function of the history preceding that round will be denoted by the minimal accepted share.

First, in the initial round $\sigma_1(\emptyset) = 1_1$ and $\sigma_2(\emptyset) = \frac{7}{8}$. The best-response property of these will be verified at the end of the argument.

If $x \neq 1_1$ is rejected in this round, then in the subsequent round $\sigma_2(x) = \left(\frac{1}{15}, \frac{14}{15}\right)$ and $\sigma_1(x) = \frac{1}{15}$ which is rationalised by "restarting" σ in the subsequent subgame G_1 with outcome $\left(\left(\frac{2}{15}, \frac{13}{15}\right), 1\right)$ for any rejected proposal, because then (i) rejection has value $\alpha\beta\frac{2}{15} = \frac{1}{15}$ to responding player 1, which equals that of acceptance, and (ii) making an offer that is rejected has value $\alpha\beta\frac{13}{15} = \frac{13}{30}$ to proposing player 2, which is less than that of making the most preferred accepted proposal which equals $\frac{13}{15}$.

If $x = 1_1$ is rejected in the initial round, then $\sigma_2(1_1) = \left(\frac{2}{15}, \frac{13}{15}\right)$ and $\sigma_1(1_1) = \frac{2}{15}$, and if x with $x_1 \geq \frac{2}{15}$ is rejected, the strategy is "restarted" so outcome $\left(\left(\frac{2}{15}, \frac{13}{15}\right), 1\right)$ in subsequent subgame G_1 obtains, whereas if x with $x_1 < \frac{2}{15}$ is rejected, then the subsequent subgame G_1 is played with the "mirror strategy" of σ where outcome $\left(\left(\frac{13}{15}, \frac{2}{15}\right), 1\right)$ obtains. Responding player 1's acceptance rule is a best response because rejecting shares $x_1 \geq \frac{2}{15}$ results in a payoff of $\frac{1}{15}$ which is less than that of acceptance of at least $\frac{2}{15}$, and accepting shares $x_1 < \frac{2}{15}$ is worth less than $\frac{2}{15}$ whereas rejection has value $\alpha\beta\frac{13}{15} = \frac{13}{30} > \frac{2}{15}$.

Finally, reconsider the initial round's specification: if responding player 2 were to accept a share $0 < x_2 < \frac{7}{8}$ her payoff would be less than $\frac{7}{8}$ which is the value of rejecting any such offer because $\alpha\frac{14}{15} = \frac{7}{8}$, making acceptance of any share greater than that optimal; and accepting a zero share would be worse than rejecting it in favour of share $\frac{13}{15}$ with one period of delay which is worth $\alpha\frac{13}{15} = \frac{13}{16}$. Proposing player 1 obtains a payoff of $\alpha\frac{2}{15} = \frac{1}{8}$ by proposing 1_1, whereas any proposal x with $x_2 \geq \frac{7}{8}$ would be immediately accepted and yield a payoff no greater, and any proposal x with $0 < x_2 < \frac{7}{8}$ would be rejected and result in a payoff of $\alpha\frac{2}{15} = \frac{1}{8}$ which is no greater either. $\square$

4 Empirical Evidence on Time Preferences and Implications

The empirical literature on time preferences is vast. A large body of evidence on choice from various domains of intertemporal trade-offs has accumulated in psychology since the 1970s and is summarised by Frederick et al. [2002] who conclude that "virtually every assumption underlying the (exponential-discounting) model has been tested and found to be descriptively invalid in at least some situations" (p. 352). Moreover, in comparison to empirical violations of expected-utility preferences, Loewenstein and Prelec [1992] observe that "the counterexamples to (exponential discounting) are simple, robust and bear directly on central aspects of economic behavior" (p. 574). The most convincing such evidence comes in the form of preference reversals, where e.g. the same person prefers \$20 today over \$30 in three weeks but also \$30 in 9 weeks over \$20 in 6 weeks. This is a direct violation of the stationarity axiom necessary for exponential discounting.

