Multi-period Matching*†

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Abstract

We examine a multi-period, two-sided matching market without monetary transfers. We identify sufficient conditions of the existence of a dynamically-stable matching and we investigate properties of the core. Matchings derived through repeated spot markets may be unstable if agents' preferences exhibit inertia or status quo bias. An extension of our model accommodating uncertainty and learning about future preferences is proposed. We relate our analysis to market unraveling, to the exposure problem, and to the importance of commitment, or lack thereof, in dynamic markets.

Keywords: Dynamic Matching, Two-sided Matching, Stability, Core, Market Design, Uncertainty

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Gale and Shapley (1962) elegantly tackled the problems of "College Admissions and the Stability of Marriage." By privileging stability, their analysis suggests an immutability to a match's outcome. Of course, this is not what we often observe. Consider but a few consequences of seemingly (or aspirationally) stable matches:

(1) After freshman year at Yale, a student (understandably) decides to transfer to Harvard.

(2) After ten years of marriage, a couple divorces. Each marries a new partner the following year.

Both situations feature intended long-term relationships—four years of college, a lifetime of marriage—that can be revised over time. Often such revisions occur. Sometimes, however, they do not:

(1′) After freshman year at Yale, a student continues on as a sophomore. Harvard was his first choice college, but transferring no longer seems worth it.

(2′) After ten years of marriage, a couple is more in love with each other than on their wedding day.

The above vignettes share three important characteristics that color most economic and social relationships. First, relationships have a temporal component. They can be revised with the passage of time and prevailing institutional arrangements determine the ease of re-matchings. Second, an agent's preferences over future partners are shaped by past outcomes. For example, switching costs imply an inter-temporal linkage in preferences. Finally, an agent is typically uncertain about his future preferences and refines his opinions as new information comes to light. Any analysis of a two-sided market where relationships are not ephemeral, which we contend is the vast majority of cases, must address these features. Examples include interpersonal relations, school assignment, labor markets, and business-to-business contracting, among many others.

In this study we provide a unified framework addressing the above three features. Our proposal is a simple two-period generalization of Gale and Shapley's (1962) "marriage market" model of one-to-one matching. In each period agents from one side of the market (men, students, workers) can partner ("match") with someone from the other side of the market (women, schools, firms). Each agent's preferences are defined over partnership plans, i.e. the entire sequence of partners they encounter over a lifetime. Preference heterogeneity renders the emergent pattern of reasonable and expected matchings nontrivial from the outset.

Our analysis has two parts. The first brackets uncertainty and investigates a perfect-information benchmark. We argue that calibrating the timing of a relationship is often as important as identifying the right partner(s) and both volatile outcomes (like cases 1 and 2 above) and persistent matchings (cases $1'$ and $2'$) can be "stable" arrangements in a multi-period market. To arrive at this conclusion, we first identify sufficient conditions ensuring the existence of ex ante and dynamically-stable matchings. A matching is ex ante stable if agents agree to the proposed plan from the outset under the presumption of commitment to its full execution. Dynamic stability does not presume commitment. We also examine our market's core and we provide sufficient conditions for its non-emptiness. While ex ante stable matchings always exist, the underlying feature supporting dynamically-stable (and core) matchings is inertia in agents' preferences. Roughly, preferences exhibit inertia if an agent becomes more eager to match again with their current partner in a future period. Surprisingly, matching procedures incorporating spot markets, where agents report preferences conditional on past match outcomes and re-match, can lead to dynamically-unstable outcomes.

The second part amends our baseline model by incorporating preference uncertainty. Each agent does not know his own future preferences and learns new information over time. Focusing on dynamically-stable outcomes, which have an important no regret property, we argue that interim spot markets, where agents re-match *after* learning new information about their preferences, can have at best a limited role. In our model, interim re-matchings cannot lead to a Pareto improvement relative to maintaining a matching derived with imperfect information.

Throughout we link our results to common practices and features of dynamic markets. For example, we relate several of our results to the phenomenon of market unraveling (Roth and Xing, 1994). In our environment, it emerges as an inclination to bring forward in time a future matching. Similarly, our analysis sheds light on the importance of calibrating relationship or contract length. Time is also an important dimension of market design, which is often ignored. We highlight instances when it is safe to do so; however, fully discounting the future can at times mask important market features or preclude innovative solutions.

The next section surveys related studies of dynamic markets. Section 2 introduces our model and Section 3 elaborates on agents' preferences. Sections 4 and 5 consider stable matchings and the core, respectively. Section 6 introduces uncertainty. Section 7 concludes. Several appendices collect supporting material and results.[1]

[1] Appendix A presents proofs of minor technical results. Appendices B, C, and D are available in the

1 Related Literature

Our model generalizes Gale and Shapley's (1962) model of one-to-one matching. Consequently, we rely heavily on their analysis and on Roth and Sotomayor's (1990) comprehensive synthesis. Though the literature on matching markets is expansive, relatively few studies directly address the multi-period nature of typical matching problems.

In regards to one-sided markets, several papers revisit the house allocation problem (Shapley and Scarf, 1974; Hylland and Zeckhauser, 1979) by incorporating dynamics (Abdulkadiroğlu and Loertscher, 2007; Bloch and Cantala, 2013; Kurino, 2014).[2] Our analysis does not address this class of problem directly, though we investigate complementary questions. For example, like Kurino (2014) we devote considerable attention to understanding the operation of spot markets/rules in dynamic markets.

Studies of dynamic two-sided markets have typically addressed many-to-one matching problems motivated by "school choice" applications (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, 2013). Dur (2012), Bando (2012), Pereyra (2013), and Kennes et al. (2013) propose models in this vein.[3] Among these studies, Kennes et al. (2013) is closest to our analysis. Kennes et al. (2013) examine the assignment of children to daycares and they identify several problems with well-known mechanisms. Several of our assumptions and mechanisms have direct parallels in their study, which we highlight below. While aspects of those papers are more general than we allow—i.e. many-to-one matching or overlapping generations (OLG) of agents—our model is not a special case of any of them.

Dynamic models of one-to-one matchings, like ours, are less common. When formulated, they have served as a forum for examining "stability" as a solution concept. For example, Damiano and Lam (2005) and Kurino (2009) identify inadequacies with the classic concepts of (pairwise) stability and the core and propose alternatives.[4] Despite the limitations of the classic concepts, we rely on them to keep our discussion's scope manageable and to allow for easy comparisons with results from static models.

Dynamic matching markets are closely related to (static) many-to-many matching mar-

online supplement. Appendix B provides a detailed comparison of our analysis with other studies of dynamic matching markets. Appendix C provides examples illustrating noteworthy observations and counterexamples to tempting conjectures. Appendix D reviews the Gale and Shapley (1962) deferred acceptance algorithm, which we use extensively.

[2]Closely related is Ünver's (2010) model of a dynamic kidney-exchange market.

[3]Bando (2012) examines a multi-period, many-to-many matching market.

[4]In Appendix B we compare our model in detail with those proposed by Damiano and Lam (2005), Kurino (2009), and Kennes et al. (2013). An acquaintance with our model, notation, and analysis is helpful to appreciate some of the distinctions that we highlight.

kets. Over a lifetime, each agent can have many partners. In many-to-many markets (pairwise-)stable and core assignments need not coincide (Blair, 1988) and the development of alternative solution concepts has drawn interest (Sotomayor, 1999; Konishi and Ünver, 2006; Echenique and Oviedo, 2006). Recently, Hatfield and Kominers (2012) have examined many-to-many matchings in the "matching with contracts" framework (Hatfield and Milgrom, 2005). They observe that the expressiveness of the contractual language, roughly corresponding to the number and the nature of the contractual relationships that agents may entertain, has important implications for market stability. Our results comparing ex ante and dynamic stability, which relate to agents' commitment ability, reinforce and complement their insight.

The final portion of our analysis examines matching markets with uncertainty (Roth, 1989). Roth and Rothblum (1999), Chakroborty et al. (2010), Haɫaburda (2010), and Lazarova and Dimitrov (2013) also study matching markets with uncertainty though they focus on different questions than we do.

Our study considers relationships that can last multiple periods. A complementary class of dynamic models studies how matchings arise over a period of time. Work in this vein has explored matching dynamics (Roth and Vande Vate, 1990), preference formation (Kadam, 2014), and market unravelling (Roth and Xing, 1994; Li and Rosen, 1998). Though our focus differs, at times we can reinterpret our model with an eye toward these questions as well.

2 The Model

Mindful of the applications noted above, for expositional ease we present our model using Gale and Shapley's terminology of a matching between men and women. For brevity, we sometimes state definitions or theorems only from the perspective of a typical man. Our model is symmetric and all definitions apply to women with obvious changes in notation.[5]

2.1 The One-Period Market

To establish a benchmark, we first review Gale and Shapley's (1962) (one-period) one-to-one, matching market. There are finite, disjoint sets of men, $M = \{m_1, \ldots, m_{|M|}\}$, and women, $W = \{w_1, \ldots, w_{|W|}\}$. Each man (woman) can be matched to one woman (man) or not matched at all. By convention, a man (woman) who is not matched is treated as matched

[5] Typically, it is sufficient to replace m's with w's and M's with W's, and vice-versa.

to himself (herself). Thus, $W_m \equiv W \cup \{m\}$ is the set of man m's potential partners.

Definition 1. The function $\mu\colon M \cup W \to M \cup W$ is a *(one-period) matching* if and only if

1. For all $m \in M$, $\mu(m) \in W_m$.

2. For all $w \in W$, $\mu(w) \in M_w$.

3. For all $i \in M \cup W$, $\mu(\mu(i)) = i$.

Each agent i has a strict preference ranking, P_i, of all potential partners. If i prefers j to k, then jP_ik. The confluence of agents' preferences determine the stability of a matching. Specifically, a matching μ is stable if (i) each agent weakly prefers his assigned partner to being not matched; and, (ii) no pair of agents can block the matching by preferring to be together in lieu of their assigned partners.

Theorem 1 (Gale and Shapley (1962))**.** *There exists a stable (one-period) matching.*

To prove Theorem 1, Gale and Shapley (1962) outline the (one-period) man-proposing deferred acceptance algorithm.[6] The algorithm identifies a stable matching in every market. We review its details in Appendix D.

2.2 A Multi-Period Market

Building on the one-period model, suppose that agents interact over two periods, $t \in \{1, 2\}$. In every period each man (woman) can be matched with exactly one woman (man) or not matched at all. An agent's partner in period t need not be his partner in period t'. We call this sequence of matches a *partnership plan.* More formally, $(i, j) \in W_m \times W_m$ is a partnership plan for man m where he is matched with i in period 1 and with j in period 2.[7] When confusion is unlikely, we abbreviate a partnership plan as $ij \equiv (i, j)$. A plan ij is *persistent* if $i = j$. Else, it is *volatile.*

Each agent's preferences are defined over partnership plans. All preferences are strict and complete. Thus, a strict preference $\succ_m$ for m is an asymmetric and negatively-transitive binary relation defined on $W_m \times W_m$. If m prefers plan ij to plan kl, we write $ij \succ_m kl$. We specify a full preference ranking for m by writing

$$\succ_m\colon ij, kl, \ldots.$$

[6] Roth (2008) explains this algorithm's broader practical and theoretical importance.

[7] If $i = m$ or $j = m$, then m is not matched to any woman in the corresponding period.

Here ij is m's most preferred plan, kl is second best, and so on. The associated weak preference relation, $\succsim_m$, is defined as $ij \succsim_m kl \iff kl \not\succ_m ij$.

A multi-period matching is a sequence of one-period matchings.

Definition 2. The function $\mu\colon M \cup W \to (M \cup W)^2$ is a *(multi-period) matching* if for all i, $\mu(i) = (\mu_1(i), \mu_2(i))$ where μ_t is a one-period matching implemented in period t.

Henceforth, we refer to a multi-period matching simply as a matching.

2.3 Stability

In the one-period market, stability combines an individual-rationality requirement and a no-blocking condition. We propose two definitions of stability in our multi-period market reflecting both ideas. Ex ante stability is motivated by situations where agents can commit to a partnership plan. Dynamic stability presumes no commitment.

2.3.1 Ex Ante Stability

Ex ante stability considers an agent's incentive to accept or block a matching before it is implemented. If the plan is better than each agent's unilateral outside option (i.e. being single for all periods) and no pair of agents can craft a better plan only among themselves, the matching is ex ante stable. Thus, ex ante stability implicitly assumes that agents can commit to the proposed plan as interim revisions are impossible. More formally, we have the following definitions.

Definition 3. The partnership plan ij is *ex ante individually rational* for m if $ij \succsim_m mm$.

Definition 4. The pair $(m, w) \in M \times W$ can *period-1 block* the matching μ if any of the following conditions hold:

1. $ww \succ_m \mu(m)$ and $mm \succ_w \mu(w)$;

2. $wm \succ_m \mu(m)$ and $mw \succ_w \mu(w)$;

3. $mw \succ_m \mu(m)$ and $wm \succ_w \mu(w)$; or,

4. $mm \succ_m \mu(m)$ and $ww \succ_w \mu(w)$.[8]

[8]Noting Definition 3, some readers may view condition (4) as redundant. It is included because individual rationality, blocking, and stability are special cases of the coalition-based definitions proposed in Section 5.

Definition 5. The matching μ is *ex ante stable* if

1. $\mu(i)$ is ex ante individually rational for all i; and,

2. μ cannot be period-1 blocked by any pair.

Echoing the one-period case, an ex ante stable matching always exists.

