Stochastic games and reputation cycles

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Abstract

This paper studies a model of reputation in which reputation is modeled as a capital stock accumulated by past investments and can have persistent effects on future payoffs. The setting is a class of discrete-time stochastic games between a long-run firm and a sequence of short-run buyers under different transition rules. If reputation is only influenced by the firm, reputation dynamics is cyclically built and exploited. In the reputation building phase, the buyers buy the product with positive probability to provide the firm with the incentives to invest and the firm plays a mixed strategy to make the buyers indifferent between buying and not buying. In the reputation exploitation phase, the reputation is so high that it is a dominant strategy for the buyers to buy, and as a consequence there is no incentives for the firm to build reputation any more. If reputation can also be affected by the buyers, it is possible that the firm is deprived of the chance of building reputation by the buyers and reputation is stagnant.

1 Introduction

Reputation plays an important role in the creation of inter-temporal incentives in a long-term relationship. We interpret reputation as a capital stock accumulated by the past investments which can generate better future payoffs. For instance, firms build reputation for high quality and popularity, government for low taxation, and workers for productive capacity. Reputation comes from not just today's effort but also from the past. Firms' investments in R&D and marketing improve future product quality and consumer satisfaction. Governments avoid short-sighted high taxation because of its long-run adverse effect on economic growth (Romer, 1986, 1989; Jones and Manuelli, 1990; King and Rebelo, 1990). Workers usually receive costly on-the-job training and learning-by-doing to increase their human capital because they are concern about their future careers (Camargo and Pastorino, 2001).

If reputation is treated as a capital stock, there are some features that can be captured by the following real-life example of firm reputation. Toyota is famous for producing high-quality cars by its continuous investments in quality. However, in 2010, millions of Toyotas were recalled with accelerator or brake problems. But Thomas, a researcher from University of Iowa thinks that "Toyota will escape long-term damage to reputation because the problem is a design flaw, not a mistake in the manufacturing".[1] The first feature is that a firm's one-time low effort depreciates but does not eliminate the accumulated reputation. The design flaw was believed to be "the result of the company growing too big, too quickly in its quest to become the largest car company in the world".[1] The second feature is that a firm is more likely to make low effort and exploit its reputation when reputation is high enough. Now Toyota hasn't just bounced back, with one of the strongest reputations among automakers in the U.S. It is the most highly regarded auto-sector company in the U.S.[2] The third feature is that a firm usually has a second chance of recovery from the depreciated reputation if enough effort is made. As a consequence, reputation can be maintained in the long run.

In the literature, reputation is typically modeled as as the belief held by the customers over non-opportunistic firms. In the perfect monitoring environment, an opportunistic behavior will totally ruin the reputation instead of merely depreciating it. Under imperfect monitoring, even though opportunistic behavior does not totally ruin the reputation, the type is eventually learned, so reputation is a short-term phenomenon (Cripps, Mailath and Samuelson, 2007) and the firm has no chance of rebuilding the reputation unless there is exogenous replacement of types (Holmstrom, 1999; Mailath and Samuelson, 2001; Phelan, 2006; Ekmekci, Gossner and Wilson, 2012) or erasing of history (Liu and Skrzypacz, 2009; Liu, 2011; Ekmekci, 2011). However, in many situations, permanent reputation exists because the firm has the ability of endogenously improving their reputation. Board and Meyer-ter-Vehn (2013) and Dilme

[1] http://news-releases.uiowa.edu/2010/february/020510toyota-researcher.html

[2] 2013 Harris Poll Reputation Quotient (RQ), from Harris Interactive

(2012) model reputation as a belief of product quality which can be changed by firm's past investments. Bohren (2012) interprets reputation as a stock which is stochastically influenced by past effort. But her study relies on the existence of absorbing states in which reputation is eventually trapped and long-run incentives vanishes, thus reputation cycles are not possible.

Motivated by the above examples, this paper models reputation as a state variable in a setting of discrete-time stochastic games in which a long-run firm interacts with a sequence of short-run buyers. The firm's current investment decisions have an impact on the reputation which influences future payoffs of the buyers. We investigate stationary Markov equilibria under different transition rules and determine when the firm builds its reputation and when it decides to exploit the reputation. Furthermore, we study how the short-run players' purchase behavior provide incentives for the firm to build reputation.

We establish that under the transition rules that reputation is only influenced by the firm, reputation dynamics is cyclic, characterized by reputation building and exploitation. In the reputation building phase where the reputation is not high enough, the buyers buy the product with positive probability to provide the firm with the incentives to invest. Moreover, the firm plays a mixed strategy to make the buyers indifferent between buying and not buying. In the reputation exploitation phase, the reputation is so high that it is the dominant strategy for the buyers to buy. Therefore, there is no incentive for the firm to build reputation any more.

Sometimes the future state of a long-term relationship can be impacted also by the decision of the short-run players, not only by the long-run players. For example, a firm's word-of-mouth advertisement today may not effectively improve its reputation if the consumers do not buy the product, experience the good and give high customer ratings to influence the decision of future customers. A worker has no chance of learning-by-doing without being hired in the first place. Therefore, it is useful to investigate reputation building if the buyers also have the power of controlling reputation. In this case, it is possible that reputation is stagnant at low levels since the firm is deprived of the chance of building reputation by the buyers.

1.1 Overview of results

In section 2.1, we consider the one step transition rule. In the one-step transition rule, the firm's one-time investment leads to a one-step upward shift of reputation and no investment causes a probability $1-p$ of one-step decrease of reputation and a small probability p of one-step increase of reputation. For small time interval, reputation dynamics is characterized by a reputation building phase and a reputation exploitation phase. In the reputation building phase, the firm plays the mixed strategy to make the buyers indifferent and buyers' purchase behavior can be characterized by a second order difference equation. In the reputation exploitation phase, the buyers buy the product for sure and the firm does not invest. A limiting result when time interval converges to 0 is deduced. The buyers' purchase behavior needs to increase exponentially as the reputation increases in the reputation phase.

In section 2.2, we generalize the one-step transition rule to stochastic one-step transition rule in which the firm's one-time investment may lead to one-step decrease of reputation with a small probability q. When time interval is small enough, reputation dynamics is similar to that in the one-step transition rule for states not too close to the lower bound. Furthermore, we can solve the unique stationary Markov equilibrium in the limit when time interval converges to 0. The limiting results suggest that higher noise p and q lower the buyer's probability of buying the product in each state, which means that it is more difficult for the firm to build reputation.

In section 2.3, we study the lower bound transition rule in which that no investment leads to a non-absorbing lower bound and investment results in a one-step increase of reputation with small noises. For any fixed time interval and high discount factor, reputation dynamic is also cyclic. In the reputation building phase, the buyers buy with positive probability in an increasing order with respect to reputation and the buyers' purchase behavior is characterized by a first order difference equation. The firm plays a mixed strategy so that the buyers are different between buying or not buying. The result of no investment is a high probability to ruin the reputation to the lowest level. After the ruin, the firm will start over and continue to build the reputation. In the reputation exploitation phase, the buyers buy the product for certain and the firm does not invest. We show that larger noises lower the buyer's probability of buying the product in each state, which means that it is more difficult for the firm to build reputation.