This section briefly discusses the evidence and implications of alternative models of discounting put forward in the literature. Most importantly, it serves to demonstrate the implication of corollary 1 that, under standard assumptions on the curvature of the instantaneous utility function, uniqueness of equilibrium, and thus immediate agreement (efficient bargaining), is a robust implication across various forms of present bias.

4.1 Separable Time Preferences (Discounting)

While assumption 1 only assumes stability of time preferences across different points in time in the sense that delayed rewards are evaluated the same way (Halevy [2012] terms this time-invariance), time preferences are most meaningful if stable also across domains of choice, i.e. if they are separable:

$$U_i(q,t) = d_i(t)\, u_i(q).$$

In this case $d_i(t)$ is individual i's "discounting" function which captures individual i's general impatience about the timing of various kinds of rewards.

Especially in the present context, it is instructive to define a function $\delta_i : \mathbb{N} \to [0,1]$ such that $\delta_i(t) \equiv \frac{d_i(t)}{d_i(t-1)}$ which expresses i's discount factor for each period of delay from the present, so that $d_i(t) \equiv \prod_{t'=1}^{t} \delta_i(t')$.[17] Note that stationarity—equivalently, exponential discounting and dynamic consistency—would require here that $\delta_i(t)$ be constant. The (β,δ)-model of quasi-hyperbolic discounting, in its strict version, would have $\delta_i(1) = \beta\delta < \delta = \delta_i(t)$ for any $t > 1$, and hyperbolic discounting, again in its strict version, would have δ_i increasing.

All of these examples satisfy condition (ii) of corollary 1: assuming separability, it says that

$$[u_i(q) = d_i(t)\, u_i(q') \Rightarrow d_i(1)\, u_i(q) \leq d_i(t+1)\, u_i(q')] \Leftrightarrow \delta_i(1) \leq \delta_i(t+1)$$

for any $t \in T$. The psychological content of this property is straightforward: the first period of delay from the present looms largest, and this confirms its interpretation as a weak form of present bias. Moreover, condition (i) is met by any instantaneous utility function which is concave—a standard property in economics—whence for the most important alternative models to exponential discounting, the prediction for the Rubinstein [1982] bargaining model is unique. As shown, this unique equilibrium satisfies all the familiar properties: it is stationary, characterised by the players' attitudes to the first period of delay together with their instantaneous utility functions $(\delta_i(1), u_i)_{i\in I}$, implies immediate agreement (efficiency) and exhibits the usual comparative statics where an increase in $\delta_i(1)$ can only increase a player i's equilibrium share for given u_i and opponent preferences.[18]

[17] I follow the convention that the empty product for $t = 0$ equals one. In order to relate this definition to "discount rates", simply let $\delta_i(t) = \frac{1}{1+\rho_i(t)}$, where $\rho_i(t)$ is the discount rate for the t-th period of delay from the present.

[18] Given an individual's preferences satisfy exponential discounting with some discount factor $\delta \in (0,1)$ and instantaneous utility function u, for any $\epsilon \in (0,1)$, there then exists an instantaneous utility function v such that $\epsilon^t \cdot v(q)$ also represents such an individual's preferences (see Fishburn and Rubinstein [1982, theorem 2] and Osborne and Rubinstein [1990, footnote 5]). Thus the discount factor alone is ill-defined as a measure of impatience, and this has lead to different concepts of impatience or delay aversion for such preferences in bargaining (Osborne and