Theorem 2. *There exists an ex ante stable matching.*

Proof. We prove this theorem by adapting the deferred acceptance algorithm (Gale and Shapley, 1962). Our adaptation posits that each man proposes to one woman at a time but includes terms concerning their relationship's timing.

Start by defining a restricted market where each man's preference $\succ_m$ is transformed into $\succ'_m$ as follows: All plans of the form ww', where $w, w' \in W$ and $w \neq w'$ are ranked below mm. The relative rankings of all remaining plans (of the form ww, wm, mw, or mm) are unchanged relative to $\succ_m$. For example, if

$$\succ_m : mw_1, w_2w_1, w_1m, w_1w_2, w_1w_1, mm, \ldots,$$

then we can define $\succ'_i$ as

$$\succ'_m : mw_1, w_1m, w_1w_1, mm, w_2w_1, w_1w_2, \ldots.$$

For each woman, define $\succ'_w$ analogously.

We construct a matching μ^* via a deferred acceptance procedure where agents make and accept proposals by following the preference $\succ'_i$:

1. In round 1, each man proposes to the woman implicated in his most-preferred partnership plan according to $\succ'_m$ and offers her the terms of that plan. For example, if m ranks wm as the best plan, he proposes to w the idea that they partner for period 1 only and be single in period 2. If mm is m's most preferred plan according to $\succ'_m$ then he does not make any proposals.

   Given all received proposals, each woman tentatively accepts her most preferred proposed plan. She rejects all proposals if she prefers ww to any of them.

2. More generally, in round t, each man whose proposal was rejected in the previous round proposes to the woman implicated in his most preferred plan that has not yet been

rejected. He offers her the terms of that agreement. If mm is the man's most preferred plan according to $\succ'_m$ among those not rejected, he does not make any proposals.

Out of any new proposals and her current tentatively accepted plan (if any), each woman tentatively accepts her most preferred plan and rejects the others. She rejects all proposals if she prefers ww to any of them.

3. The process terminates when no further rejections occur. The tentatively-accepted plans are implemented as the matching μ^*. Agents without an accepted plan remain single for both periods.

We must check two conditions to confirm that μ^* is ex ante stable. First, it is ex ante individually rational as the algorithm ensures no agent i is ever tied-up in a plan worse than ii. Second, μ^* cannot be period-1 blocked by any pair (m, w). To prove this, suppose the contrary. This implies that there exists a plan $x_m \in \{wm, mw, ww\}$ for m and a compatible plan $x_w \in \{mw, wm, mm\}$ for w such that $x_m \succ_m \mu^*(m)$ and $x_w \succ_w \mu^*(w)$.[9] If $x_m \succ_m \mu^*(m)$, then $x_m \succ'_m \mu^*(m)$. Hence, m must have proposed the plans $\{x_m, x_w\}$ to w at some round before he made his proposal defined in $\mu^*(m)$. w must have rejected that original proposal; thus, $\mu^*(w) \succ'_w x_w$. This implies that $\mu^*(w) \succ_w x_w$—a contradiction. □

To conclude, we comment on Theorem 2's proof. First, the constructed matching may not be Pareto-optimal.[10] Second, the contracts that men propose specify the dates of the relationship and the binding dates of single-hood. Though atypical in interpersonal relationships, such arrangements bear resemblance to a non-compete clause in an employment contract.[11] Finally, if agents are restricted (for example, by legal institutions) to choose only among persistent plans, ex ante stability reduces to stability in the sense of Gale and Shapley (1962). Such restrictions can be encoded in agents' preferences.

2.3.2 Dynamic Stability

Though some applications feature binding long-term arrangements, many do not. Instead, agents can renege or block a plan at an interim stage before its conclusion. Some agents might re-match between periods. A dynamically-stable plan is both ex ante stable and immune to blocking actions conditional on the passage of time.

[9] By "compatible plan" we mean the timings coincide appropriately.

[10] See Example C.1 in Appendix C. The matching μ Pareto-dominates μ' if for all i, $\mu(i) \succsim_i \mu'(i)$ and $\mu(i) \succ_i \mu'(i)$ for at least one i. A Pareto-optimal matching is not Pareto-dominated by any other matching.

[11] As another example, most colleges allowing deferred matriculation do not allow the student to enroll in a different degree program during the deferment.

Definition 6. The partnership plan ij is *dynamically individually rational* for m if $ij \succsim_m mm$ and $ij \succsim_m im$.

Definition 7. The pair $(m, w) \in M \times W$ can *period-2 block* the matching μ if any of the following conditions hold:

1. $(\mu_1(m), w) \succ_m \mu(m)$ and $(\mu_1(w), m) \succ_w \mu(w)$; or,
2. $(\mu_1(m), m) \succ_m \mu(m)$ and $(\mu_1(w), w) \succ_w \mu(w)$.

Definition 8. The matching μ is *dynamically stable* if

1. $\mu(i)$ is dynamically individually rational for all i; and,
2. for all t, μ cannot be period-t blocked by any pair.

While dynamic stability has intuitive appeal due to most people's limited commitment ability, it is too strong to always point to a stable outcome.

Example 1. Suppose there is one man m and one woman w. Their preferences are

$$\succ_m : wm, ww, mm, \ldots$$
$$\succ_w : mm, ww, \ldots$$

All partnership plans not listed are inferior to those listed.

There are only two candidate stable matchings. The matching where $\mu(w) = ww$ is not ex ante stable. The couple can period-1 block it as both prefer a long-term relationship to no relationship at all. The matching where $\mu'(w) = mm$ is ex ante stable. However, it is not dynamically stable since m will renege after period 1.

Example 1 suggests that agents' preferences must exhibit additional structure to guarantee the existence of a dynamically-stable matching. We develop this structure in the following section. Before doing so however, we remark on the idea of "stability" in dynamic markets. Notably, dynamic stability is distinct from "autarkic stability" and "stability" as proposed by Kennes et al. (2013), though it is closer to the latter. Damiano and Lam (2005) and Kurino (2009) emphasize the importance of credibility in agents' blocking actions. We do not entertain such higher-order concerns. We further elaborate on the preceding observations and relations in Appendix B.

3 Preferences

We propose two preference restrictions that together ensure the existence of a dynamically-stable matching in a wide class of situations. The restrictions encode intuitive properties that are commonly encountered in practice. The first links an agent's preferences (defined over plans) to a ranking of partners viewed in isolation, not unlike in a single-period market. Variants of such preferences are sometimes called separable or time-invariant. Regrettably, preferences in this class preclude many economically-meaningful situations, particularly those associated with inter-temporal complementarities. Thus, our second qualification introduces preference inertia, which is analogous to status quo bias. Combining these conditions results in preferences satisfying the rankability condition of Kennes et al. (2013). Our construction decouples the ranking and the inertia elements implied by rankability. This separation cleanly illustrates the implications of each component concerning the existence of stable and core matchings and the relationships between said concepts. For example, anticipating the argument of Section 5, by "magnifying" the degree of preference inertia, we better align the sets of stable and core matchings. Therefore, the separation affords increased analytic flexibility.

3.1 Spot Rankings

We introduced our model by considering the one-period marriage market. In that model agents had a ranking of potential partners. In our dynamic model, they have preferences over partnership plans. Thus far we have not presumed any connection between one-period rankings and preferences over plans. Arguably, some connection ought to exist.

Adapting the terminology and notation from the one-period model, we henceforth call a (strict) ranking of potential partners when viewed in isolation a *spot ranking*. We let P_m denote such a ranking for agent m. Thus, P_m is a strict linear order of W_m. If $i, j \in W_m$ and i *is superior to* j, we write iP_mj. Given a spot ranking, we can identify a family of preferences that capture that ranking's essence. Intuitively, plans combining individually higher-ranked options are preferred.

Definition 9. Let P be a spot ranking. The preference $\succ$ *reflects* P if and only if

1. iPi' and $jPj' \implies ij \succ i'j'$;

2. $\forall i$, $jPj' \implies ij \succ ij'$; and,

3. $\forall j$, $iPi' \implies ij \succ i'j$.

Let $\mathcal{S}_i$ be the set of preference profiles for agent i that reflect some spot ranking.

Example 2. Consider the case of a man m where $W = \{w_1, w_2\}$. The preferences

$$\succ: w_1w_1, w_1w_2, w_2w_1, w_2w_2, mw_1, mw_2, w_1m, w_2m, mm \tag{1}$$

and

$$\succ': w_1w_1, w_2w_1, w_1w_2, w_2w_2, mw_1, mw_2, w_1m, w_2m, mm \tag{2}$$

both reflect w_1Pw_2Pm. On the other hand,

$$\succ'': w_1w_1, w_1w_2, w_2w_2, w_2w_1, mw_1, mw_2, w_1m, w_2m, mm \tag{3}$$

does not reflect any spot ranking.

Preferences in $\mathcal{S}_i$ feature in many related studies. These preferences are called "time invariant" by Kurino (2014) and are also used by Pereyra (2013). They also often emerge when preferences are defined with an additively-separable utility function (Damiano and Lam, 2005; Bloch and Cantala, 2013). Bando (2012) employs a closely-related history independence condition.

Definition 9 identifies a family of preferences for each spot ranking. To go in the opposite direction—from a $\succ$ to a P—we focus on persistent plans. The next definition follows an analogous exercise performed by Kennes et al. (2013) in their definition of an "isolated preference relation."

Definition 10. The *ex ante spot ranking induced by preference* $\succ$, denoted $P_\succ$, is a spot ranking defined as $iP_\succ j \iff ii \succ jj$.

The ex ante spot ranking stems from an agent's preferences in the hypothetical case where he is matched with the same partner for all periods. The construction parallels the manner in which "per-period preferences" might be identified when examining consumption over time. For example, if an agent maximizes a discounted sum of utilities, we can identify his per-period preferences with his preferences over constant consumption streams. Similarly, when examining risk preferences, we can extract an agent's assessment of the available prizes by eliciting preferences for degenerate lotteries promising the same prize in every state of the world.

We rely on the following technical result tying Definitions 9 and 10 together.

Lemma 1. *Let $\succ$ be a preference for agent m. If $\succ$ reflects P, then $P = P_{\succ}$.*

Proof. See Appendix A. $\square$

3.2 Preference Inertia

A limitation of preferences in $\mathcal{S}_i$ is that an agent's spot ranking of partners in period 2 is independent of his partner in period 1. This misses a key feature of dynamic choice problems where implicit or explicit switching costs often matter. For example, after a year of college a student may become more enthusiastic about his school given his new friends. Similarly, a worker performing a task may wish to continue at that task since learning-by-doing renders it easier over time. Additionally, many psychological biases imply inertia in observed decision making even if "true preferences" lack such inclinations or if switching costs are small. For example, Samuelson and Zeckhauser (1988) examine status-quo bias across a variety of economic domains. Similarly, Kahneman et al. (1990) examine the endowment effect where an initial allocation of goods (the period 1 matching) tilts preferences toward maintaining that same allocation in the future.

A simple way to capture preference inertia is to allow persistent plans to rise in their relative ranking.

Definition 11. Let $\succ$ and $\succ'$ be preferences. $\succ$ *exhibits inertia relative to* $\succ'$ if and only if

1. $ii \succ' jk \implies ii \succ jk$;
2. $ii \succ' jj \iff ii \succ jj$; and,
3. If $i \neq i'$ and $j \neq j'$, then $ii' \succ' jj' \iff ii' \succ jj'$.

When a preference exhibits inertia relative to another, it shares the same ex ante spot ranking and the relative ordering of volatile plans is unchanged. Only persistent plans (weakly) "move-up" in rank. For instance, in Example 2 above, $\succ''$ exhibits inertia relative to $\succ$. Conditional on being matched with w_2 in period 1, m is more enthusiastic about continuing that relationship than switching to w_1. According to $P_{\succ''}$, however, w_1 is superior to w_2. Thus, preferences with inertia can exhibit interim preference reversals.

While any preference can be seasoned with extra inertia, we focus on inertia's implications for preferences that reflect a spot ranking. More formally, we let $\Upsilon(\succ)$ be the set of all preference profiles that exhibit inertia relative to $\succ$.[12] We call

$$\bar{\mathcal{S}}_i \equiv \bigcup_{\succ \in \mathcal{S}_i} \Upsilon(\succ)$$

the set of *preferences with inertia relative to* $\mathcal{S}_i$.[13] Preferences with inertia relative to $\mathcal{S}_i$ satisfy the rankability condition of Kennes et al. (2013, Assumption 1). Though preferences with inertia relative to $\mathcal{S}_i$ are motivated by the merger of two common behavioral characteristics (spot rankings and inertia), they also enjoy several useful technical properties.

Lemma 2. *Suppose* $\succ \in \bar{\mathcal{S}}_m$.

1. $ii \succ jj \implies ii \succ ij$ *and* $ii \succ jj \implies ii \succ ji$.

2. $ij \succ jj \implies ii \succ jj$ *and* $ji \succ jj \implies ii \succ jj$.

3. *Whenever* $j \neq i$ *and* $k \neq i$, *then* $jj \succ kk \iff ij \succ ik$ *and* $jj \succ kk \iff ji \succ ki$.

Proof. See Appendix A. □

4 Dynamically-Stable Matchings

Example 1 showed that dynamically-stable matchings need not exist, even in very small markets. By restricting agents' preference to $\bar{\mathcal{S}}_i$ existence is restored. We prove this result in Theorem 3 below by employing the following multi-period adaptation of Gale and Shapley's deferred acceptance algorithm.