In section 2.4, we study a transition rule that can also influenced by the buyers. It is relatively difficult for the firm to build reputation. If the lower bound of the reputation is high enough, reputation is still cyclic between building and exploitation. If the lower bound of the reputation is not high enough, reputation is not a long-term phenomenon. There are three phases in reputation dynamics: reputation stagnation, reputation building and reputation exploitation. Players behave the same as before in the last two phases. However, if the reputation is very low, the firm is deprived of the chance of building reputation by the buyers. Eventually, reputation goes down to the phase of stagnation after a sequence of unlucky draws and stays there forever.

In section 2.5, we extend our model to a more general setting that the firm has multiple investment choices rather than binary choices: investment or no investment. We propose a setting in which there is a stationary Markov equilibrium in which the firm only chooses between no investment and an "efficient" investment level with the smallest marginal cost relative to marginal benefit. In the reputation building case, the firm mixes between no investment and the "efficient" investment so that the buyers is indifferent between buying and not buying. In the reputation exploitation phase, the buyers buy for sure and the firm has no incentive to invest. However, there may be other equilibria as well.

1.2 Literature

The paper relates to the reputation papers which try to explain permanent reputation and deliver reputation cycles. Liu and Skrzypacz (2013) introduce limit record-keeping ability of the short-run players. The state variable is the clean history: the number of the most recent reputation-building behavior. The long-run player's exploitation will drive the clean history to a non-absorbing lower bound: zero. Liu (2011) provides a model where the limit record is endogenously determined because of costly information acquisition. Phelan (2006) analyzes the reputation model where the type of the long-run player is replaced with certainty probability in each round. The state variable is the belief of the long-run player's being a commitment type. If the long-run player refuses to build reputation, the belief will move to a non-absorbing lower bound: the belief that a long-run player is replaced with a commitment type. All above models show that the reputation dynamics has two phases as in this paper: a reputation-building phase and a reputation-exploitation phase. In the first phase, investment leads to an increase of the updated belief (Phelan, 2006) or a better record of history (Liu and Skrzypacz, 2013; Liu, 2011). The long-run player has the exact incentive to mix between investing and not investing and the short-run players will mix between trust and distrust in an increasing order to provide the long-run player with the exact incentive to mix between investment and no investment. In the latter phase, the long-run player has incentive to exploit the reputation because he can not create cleaner history (Liu and Skrzypacz, 2013; Liu, 2011) or the belief of commitment type is high enough for the trust behavior is a dominant strategy for the short-run player, who has no room to provide incentive to build reputation (Phelan, 2006). My model under the lower bound transition rule delivers similar reputation cycles as papers above except that we allow for the existence of noise. More importantly, we can characterize the reputation cycles when a low effort only leads to a depreciation but not a ruin of reputation under one-step transition rule.

Other explanations of permanent reputation also involve the incomplete record of history such as the information censoring in Ekmekci (2011) and the bounded memory in Monte (2013). In both papers, there exists a finite set of ratings and a transition rule that can explain permanent reputation. In the equilibrium, each rating represents a belief of non-opportunistic type (reputation). The transition rule gets rid of the restriction that a bad behavior leads to reputation vanishing and allows richer transitions. My paper tries to explore the reputation dynamic under different exogenously given transition rules.

As mentioned before, this paper is set up in a stochastic game framework, which is related to Bohren (2011). Bohren studies a continuous-time model with persistent actions and imperfect monitoring (Brownian information) and identifies the conditions for the existence of Markov equilibria, and conditions for the uniqueness of a Markov equilibrium in the class of all perfect public equilibria. We analyze the Markov equilibria of a discrete-time model with persistent actions, sub-modular payoff structure and various transition rules. We establish the uniqueness of a Markov equilibrium among all Markov equilibria under different transition rules in the limit where time interval converges 0. Firstly, this paper allows for an explicit characterization of equilibrium continuation payoffs and actions in the discrete-time setting under the assumption of submodularity. Secondly, Bohren needs Brownian information and uniqueness of the static Nash equilibrium of the auxiliary game considering the long-run incentives. The sub-modular stage game which is typical in the reputation literature does not satisfy this assumption. We can study the sub-modular stage game without assuming Brownian information. Thirdly, in order to guarantee uniqueness of a Markov equilibrium, Bohren (2011) assumes that boundaries of the state space are absorbing points. Therefore, the agency's incentive constraint is reduced to the myopic optimization of its instantaneous flow payoffs at the boundary points and the state will eventually converge to the lower bound. In this paper, the lower bound of the state space is not absorbing state. In the equilibrium the long-run player still has incentive to build reputation at the lower bound and leaves the lower bound. Finally, we only study the Markov equilibria and there are no results for non-Markov equilibria. However, in the continuous-time model with Brownian information, there are only Markov equilibria.

Our work is also conceptually related to Board and Meyer-ter-Vehn (2013) and Dilme (2012). Board and Meyer-ter-Vehn (2013) models reputation as a belief of product quality which can be high or low and reputation can be changed by past investment. Consumers learn about product quality through Poisson noisy signal so that the reputation is smoothly drifted and accompanied with reputation jumps. For a class of imperfect Poisson learning processes and low investment costs, they show that there exists a work-shirk equilibrium in which firm invests when its reputation lies below some cutoff and does not invest above the cutoff. Dilme (2012) models the reputation as a moral-hazard phenomenon. The firm can switch its product quality at each period by paying a switching cost. Therefore, quality can be interpreted as a stock, so the cost of achieving a given stock next period depends on the current period's stock level. Under perfect good news, there exists an ergodic reputation cycle in which the low firm is willing to switch its product to high quality when reputation is low enough and the high firm wants to switch to a low type when reputation is very high. In the intermediate level, there is no switching decision. Under certain imperfect monitoring conditions, both papers have similar reputation cycles as in our paper: reputation dynamics is ergodic and cyclic between reputation building phase and reputation exploitation phase.

2 Model

There is a long-run player 1 who plays with a infinite sequence of short-run players 2 in time period $0, 1, \ldots, \infty$. A short-run player 2 who arrives at time t plays one stage game with player 1 and exits the game. In the stage game, two players move simultaneously. There are two pure actions for player 1: I and NI, which represent investment and no investment. $a \in [0, 1]$ is the mixed strategy of player 1: the probability of playing I. a can be interpreted as the degree of investment. Player 2 chooses between two actions in each period: B and NB, which represents buying the product and not buying the product. Player 2 choose a mix strategy $y \in [0, 1]$ which is the probability of playing B.

Consider the following framework of a stochastic game. The state variable X is called reputation. The state variable $X \in \{0, \Delta, 2\Delta, \ldots\}$. Reputation can only been built smoothly in the sense that any increase of reputation is proportional to the time interval. The past actions of player 1 influence X on which player 2's future payoffs depend. Denote $g_1(a, y)$ as player 1's stage game expected payoff and $g_2(a, y, X)$ as player 2's stage game expected payoff. We assume that the reputation only has a direct impact on short run player 2's future payoff. The time length is Δ. Player 1 discounts the future payoff by $\beta = e^{-b\Delta}$ and maximizes the expected sum of discounted payoffs. Each player 2 maximizes his stage game payoffs. Without loss of generality, assume that $g_1(0, 0) = g_2(0, 0) = 0$. We make the following assumptions on the stage game.