However, while some form of (non-exponential) hyperbolic discounting appears empirically well-established regarding "primary" rewards (actual consumption), there is some controversy about whether this is true (or should even be expected) also about time preferences over money (see for instance the discussions in Read [2001a], McClure et al. [2007], Andreoni and Sprenger [2012], Read et al. [2012] and Augenblick et al. [2013]). Interestingly, there are several recent experimental studies, investigating such monetary rewards over also particularly short horizons of less than a week which find behaviour that suggests impatience, as measured by δ_i, may actually increase with delay for short delays and then switch to being decreasing (at least weakly) only for delays of more than at least a few days (see in particular the findings and discussions in Sayman and Öncüler [2009], Attema et al. [2010] and Takeuchi [2011], and also further references in Attema [2012, section 3.1] among which Read [2001a] is first to explicitly advance such evidence, and Halevy [2012] and Olea and Strzalecki [2014] are most recent examples). Qualitatively, in a way similar to how (β, δ)-discounting captures the most salient property of hyperbolic discounting, this is for instance modelled by the discounting function in section 3.4, for which a novel kind of equilibria with delayed agreement is shown to arise. While the domain for which such time preferences have been documented appears most pertinent for bargaining, such a "discounting-based explanation" for inefficient delay in real bargaining over money—in spite of perfect information and full sophistication—awaits further validation of these findings.

4.2 Non-separable Time Preferences

Rather naturally, the economics literature on time-preferences has focussed on separable preferences (discounting). There are a few exceptions, however, which this section deals with. First, Benhabib et al. [2010] find strong experimental evidence for present bias in choice over monetary rewards but estimate that such bias has a fixed-cost component. They suggest the following time-preference specification (here in the most general form they envisage, in slightly different notation):

$$U_i(q,t) = \delta_i^t u_i(q) - b_i\left(1 - (\delta_i \epsilon_i)^t\right),$$

Rubinstein [1990, section 3.10.2]) and more generally (Benoît and Ok [2007]).

where $(\delta_i, \epsilon_i) \in (0,1)^2$ and $b_i > 0$ is a fixed cost to delay. This specification also satisfies condition (ii) of corollary 1 whence, when combined with concave u_i, it yields a unique equilibrium prediction in the bargaining model studied here.

Second, Noor [2011], referring to the empirical finding that imputed discount rates for a given delay appear decreasing in the size of the (monetary) reward (the so-called "magnitude effect"), develops an axiomatic decision theory for the following representation:

$$U_i(q,t) = (\delta_i(q))^t \, u_i(q),$$

where $\delta_i(\cdot)$ is an increasing function. This model can rationalise evidence on hyperbolic discounting but, contrary to the latter, does not require calibrationally implausible degrees of concavity on the instantaneous utility function to generate the magnitude effect. Once more, condition (ii) is satisfied for such decreasing δ_i, which may therefore be thought of another form of present bias.[19] Contrary to any separable time preferences or also the fixed-cost model of Benhabib et al. [2010], concavity of u_i is now not sufficient for condition (i) to hold, however, because δ_i is a function of q and a "sufficiently convex" $\delta_i(q)\, u_i(q)$ may still permit multiple fixed points of f_i (see Noor [2011, section 4]).

4.3 Other "Time Preferences"

Recently, Ok and Masatlioglu [2007] have developed an axiomatic decision theory of "relative discounting". Assuming separability but allowing for cycles in the ranking of rewards that come with different delays, they obtain a representation of the following form: a reward q with delay t is weakly preferred by individual i to a reward q' with delay t' if and only if

$$u_i(q) \geq \eta_i(t,t')\, u_i(q'),$$

[19] Noor [2011] actually applies his theory to the bargaining problem analysed here, but under the restriction to stationary equilibrium. My results show that this is indeed without loss of generality with regards to characterising the set of equilibrium divisions.

where $\eta_i(t,t') = \frac{1}{\eta_i(t',t)}$ reflects how discounting is relative to the particular delays under consideration. The authors also show how their theory can accommodate formal versions of two procedural modes of choice in the domain of delayed money rewards which have been suggested: subadditive discounting by intervals (Read [2001a]) and similarity-based choice rules (Rubinstein [2003a]).