Definition 12. The *ex ante deferred acceptance procedure* defines a matching μ^* as follows:

1. For all t, let $\mu_t^*(\cdot)$ be the matching identified by the one-period, man-proposing deferred acceptance algorithm when each agent i reports the ranking $P_{\succ_i}$ as his/her preferences.[14]

2. For each i, assign the partnership plan $\mu^*(i) = (\mu_1^*(i), \mu_2^*(i))$.[15]

Theorem 3. *If* $\succ_i \in \bar{\mathcal{S}}_i$ *for all* $i \in M \cup W$, *then there exists a dynamically-stable matching.*

[12] The intended mnemonic is that Υ (*up*silon) moves persistent plans up in the preference ranking.
[13] Since $\succ$ exhibits inertia relative to itself, $\mathcal{S}_i \subset \bar{\mathcal{S}}_i$.
[14] That is, $\mu_t^* = \mu_{t'}^*$.
[15] This procedure differs from the DA-IP mechanism proposed by Kennes et al. (2013). See Example 5.

Proof. Let μ^* be the matching identified by the ex ante deferred acceptance procedure. We argue that μ^* is dynamically stable by checking three conditions.

First, we verify that μ^* is dynamically individually rational for all $m \in M$. (The same reasoning applies if $w \in W$.) Suppose $\mu^*(m) \neq mm$. Since $\mu^*(m)$ is a persistent allocation, $\mu_1^*(m) = \mu_2^*(m) = w$ for some $w \in W$. The deferred acceptance algorithm always identifies an individually-rational matching. Thus, $wP_{\succ_m}m$. This implies $ww \succ_m mm$. Also, $ww \succ_m mm \implies ww \succ_m wm$ (Lemma 2).

Second, we show that μ^* cannot be period-2 blocked. Suppose the contrary and assume $m \in M$ and $w \in W$ can period-2 block μ^*. If $\mu^*(m) = ii$ this means that $iw \succ_m ii$. By Lemma 2, $ww \succ_m ii \implies wP_{\succ_m}i$. By the same reasoning as in the above paragraph, we can conclude that $mP_{\succ_w}j$ where $\mu^*(w) = jj$. However, this implies that the matching identified by the deferred acceptance algorithm given the reported rankings was not (pairwise) stable, contradicting Gale and Shapley (1962).

Finally, we confirm that μ^* cannot be period-1 blocked. Suppose the contrary and assume m and w can period-1 block μ^*. As above, let $\mu^*(m) = ii$ and $\mu^*(w) = jj$. There are three sub-cases:

1. Suppose $ww \succ_m ii$ and $mm \succ_w jj$. This implies $wP_{\succ_m}i$ and $mP_{\succ_w}j$. However, this implies that the single period matching $\mu_t^*(\cdot)$ is not a (pairwise) stable matching in the sense of Gale and Shapley (1962)—a contradiction.

2. Suppose $wm \succ_m ii$ and $mw \succ_w jj$. Since $ii \succsim_m mm$, if $ii \succ_m ww$ then $ii \succ_m wm$, which is a contradiction. Thus, $ww \succ_m ii$. Likewise $mm \succ_w jj$. Hence, case 1 above applies.

3. Suppose $mw \succ_m ii$ and $wm \succ_w jj$. Since $ii \succsim_m mm$, if $ii \succ_m ww$ then $ii \succ_m mw$, which is a contradiction. Thus, $ww \succ_m ii$. Likewise $mm \succ_w jj$. Hence, case 1 above applies.

Therefore, our assumption concerning the existence of a blocking pair is incorrect. □

As follow-up to Theorem 3, we note four observations. First, volatile plans can be dynamically stable. Second, expediting a future one-period matching generates a new dynamically-stable matching. Third, preferences not in $\bar{\mathcal{S}}_i$ do not preclude the existence of a stable matching. Finally, an alternative, intuitively appealing, matching procedure relying on successive "spot markets" to facilitate re-matchings need not yield a stable matching.

4.1 Volatile Plans

The proof of Theorem 3 constructs a dynamically-stable, but persistent, plan. Volatile plans can also be dynamically stable.

Example 3. Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. Agents' preferences are:

$$\succ_{m_1}: w_1w_1, w_1w_2, w_2w_2, w_2w_1, m_1m_1, \ldots$$
$$\succ_{m_2}: w_2w_2, w_2w_1, w_1w_1, w_1w_2, m_2m_2, \ldots$$

$$\succ_{w_1}: m_2m_2, m_2m_1, m_1m_2, m_1m_1, w_1w_1, \ldots$$
$$\succ_{w_2}: m_1m_1, m_1m_2, m_2m_1, m_2m_2, w_2w_2, \ldots$$

In this case, $\succ_i \in \bar{\mathcal{S}}_i$ for all i. There are exactly three dynamically-stable matchings, as summarized in Table 1. To read the table, under μ^1 agent m_1 is matched to w_1 in both periods. The matching μ^2 is volatile as agents swap partners between period 1 and 2.

Table 1: All dynamically-stable matchings in Example 3.

Matching	m_1	m_2	w_1	w_2
μ^1	w_1w_1	w_2w_2	m_1m_1	m_2m_2
μ^2	w_1w_2	w_2w_1	m_1m_2	m_2m_1
μ^3	w_2w_2	w_1w_1	m_2m_2	m_1m_1

4.2 "If we are going to get married eventually, . . ."

". . . we might as well get married today!" Likely some readers have proposed that idea to a future spouse. Though that suggestion may be met with either unease or delight, by bringing forward in time one-period matchings that are constituents of a dynamically-stable matching, we can actually construct a new dynamically-stable matching.

Theorem 4. *Suppose $\succ_i \in \bar{\mathcal{S}}_i$ for all i and let $\mu = (\mu_1, \mu_2)$ be a dynamically-stable matching. Then $\bar{\mu} = (\mu_2, \mu_2)$ is also a dynamically-stable matching.*

Proof. Consider the dynamically-stable matching $\mu = (\mu_1, \mu_2)$ and let $\bar{\mu} = (\mu_2, \mu_2)$. If $\mu_1 = \mu_2$, then the theorem is trivially true. Henceforth, suppose $\mu_1 \neq \mu_2$. We argue by contradiction. If $\bar{\mu}$ is not dynamically stable, at least one of four possible situations must be true.

1. Suppose $ii \succ_i \bar{\mu}(i)$ for some m. Since $\succ_i \in \bar{\mathcal{S}}_i$, this implies $(\mu_1(i), i) \succ_i (\mu_1(i), \mu_2(i))$. Thus, μ is not dynamically stable—a contradiction.

2. Suppose $(\mu_2(i), i) \succ_i \bar{\mu}(i)$ for some i. This implies $ii \succ_i (\mu_2(i), \mu_2(i))$, which reduces to case 1 above.

3. Suppose m and w can period-1 block $\bar{\mu}$. There are four sub-cases:

   (a) Suppose $mm \succ_m \bar{\mu}(m)$ and $ww \succ_w \bar{\mu}(w)$. This case reduces to case (1) examined above.

   (b) Suppose $ww \succ_m \bar{\mu}(m)$ and $mm \succ_w \bar{\mu}(w)$. This implies that $wP_{\succ_m}\mu_2(m)$ and $mP_{\succ_w}\mu_2(w)$. Hence, $(\mu_1(m), w) \succ_m (\mu_1(m), \mu_2(m))$ and $(\mu_1(w), m) \succ_w (\mu_1(w), \mu_2(w))$. But this implies m and w can period-2 block μ—a contradiction.

   (c) Suppose $wm \succ_m \bar{\mu}(m)$ and $mw \succ_w \bar{\mu}(w)$. From (1) and (2) above, we know that $\bar{\mu}$ is dynamically individually rational. Hence, $\bar{\mu}(m) \succsim_m mm$ and $\bar{\mu}(w) \succsim_w ww$. Noting case (a) above, we may assume that $\bar{\mu}(m) \succ_m mm$ and $\bar{\mu}(w) \succ_w ww$. But this implies $ww \succ_m wm \succ_m \bar{\mu}(m)$ and $mm \succ_w mw \succ_w \bar{\mu}(w)$. Hence, this case reduces to case (2) above.

   (d) Suppose $mw \succ_m \bar{\mu}(m)$ and $wm \succ_w \bar{\mu}(w)$. An analogous argument to the preceding case applies, again leading to a contradiction.

4. Suppose m and w can period-2 block $\bar{\mu}$. Then $(\mu_2(m), w) \succ_m (\mu_2(m), \mu_2(m)) \implies ww \succ_m \bar{\mu}(m)$ and $(\mu_2(w), m) \succ_w (\mu_2(w), \mu_2(w)) \implies mm \succ_w \bar{\mu}(w)$. Hence, the reasoning from case (3) applies.

This eliminates all the possibilities which can imply that $\bar{\mu}_2$ is not dynamically stable. Hence, $\bar{\mu}_2$ is dynamically stable. $\square$

In the introduction, we noted that aspects of our model tie into the phenomenon of market unraveling (Roth and Xing, 1994). In markets that unravel, parties contract at earlier-and-earlier dates. Theorem 4 offers one perspective on this issue as bringing forward relationships that will happen eventually constitutes a stable arrangement. For example, if it is known that a star athlete will play for a professional team (in period 2), then them skipping college or junior-league play (in period 1) may be reflective of a stable outcome. Of course this preliminary observation downplays the role of uncertainty in market unraveling. We therefore revisit this question again below.

The complementary case to Theorem 4 need not be true. Prolonging the period-1 matching, i.e. $\bar{\mu} = (\mu_1, \mu_1)$, can generate an unstable outcome. To anticipate the discussion to follow, however, we qualify this observation in Section 6.

Example 4. Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. Agents' preferences are:

$$\succ_{m_1}: w_1w_1, w_1m_1, w_1w_2, m_1w_1, w_2w_1, m_1m_1, w_2w_2, \ldots$$
$$\succ_{m_2}: w_2w_2, w_2m_2, w_2w_1, m_2w_2, w_1w_2, m_2m_2, w_1w_1, \ldots$$

$$\succ_{w_1}: m_2m_2, m_2m_1, m_1m_2, m_1m_1, w_1w_1, \ldots$$
$$\succ_{w_2}: m_1m_1, m_1m_2, m_2m_1, m_2m_2, w_2w_2, \ldots$$

There are exactly two dynamically-stable matchings in this economy, as summarized in Table 2.

Table 2: All dynamically-stable matchings in Example 4.

Matching	m_1	m_2	w_1	w_2
μ^1	w_2w_1	w_1w_2	m_2m_1	m_1m_2
μ^2	w_1w_1	w_2w_2	m_1m_1	m_2m_2

The matching (μ_1^1, μ_1^1) is not individually rational for m_1 as $m_1m_1 \succ_{m_1} w_2w_2$. Confirming Theorem 4, however, we see that $\mu^2 = (\mu_2^1, \mu_2^1)$. This matching is dynamically-stable.

4.3 Dynamic Stability and General Preferences

An intriguing set of stable matchings can emerge if agents' preferences fall outside of $\bar{\mathcal{S}}_i$.

Example 5. Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. Agents' preferences are:

$$\succ_{m_1}: w_1w_2, w_2w_2, m_1w_1, w_2m_1, m_1m_1, \ldots$$
$$\succ_{m_2}: w_2w_1, m_2w_1, w_2w_2, m_2m_2, \ldots$$

$$\succ_{w_1}: m_2m_1, w_1m_2, m_2m_2, m_1w_1, w_1w_1, \ldots$$
$$\succ_{w_2}: w_2m_1, m_2w_2, m_1m_2, w_2w_2, \ldots$$

In this economy there are exactly three dynamically-stable matchings, as summarized in Table 3. Two facts are noteworthy. First, m_1 and m_2 disagree on which matching is the best among the stable set. m_1 prefers μ^3 while m_2 prefers μ^1. In a single-period setting, all men agree on their preferred stable matching (Gale and Shapley, 1962). Second, an agent may not have any partners in some stable matchings and be matched for all periods in others. This contrasts the "rural hospital theorem" from the single-period market (cf. Roth and Sotomayor, 1990, Theorem 2.16).

Table 3: All dynamically-stable matchings in Example 5.

Matching	m_1	m_2	w_1	w_2
μ^1	m_1m_1	w_2w_1	w_1m_2	m_2w_2
μ^2	m_1m_1	m_2w_1	w_1m_2	w_2w_2
μ^3	w_1w_2	m_2m_2	m_1w_1	w_2m_1

4.4 Repeated Spot Markets

To prove Theorem 3 we employed the ex ante deferred acceptance procedure. This procedure only employed agents' ex ante spot rankings to generate multi-period matching. An intuitively-appealing alternative procedure mimics the operation of successive spot markets. In period 1, the matching is defined by each agent's ex ante spot ranking, as presently. For period 2, however, the matching is based on each agent's spot ranking *conditional* on the period-1 match. This dynamic mechanism leverages information about preferences pinned-down by the period-1 match, a possibly valuable feature. Surprisingly, however, we show that an unstable matching can result from such a mechanism. To arrive at this conclusion, we first define an agent's conditional spot ranking.

Definition 13. If $\succ$ are an agent's preferences, his *conditional spot ranking at* i, denoted $P^i_\succ$, is a spot ranking defined as $jP^i_\succ k \iff ij \succ ik$.[16]

The following matching procedure is a specialization of the DA-IP procedure proposed by Kennes et al. (2013) for an OLG school-choice environment.