Assumption 1 (Myopic incentive of player 1): $g_1(0, y) > g_1(1, y)$ for any $0 < y \leq 1$. $g_1(a, 1) > g_1(a, 0)$ for any $0 \leq a \leq 1$.

Assumption 2 (Submodularity of player 1): $g_1(0, 0) - g_1(1, 0) < g_1(0, 1) - g_1(1, 1)$.

Player 1 has a myopic incentive not to invest and the incentive is highest when player 2 buys the product for sure. Think about the situation that if a short-run buyer 2 refuses to buy the product, player 1 gets nothing independent of the investment decision. However, there is a cost to invest if the product is bought by the player 2.

Assumption 3 (Myopic incentive of player 2): $g_2(1, y, X) > g_2(0, y, X)$ for any $0 < y \leq 1$. $g_2(1, 1, X) > g_2(1, 0, X)$.

Assumption 4 (Reputation is valuable for player 2): $g_2(a, y, X)$ is strictly increasing in X.

Assumption 5 (Buy for sure for high reputation): There is X^* such that if $X \geq X^*$ then $g_2(0,1,X) \geq g_2(0,0,X)$. Otherwise, $g_2(0,1,X) < g_2(0,0,X)$.

Assumption 5 tells us that if $X \geq X^*$, it is strictly dominant strategy for player 2 to play B. If $X < X^*$, then there is a mixed strategy $\pi(X) \in (0,1)$ of playing I for player 1 to make player 2 indifferent between B and NB. It is reasonable to assume that player 2 will buy the product for sure independent of player 1's current behavior if player 1 has done good enough in the past.

An example of the stage game payoff matrix in my mind is

	B	NB
I	1,$\lambda + (1-\lambda)X$	0,0
NI	2,$-\lambda + (1-\lambda)X$	0,0

In this example, we can show that $\pi(X) = \frac{1}{2} - \frac{1-\lambda}{2\lambda}X$ and $X^* = \frac{\lambda}{1-\lambda}$.

At last, we need to specify the transition rule of state variable to tell how the past actions have an impact on the future payoffs of player 2. The transition rule is important because it provides player 1 the long-run incentive to make an investment in the current period, versus the myopic incentive not to invest duet to the short-term cost. We will study the Stationary Markov equilibrium and reputation dynamics under different transition rules. Denote the tradition rule as $X' = F(a, y, X)$.

In this paper, the short-run player 2 plays a *stationary Markov strategy* if his actions only depend on state variable X, not on other past history and calendar time. In the equilibrium, it is without loss of generality to assume that player 1 also plays stationary Markov strategy since player 1 is best response to player 2's stationary Markov strategy. We pay attention to stationary Markov strategy for several reasons. Firstly, Markov strategy only depends on payoff relevant variables to specify incentives. In this paper, we wants to isolate and study the role of the action persistence in creating a channel for effective inter-temporal incentive provision. Secondly, stationary strategy is a more appropriate concept if we focus on behaviors in the long-run. Thirdly, Markov stationary strategy is simple and in reality the short-run player may not be able or not be willing to get access to all past history of the game, but only one simple state variable: reputation. At last, we can reach tractable solution of Markov stationary equilibrium and show its uniqueness among all Markov stationary equilibria under some assumptions.

Definition 1: $(a(X), y(X), V(X))$ is a *stationary Markov Equilibrium* if $a(X)$ and $y(X)$ are best response to each other in each state X and state variable X is subject to the transition rule. $V(X)$ is player 1's continuation payoff at state X.

$$V(X) = \max_{a\in[0,1]} g_1(a, y(X)) + \beta V(X')$$

$$a(X) \in \arg\max_{a\in[0,1]} g_1(a, y(X)) + \beta V(X')$$

$$s.t.\ X' = F(a, y, X)$$

$$y(X) \in \arg\max_{y\in[0,1]} g_2(a(X), y, X)$$

2.1 One-step transition rule

In one-step transition rule, the domain of next state is either one-step up or one-step down. Compared with investment I, no investment NI is less likely to reach a higher reputation in the next period. However, the consequence of one-period no investment is not so severe as in the lower bound transition rule. Specifically, if the outcome of player 1's action is I, then the next state X' is $X + \Delta$ for certain.

$$P(X'|I) = \begin{cases} 1 & X' = X + \Delta \\ 0 & X' = \max\{X - \Delta, 0\} \end{cases}$$

If the outcome of player 1's action is NI, then the probability that $X' = \max(X - \Delta, 0)$ is $1 - p$ and the probability that $X' = X + \Delta$ is p.

$$P(X'|NI) = \begin{cases} 1-p & X' = \max\{X - \Delta, 0\} \\ p & X' = X + \Delta \end{cases}$$

$p = 0$ corresponds to the determinate case where not investing certainly leads to a one-step depreciation of reputation and investment implies a one-step increase of reputation for sure.

Assumption 1.1: $g_1(1, 0) = g_1(0, 0) = 0$.

Assumption 1.1 is mainly for simplicity and can be relaxed in the later sections. However, it is a proper assumption because in the product-choice game, if the short-run player does not buy the product, the long-run player will get nothing independent of the investment decision. Define $A = \frac{g_1(1,1)}{g_1(0,1)}$, $A_p = \frac{A(1-p)}{1-Ap}$ and $\epsilon = \frac{1}{2\beta}(1 - A_p + \sqrt{(1-A_p)^2 + 4A_p\beta^2})$. Define K such that $K\Delta > X^*$ and $(K-1)\Delta \leq X^*$. In other words, $K = [\frac{X^*}{\Delta}] + 1$.

Assumption 1.2 : $K = [\frac{X^*}{\Delta}] + 1 > 3 + \frac{\log \frac{\epsilon-1}{A_p}}{\log \frac{A_p}{\epsilon}}$.

Assumption 1.2 is satisfied if time interval Δ is small enough. We can characterize the stationary Markov equilibria when time interval Δ is small enough.

Theorem 1: Under Assumption 1-5 and Assumption 1.1-1.2, the stationary Markov equilibria display reputation cycles as below:

(1) In state $X \in \{0, 2\Delta, 3\Delta, \ldots, (K-1)\Delta\}$, player 1 plays mixed strategy $a(k\Delta) = \pi(k\Delta)$. Player 2 also plays strict mixed strategy $y(X) \in (0,1)$. $y(k\Delta)$ is characterized by a second-order difference equation:

$$y((k+1)\Delta) = \frac{1}{\beta}(1 - A_p)y(k\Delta) + A_p y((k-1)\Delta) \ \forall 1 \leq k \leq K-2$$

(2) Under the condition that strict mixed strategy by player 1 in state Δ leads to a solution satisfying $y(\Delta) > 1$, then in state 1, player 1 plays strictly prefers I: $a(X) = 1$ and player 2 buys for sure: $y(\Delta) = 1$. Otherwise, player 1 plays mixed strategy $a(\Delta) = \pi(\Delta)$ and player 2 plays strict mixed strategy $y(\Delta) \in (0,1)$.