While assumption 1 imposes transitivity also in binary comparisons across any different delays, the set of equilibrium outcomes is characterised by means of optimal penal codes which mean that each player, at any stage, faces only a problem of choosing between a pair of outcomes: an immediate agreement and a known delayed agreement (given perfect information, in equilibrium there is no uncertainty about which outcome would obtain upon a contemplated deviation); hence only the functions $\eta_i(0,t)$ are pertinent to the characterisation. But then one can simply set $\eta_i(0,t) = d_i(t)$ and analyse them as a separable time preference (see section 4.1); the type of intransitivity permitted by Ok and Masatlioglu [2007] is immaterial in this application.[20]

5 Conclusion

The contribution of this paper is two-fold: on the one hand, I characterise the equilibrium of the classic alternating-offers bargaining game proposed by Rubinstein [1982] for time preferences which are of increasing interest in economics, in particular hyperbolic and quasi-hyperbolic discounting. Corollary 1 presents sufficient conditions for equilibrium uniqueness which are satisfied by the latter. On the other hand, I employ a novel analytical approach to the equilibrium characterisation which reveals the generalisability of the powerful framework that Abreu [1988] developed for repeated games with discounting to other stochastic games. This insight may be useful for related theoretical work.

In the bargaining model studied here, this approach uncovers a novel kind of non-stationary delay equilibria for time preferences where a near-future period is "discounted" more heavily than the first period from the immediate present. Interestingly, time preferences with this property have recently been documented experimentally for a significant portion of subjects on a domain which

[20] In fact, the authors also apply their decision theory to the exact same bargaining problem as investigated here, claiming a certain uniqueness result, but they fail to recognise how the most widely used method of proof for uniqueness cannot be applied with dynamically discounting (see section 3.1).

appears most relevant for bargaining: short-horizon monetary trade-offs. It remains, however, to be seen whether these preferences elicited at a single point in time are due to transient individual environmental factors or in fact inherently stable across time as assumed here; Halevy [2012] provides an investigation of this question.

The existence of such delay equilibria comes with multiplicity, which raises the question of refinement: in particular, how much and what kind of intra-personal coordination is required to obtain a unique prediction. Given the robustness of stationary equilibrium, it seems that this would be the prediction of any refinement that achieves this.

While the results presented here were derived for a bargaining environment without uncertainty, their extension to alternating-offers bargaining under the shadow of exogenous breakdown risk is straightforward. Hence the pure equilibria for various non-expected-utility risk preferences are characterised, e.g. those proposed by Halevy [2008]. Once there is risk, however, randomisation may be of strategic use. But it seems unlikely that allowing for mixed strategies would affect the sets of equilibrium outcomes and payoffs which are found to be convex already here (regarding outcomes, this holds for any given delay only since time is discrete).

Finally and relatedly, the relationship between the predictions derived here and axiomatic bargaining solutions deserve further attention, especially in view of the work of Rubinstein et al. [1992] who extended the Nash bargaining solution to (dynamically inconsistent) non-expected-utility preferences.

References

Dilip Abreu. On the theory of infinitely repeated games with discounting. *Econometrica*, 56(2): 383–396, 1988.

Zafer Akin. Time inconsistency and learning in bargaining games. *International Journal of Game Theory*, 36(2):275 – 299, 2007.

James Andreoni and Charles Sprenger. Estimating time preferences from convex budgets. *The American Economic Review*, 102(7):3333–3356, 2012.

Geir B. Asheim. Individual and collective time-consistency. *The Review of Economic Studies*, 64 (3):427–443, 1997.

Nava Ashraf, Dean Karlan, and Wesley Yin. Tying odysseus to the mast: Evidence from a commitment savings product in the philippines. *The Quarterly Journal of Economics*, 121(2): 635–672, 2006.

Arthur E. Attema. Developments in time preference and their implications for medical decision making. *Journal of the Operational Research Society*, 63(10):1388–1399, 2012.

Arthur E. Attema, Han Bleichrodt, Kirsten I. M. Rohde, and Peter P. Wakker. Time-tradeoff sequences for analyzing discounting and time inconsistency. *Management Science*, 56(11):2015–2030, 2010.

Ned Augenblick, Muriel Niederle, and Charles Sprenger. Working over time: Dynamic inconsistency in real effort tasks. January 2013.