Definition 14. The *spot-market deferred acceptance procedure* defines a matching $\tilde{\mu}$ as follows:

1. The period-1 matching $\tilde{\mu}_1$ is the one-period matching identified by the one-period, man-proposing deferred acceptance algorithm where each agent i makes/accepts proposals according to his/her ex ante spot ranking, $P_{\succ_i}$.

2. The period-2 matching $\tilde{\mu}_2$ is the one-period matching identified by the one-period, man-proposing deferred acceptance algorithm where each agent i makes/accepts proposals according to his/her conditional spot ranking at $\tilde{\mu}_1(i)$, $P^{\tilde{\mu}_1(i)}_{\succ_i}$.

At first glance, $\tilde{\mu}$ should identify a stable matching, particularly when agents' preferences exhibit inertia. Given the period 1 matching, each agent's partner rises in rank and should be identified again if the deferred acceptance procedure is repeated. Surprisingly, this intuition can be misleading.

Example 6. Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Agents' preferences are:

$$\succ_{m_1}: w_2w_2, w_1w_2, w_1w_1, \ldots$$
$$\succ_{m_2}: w_1w_1, w_3w_3, w_3w_1, \ldots$$
$$\succ_{m_3}: w_1w_1, w_2w_1, w_2w_2, \ldots$$

$$\succ_{w_1}: m_1m_1, m_2m_2, m_3m_3, m_1m_2, m_1m_3, \ldots$$
$$\succ_{w_2}: m_3m_3, m_1m_1, m_3m_1, \ldots$$
$$\succ_{w_3}: m_2m_2, \ldots$$

Given $\succ_i$ we can define each agent's ex ante spot preference:

$$\begin{array}{ll} P_{\succ_{m_1}}: w_2, w_1, \ldots & P_{\succ_{w_1}}: m_1, m_2, m_3, \ldots \\ P_{\succ_{m_2}}: w_1, w_3, \ldots & P_{\succ_{w_2}}: m_3, m_1, \ldots \\ P_{\succ_{m_3}}: w_1, w_2, \ldots & P_{\succ_{w_3}}: m_2, \ldots \end{array}$$

[16]Kennes et al. (2013) present an analogous definition when introducing the "isolated preference relation."

Using the above spot rankings we can construct $\tilde{\mu}_1$ via the man-proposing deferred acceptance algorithm (see Appendix D):

$$\begin{array}{lll} \tilde{\mu}_1(m_1) = w_1 & \tilde{\mu}_1(m_2) = w_3 & \tilde{\mu}_1(m_3) = w_2 \\ \tilde{\mu}_1(w_1) = m_1 & \tilde{\mu}_1(w_2) = m_3 & \tilde{\mu}_1(w_3) = m_2 \end{array}$$

At $\tilde{\mu}_1(\cdot)$, agents' conditional spot rankings are:

$$\begin{array}{ll} P^{w_1}_{\succ_{m_1}} : w_2, w_1, \ldots & P^{m_1}_{\succ_{w_1}} : m_1, m_2, m_3, \ldots \\ P^{w_3}_{\succ_{m_2}} : w_3, w_1, \ldots & P^{m_3}_{\succ_{w_2}} : m_3, m_1, \ldots \\ P^{w_2}_{\succ_{m_3}} : w_1, w_2, \ldots & P^{m_2}_{\succ_{w_3}} : m_2, \ldots \end{array}$$

Using the above spot rankings we can construct $\tilde{\mu}_2$ via the man-proposing deferred acceptance algorithm:

$$\begin{array}{lll} \tilde{\mu}_2(m_1) = w_2 & \tilde{\mu}_2(m_2) = w_3 & \tilde{\mu}_2(m_3) = w_1 \\ \tilde{\mu}_2(w_1) = m_3 & \tilde{\mu}_2(w_2) = m_1 & \tilde{\mu}_2(w_3) = m_2 \end{array}$$

The resulting matching is:

$$\begin{array}{lll} \tilde{\mu}(m_1) = w_1 w_2 & \tilde{\mu}(m_2) = w_3 w_3 & \tilde{\mu}(m_3) = w_2 w_1 \\ \tilde{\mu}(w_1) = m_1 m_3 & \tilde{\mu}(w_2) = m_3 m_1 & \tilde{\mu}(w_3) = m_2 m_2 \end{array}$$

This matching is neither ex ante nor dynamically stable. For example, m_1 and w_2 can period-1 block $\tilde{\mu}$ since $w_2 w_2 \succ_{m_1} \tilde{\mu}(m_1)$ and $m_1 m_1 \succ_{w_2} \tilde{\mu}(w_2)$. Similarly, m_2 and w_1 can also period-1 block $\tilde{\mu}$.

Example 5 highlights three related ideas: exposure, strategic behavior, and unraveling.

The Exposure Problem In Example 5, w_1 faces an exposure problem if she pursues a relationship with m_1, her favorite partner. In period 1 she is able to match with m_1, seemingly making progress toward her most preferred outcome, $m_1 m_1$. Nevertheless, the spot market exposes w_1 to (loosely speaking) risk concerning the durability of others' preferences. Others' changing opinions impose an externality on w_1 ultimately leading to disappointment. An analogous situation arises in a multi-item auction where complementary goods are sold independently, often through multi-round procedures. In that setting, it is known that complementarities often inhibit equilibrium existence and agents face the risk of not securing items at acceptable prices (Milgrom, 2000). This concern features prominently in the design of auctions for radio spectrum, for example, and impinges upon bidding strategies (Bulow et al., 2009).

To mitigate the exposure problem, an auctioneer may bundle related goods or allow package bidding (Ausubel and Milgrom, 2002). That is, solutions involve adopting mechanisms sensitive to complementarities. The spot-market procedure, considered above, ignores such complementarities in its design. The end result can be a matching that fails to accord with a stable ("equilibrium") outcome. Mimicking the idea of bundling, the ex ante deferred acceptance procedure defines an assignment for both periods from the outset thereby limiting exposure and instability.

Strategic Manipulation To manage exposure risk in a spot mechanism, an agent may wish to adopt a more strategic approach than the naive behavior we have implicitly assumed. In a single-period market operating through a deferred acceptance procedure, it is not a dominant strategy for all agents to behave straightforwardly (Roth, 1982).[17] Instead, the side receiving proposals (women in our presentation) has an incentive to strategize. Roth and Rothblum (1999), Ehlers (2004), and Coles and Shorrer (2014) observe that an often worthwhile strategy (with uncertainty concerning others' preferences) involves on an agent "truncating" her preferences by claiming that the least-desirable, but still acceptable, partners are unacceptable.

In a dynamic market, an agent's strategic opportunities are far richer. In Example 5, for instance, w_1 is better off settling for a relationship with a lesser-ranked partner (m_2) from the outset rather than pursuing a relationship with m_1.[18]

Thus, a woman receiving proposals may wish to top-truncate her preferences by claiming the most attractive partners are not acceptable.[19] For example, had w_1 shunned the period-1 proposal of m_1—her favorite partner—she would have been matched with m_2 in both periods and $m_2m_2 \succ_{w_1} \tilde{\mu}(w_1)$. Therefore, the well-known proverb "A bird in the hand is worth two in the bush" provides some strategic guidance in such a market.

Unraveling Finally, the instability of the spot-market outcome suggests that inter-temporal complementarities may contribute to or reinforce market unraveling.

[17]The single-period market is a special case of the multi-period market; hence, existing impossibility results apply to our model. For brevity, we do not investigate the strategic implications of particular matching mechanisms in depth, though this may be an interesting question for future research.

[18]In fact, in the typical implementation of the deferred acceptance procedure, m_2 will propose to w_1 before m_1 does (see Appendix D).

[19]Top-truncation is a common strategy usually attributed to a resource constraint during an interviewing process (Coles et al., 2010). See also Kadam (2014).

Inter-temporal links in preferences allow agents to offer enticing preemptive carrots that can lock-in prospective partners. For example, the job-market for entry-level lawyers in the United States is often viewed as de facto operating through the market for summer law interns in the preceding year.[20] Most firms have a summer program and extend associate job offers for the following year to a high fraction (>90%) of summer interns (National Association for Law Placement, 2014). For firms, there exists a complementarity between summer and permanent positions as training cost can be lessened. For the student, a secured job lessens the burden of the final year of school.

5 The Core and Strong Inertia

It is sometimes claimed that a focus on pairwise blocking renders stability too weak a solution concept. In this section we strengthen our solution concepts by allowing collective blocking actions and we investigate the market's core. While the core may at times be empty, a slight strengthening of the degree of inertia in agents' preferences is enough to ensure stable and core matchings coincide. When the preferences of one side of the market feature sufficient inertia our dynamic problem essentially collapses into a static problem, an observation that may be useful in practical exercises of market design.

A *coalition* C is a non-empty subset of agents, $C \subset M \cup W$. A coalition can block a matching μ if it can leave the market and define a within-coalition matching that its members find preferable to μ. A core matching is immune to such collective deviations. More formally, we have the following analogues of previous definitions.

Definition 15. The function $\mu_t^C : C \to C$ is a *one-period matching for coalition C (at date t)* if and only if

1. For all $m \in M \cap C$, $\mu_t^C(m) \in (W \cap C) \cup \{m\}$.
2. For all $w \in W \cap C$, $\mu_t^C(w) \in (M \cap C) \cup \{w\}$.
3. For all $i \in C$, $\mu_t^C(\mu_t^C(i)) = i$.

Definition 16. A coalition C can *period-1 block* the matching μ if there exist one-period matchings for the coalition C, μ_1^C and μ_2^C, such that for all $i \in C$, $(\mu_1^C(i), \mu_2^C(i)) \succ_i \mu(i)$.

[20] Roth and Xing (1994) and Ginsburg and Wolf (2004) provide detailed descriptions of this market. Avery et al. (2001) examine unraveling in the closely-related market for judicial law clerks.

Definition 17. A coalition C can *period-2 block* the matching μ if there exists a one-period matching for coalition C, μ_2^C, such that for all $i \in C$, $(\mu_1(i), \mu_2^C(i)) \succ_i \mu(i)$.

As above, we can distinguish between an ex ante core and a dynamic core.

Definition 18. The matching μ is in the *ex ante core* if it cannot be period-1 blocked by any coalition.

Definition 19. The matching μ is in the *dynamic core* if for all t it cannot be period-t blocked by any coalition.

Remark 1. The definitions of ex ante and dynamic core collapses to those of ex ante and dynamic stability when only one-agent or couple coalitions are allowed. Single-agent coalitions subsume the individual-rationality requirements.[21]

For context, we note that the ex ante core corresponds to the "core" in Damiano and Lam (2005, Definition 3). What we call the dynamic core is sometimes called the "recursive core."[22] The dynamic core differs from Kurino's (2009) "dynamic group-stability," which allows deviating agents to be matched with non-coalition members in future periods.

In a one-period market, the core is not empty and corresponds to the set of stable matches (Gale and Shapley, 1962). In the multi-period setting, both the ex ante and the dynamic core can be empty, even when all agents have preferences with inertia relative to $\mathcal{S}_i$.

Example 7. Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Agents' preferences are:

$$\succ_{m_1}: w_2w_2, w_3w_3, w_3w_2, m_1w_2, w_1w_2, w_1w_3, m_1m_1, w_1w_1, \ldots$$
$$\succ_{m_2}: w_3w_3, w_1w_1, w_1w_3, m_2w_3, w_2w_3, w_2w_1, m_2m_2, w_2w_2, \ldots$$
$$\succ_{m_3}: w_1w_1, w_2w_2, w_2w_1, m_3w_1, w_3w_1, w_3w_2, m_3m_3, w_3w_3, \ldots$$

$$\succ_{w_1}: m_1m_1, m_1w_1, m_1m_2, m_1m_3, w_1w_1, m_2m_2, m_3m_3, \ldots$$
$$\succ_{w_2}: m_2m_2, m_2w_2, m_2m_3, m_2m_1, w_2w_2, m_3m_3, m_1m_1, \ldots$$
$$\succ_{w_3}: m_3m_3, m_3w_3, m_3m_1, m_3m_2, w_3w_3, m_1m_1, m_2m_2, \ldots$$

[21] There is a generalization of "period-t blocking," "ex ante/dynamic stability," and "ex ante/dynamic core" to economies with more than two periods. A coalition C can *period-t block* μ if there exists a sequence of one-period matchings $(\mu_t^C, \mu_{t+1}^C, \ldots)$ such that for all $i \in C$,

$$(\ldots, \mu_{t-1}(i), \mu_t^C(i), \mu_{t+1}^C(i), \ldots) \succ_i (\ldots, \mu_{t-1}(i), \mu_t(i), \mu_{t+1}(i), \ldots).$$

The definitions of stability and the core follow similarly.

[22] See Damiano and Lam (2005, Definition 4) and Becker and Chakrabarti (1995).

This market has four ex ante stable matchings, noted in Table 4. The table identifies the coalition that can period-1 block each matching. Since all matchings in the ex ante and the dynamic core must be ex ante stable, the ex ante and dynamic cores are empty.[23]

Table 4: All ex ante stable matchings in Example 7 and blocking coalitions.