(3) State $k\Delta \geq K\Delta$ is a reputation-exploitation phase. In state $k\Delta$, player 1 plays NI for sure and player 2 plays B for sure, i.e. $y(k\Delta) = 1$ and $a(k\Delta) = 0$.

Reputation dynamics is characterized by a reputation building phase and a reputation exploitation phase. In the reputation building phase, player 2 mixes between buying or not buying in order to provide player 1 with the incentive to invest. Player 1 plays a mixed strategy so that player 2 will buy with positive probability. In the reputation exploitation phase, it is dominant strategy for player 2 to buy the product. Therefore, player 2 can not reward player 1, so there is no incentive for player 1 to build reputation any more.

Next, consider the limiting equilibrium when $\Delta \to 0$. We can deduce an analytic solution of the stationary Markov equilibrium. For $X > 0$, define $y(X) = \lim_{\Delta \to 0, k\Delta \to X} y(k\Delta)$, $a(X) = \lim_{\Delta \to 0, k\Delta \to X} a(k\Delta)$, $V(X) = \lim_{\Delta \to 0, k\Delta \to X} V(k\Delta)$.

Proposition 1 : Under Assumption 1-6 and Assumption 1.1, in the limit where $\Delta \to 0$, there is a unique stationary Markov equilibrium.

If $X = 0$, then $y(X) = \frac{1+(1-2p)A}{2-2p} e^{-b\frac{1-A}{1+(1-2p)A}X^*}$, $a(X) = \pi(X)$.

If $0 < X < X^*$, then $y(X) = e^{-b\frac{1-A}{1+(1-2p)A}(X^*-X)}$, $a(X) = \pi(X)$.

If $X \geq X^*$, then $y(X) = 1$,$a(X) = 0$.

$$V(X) = \begin{cases} \frac{1+(1-2p)A}{2-2p} e^{-b\frac{1-A}{1+(1-2p)A}(X^*-X)} g_1(0,1) & 0 \leq X < X^* \\ \left(1 - \frac{(1-A)(1-2p)}{2-2p} e^{-\frac{b}{1-2p}(X-X^*)}\right) g_1(0,1) & X \geq X^* \end{cases}$$

Proposition 1 gives an explicit analytic solution of the stationary Markov equilibrium. In the reputation building phase, the value function $V(X)$ and trust behavior $y(X)$ is convex, which means that player 2 needs to provide higher incentive for player 1 as player 1's reputation increases. Furthermore, the investment behavior $a(X)$ is decreasing because it is easier for player 1 to make player 2 to trust as reputation increases.

Moreover, when player 1 cares less about future (large b) and the cost of reputation building is higher (small A), player 2 need to provide more incentive in the future, a high growth rate in trust $y(X)$ and a lower level of trust at each state X.

2.2 Stochastic one-step transition rule

In this section, we generalize the transition rule such that a one-period investment may cause reputation to go one-step down with small probability $q > 0$. Specifically, if the outcome of player 1's action is I, then the probability that the next state $X' = X + \Delta$ is $1 - q$ and the probability that $X' = X - \Delta$ is q.

$$P(X'|I) = \begin{cases} 1 - q & X' = X + \Delta \\ q & X' = \max\{X - \Delta, 0\} \end{cases}$$

If the outcome of player 1's action is NI, then the probability that $X' = \max(X - \Delta, 0)$ is $1-p$ and the probability that $X' = X + \Delta$ is p.

$$P(X'|NI) = \begin{cases} 1-p & X' = \max\{X-\Delta, 0\} \\ p & X' = X + \Delta \end{cases}$$

Make a few definitions:

$M = 3 + \frac{\log(\beta(\epsilon-1)/\epsilon)}{\log(A_{pq}/\epsilon)}$, $A_{pq} = \frac{(1-p)A-q}{1-q-Ap}$, $\epsilon = \frac{1}{2\beta}(1 - A_{pq} + \sqrt{(1-A_{pq})^2 + 4A_{pq}\beta^2})$.
$x_1 = \frac{1-\sqrt{1-4\beta^2 q(1-q)}}{2\beta(1-q)} < 1$, $x_2 = \frac{1+\sqrt{1-4\beta^2 q(1-q)}}{2\beta(1-q)} > 1$. $D = \frac{(\frac{1-A}{\beta(1-p-q)})(1-\beta(1-q)x_1)}{(\frac{q(1-A)\beta}{1-p-q}+A(1-\beta))(x_2-x_1)}$.

Assumption 2.1 : $A > \frac{q}{1-p}$ and $\frac{1-A}{1-q-Ap} > q$.

Assumption 2.1 means that the noise p and q can not be too large.

Assumption 2.2 : $K - M > (\frac{\log(\frac{2}{2+\beta})}{\log(A_{pq})} + 2)(\frac{\log(D(x_2-1))}{\log(x_1)} + 3)$.

Assumption 2.2 is satisfied if $\Delta \to 0$.

The stationary Markov equilibria also display reputation cycles, but the equilibrium behavior around state 0 is not characterized and may not be unique. When the state is away from 0, both players play mixed strategy as in the transition rule where $q = 0$. $K\Delta$ is still the threshold of reputation building phase and reputation exploitation phase.

Theorem 2 : Under Assumption 1-5 and Assumption 1.1, 2.1 and 2.2, the stationary Markov equilibria display reputation cycles as below:

(1) In state $X \in \{0, M\Delta, (M+1)\Delta, \ldots, (K-1)\Delta\}$, player 1 plays cutoff strategies $a(k\Delta) = \pi(k\Delta)$. Player 2 also plays strict mixed strategy $y(X) \in (0,1)$. $y(k\Delta)$ is characterized by a second-order difference equation:

$$y((k+1)\Delta) = \frac{1}{\beta}(1-A_{pq})y(k\Delta) + A_{pq}y((k-1)\Delta) \ \forall 1 \le k \le K-2$$

(3) State $k\Delta \ge K\Delta$ is a reputation-exploitation phase. In state $k\Delta$, player 1 plays NI for sure and player 2 plays B for sure, i.e. $y(k\Delta) = 1$ and $a(k\Delta) = 0$.

Proposition 2 : Under Assumption 1-5 and Assumption 1.1, in the limit where $\Delta \to 0$, there is a unique stationary Markov equilibrium.

If $X = 0$, then $y(X) = \frac{1-2q+(1-2p)A}{2(1-p-q)} e^{-b\frac{1-A}{1+(1-2p)A}X^*}$, $a(X) = \pi(X)$.

If $0 < X < X^*$, then $y(X) = e^{-b\frac{1-A}{1-2q+(1-2p)A}(X^*-X)}$, $a(X) = \pi(X)$.