Jess Benhabib, Alberto Bisin, and Andrew Schotter. Present-bias, quasi-hyperbolic discounting, and fixed costs. *Games and Economic Behavior*, 69(2):205–223, 2010.

Jean-Pierre Benoît and Efe A. Ok. Delay aversion. *Theoretical Economics*, 2(1):71–113, 2007.

Ken Binmore, Ariel Rubinstein, and Asher Wolinsky. The nash bargaining solution in economic modelling. *The RAND Journal of Economics*, 17(2):176–188, 1986.

Ken Binmore, Martin J. Osborne, and Ariel Rubinstein. Noncooperative models of bargaining. In Robert J. Aumann and Sergiu Hart, editors, *Handbook of Game Theory with Economic Applications*, volume 1, chapter 7. North Holland, 1992.

Han Bleichrodt, Kirsten I. M. Rohde, and Peter P. Wakker. Non-hyperbolic time inconsistency. *Games and Economic Behavior*, 66(1):27–38, 2009.

Lasse Brune, Xavier Giné, Jessica Goldberg, and Dean Yang. Commitments to save: A field experiment in rural malawi. October 2013.

Lutz-Alexander Busch and Quan Wen. Perfect equilibria in a negotiation model. *Econometrica*, 63(3):545–565, 1995.

Hector Chade, Pavlo Prokopovych, and Lones Smith. Repeated games with present-biased preferences. *Journal of Economic Theory*, 139(1):157–175, 2008.

Olivier Compte and Philippe Jehiel. Bargaining with reference dependent preferences.

Olivier Compte and Philippe Jehiel. Gradualism in bargaining and contribution games. *Review of Economic Studies*, 71(4):975–1000, 2004.

Pascaline Dupas and Jonathan Robinson. Why don't the poor save more? evidence from health savings experiments. *American Economic Review*, 103(4):1138–1171, 2013.

Chaim Fershtman and Daniel J. Seidmann. Deadline effects and inefficient delay in bargaining with endogenous commitment. *Journal of Economic Theory*, 60(2):306–321, 1993.

Peter C. Fishburn and Ariel Rubinstein. Time preference. *International Economic Review*, 23(3): 677–694, 1982.

Shane Frederick, George Loewenstein, and Ted O'Donoghue. Time discounting and time preference: A critical review. *Journal of Economic Literature*, 40(2):351–401, 2002.

Drew Fudenberg and Jean Tirole. *Game Theory*. The MIT Press, 1991.

Xavier Giné, Jessica Goldberg, Dan Silverman, and Dean Yang. Revising commitments: Field evidence on the adjustment of prior choices. August 2013.

Steven M. Goldman. Consistent plans. *The Review of Economic Studies*, 47(3):533–537, 1980.

Yoram Halevy. Strotz meets allais: Diminishing impatience and the certainty effect. *The American Economic Review*, 98(3):1145–1162, 2008.

Yoram Halevy. Time consistency: Stationarity and time invariance. June 2012.

John Kennan and Robert Wilson. Bargaining with private information. *Journal of Economic Literature*, 31(1):45–104, 1993.

Narayana R. Kocherlakota. Reconsideration-proofness: A refinement for infinite horizon time inconsistency. *Games and Economic Behavior*, 15(1):33–54, 1996.

David M. Kreps. *A Course in Microeconomic Theory.* Princeton University Press, 1990.

David I. Laibson. Golden eggs and hyperbolic discounting. *The Quarterly Journal of Economics*, 112(2):443–478, 1997.

Duozhe Li. Bargaining with history-dependent preferences. *Journal of Economic Theory*, 136(1): 695–708, 2007.

George Loewenstein and Drazen Prelec. Anomalies in intertemporal choice: Evidence and an interpretation. *The Quarterly Journal of Economics*, 107(2):573–597, 1992.

Shih En Lu. Quasi-hyperbolic discounting and the dual-self model in rubinstein-ståhl type bargaining. March 2006.