Matching	m_1	m_2	m_3	w_1	w_2	w_3	Blocking Coalition
μ^1	w_1w_2	w_2w_1	m_3m_3	m_1m_2	m_2m_1	w_3w_3	$\{m_2, m_3, w_2, w_3\}$
μ^2	m_1m_1	w_2w_3	w_3w_2	w_1w_1	m_2m_3	m_3m_2	$\{m_1, m_3, w_1, w_3\}$
μ^3	w_1w_3	m_2m_2	w_3w_1	m_1m_3	w_2w_2	w_3w_1	$\{m_1, m_2, w_1, w_2\}$
μ^4	m_1m_1	m_2m_2	m_3m_3	w_1w_1	w_2w_2	w_3w_3	$\{m_2, m_3, w_2, w_3\}$

The core's emptiness has been noted in other dynamic models under different assumptions (Damiano and Lam, 2005; Kurino, 2009) and, more broadly, in models of many-to-many matching (Blair, 1988). To restore the core's non-emptiness, we modify the set of admissible preferences by strengthening the degree of inertia.[24]

Definition 20. Let $\succ$ be a preference for agent i.

1. $\succ$ exhibits *strong inertia* if for all j, k, $j \neq k$, $jj \succ jk$ and $jj \succ kj$. Let $\mathcal{I}_i$ denote the set of preference profiles for i that exhibit strong inertia.

2. $\succ$ exhibits *very strong inertia* if $jj \succ kl$ for all j, k, l, $k \neq l$. Let $\mathcal{I}_i^*$ denote the set of preference profiles for i that exhibit very strong inertia.

Preferences with strong inertia are reasonable in cases where an agent's enthusiasm for a particular partnership plan is most strongly influenced by his least-favorite partner in that plan.[25] Preferences with very strong inertia imply an overwhelming desire for persistent plans. Such preferences (or priorities) feature in most school-choice applications. Typically, a school guarantees currently-enrolled students a spot for the following year. In this case, the priority structure exhibits very strong inertia (Kennes et al., 2013).[26]

[23] Example C.2 in Appendix C show that even if the ex ante core is not empty, the dynamic core can be empty.

[24] Weakening the degree of inertia does not always lead to a non-empty dynamic core (Example C.3).

[25] A switching cost can rationalize the strict inferiority of variable plans. A moment of reflection points to a link between these preferences and Leontief preferences from consumer theory.

[26] See also Appendix B. Guaranteed future attendance is not a universal policy. Progression in an educational program may be contingent on performance, implying spots in future years are not guaranteed.

Preferences with strong and very strong inertia are a large class as they place very mild restrictions on the relative rankings of volatile plans. Neither $\mathcal{I}_i$ nor $\mathcal{I}_i^*$ encompass or whittle-down $\bar{\mathcal{S}}_i$, but an overlap exists. For perspective, Figure 1 illustrates the relationships among the preference domains that we have introduced. Intuitively, $\mathcal{I}_i \cap \bar{\mathcal{S}}_i$ and $\mathcal{I}_i^* \cap \bar{\mathcal{S}}_i$ are refinements of $\bar{\mathcal{S}}_i$ that place a more uniform emphasis on inter-temporal complementarities.

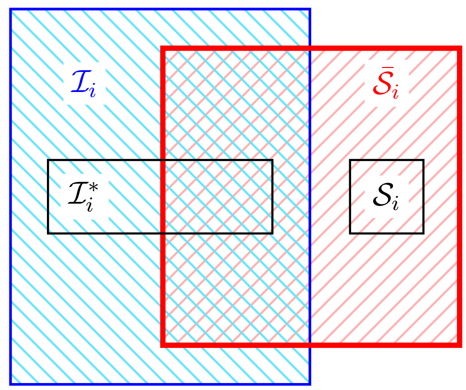

Figure 1: Preference domains. $\mathcal{S}_i$ – preferences that reflect a spot ranking; $\bar{\mathcal{S}}_i$ – preferences that exhibit inertia relative to $\mathcal{S}_i$; $\mathcal{I}_i$ – preferences with strong inertia; $\mathcal{I}_i^*$ – preferences with very strong inertia.

We have already noted that our multi-period model reduces to the one-period model in many special cases. By strengthening inertia, and moving agents' preferences toward $\mathcal{I}_i$ or $\mathcal{I}_i^*$, we again move closer to Gale and Shapley's (1962) setting. Indeed, when each agent's preferences exhibit very strong inertia, our model trivially reduces to their model. However, such a drastic restriction is not required to recover many classic results first observed in the one-period setting. In fact, it is sufficient to nudge the preferences of only one side of the market into $\mathcal{I}_i$ to conclude that the distinction between ex ante and dynamic stability matters no more and all stable matchings are also core matchings.

Theorem 5. *Suppose $\succ_m \in \bar{\mathcal{S}}_m \cup \mathcal{I}_m$ for all $m \in M$ and suppose $\succ_w \in \mathcal{I}_w$ for all $w \in W$.*

1. *All ex ante stable matchings are persistent.*
2. *All ex ante stable matchings are dynamically stable.*
3. *A matching is dynamically stable if and only if it is in the dynamic core.*

Proof. 1. Let μ^* be an ex ante stable matching. Suppose $m \in M$ is assigned a volatile plan: $\mu_1^*(m) \neq \mu_2^*(m)$. Since $\succ_m \in \bar{\mathcal{S}}_m$, $(\mu_t(m), \mu_t(m)) \succ_m \mu^*(m)$ for some $t \in \{1, 2\}$. There are two subcases:

   (a) If $\mu_t(m) = m$, then $mm \succ_m \mu^*(m)$. This contradicts individual rationality.

   (b) If $\mu_t(m) = w \in W$, then $ww \succ_m \mu^*(m)$. For w however, $\mu_t(w) = m$ and since $\succ_w \in \mathcal{I}_w$, $mm \succ_w \mu^*(w)$. Therefore, m and w can period-1 block μ^*—a contradiction.

   Hence, m cannot be assigned a volatile plan in an ex ante stable matching.

   Suppose instead some $w \in W$ is assigned a volatile plan. If w is single in any period, then μ^* is not ex ante individually rational for w. If instead w is never single, each of her male partners must have also been assigned a volatile plan and the above argument applies.

2. Let μ^* be an ex ante stable matching. Hence, it is ex ante individually rational and immune to period-1 blocking by any pair of agents. Two additional conditions need to be verified:

   (a) Suppose that for some $m \in M$, $(\mu_1^*(m), m) \succ_m \mu^*(m)$. Since $\mu^*(m)$ is persistent and $\succ_m \in \bar{\mathcal{S}}_m$, $mm \succ_m (\mu_1^*(m), m) \succ_m \mu^*(m)$. Hence, $\mu^*(m)$ is not ex ante individually rational—a contradiction. If instead $(\mu_1^*(w), w) \succ_w \mu^*(w)$ for some $w \in W$, then $ww \succ_w \mu^*(w)$. This also contradicts ex ante individual rationality. Hence, μ^* must be dynamically individually rational for all i.

   (b) Suppose there exists a pair $(m, w) \in M \times W$ that can period-2 block μ^*. Thus, $(\mu_1^*(m), w) \succ_m \mu^*(m)$. Since $\mu^*(m)$ is time invariant and $\succ_m \in \bar{\mathcal{S}}_m$, $ww \succ_m \mu^*(m)$. Similarly, since $(\mu_1^*(w), m) \succ_w \mu^*(w)$ and $\succ_w \in \mathcal{I}_w$, $mm \succ_w \mu^*(w)$. Hence, m and w can period-1 block μ^*, which contradicts ex ante stability.

   Therefore, all ex ante stable matchings are dynamically stable.

3. ($\Leftarrow$) Follows from the definition of the core. ($\Rightarrow$) We prove the contrapositive: If μ is not the in the dynamic core, then it is not dynamically stable.

   If μ can be blocked by a single-agent coalition, then it is not dynamically individually rational. Hence, it cannot be dynamically stable.

Suppose, instead, that μ is dynamically individually rational for all agents, but it can be period-1 blocked by coalition C. C must contain at least one man and one woman. Choose $m \in M \cap C$. Then $(\mu_1^C(m), \mu_2^C(m)) \succ_m \mu(m) \succ_m mm$. There are two sub cases:

(a) If $\mu_1^C(m) = \mu_2^C(m) = w$, then $(\mu_1^C(w), \mu_2^C(w)) \succ_w \mu(w)$. Hence, m and w can period-1 block μ and it is not dynamically stable.

(b) If $\mu_1^C(m) \neq \mu_2^C(m)$ then since $m \in \bar{\mathcal{S}}_i$, $(\mu_t^C(m), \mu_t^C(m)) \succ \mu(m)$ for some t. If $\mu_t^C(m) = m$, then $\mu(m)$ is not dynamically individual rational. Therefore, $\mu_t^C(m) = w$ for some $w \in C \cap W$. For w however, $\mu^C(w) \succ_w \mu(w)$ and since $\succ_w \in \mathcal{I}_w$ and $\mu_t^C(w) = m$, we conclude that $mm \succ_w \mu(w)$. Therefore, m and w can period-1 block μ and it is not dynamically stable.

Hence, if some coalition can period-1 block μ, μ is not dynamically stable.

Finally, suppose μ is dynamically individually rational for all agents but can be period-2 blocked by some coalition C. C must contain at least one man m and one woman w such that $(\mu_1(m), w) \succ_m \mu(m)$ and $(\mu_1(w), m) \succ_w \mu(w)$. If $\mu_1(m) \neq \mu_2(m)$, then by parts 1 and 2 above μ cannot be dynamically stable. Thus, suppose $\mu(m)$ is persistent. Since $\succ_m \in \bar{\mathcal{S}}_m$, $ww \succ_m \mu(m)$. Since $\succ_w \in \mathcal{I}_w$, $mm \succ_w \mu(w)$. Therefore, m and w can period-1 block μ. Thus, μ is not dynamically stable.

□

Since ex ante stable matchings exist, the existence and equivalence of stable and core matchings follows as a corollary.

Corollary 1. *Under the conditions of Theorem 5, the sets of ex ante stable matchings (ES), dynamically-stable matchings (DS), ex ante core matchings (EC), and dynamic core matchings (DC) coincide and are not empty.*

Proof. By definition $DS \subset ES$. From Theorem 5, $ES \subset DS$; hence $ES = DS$. Also from Theorem 5, $DS = DC$. Finally, since $EC \subset ES$, $DS = DC \subset EC \subset ES = DS$. Hence, $EC = DS$. These sets are not empty as $ES \neq \emptyset$. □

6 Limited Information and Learning

We have focused thus far on preference inertia, which is but one feature of multi-period markets. A second distinguishing feature is agents' preference uncertainty due to limited information about the future. Fresh facts come to light, opinions are refined, and changes are implemented on account of new information. Students transfer colleges, marriages are announced (or dissolved), employees are fired or quit, and contracts are breached as relationship-relevant information becomes known.

To accommodate limited information and learning we amend our model in two ways. First, we limit what each agent i knows about his preferences and we specify what he learns as time unfolds. Second, we examine not only the final matching, but we also consider the procedure that led to that matching. The constrained information structure justifies this complementary focus. If agents learn about their preferences over time, whatever matching is implemented in period t should be reliably arrived at by some method using only information available up to period t. This requirement imposes additional structure on the set of matchings we might reasonably expect to see. In our discussion we pay particular attention to the use of an interim market for re-matching between periods. Learning offers a justification for re-matchings and an improvement relative to the initial assignment should be possible. Though we are sympathetic to such an intuition, our analysis qualifies it considerably.

6.1 Uncertain Preferences and Dynamic Stability

Consider agent i who has preferences over partnership plans $\succ$. Suppose, however, that i does not know the full ranking as specified by $\succ$. Instead, his initial knowledge is partial, but it will become more complete with experience in the market. To be precise, suppose that before period 1 agent i knows the following:

(L1) The agent's preferences have inertia in the sense that $\succ \in \bar{\mathcal{S}}_i$.

(L2) The agent knows his ex ante spot ranking of potential partners, $P_{\succ}$.

Given this limited information, there are many preferences that the agent may actually hold. For example, the agent may know that $ll \succ jj \succ kk \succ \cdots$ but the relative ranking of jk, for example, is unknown. As time passes, agent i learns more about his preferences.

(L3) If in period 1 agent i is assigned to j, he learns his preferences for plans of the form jl', for all l'.

Continuing the above illustration, after being matched with j, agent i could learn that $\succ$ in fact satisfies

$$ll \succ jl \succ jj \succ kk \succ jk \succ \cdots ,$$

which supplements his initial knowledge. Together, (L1)–(L3) outline a simple model of path-dependent learning.

Remark 2. (L1)–(L3) is consistent with an agent learning about switching costs or the strength of preference inertia. For example, i knows that $ll \succ jj$, but is initially unsure whether switching to l in period 2 after being matched with j in period 1 is worthwhile. Given his period 1 knowledge, $ll \succ jl \succ jj$ and $ll \succ jj \succ jl$ are both plausible. The agent recognizes the true case only after a period-1 match to j.

Remark 3. Beyond the restrictions in (L1)–(L3), we do not introduce further probabilistic beliefs or priors.

Though we have modified our model, we maintain dynamic stability as our preferred solution concept. At first glance, this claim appears at odds with agents' knowledge and with the feasibility of relevant blocking actions. For example, suppose i is matched to j in period 1 and then to $k \neq j$ in period 2. From a period-1 point of view, i does not know his preference for the plan jk. Thus, he may be uneasy to initiate a period-1 block if this plan is proposed to him ex ante. Similarly, suppose the matching jk is generated by a sequence of spot markets (e.g. Definition 14). Once period 2 has arrived and i can rank jk relative to other options, he cannot turn back the clock to initiate a period-1 block.