If $X \geq X^*$, then $y(X) = 1$,$a(X) = 0$.

$$V(X) = \begin{cases} \frac{1-2q+(1-2p)A}{2(1-p-q)} e^{-b\frac{1-A}{1-2q+(1-2p)A}(X^*-X)} g_1(0,1) & 0 \leq X < X^* \\ \left(1 - \frac{(1-A)(1-2p)}{2(1-p-q)} e^{-\frac{b}{1-2p}(X-X^*)}\right) g_1(0,1) & X \geq X^* \end{cases}$$

Proposition 2 tells that as noise p and q becomes larger, the incentive in the future is weakened because a one-time no investment can cause the reputation to increase with probability p rather than a certain depreciation of reputation and a one-time investment can cause the reputation to decrease with probability q rather than a certain increase of reputation Therefore, player 2 needs to compensate the weakening of incentive by increasing the growth rate of purchase probability. Since $y(X)$ reaches 1 at state X, a higher growth rate leads to a lower level of $y(X)$ in each state X. In all, a higher noise p and q makes it more difficult to build reputation.

2.3 Lower bound transition rule

The lower bound of the state space is 0. In lower bound transition rule, the domain of next state is either one-step up or 0. The consequence of no investment is a higher probability to reach a complete ruin of reputation to the lowest level. Specifically, if the outcome of player 1's action is I, then the probability that next state $X' = 0$ is q and the probability that $X' = X + \Delta$ is $1 - q$:

$$P(X' = 0|I) = q,\ P(X' = X + \Delta|NI) = 1 - q$$

If the outcome of player 1's action is NI, then the probability that $X' = 0$ is $1 - p$ and the probability that $X' = X + \Delta$ is p:

$$P(X' = 0|NI) = 1 - p,\ P(X'|NI) = X + \Delta|NI) = p$$

$p = q = 0$ corresponds to the determinate case where no investment certainly leads to a complete milking of reputation and investment implies a one-step increase of reputation for sure. Define $A = \frac{g_1(1,1)-g_1(1,0)}{g_1(0,1)-g_1(0,0)}$, $\gamma = \frac{g_1(0,0)-g_1(1,0)}{g_1(0,1)-g_1(0,0)}$.

Assumption 3.1 :

$$A > \frac{q}{1-p}$$

Assumption 3.2 (High discount factor):

$$\beta > \frac{1-A+\gamma}{1-q-pA}$$

Under Assumption 2.1, $\frac{1-A+\gamma}{1-q-pA} \in (0,1)$ and Assumption 2.2 is well defined. Note that both assumptions still include a large amount of possibility of imperfect monitoring and discount factor.

Next, we characterize the reputation dynamics under lower bound transition rule. In this section, we find out that the characterization works for all fixed time intervals and high discount factors. Without loss of generality, fix the time interval at $\Delta = 1$ and use integer number k to denote reputation. Denote $K \equiv [X^*]+1$. Furthermore, player 2's trust behavior is strictly increasing in reputation.

Theorem 3: Under Assumptions 1-5 and Assumption 2.1-2.2, the stationary Markov equilibrium is unique and displays a reputation cycle as below:

(1) The state $k \in \{0, 1, 2, \ldots, K-1\}$ forms a reputation building phase in which player 1 plays mixed strategy $a(k) = \pi(k)$ in state k . Player 2 also plays strict mixed strategy $y(k) \in (0,1)$ where $y(k)$ is strictly increasing in k.

$$y(k) = \begin{cases} \frac{\eta_3+\eta_2\eta_1^K}{\eta_3+\eta_1^K(1-\eta_1-\eta_3)} - \frac{\eta_1+\eta_2+\eta_3-1}{\eta_3+\eta_1^K(1-\eta_1-\eta_3)}\eta_1^k & 0 \leq k \leq K-1 \\ 1 & k \geq K \end{cases}$$

where $\eta_1 = \frac{1-A}{\beta(1-q-pA)}$, $\eta_2 = \frac{\gamma(1-\beta p)}{\beta(1-q-pA)}$, $\eta_3 = \frac{1-p-q}{1-q-pA}$.

(2) Any state $k \geq K$ belongs to a reputation exploitation phase. In state $k \geq K$, player 1 plays NI for sure and player 2 plays B for sure, i.e. $y(k) = 1$ and $a(k) = 0$.

Similar the one-step transition rule, there is also a reputation cycle with a reputation building phase and a reputation exploitation phase. In the reputation building phase, player 2 buys the product with increasing probability with respect to reputation to provide player 1 with the incentives to invest. Player 1 plays a mixed strategy so that player 2 will be indifferent between B and NB. The result of a bad outcome is a high probability to ruin the reputation to the lowest level. After the ruin, player 1 will start over and continues to build the reputation. In the reputation exploitation phase, it is dominant strategy for player 2 to buy for sure. Therefore, player 2 can not reward player 1, so there is no incentives for player 1 to build reputation any more.

Proposition 3: Consider the limit case where $K \to +\infty$,

$$y(k) = 1 - \frac{\eta_1 + \eta_2 + \eta_3 - 1}{\eta_3} \eta_1^k \ \forall k \geq 0$$

$$\frac{\partial y(k)}{\partial p} < 0, \ \frac{\partial y(k)}{\partial q} < 0 \ \forall k \geq 0$$

When a noise p and q is introduced, the incentive in the future is weakened because a one-time no investment can cause the reputation to increase with probability p rather than a ruin of reputation for sure and a one-time investment can lead to the ruin of reputation with probability q instead of an increase of reputation for certain. Therefore, player 2 needs to compensate the weakening of incentive by increasing the growth rate of purchase probability, thus lowers purchase probability $y(k)$ in each state k.

Claim 1: If Assumption 3.1 does not hold, the stationary Markov equilibrium in the limit $\Delta \to 0$ is

$$y(k) = 0 \ \forall \ 0 \leq k \leq [k^*]$$

$$y(k) = 1 - (1 - y^*)\eta_1^k \ \forall \ k \geq [k^*] + 1$$

where $k^* = \frac{\log(1-y^*)}{\log 1/\eta_1} > 0$ and $y^* = \frac{A(1-p)-\gamma-q}{1-p-q} < 0$

However, if the noise p and q is too large and the cost of building reputation is too high (small A), player 2 can not provide enough incentive for player 1 to build reputation if reputation is very small. When $0 \leq k \leq [k^*]$, player 1 will not invest and player 2 will not trust player 1 until a sequence of good luck may push reputation to a high level such that $k \geq [k^*] + 1$ and player 1 starts to build reputation and player 2 starts to trust player 2.

2.4 A transition rule also influenced by player 2

In previous sections, player 2 has no impact on the accumulation of reputation. In reality, this is not always a proper assumption. For example, a firm's marketing strategy today may not effectively improve its reputation if the consumers do not buy the product, experience the good and give high customer ratings. Therefore, in the situation of word-of-mouth advertisement, it is appropriate to model the transition rule that can be influence also by the short-run players. In all, it is useful to analyze the reputation dynamics if the decisions of player 2 have an impact on the transition of the states. In particular, if player 2 chooses NB in state X, then the state will remain the same no matter what player 1 does.

$$P(X' = X|I, NB) = P(X' = X|NI, NB) = 1$$

If player 2 chooses B, then investment I of the long-run player will bring the state one-step up and no investment NI will bring the state one-step down.

$$P(X' = X + \Delta|I, B) = 1,\ P(X' = \max(X - \Delta, 0)|NI, B) = 1$$

There are two type of equilibria. *Absorbing equilibrium*: The short-run player 2 strictly prefers NB at state 0: $y_0 = 0$. *Non-absorbing equilibrium*: The short-run player weakly prefer B at state 0: $0 < y_0 < 1$.