Paola Manzini and Marco Mariotti. Sequentially rationalizable choice. *The American Economic Review*, 97(5):1824–1839, 2007.

Samuel M. McClure, Keith M. Ericson, David I. Laibson, George Loewenstein, and Jonathan D. Cohen. Time discounting for primary rewards. *The Journal of Neuroscience*, 27(21):5796–5804, 2007.

Antonio Merlo and Charles Wilson. A stochastic model of sequential bargaining with complete information. *Econometrica*, 63(2):371–399, 1995.

Jawwad Noor. Intertemporal choice and the magnitude effect. *Games and Economic Behavior*, 72 (1):255–270, 2011.

Ted O'Donoghue and Matthew Rabin. Doing it now or later. *The American Economic Review*, 89 (1):103–124, 1999.

Efe A. Ok and Yusufcan Masatlioglu. A theory of (relative) discounting. *Journal of Economic Theory*, 137(1):214–245, 2007.

José Luis Montiel Olea and Tomasz Strzalecki. Axiomatization and measurement of quasi-hyperbolic discounting. January 2014.

Martin J. Osborne and Ariel Rubinstein. *Bargaining and Markets*. Academic Press, Inc., 1990.

Martin J. Osborne and Ariel Rubinstein. *A Course in Game Theory*. The MIT Press, 1994.

Jinrui Pan, Craig S. Webb, and Horst Zank. Discounting the subjective present and future. April 2013.

Bezalel Peleg and Menahem E. Yaari. On the existence of a consistent course of action when tastes are changing. *The Review of Economic Studies*, 40(3):391–401, 1973.

Michele Piccione and Ariel Rubinstein. On the interpretation of decision problems with imperfect recall. *Games and Economic Behavior*, 20:3–24, 1997.

Asaf Plan. Weakly forward-looking plans. May 2010.

Robert A. Pollak. Consistent planning. *The Review of Economic Studies*, 35(2):201–208, 1968.

Daniel Read. Is time-discounting hyperbolic or subadditive? *Journal of Risk and Uncertainty*, 23 (1):5–32, 2001a.

Daniel Read, Shane Frederick, and Mara Airoldi. Four days later in cincinnati: Longitudinal tests of hyperbolic discounting. *Acta Psychologica*, 140(2):177–185, 2012.

Ariel Rubinstein. Perfect equilibrium in a bargaining model. *Econometrica*, 50(1):97–109, 1982.

Ariel Rubinstein. A sequential strategic theory of bargaining. In Truman F. Bewley, editor, *Advances in Economic Theory: Fifth World Congress*, chapter 5, pages 197–224. Cambridge University Press, 1987.

Ariel Rubinstein. On the interpretation of two theoretical models of bargaining. In Kenneth J. Arrow, Robert H. Mnookin, Lee Ross, Amos Tversky, and Robert B. Wilson, editors, *Barriers to Conflict Resolution*, chapter 7. W. W. Norton & Company, 1995.

Ariel Rubinstein. "economics and psychology"? the case of hyperbolic discounting. *International Economic Review*, 44(4):1207–1216, 2003a.

Ariel Rubinstein, Zvi Safra, and William Thomson. On the interpretation of the nash bargaining solution and its extension to non-expected utility preferences. *Econometrica*, 60(5):1171–1186, 1992.

Serdar Sayman and Ayse Öncüler. An investigation of time inconsistency. *Management Science*, 55(3):470–482, 2009.

Avner Shaked and John Sutton. Involuntary unemployment as a perfect equilibrium in a bargaining model. *Econometrica*, 52(6):1351–1364, 1984.

Ingolf Ståhl. *Bargaining Theory*. EFI The Economics Research Institute, Stockholm, 1972.

Robert H. Strotz. Myopia and inconsistency in dynamic utility maximization. *The Review of Economic Studies*, 23(3):165–180, 1955-1956.

Kan Takeuchi. Non-parametric test of time consistency: Present bias and future bias. *Games and Economic Behavior*, 71(2):456–478, 2011.