Both considerations above draw on a "forward-looking" interpretation of blocking and dynamic stability, which we have thus far emphasized. However, dynamic stability also has a complementary "backward-looking" interpretation as an absence of regret. At the end of period 2, agent i knows his ranking of jk relative to all persistent plans. If jk is not dynamically stable, i may regret not eloping with l (say) at an earlier opportunity. Thus, even with uncertain preferences, dynamically-stable outcomes offer an appealing normative benchmark.

6.2 Matching Procedures

When agents' preferences were known, our analysis emphasized only the final matching, $\mu = (\mu_1, \mu_2)$. We examined the properties of reasonable market outcomes, such as stable or core matchings, without emphasizing how they arose. In an environment with learning and uncertainty, however, preferences and information evolve over time. Arguably, what is considered a "reasonable market outcome" ought to account for these information imperfections and dynamics. We therefore focus our analysis by considering matchings that are compatible with specific procedures sensitive to these environmental features.

A *matching procedure* identifies a matching in every market. We have already encountered many matching procedures in the preceding discussion. The ex ante deferred acceptance procedure and the spot-market deferred acceptance procedure are but two examples. To introduce some notation, an economy is a tuple $e = (M, W, (\succ_i)_{i \in M \cup W})$ encompassing the market's men, women, and their preferences. Let $\mathscr{E}(M, W)$ be the set of all economies with agents in M and W and let $\mathscr{M}(M, W)$ be the set of all matchings of these agents.We call the function $A(\cdot)$ a *matching procedure* if it assigns to each economy a matching. That is, if $e \in \mathscr{E}(M, W)$, then $A(e) \in \mathscr{M}(M, W)$. Let $A_t(e)$ be the one-period matching identified by procedure A in economy e.

Matching procedures may differ along many dimensions. Some always generate Pareto-optimal assignments, others are strategy-proof, and still others are dictatorial—the space of all procedures is expansive.[27] In our case, we are interested in procedures with two pertinent properties. First, we specialize to procedures that do not employ information that is unknown to agents. This implies a procedure cannot use information revealed in period 2 to influence the period-1 matching.

Definition 21. The matching procedure A is *non-prophetic* if for all economies $e, e' \in \mathscr{E}(M, W)$ such that $P_{\succ_i} = P_{\succ'_i}$ for each i , $A_1(e) = A_1(e')$.

A non-prophetic matching procedure uses at most the information contained in agents' ex ante spot ranking in determining a period-1 matching. Both the ex ante and spot-market deferred acceptance procedures are non-prophetic. A trivial non-prophetic procedure always assigns each agent to single-hood in period 1.

Our second restriction is to procedures that generate dynamically-stable matchings whenever $\succ_i \in \bar{\mathcal{S}}_i$ for all i. We call such a procedure *dynamic-stability inclined*; however, we do not impose restrictions on its operation when preferences are fully general. As noted above, such procedures insulate agents from ex post regret.

We offer two theorems drawing on the above restrictions. Abstracting from qualifications, Theorem 6 shows that if $\mu = (\mu_1, \mu_2)$ is a dynamically-stable matching generated by some non-prophetic procedure, then $\bar{\mu} = (\mu_1, \mu_1)$ is also dynamically-stable. In principle, a non-prophetic procedure may generate a dynamically-stable matching where $\mu_1 \neq \mu_2$. μ_2 may reflect new information about agents' preferences thereby justifying the re-matching. We can interpret the interim swapping as the operation of some spot market. The theorem concludes,

[27] In general, a procedure can lead to a random matching, i.e. a distribution over $\mathscr{M}(M, W)$. For simplicity, we consider non-random procedures.

however, that if we are only interested in dynamically-stable matchings, interim re-matchings are unnecessary. Prolonging the period-1 matching leads to a dynamically-stable outcome.

Theorem 6. *Let A be a non-prophetic, dynamic-stability inclined matching procedure.*[28] *Suppose that in economy e where $\succ_i \in \bar{\mathcal{S}}_i$ for all i, $A(e) = \mu = (\mu_1, \mu_2)$. Then $\bar{\mu} = (\mu_1, \mu_1)$ is also a dynamically-stable matching in economy e.*

Proof. Suppose $\mu_1 \neq \mu_2$. Else, the theorem is trivially true. First, we verify that $\bar{\mu}$ is dynamically individually rational. Suppose for some $m \in M$, $mm \succ_m \bar{\mu}(m)$. This implies $\mu_1(m) = w_1 \in W$. If $\mu_2(m) = m$, then $w_1 m \succ_m mm \succ_m w_1 w_1$, which is a contradiction as $\succ_m \in \bar{\mathcal{S}}_m$. Therefore, $\mu_2(m) = w_2 \neq w_1$ and hence

$$w_2 w_2 \succ_m \underbrace{w_1 w_2}_{\mu(m)} \succ_m mm \succ_m w_1 w_1.$$

Now consider an alternative economy, e', with the same agents and where the preferences of all $i \neq m$ are exactly as in e, i.e. $\succ_i = \succ'_i$ for all $i \neq m$. However, the preferences of agent m, $\succ'_m$, exhibit very strong inertia but they maintain the same ex ante ranking as in e, i.e. $P_{\succ'_m} = P_{\succ_m}$. The relative rankings of other partnership plans is unchanged relative to $\succ_m$ as well. Clearly, $\succ'_m \in \bar{\mathcal{S}}_m \cap \mathcal{I}^*_m$ as all persistent plans were shifted to the top of the preference list. Moreover, $mm \succ'_m w_1 w_1 \succ'_m w_1 i$ for all $i \in W_m \setminus \{w_1\}$.

As the procedure A is non-prophetic, $A_1(e) = A_1(e')$. This implies that in the matching $A_1(e')$, agent m is matched to w_1 in period 1. But this contradicts A always generating a dynamically-stable matching when agents' preferences are in $\bar{\mathcal{S}}_i$. Thus, $\bar{\mu}(m) \succsim_m mm$. Noting this fact, it follows that $\bar{\mu}(m) \succsim_m \mu_1(m)m$ as well. Hence, $\bar{\mu}$ is dynamically individually rational.

Suppose some pair, m and w, can period-1 block $\bar{\mu}$. Since $\bar{\mu}$ is dynamically individually rational, there three possible cases:

1. Suppose $ww \succ_m (\mu_1(m), \mu_1(m))$ and $mm \succ_w (\mu_1(w), \mu_1(w))$. Clearly, $w \neq \mu_1(m)$ and $m \neq \mu_1(w)$.

   Now consider an alternative economy e' where the preferences of all agents other than m and w are identical to those in e. However, the preferences of m, $\succ'_m$, are identical to $\succ_m$ except that all persistent partnership plans are shifted to the very top of the preference ranking and $P_{\succ'_m} = P_{\succ_m}$. Define $\succ'_w$ similarly. In this alternative economy, matching

[28] For example, the ex ante deferred acceptance procedure satisfies both conditions. It is not the only procedure to do so.

procedure A must assign agent m to $\mu_1(m)$ in period 1. However $ww \succ'_m \mu_1(m)i$ for all $i \in W_m$. Likewise, w must be assigned to $\mu_1(w)$, but $mm \succ'_w \mu_1(w)j$ for all $j \in M_w$. Hence, m and w would be able to period-1 block the matching generated by A in the economy e', contradicting μ being dynamically stable.

2. Suppose $mw \succ_m (\mu_1(m), \mu_1(m))$ and $wm \succ_w (\mu_1(w), \mu_1(w))$. Since $\succ_i \in \bar{\mathcal{S}}_i$, $ww \succ_m mw \succ_m (\mu_1(m), \mu_1(m))$ and $mm \succ_w wm \succ_w (\mu_1(w), \mu_1(w))$. Thus, case (1) above applies.

3. Suppose $wm \succ_m (\mu_1(m), \mu_1(m))$ and $mw \succ_w (\mu_1(w), \mu_1(w))$. The same reasoning as case (2) and (1) applies.

Therefore, no pair wishes to period-1 block $\bar{\mu}$.

Finally, suppose m and w can period-2 block $\bar{\mu}$. Then $(\mu_1(m), w) \succ_m \bar{\mu}(m) \implies ww \succ_m (\mu_1(m), w) \succ_m \bar{\mu}(m)$. Likewise, $(\mu_1(w), m) \succ_w \bar{\mu}(w) \implies mm \succ_w (\mu_1(w), m) \succ_w \bar{\mu}(w_1)$. But this implies m and w could period-1 block $\bar{\mu}$, which by the previous argument is not possible. Thus, no pair wishes to period-2 block $\bar{\mu}$. □

Remark 4. Theorem 4 showed that period-2 matchings can be brought forward in time to yield a new stable matching. Expediting future matchings that are (possibly) contingent on *unknown* information is often not feasible. Theorem 6 shows that if period-1 matchings are generated by a non-prophetic procedure, they can be prolonged while preserving dynamic stability. This conclusion does not contradict Example 4 as the theorem restricts attention to a subset of dynamically-stable matchings.

Theorem 6 shows that prolonging a period-1 relationship leads to a dynamically-stable outcome overall. This result is important as it provides a fair benchmark for comparison if we consider the gains that may arise if we consider the operation of a market where agents re-match between periods 1 and 2.[29] Theorem 7 offers a welfare comparison of $\mu = (\mu_1, \mu_2)$ and $\bar{\mu} = (\mu_1, \mu_1)$ when both matchings are dynamically-stable. It shows that if $\mu_1 \neq \mu_2$, i.e. a non-trivial interim re-matching occurs, then μ cannot Pareto-dominate $\bar{\mu}$. Thus, a situation where "everyone wins" if they find a new partner for period 2 is impossible.

Theorem 7. *Assume $\succ_i \in \bar{\mathcal{S}}_i$ for all i and suppose $\mu = (\mu_1, \mu_2)$ is a dynamically-stable matching such that $\mu_1 \neq \mu_2$. Let $\bar{\mu} = (\mu_1, \mu_1)$. If for $m \in M$, $\mu(m) \succ_m \bar{\mu}(m)$ then $\mu_2(m) = w \in W$ and $\bar{\mu}(w) \succ_w \mu(w)$.*

[29]It ensures that we compare a dynamically-stable matching with another dynamically-stable matching.

Proof. Let $m \in M$ be such that $\mu(m) = (\mu_1(m), \mu_2(m)) \succ_m (\mu_1(m), \mu_1(m)) = \bar{\mu}(m)$. Hence,

$$(\mu_2(m), \mu_2(m)) \succ_m \mu(m) \succ_m \bar{\mu}(m).$$

If $\mu_2(m) = m$, then μ would not by individually rational for m. Hence, $\mu_2(m) = w \in W$. Thus, $ww \succ_m \mu(m) \succ_m \bar{\mu}(m)$.

Now consider w, from above. We know $\mu_1(w) \neq m$, $\mu_2(w) = m$ and that μ is dynamically stable. Thus, $\mu(w) \succ_m mm$. But this implies $(\mu_1(w), \mu_1(w)) \succ_w \mu(w)$, which is the desired conclusion. □

Remark 5. Theorem 7 is independent of the matching procedure that led to μ.

Theorem 7 allows for a welfare assessment of interim re-matching. It has particular bite in markets with preference uncertainty. As argued above, in such markets we may reasonably expect matchings to be compatible with some non-prophetic procedure due to the environment's limited information. Theorem 6 shows that prolonging an initial matching preserves dynamic stability. Theorem 7 shows that any further re-matching necessarily leads to a welfare loss for half of the agents who receive a new partner. Thus, the operation of an interim market to leverage what agents' learn about their preferences cannot be universally beneficial.

We can relate the above analysis to two practical considerations in the functioning of matching markets. First, uncertainty has been identified as a key reason contributing to market unraveling (Roth and Xing, 1994) and early contracting can serve as insurance (Li and Rosen, 1998).[30] Our model is consistent with such an interpretation in the following sense. If initial matchings are determined by a non-prophetic procedure that is dynamic-stability inclined, agents will generally be averse to revisions of this matching once period 2 arrives. Prolonging the initial matching is dynamically stable and Pareto improvements in relation to that period-1 status quo cannot be realized. The absence of a Pareto improvement renders the functioning of a meaningful period-2 market rather precarious. Hence, the period-2 market, which could exist and be quite lively, has a natural inclination to fold into the period-1 market. Thus, it is not surprising that the law intern gets a permanent job offer a year in advance.

[30]Nearly all theoretical studies of market unravelling focus on information imperfections, or uncertainty about preferences or others' interests. See, for example, Hałaburda (2010), Ostrovsky and Schwarz (2010), Echenique and Pereyra (2013). Avery et al. (2001) describe the market for law clerks in the United States. Their survey evidence corroborates many of these explanations.

Building on the above observation, a second consideration touches on advice to participants in dynamic, multi-period markets, where preferences may be uncertain. While we have only briefly highlighted the strategic concerns in multi-period markets, one emergent theme is that it may be unwise to anticipate a successful re-matching opportunity in a future period. We suggest that the majority of agents, at least in markets approximating our setting, ought to approach multi-period problems anticipating that their initial matching will be in effect for a very long time. This advice stems from the reality of status quo bias in preferences (captured by inertia) and is corroborated by the fragility of future markets where re-matching could occur. This advice is reflected in many institutional arrangements surrounding multi-period relationships. Marriages are presumed to be permanent; labor contracts are often for a time unspecified; and most colleges admit students for a whole degree rather than a year or two. Though a couple may divorce, an employee may quit (be made redundant), or a student may transfer (flunk-out), few parties approach such relationships anticipating those outcomes.