Next, we characterize the unique stationary Markov Equilibrium under any interval and discount factor. Define $\hat{K} = [\frac{1+A}{1-A}\frac{\beta}{1-\beta} - \frac{1}{1+\beta}] + 1$ and $K = [\frac{X^*}{\Delta}] + 1$. Define $K^* = K$ if K is even and $K^* = K + 1$ if K is odd.

Theorem 4: Under Assumption 1-5 and Assumption 1.1, the stationary Markov equilibrium is unique.

Non-absorbing equilibrium: $K \leq \hat{K} - 1$.

If $0 \leq k \leq \max(K^* - 2[\frac{\beta^2}{1-\beta^2}] - 2, 0)$, $0 < y(k) < 1$ and $a(k) = \pi(k)$.

If $\max(K^* - 2[\frac{\beta^2}{1-\beta^2}] - 2, 0) \leq k \leq K^* - 1$, $y(k) = a(k) = 1$ for odd k, $0 < y(k) < 1$ and $a(k) = \pi(k)$ for even k.

If $k \geq K^*$, then $y_k = 1$ and $a(k) = 0$.

Absorbing equilibrium: $K \geq \hat{K}$.

If $0 \leq k \leq K - \hat{K}$, $y(k) = 0$ and $a(k) \leq \pi(k)$.

If $K - \hat{K} + 1 \leq k \leq K - 1$, $0 < y(k) < 1$ and $a(k) = \pi(k)$

If $k \geq K$, $y(k) = 1$ and $a(k) = 0$.

If $K \leq \hat{K} - 1$, then it is easy for player 1 to build reputation from nothing (y=0) to everything (y=1). There is a unique non-absorbing equilibrium characterized by two stages:

reputation building and reputation exploitation. In the reputation building phase, the the probability of buying needs to be increasing for all odd states and increasing for all even states. And The probabilities of buying in the odd states are relatively higher than in the even states. If reputation is close to $K^*\Delta$, player 2 buys for sure and player 1 invests for sure in odd states. In the reputation exploitation stage when state is larger than $K^*\Delta$, player 1 has no reward of building reputation since player 2 will buy for sure in all states larger than $K^*\Delta$.

If $K \geq \hat{K}$, then it is difficult for player 1 to build reputation from nothing (y=0) to everything (y=1). There is a unique absorbing equilibrium characterized by three stages: reputation stagnation, reputation building and reputation exploitation. There will be a threshold level $(K - \hat{K})\Delta$ at which the future continuation payoff is just not enough for player 1 to build reputation. For all states below this threshold (reputation stagnation stage), reputation remains constant because of coordination failure. In state X, player 2 does not buy the product because he knows that player 2 in state $X+1$ won't buy the product so that player 1 in state X has no incentive to build reputation. For all states larger than the threshold and less than $K\Delta$ (reputation building stage), player 2 will buy with positive probability in an increasing order to provide player 1 the incentives to build reputation and player 1 will play a mix strategy to make player 2 just indifferent between B and NB. In the reputation exploitation stage when state is larger than $K\Delta$, player 1 has no reward of building reputation since player 2 will buy for sure in all states larger than $K\Delta$.

We can deduce an analytic solution when Δ converges to 0.

Proposition 4 : Under Assumption 1-6 and Assumption 1.1, the stationary Markov equilibrium in the limit $\Delta \to 0$ is characterized as:

Non-absorbing equilibrium: $bX^* \leq \frac{1+A}{1-A}$.

If $0 \leq X \leq \max(X^* - \frac{1}{b}, 0)$, then

$$(a(X), y(X)) = \begin{cases} (\pi(X), \frac{(1+A)-b(1-A)(X^*-X)}{2}) & X = \lim_{\Delta\to 0}(2k+1)\Delta \\ (\pi(X), \frac{(1+A)-b(1-A)(X^*-X)}{2A}) & X = \lim_{\Delta\to 0} 2k\Delta \end{cases}$$

If $\max(X^* - \frac{1}{b}, 0) < X \leq X^*$, then

$$(a(X), y(X)) = \begin{cases} (1,1) & X = \lim_{\Delta \to 0}(2k+1)\Delta \\ (\pi(X), \frac{1+A-b(1-A)(X^*-X)}{1+A+b(1-A)(X^*-X)}) & X = \lim_{\Delta \to 0} 2k\Delta \end{cases}$$

If $X \geq X^*$, $(a(X), y(X)) = (0,1)$

Absorbing equilibrium: $bX^* > \frac{1+A}{1-A}$.

If $0 \leq X \leq X^* - \frac{1+A}{b(1-A)}$, $y(X) = 0$ and $a(X) \leq \pi(X)$.

If $X^* - \frac{1+A}{b(1-A)} < X \leq X^*$, $(a(X), y(X)) = (\pi(X), 1 - \frac{1-A}{1+A}b(X^* - X))$.

If $X \geq X^*$, $(a(X), y(X)) = (0,1)$

2.5 Multiple investment levels

In the basic model, we assume that the long-run player has only two choices: investment I and no investment NI. In this section, we relax the assumption that there is only one investment choice. Instead, there are n investment choices: $\{I_i\}_{i=1}^n$ and a no investment choice $NI \equiv I_0$. Assume that in the next period, reputation can only go one-step up or down. c_i is the cost of investment I_i. Therefore, if the short-run player 2 chooses B and the long-run player chooses I_i. Denote $g_i(I_i, B)$ $(g_i(I_i, NB))$ as player i's stage game payoff if investment I_i and $B(NB)$ are chosen.

Assumption 5.1 (Myopic incentive of player 1): $g_1(I_i, B) > g_1(I_j, B)$, $g_1(I_i, NB) \geq g_1(I_j, NB)$ for any $i < j$.

Assumption 5.2 (Submodularity of player 1): $g_1(I_0, NB) - g_1(I_i, NB) < g_1(I_0, NB) - g_1(I_i, NB)$ for any $1 \leq i \leq n$.

Assumption 5.3 (Myopic incentive of player 2): $g_2(I_i, B, X) > g_2(I_i, NB, X)$ for any $1 \leq i \leq n$.

Assumption 5.4 (Reputation is valuable for player 2): $g_2(I_i, B, X)$ and $g_2(I_i, NB, X)$ is strictly increasing in X.

Assumption 5.5 (Buy for sure for high reputation): There is X^* such that if $X \geq X^*$ then $g_2(I_0, B, X) \geq g_2(I_0, NB, X)$. Otherwise, $g_2(I_0, B, X) < g_2(I_0, NB, X)$.

Assumption 5 tells us that if $X \geq X^*$, it is strictly dominant strategy for player 2 to play B. If $X < X^*$, then there is a mixed strategy $\pi_i(X) \in (0,1)$ of playing I_i and $1 - \pi_i(X)$ of playing I_0 to make player 2 indifferent between B and NB. It is reasonable to assume that player 2 will buy the product for sure independent of player 1's current behavior if player 1 has done good enough in the past.

Assumption 5.6: $g_1(I_i, NB) = g_1(I_j, NB)$ for any $0 \leq i \leq n$.

Next, define the transition rule. $p \in (0,1)$ is the probability of one-step increase of reputation in the next period if no investment I_0 is chosen and $q_i \in (1-p, 1)$ is the probability of one-step decrease of reputation in the next period if investment I_i is chosen:

$$P(X'|I_i) = \begin{cases} 1-q_i & X' = X + \Delta \\ q_i & X' = \max(X-\Delta, 0) \end{cases}$$

Define $c_i = g_1(I_0, B) - g_1(I_i, B)$ as the cost of investment I_i. We assume that $1-q_i > 1-q_j$ for $i > j$ because an investment with large cost will lead to a higher probability of one-step increase of reputation in the next period. Denote $i^* = \arg\min_{i\geq 1}\{\frac{c_i}{q_0-q_i}\}$. Therefore, c_{i^*} is the most "efficient" investment level in the sense that the marginal cost is minimized relative to marginal benefit. We can show that there is a stationary Markov equilibrium that player 1 ignores investments different from I_{i^*}.

Proposition 5 : Under Assumption 5.1-5.6, there is a stationary Markov equilibrium as below:

(1) In state $X \in \{0, 2\Delta, 3\Delta, \ldots, (K-1)\Delta\}$, player 1 puts probability $\pi_{i^*}(k\Delta)$ on I_{i^*} and $1 - \pi_{i^*}(k\Delta)$ on I_0. Player 2 also plays strict mixed strategy $y(X) \in (0,1)$. $y(k\Delta)$ is characterized by a second-order difference equation:

$$y((k+1)\Delta) = \frac{1}{\beta}(1-A_i)y(k\Delta) + A_i y((k-1)\Delta) \; \forall 1 \leq k \leq K-2$$

where $A_i = \frac{q_0 A - q_i}{1-q_{i^*}-A(1-q_0)}$ and $A = \frac{g_1(I_{i^*}, B)}{g_1(I_0, B)}$.

(3) State $k\Delta \geq K\Delta$ is a reputation-exploitation phase. In state $k\Delta$, player 1 plays I_0 for sure and player 2 plays B for sure, i.e. $y(k\Delta) = 1$ and $a(k\Delta) = 0$.

3 Conclusion

In this paper, we study the reputation dynamics in a setting of stochastic games in which reputation is modeled as a state variable. The key feature distinguishing our paper from classical models of reputation is that reputation is a function of past investments rather than current effort. Under a rich class of transition rules, stationary Markov equilibria can be characterized as reputation cycles with a reputation building phase and a reputation exploitation phase.

An important assumption which gives us the equilibrium results is *submodularity*. This assumption is common in dynamic setups and in the reputation literature (Liu, 2011; Liu and Skrzypacz, 2014; Phelan, 2006). Intuitively, we study *submodularity* because we are interested in situations where two parties have severe conflicting interests. However, it is possible that there are examples where *supermodularity* or fixed investment cost should be assumed. We will work on it in the future research. Another key assumption is that both players have a binary choice. The implicit assumption is that player 1's stage game payoff is linear in investment. In section 2.5, we extend the model to multiple investment choices and establish stationary Markov equilibrium in which player 1 pays attention to a binary choice. However, it remains to investigate the existence of other equilibria.

There are several interesting ways to extend this model. Faced with competition, a firm builds reputation because it wants to differentiate its product from other firms. Therefore, we can study the industry dynamics when there are multiple firms in the market. It is interesting to investigate firms' exit and entry decisions and the stationary distribution of reputation in a steady-state equilibrium. Furthermore, a car company may have multiple sub-brands to sell or may have only a brand to sell but consumers care about different dimensions of the car quality: performance, reliability or appearance. As a sequence, it is useful to study how a car company to allocate its resource on R&D in order to optimally manage its reputation for different qualities. In a companion paper, we establish that in a model of two dimensions of reputation, a firm will focus on a certain dimension with higher reputation and build this dimension to a very high level and then starts to allocate resource to a new dimension because a low effort is enough to maintain the reputation of the old dimension.

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4 Appendices

Appendix 1

In Appendix 1, $V(k)$ and y_k denote$V(k\Delta)$ and $y(k\Delta)$.

Lemma 1.1: $0 < y_0 < 1$.

Proof. If $y_0 = 1$, then $V(0) = g_1(1,1)+\beta V(1) \geq g_1(0,1)+\beta(pV(1)+(1-p)V(0))$. Therefore, $(1-p)(V(1)-V(0)) \geq g_1(0,1) - g_1(1,1) > 0$, so $V(1) > V(0)$.

$$\therefore V(0) \geq g_1(0,1) + \beta(pV(1) + (1-p)V(0)) > g_1(0,1) + \beta V(0)$$

Therefore, $V(0) \geq \frac{g_1(0,1)}{1-\beta}$, a contradiction to the fact that $\frac{g_1(0,1)}{1-\beta}$ is the upper bound of the continuation payoff.

If $y_0 = 0$, then $V(0) = g_1(0,0) + \beta(pV(1) + (1-p)V(0)) \geq g_1(1,0) + \beta V(1)$. Therefore, $V(1) \leq V(0) = 0$, thus $V(1) = 0$.

$$\therefore 0 = V(1) \geq g_1(0, y_1) + \beta(pV(2) + (1-p)V(0)) = g_1(0, y_1) + \beta p V(2)$$

Therefore, $V(2) = 0$ and $y_1 = 0$.

By induction, assume that $V(i+1) = 0$ and $y_i = 0$ for some $i \geq 1$.

$$0 = V(i+1) = g_1(0, y_{i+1}) + \beta(pV(i+2) + (1-p)V(i)) = g_1(0, y_{i+1}) + \beta p V(i+2)$$

So, $V(i+2) = 0$ and $y_{i+1} = 0$. Therefore, $V(i+1) = 0$ and $y_i = 0$ for all $i \geq 1$, a contradiction to $y_T = 1$.

□

Lemma 1.2: $y_k > 0$ for all $0 \leq k \leq K-1$.