7 Concluding Remarks

We have proposed a conservative generalization of the classic model of one-to-one matching to a multi-period setting. The restrictions on preferences that we identify as supportive of stable outcomes are behaviorally-plausible and, we argue, quite common in practical situations. Though our model is restricted to two periods, the intuition supporting our results generalizes. Our model readily accommodates cases where agents are uncertain about their future preferences. Though agents learn relevant information, leveraging this new information so as to yield a Pareto-improvement relative to a status quo can be difficult.

For tractability we have abstracted away from many features of dynamic markets. For example, we have only sketched some of the strategic nuances at play in dynamic markets. Similarly, we have not addressed the arrival/departure of agents from the market, an important feature in practice (Kennes et al., 2013; Pereyra, 2013; Kurino, 2014). We hope to address these questions, among others, in future work.

A Appendix: Proofs

Proof of Lemma 1. By definition, $P = P_{\succ}$ if and only if for all $i, j \in W_m$, $iPj \iff iP_{\succ}j$. Suppose $\succ$ reflects P. Given $\succ$, $P_{\succ}$ is uniquely defined. Let $i, j \in W_m$ and without loss of generality suppose $iP_{\succ}j$. Suppose, for contradiction, that jPi. Then, $jPi \implies jj \succ ii \implies jP_{\succ}i \implies \neg[iP_{\succ}j]$. The first implication is from Definition 9. The second follows from Definition 10. The final implication is because $P_{\succ}$ is asymmetric. However, the final implication is a contradiction. Therefore, $iP_{\succ}j \implies iPj$. The converse implication follows similarly. Therefore, $P = P_{\succ}$. □

Proof of Lemma 2. Since $\succ \in \bar{\mathcal{S}}_m$, there exists $\succ' \in \mathcal{S}_m$ such that $\succ$ exhibits inertia relative to $\succ'$ and $\succ'$ reflects $P_{\succ'}$

1. $ii \succ jj \iff ii \succ' jj$. Hence $iP_{\succ'}j$ and since $\succ'$ reflects $P_{\succ'}$, $ii \succ' jj \implies ii \succ' ij$. Since $\succ$ exhibits inertia relative to $\succ'$, $ii \succ ij$. To arrive at the second implication, note that $ii \succ' jj \implies ii \succ' ji$. Hence, $ii \succ ji$.

2. Suppose $ij \succ jj$. Then $ij \succ' jj \implies iP_{\succ'}j \implies ii \succ' jj \implies ii \succ jj$. Similarly, assume $ji \succ jj$. Then $ji \succ' jj \implies iP_{\succ'}j \implies ii \succ' jj \implies ii \succ jj$.

3. Suppose $jj \succ kk$. Then $jj \succ' kk$ and thus $jP_{\succ'}k$. Therefore, for all i, $ij \succ' ik$. Since $\succ$ exhibits inertia relative to $\succ$, the relative ranking of volatile partnership plans is unchanged. Thus, $ik \succ ik$. Conversely, $ij \succ ik \implies ij \succ' ik$ as $\succ$ and $\succ'$ agree on the relative rankings of volatile plans. If $kP_{\succ'}j$ then $ik \succ' iyj$, which is not true. Therefore, $jP_{\succ'}k$ and thus $jj \succ' kk \implies jj \succ kk$.

   The second implication follows similarly. $jj \succ kk \implies jP_{\succ'}k \implies ji \succ' ki \implies ji \succ ki$. Conversely, $ji \succ ki \implies ji \succ' ki \implies jP_{\succ'}k \implies jj \succ' kk \implies jj \succ kk$.

□

B Appendix: Comparisons and Contrasts

There are several recent studies of dynamic, two-sided matching markets. These studies differ from our analysis along several dimensions. Principally, these differences can be traced back to the admissible preference domains and to the definition(s) of market stability.

B.1 Damiano and Lam (2005)

Damiano and Lam (2005) develop a multi-period matching model that they use to examine the notion of "stability" in a dynamic, two-sided markets. They identify drawbacks of well-known concepts—such as the core—and they propose alternative definitions, such as self-sustaining stability.

B.1.1 Preferences

Damiano and Lam (2005) adopt a preference specification over "matching plans" (equivalent to our partnership plan) that is based on a discounted sum of per-period utilities. Our preference specification is ordinal over partnership plans.

B.1.2 Stability ("Core" and "Recursive Core")

Damiano and Lam (2005, Definition 3) say a matching μ is in the core if no coalition of agents can propose an alternative matching plan (only among themselves) that they prefer to μ. A key element in this definition is that the alternative within-coalition matching is binding among the coalition members and formed at period 1. Therefore, this definition corresponds to our definition of the ex ante core and is weaker than our definition of dynamic core. The "recursive core," applied to a matching market by Damiano and Lam (2005) but proposed by Becker and Chakrabarti (1995) in a different application, is a stronger definition and corresponds to our definition of the dynamic core. Beyond the requirements identified above, this definition allows for a coalition to form in any period t (conditional on the history of matchings) to block μ.

Beyond examining the core and the recursive core, Damiano and Lam (2005) propose several stronger notions of stability, such as self-sustaining stability and strict self-sustaining stability. These definitions are similar in spirit to the core, but they additionally require any blocking deviations pursued by a coalition to be credible. A coalition can block a proposal only if additional coalitions cannot deviate from the coalition's deviation, and so on. These additional requirements are stronger than what we consider. Imposing such credibility restrictions on blocking coalitions has been considered in many-to-many matching models (Konishi and Ünver, 2006).

B.2 Kurino (2009)

Our study shares a similar motivation to the analysis of Kurino (2009), who also examines a two-sided marriage-market model where agents match for multiple periods. We make divergent assumptions concerning preferences and our preferred specifications of stable outcomes also differ.

B.2.1 Preferences

Kurino (2009) adopts a preference specification based on a sum of per-period utilities. Adopting his notation, he assumes that agent's i utility from the matching μ is $U_i(\mu) = \sum_{t=0}^{T} u_i^t(\mu^t(i))$. While such preferences allow an agents' per-period rankings of potential partners to vary over time in an arbitrary way, such variance is independent of past match outcomes. Therefore, such preferences are neither more general nor more restrictive than our specification.

B.2.2 Stability (Dynamic Group Stability)

Kurino (2009) defines the core analogously to Damiano and Lam (2005). Therefore, our definition of the ex ante core corresponds to his definition. Kurino (2009) provides an example showing the core's emptiness given his preference specification. Kurino's (2009) definition of "dynamic group-stability" allows agents implicated in a potential blocking coalition to be sometimes matched with non-coalition members.[31] We do not allow this possibility.

B.3 Kennes et al. (2013)

Kennes et al. (2013) study a dynamic model of school choice. Therefore, their model is one of many-to-one matching. Thus, unlike our analysis, their model assumes a fundamental asymmetry between the market's two sides.

[31] Kurino's (2009) definition has the flavor of "group stability" as presented by Roth and Sotomayor (1990, Definition 5.4) for the college-admission model.

B.3.1 Preferences (and Priorities)

Kennes et al. (2013) study a school-choice problem. Therefore, they endow students with preferences over schools. Schools rank students based on administratively defined priorities. Preferences and priorities are not defined symmetrically.

Students' Preferences Preference in $\bar{\mathcal{S}}_i$ are equivalent to preferences satisfying the "rankability" assumption proposed by Kennes et al. (2013). Preferences in $\mathcal{S}_i$ satisfy their strong rankability definition.

Schools' Priorities Kennes et al. (2013) propose a priority structure that resembles the Danish daycare assignment system, which is the market motivating their analysis. Focusing on this priority structure, however, implies that they restrict one side of the market (schools) to a set of preferences that is more restrictive (in a sense that we explain below) than those that we allow.

The priority structure has several components describing how a student's priority at a school evolves. Part of their priority structure's complexity stems from the need to accommodate their model's OLG structure, which is absent from our analysis. Here we highlight the most salient feature of this priority structure. Kennes et al. (2013, p. 10) "...assume that each school ranks the children in a lexicographical manner in which children's past attendance matters the most and then some criterion based on exogenous characteristics of the child...." In our setting, which restricts their model to one cohort of children born at period 0 and to schools with capacity 1, their Assumption 2.1 implies that each school ranks persistent plans at the top of its preference ranking. Thus, each school's "preferences" exhibit very strong inertia, which is more restrictive than we assume. Kennes et al. (2013, Remark 1) note that in their many-to-one setting, this special case corresponds to a static school-choice problem.

To illustrate the implications of this restriction on preferences in our setting, if the preferences of one side of the market are in $\bar{\mathcal{S}}_i$ while the preferences of the other are in $\mathcal{I}_i^*$, all dynamically-stable matchings will only feature persistent partnership plans. Of course, this reduces our dynamic market to an (essentially) static one. Thus, our results corroborate Kennes et al. (2013, Remark 1).

Our baseline assumption concerning preferences, $\succ_i \in \bar{\mathcal{S}}_i$, allows volatile plans to be part of a dynamically stable matching.

B.3.2 Stability ("Autarkic Stability" and "Stability")

Kennes et al. (2013) propose two notions of market stability. Their definitions are tailored to a school-choice environment set in an overlapping-generation (OLG) matching model. Notwithstanding cosmetic differences (i.e. the OLG structure, the many-to-one per-period match, etc.), their definitions additionally posit an alternative set of market-destabilizing deviations than we entertain. Overall, our definition of dynamic stability is neither stronger nor weaker than their definitions. We view our definitions as complementary to their proposals.

Kennes et al.'s weakest stability concept is "autarkic stability." Without changing their notation, we reproduce their definition here.

Definition B.1. (Kennes et al., 2013, Definition 6) A matching μ satisfies *autarkic stability* if at any period $t \geq 1$, there does not exist a school-child pair (s, i) such that (1) and (2) below hold at the same time.

1. (a) $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$ or (b) $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$.
2. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

We wish to highlight but two difference between autarkic stability, as defined above, and our definition of dynamic stability.

- Autarkic stability prevents agents on the two sides of the market from "blocking" a proposed matching by committing to a two-period relationship. We call this a period-1 block.
- Autarkic stability allows for a single-period deviation by a student while maintaining the student's original assignment in the following period. We do not allow such deviations in our definition.

"Stability" (Kennes et al., 2013, Definition 7) is a strengthening of autarkic stability. A stable allocation is an autarkic stable allocation that additionally allows for the two-period deviations we discussed above. Additionally agents are forward looking in the sense that they anticipate how match outcomes affect priorities. Despite the strengthening, this definition of stability and its application is not in general stronger than our definition of dynamic stability. Specifically, one side of the market (schools) bases its acceptance/rejection decisions on a period-by-period priority structure neglecting the period 2 consequences of accepting an alternative student in period 1 as part of destabilizing action. In our model, agents on both sides of the market are treated symmetrically.

C Appendix: Additional Examples

Example C.1 (A Pareto-Dominated Ex Ante Stable Matching)**.** This example demonstrates an ex ante stable matching that is Pareto dominated by another ex ante stable matching.

Consider a market with two men and two women. Agents' preferences are:

$$\succ_{m_1}: w_1w_2, w_1w_1, m_1m_1, \ldots$$
$$\succ_{m_2}: w_2w_1, w_2w_2, m_1m_1, \ldots$$

$$\succ_{w_1}: m_1m_2, m_1m_1, w_1w_1, \ldots$$
$$\succ_{w_2}: m_2m_1, m_2m_2, w_2w_2, \ldots$$

There are exactly two ex ante stable matchings:

$$\begin{array}{ll} \mu^1(m_1) = w_1w_1 & \mu^1(m_2) = w_2w_2 \\ \mu^1(w_1) = m_1m_1 & \mu^1(w_2) = m_2m_2 \end{array}$$

$$\begin{array}{ll} \mu^2(m_1) = w_1w_2 & \mu^2(m_2) = w_2w_1 \\ \mu^2(w_1) = m_1m_2 & \mu^2(w_2) = m_2m_1 \end{array}$$

μ^1 is the matching identified by the algorithm in the proof of Theorem 2. The matching μ^2 Pareto dominates μ^1.

Example C.2 (An Empty Dynamic Core but a Non-Empty Ex Ante Core)**.** This example compares the ex ante and dynamic core in the same market. The dynamic core may be empty even when the ex ante core has multiple members.

Consider a market with four men and four women. Agents' preferences are:

$$\succ_{m_1}: w_1w_1, w_1w_3, w_1w_2, w_3w_3, w_2w_2, m_1m_1, \ldots$$
$$\succ_{m_2}: w_2w_2, w_2w_4, w_2w_1, w_4w_4, w_1w_1, m_2m_2, \ldots$$
$$\succ_{m_3}: w_1w_1, m_3m_3, \ldots$$
$$\succ_{m_4}: w_2w_2, m_4m_4, \ldots$$

$$\succ_{w_1}: m_2m_2, m_3m_2, m_1m_2, m_3m_3, m_1m_1, w_1w_1, \ldots$$
$$\succ_{w_2}: m_1m_1, m_4m_1, m_2m_1, m_4m_4, m_2m_2, w_2w_2, \ldots$$
$$\succ_{w_3}: m_1m_1, w_3m_1, w_3w_3, \ldots$$
$$\succ_{w_4}: m_2m_2, w_4w_4, \ldots$$

Though we do not list the entire preference ranking, we can assume that $\succ_i \in \bar{\mathcal{S}}_i$ for all i.

There exist exactly two ex ante stable matchings:

$$\begin{array}{llll} \mu^1(m_1) = w_3w_3 & \mu^1(m_2) = w_4w_4 & \mu^1(m_3) = w_1w_1 & \mu^1(m_4) = w_2w_2 \\ \mu^1(w_1) = m_3m_3 & \mu^1(w_2) = m_4m_4 & \mu^1(w_3) = m_1m_1 & \mu^1(w_4) = m_2m_2 \end{array}$$

$$\begin{array}{llll} \mu^2(m_1) = w_1w_2 & \mu^2(m_2) = w_2w_1 & \mu^2(m_3) = m_3m_3 & \mu^2(m_4) = m_4m_4 \\ \mu^2(w_1) = m_1m_2 & \mu^2(w_2) = m_2m_1 & \mu^2(w_3) = w_3w_3 & \mu^2(w_4) = w_4w_4 \end{array}$$

μ^1 and μ^2 are both in the ex ante core. Only μ^1 is dynamically stable. However, it is not in the dynamic core. μ^1 can be period-1 blocked by the coalition $C = \{m_1, m_2, w_1, w_2\}$. Coalition members prefer the within-coalition matching μ^C:

$$\begin{array}{ll} \mu^C(m_1) = w_1w_2 & \mu^C(m_2) = w_2w_1 \\ \mu^C(w_1) = m_1m_2 & \mu^C(w_2) = m_2m_1 \end{array}.$$

Thus, the dynamic core is empty.

Example C.3 (Dynamic Core Matchings and $\succ_i \in \mathcal{S}_i$)**.** This example demonstrates a market where $\succ_i \in \mathcal{S}_i$ for all i but the set of dynamic core matchings is a strict subset of the set of dynamically-stable matchings. Therefore, restricting preferences to $\mathcal{S}_i$ does not restore the equivalence between stable and core matchings.

Consider a market with four men and four women. Agents' preferences are:

$$\succ_{m_1}: w_4w_4, \dots, w_3w_3, w_1w_4, w_2w_3, w_3w_2, w_2w_2, \dots, w_1w_1, \dots$$
$$\succ_{m_2}: w_4w_4, w_4w_1, w_1w_4, w_1w_1, \dots, w_3w_3, \dots, w_2w_2, \dots$$
$$\succ_{m_3}: w_1w_1, w_1w_4, w_4w_1, w_4w_4, \dots, w_2w_2, \dots, w_3w_3, \dots$$
$$\succ_{m_4}: w_1w_1, \dots, w_2w_2, w_4w_1, w_3w_2, w_2w_3, w_3w_3, \dots, w_4w_4, \dots$$

$$\succ_{w_1}: m_1m_1, \dots, m_2m_2, m_1m_4, m_2m_3, m_3m_2, m_3m_3, \dots, m_4m_4, \dots$$
$$\succ_{w_2}: m_1m_1, m_1m_4, m_4m_1, m_4m_4, \dots, m_2m_2 \dots, m_3m_3, \dots$$
$$\succ_{w_3}: m_4m_4, m_4m_1, m_1m_4, m_1m_1, \dots, m_3m_3 \dots, m_2m_2, \dots$$
$$\succ_{w_4}: m_4m_4, \dots, m_3m_3, m_4m_1, m_3m_2, m_2m_3, m_2m_2, \dots, m_1m_1, \dots$$

Though we do not list the entire preference profile, we can further assert that each agent's preferences reflect their ex ante spot preferences, which are defined above. Thus, $\succ_i \in \mathcal{S}_i$ for all i.

While there are many dynamically-stable matches in this economy, consider the matching μ defined as follows:

$$\begin{array}{llll} \mu(m_1) = w_2w_3 & \mu(m_2) = w_1w_4 & \mu(m_3) = w_4w_1 & \mu(m_4) = w_3w_2 \\ \mu(w_1) = m_2m_3 & \mu(w_2) = m_1m_4 & \mu(w_3) = m_4m_1 & \mu(w_4) = m_3m_2 \end{array}$$

This matching is dynamically-stable but is not in the core. For example, consider the coalition $C = \{m_1, m_4, w_1, w_4\}$. This coalition can period-1 block via the within-coalition matching:

$$\begin{array}{ll} \mu^C(m_1) = w_1w_4 & \mu^C(w_1) = m_4m_1 \\ \mu^C(m_4) = w_4w_1 & \mu^C(w_4) = m_1m_4 \end{array}.$$

D Appendix: The Deferred Acceptance Algorithm

We often reference and employ the deferred acceptance algorithm (Gale and Shapley, 1962). Though well-known, we review this procedure below as it applies to a one-period market. Each man m (woman w) has a strict preference ranking, P_m (P_w), over potential partners in W_m (M_w).

Definition D.1. The *man-proposing deferred acceptance algorithm* constructs a (one-period) matching μ as follows:

1. In round 1, each man proposes to his most preferred partner as defined by P_m. (If mP_mw for all $w \in W$, he does not make any proposals.) Given all received proposals, each woman engages her most preferred partner as defined by P_w and rejects the others. All proposals from unacceptable partners (i.e. ranked below w by P_w) are rejected.

2. More generally, in round t, each man whose proposal was rejected in the previous round proposes to his most preferred partner who has not yet rejected him. If all such partners are unacceptable, he does not make any proposals. Out of the set of new proposals and her current engagement (if any), each woman engages her most preferred partner and rejects the others. If all proposals are unacceptable, she rejects them all.

3. The process stops once no further rejections occur. At that time all engaged pairs are matched and agents without a partner remain single (i.e. are matched to themselves).

Remark D.1. There is also a woman-proposing deferred acceptance algorithm. It is identical to the procedure described above with the roles of men and women reversed.

The next example illustrates the algorithm's operation. These same preferences feature in Example 5.

Example D.1. Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Agents' preferences are:

$$\begin{array}{ll} P_{m_1}: w_2, w_1, m_1, \ldots & \quad P_{w_1}: m_1, m_2, m_3, w_1, \ldots \\ P_{m_2}: w_1, w_3, m_2, \ldots & \quad P_{w_2}: m_3, m_1, w_2, \ldots \\ P_{m_3}: w_1, w_2, m_3, \ldots & \quad P_{w_3}: m_2, w_3, \ldots \end{array}$$

That is, m_1 prefers w_2 to w_1. He prefers either to being single. w_3 is not acceptable.

Table 5 summarizes the round-by-round operation of the man-proposing deferred acceptance algorithm. To read the table, in round 1, m_2 and m_3 propose to w_1. She engages m_2, which is underlined and m_3 is rejected. m_1 proposes to w_2 and is engaged. No one proposes to w_3. Eventually we arrive at the final matching:

$$\begin{array}{lll} \mu(m_1) = w_1 & \mu(m_2) = w_3 & \mu(m_3) = w_2 \\ \mu(w_1) = m_1 & \mu(w_2) = m_3 & \mu(w_3) = m_2 \end{array} .$$

Table 5: Round-by-round operation of the deferred acceptance algorithm in Example D.1.

Round	w_1	w_2	w_3
1	$\underline{m_2}, m_3$	$\underline{m_1}$	-
2	$\underline{m_2}$	$m_1, \underline{m_3}$	-
3	$\underline{m_1}, m_2$	$\underline{m_3}$	-
4	$\underline{m_1}$	$\underline{m_3}$	$\underline{m_2}$

References

Abdulkadiroğlu, A. (2013). School choice. In Vulkan, N., Roth, A. E., and Neeman, Z., editors, *The Handbook of Market Design*, chapter 5, pages 138–169. Oxford University Press.

Abdulkadiroğlu, A. and Loertscher, S. (2007). Dynamic house allocation. Mimeo.

Abdulkadiroğlu, A. and Sönmez, T. (2003). School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747.

Ausubel, L. M. and Milgrom, P. R. (2002). Ascending auctions with package bidding. *Frontiers of Theoretical Economics*, 1(1):Article 1.

Avery, C., Jolls, C., Posner, R. A., and Roth, A. E. (2001). The market for federal judicial law clerks. *University of Chicago Law Review*, 68(3):793–902.

Bando, K. (2012). Dynamic matching markets with choice functions. Mimeo.

Becker, R. A. and Chakrabarti, S. K. (1995). The recursive core. *Econometrica*, 63(2):401–423.

Blair, C. (1988). The lattice structure of the set of stable matchings with multiple partners. *Mathematics of Operations Research*, 13(4):619–628.

Bloch, F. and Cantala, D. (2013). Markovian assignment rules. *Social Choice and Welfare*, 40(1):1–25.

Bulow, J., Levin, J., and Milgrom, P. R. (2009). Winning play in spectrum auctions. Working Paper 14765, NBER.

Chakroborty, A., Citanna, A., and Ostrovsky, M. (2010). Two-sided matching with interdependent values. *Journal of Economic Theory*, 145:85–105.

Coles, P., Cawley, J., Levine, P. B., Niederle, M., Roth, A. E., and Siegfried, J. J. (2010). The job market for new economists: A market design perspective. *Journal of Economic Perspectives*, 24(4):187–206.

Coles, P. and Shorrer, R. (2014). Optimal truncation in matching markets. *Games and Economic Behavior*.

Damiano, E. and Lam, R. (2005). Stability in dynamic matching markets. *Games and Economic Behavior*, 52(1):34–53.

Dur, U. (2012). Dynamic school choice problem. Mimeo.

Echenique, F. and Oviedo, J. (2006). A theory of stability in many-to-many matching markets. *Theoretical Economics*, 1(2):233–273.

Echenique, F. and Pereyra, J. S. (2013). Strategic uncertainty and unraveling in matching markets. Mimeo.

Ehlers, L. (2004). In search of advice for participants in matching markets which use the deferred-acceptance algorithm. *Games and Economic Behavior*, 48(2):249–270.

Gale, D. and Shapley, L. S. (1962). College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15.

Ginsburg, T. and Wolf, J. A. (2004). The market for elite law firm associates. *Florida State University Law Review*, 31(4):909–965.

Hałaburda, H. (2010). Unravelling in two-sided matching markets and similarity of preferences. *Games and Economic Behavior*, 69(2):365–393.

Hatfield, J. W. and Kominers, S. D. (2012). Contract design and stability in many-to-many matching. Mimeo.

Hatfield, J. W. and Milgrom, P. R. (2005). Matching with contracts. *American Economic Review*, 95(4):913–935.

Hylland, A. and Zeckhauser, R. J. (1979). The efficient allocation of individuals to positions. *Journal of Political Economy*, 87(2):293–314.

Kadam, S. V. (2014). Interviewing in many-to-one matching market. Mimeo.

Kahneman, D., Knetsch, J. L., and Thaler, R. H. (1990). Experimental tests of the endowment effect and the coase theorem. *Journal of Political Economy*, 98(6):1325–1348.

Kennes, J., Monte, D., and Tumennasan, N. (2013). The daycare assignment: A dynamic matching problem. Mimeo.

Konishi, H. and Ünver, M. U. (2006). Credible group stability in many-to-many matching problems. *Journal of Economic Theory*, 129(1):57–80.

Kurino, M. (2009). Credibility, efficiency, and stability: A theory of dynamic matching markets. Mimeo.

Kurino, M. (2014). House allocation with overlapping generations. *American Economic Journal: Microeconomics*, 6(1):258–289.

Lazarova, E. and Dimitrov, D. (2013). Paths to stability in two-sided matching under uncertainty. Mimeo.

Li, H. and Rosen, S. (1998). Unraveling in matching markets. *American Economic Review*, 88(3):371–387.

Milgrom, P. (2000). Putting auction theory to work: The simultaneous ascending auction. *Journal of Political Economy*, 108(2):245–272.

National Association for Law Placement (2014). Perspectives on fall 2013 law student recruiting.

Ostrovsky, M. and Schwarz, M. (2010). Information disclosure and unravelling in matching markets. *American Economic Journal: Microeconomics*, 2:34–63.

Pereyra, J. S. (2013). A dynamic school choice model. *Games and Economic Behavior*, 80:100–114.

Roth, A. E. (1982). The economics of matching: Stability and incentives. *Mathematics of Operations Research*, 7(4):617–628.

Roth, A. E. (1989). Two-sided matching with incomplete information about others' preferences. *Games and Economic Behavior*, 1(2):191–209.

Roth, A. E. (2008). Deferred acceptance algorithms: History, theory, practice, and open questions. *International Journal of Game Theory*, 36(3–4):537–569.

Roth, A. E. and Rothblum, U. G. (1999). Truncation strategies in matching markets—in search of advice for participants. *Econometrica*, 67(1):21–43.

Roth, A. E. and Sotomayor, M. A. O. (1990). *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Number 18 in Econometric Society Monographs. Cambridge University Press, New York.

Roth, A. E. and Vande Vate, J. H. (1990). Random paths to stability in two-sided matching. *Econometrica*, 58(6):1475–1480.

Roth, A. E. and Xing, X. (1994). Jumping the gun: Imperfections and institutions related to the timing of market transactions. *American Economic Review*, 84(4):992–1044.

Samuelson, W. F. and Zeckhauser, R. J. (1988). Status quo bias in decision making. *Journal of Risk and Uncertainty*, 1(1):7–59.

Shapley, L. S. and Scarf, H. (1974). On cores and indivisibility. *Journal of Mathematical Economics*, 1(1):23–37.

Sotomayor, M. (1999). Three remarks on the many-to-many stable matching problem. *Mathematical Social Sciences*, 38(1):55–70.

Ünver, M. U. (2010). Dynamic kidney exchange. *Review of Economic Studies*, 77(1):372–414.