Proof. By lemma 1, we have $0 < y_0 < 1$. Therefore, $V(0) = g_1(0, y_0) + \beta(pV(1) + (1-p)V(0)) = g_1(1, y_0) + \beta V(1)$. Then, $V(1) > V(0)$. Assume $y_1 = 0$, then $V(1) = g_1(0,0) + \beta(pV(2) + (1-p)V(0)) \geq g_1(1,0) + \beta V(2)$. Then, $V(2) \leq V(0)$.

$$\therefore V(1) = g_1(0,0) + \beta(pV(2) + (1-p)V(0)) \leq g_1(0,0) + \beta V(0) < V(0)$$

a contradiction. Therefore, $y_1 > 0$. Prove by induction and assume that $y_t > 0$ for all $t \leq k$.

$$V(k-1) = g_1(1, y_{k-1}) + \beta V(k)$$

$$V(k) = g_1(1, y_k) + \beta V(k+1) \geq g_1(0, y_k) + \beta(pV(k+1) + (1-p)V(k-1))$$

$$\therefore \beta(V(k+1) - V(k-1)) \geq \frac{1}{1-p}(g_1(0, y_k) - g_1(1, y_k))$$

Assume that $y_{k+1} = 0$, then

$$V(k+1) = g_1(0,0) + \beta(pV(k+2) + (1-p)V(k)) \geq g_1(1,0) + \beta V(k+2)$$

Therefore, $V(k+2) \leq V(k)$ and $V(k+1) \geq g_1(0,0) + \beta V(k) = \beta V(k)$.

$$\therefore (1-\beta^2)V(k) \leq g_1(1, y_k)$$

Furthermore,

$$V(k) \geq g_1(0, y_k) + \beta V(k-1) + \beta p(V(k+1) - V(k-1))$$

$$\geq g_1(0, y_k) + \beta g_1(1, y_{k-1}) + \beta^2 V(k) + \frac{p}{1-p}(g_1(0, y_k) - g_1(1, y_k))$$

$$\therefore g_1(1, y_k) \geq (1-\beta^2)V(k) \geq g_1(0, y_k) + \beta g_1(1, y_{k-1}) + \frac{p}{1-p}(g_1(0, y_k) - g_1(1, y_k))$$

$$\therefore 0 \geq \frac{1-A}{1-p} y_k + \beta A y_{k-1}$$

a contradiction. Therefore, $y_{k+1} > 0$. In all, we have shown that $y_k > 0$ for all $0 \leq k \leq K-1$.

□

Lemma 1.3: It is impossible to have two consecutive complete trust: $y_k = y_{k+1} = 1$ for some $1 \leq k \leq K-2$.

Proof. If $y_k = y_{k+1} = 1$ for some $0 \leq k \leq K-2$, then

$$V(k) = g_1(1,1) + \beta V(k+1) \geq g_1(0,1) + \beta(pV(k+1) + (1-p)V(k-1))$$

$$V(k+1) = g_1(1,1) + \beta V(k+2) \geq g_1(0,1) + \beta(pV(k+2) + (1-p)V(k))$$

Assume that player 1 is indifferent between I and NI at period $k+2$, then

$$V(k+2) = g_1(1, y_{k+2}) + \beta V(k+3) = g_1(0, y_{k+2}) + \beta(pV(k+3) + (1-p)V(k+1))$$

$$\therefore V(k+3) - V(k+1) = \frac{1}{\beta(1-p)}(g_1(0, y_{k+2}) - g_1(1, y_{k+2}))$$

$$< \frac{1}{\beta(1-p)}(g_1(0,1) - g_1(1,1)) = V(k+2) - V(k)$$

$$\therefore V(k+2) = V(k+1) + (V(k+2) - V(k+1)) \geq g_1(0,1) + \beta(pV(k+2) + (1-p)V(k)) + (V(k+2) - V(k+1))$$

$$= g_1(0,1) + \beta V(k) + (V(k+2) - V(k+1)) + \beta p(V(k+2) - V(k))$$

$$> g_1(0,1) + \beta V(k+1) + \beta p(V(k+3) - V(k+1))$$

$$= g_1(0,1) + \beta(pV(k+3) + (1-p)V(k+1))$$

where we use the fact that $V(k+1)-V(k)=\beta(V(k+2)-V(k+1))<V(k+2)-V(k+1)$. Therefore, player 1 strictly prefers NI to I at period $k+2$, a contradiction. Since the long-run player weakly prefers I to NI for any $t \geq 0$ by lemma 2 and he can not be indifferent between C and N at period $k+2$, then

$$V(k+2)=g_1(1,1)+\beta V(k+3)>g_1(0,1)+\beta(pV(k+3)+(1-p)V(k+1))$$

By induction, we have shown that

$$V(t)=g_1(1,1)+\beta V(t+1)\geq g_1(0,1)+\beta(pV(t+1)+(1-p)V(t-1))\ \forall t\geq k$$

Since $\{V(t)\}_{t\geq k}$ is a strictly increasing and bounded sequence, there is a limit V^* such that $V^*=g_1(1,1)+\beta V^*$. Therefore, $V(t+1)<V^*=\frac{g_1(1,1)}{1-\beta}$ for any $t\geq k$. However, $V(t+1)>V(t)=g_1(1,1)+\beta V(t+1)$ and hence $V(t+1)>\frac{g_1(1,1)}{1-\beta}$, a contradiction.

□

Lemma 1.4 : (1) If $y_k<1$ and $y_{k+1}<1$, then $y_{k+1}=\frac{1}{\beta}(1-A_p)y_k+A_py_{k-1}$.

(2) If $y_{k+1}=1$, then $y_{k+1}\leq\frac{1}{\beta}(1-A_p)y_k+A_py_{k-1}$.

(3) If $y_{k+1}=1$ and $V(k+2)=g_1(0,y_{k+2})+\beta V(k+1)$, then $y_{k+2}\geq\frac{1}{\beta}(1-A_p)y_{k+1}+A_py_k$.

Proof. We know that $y_k>0$ for all $0\leq k\leq K-1$. Therefore, $V(k)=g_1(1,y_k)+\beta V(k+1)$ for all $0\leq k\leq K-1$.

(1) $y_k<1$ and $y_{k+1}<1$.

$$V(k-1)=g_1(1,y_{k-1})+\beta V(k)$$

$$V(k)=g_1(1,y_k)+\beta V(k+1)=g_1(0,y_k)+\beta(pV(k+1)+(1-p)V(k-1))$$

$$V(k+1)=g_1(1,y_{k+1})+\beta V(k+2)=g_1(0,y_{k+1})+\beta(pV(k+2)+(1-p)V(k))$$

$$\therefore (1-\beta^2)V(k)=g_1(0,y_k)+\beta g_1(1,y_{k-1})+\beta p(V(k+1)-V(k-1))$$

$$\therefore (1-\beta^2)V(k)=g_1(1,y_k)+\beta g_1(0,y_{k+1})+\beta^2p(V(k+2)-V(k))$$

$$\therefore g_1(0,y_k)+\beta g_1(1,y_{k-1})+\frac{p}{1-p}(g_1(0,y_k)-g_1(1,y_k))$$
$$=g_1(1,y_k)+\beta g_1(0,y_{k+1})+\beta\frac{p}{1-p}(g_1(0,y_{k+1})-g_1(1,y_{k+1}))$$

$$\therefore y_{k+1} = \frac{1}{\beta}\frac{1-A}{1-Ap}y_k + \frac{A(1-p)}{1-Ap}y_{k-1}$$

$$\therefore y_{k+1} = \frac{1}{\beta}(1-A_p)y_k + A_p y_{k-1}$$

(2) $y_{k+1} = 1$.

By lemma 1.2, we have $y_k < 1$.

$$V(k-1) = g_1(1, y_{k-1}) + \beta V(k)$$

$$V(k) = g_1(1, y_k) + \beta V(k+1) = g_1(0, y_k) + \beta V(k-1) + \beta p(V(k+1) - V(k-1))$$

$$V(k+1) = g_1(1, y_{k+1}) + \beta V(k+2) \geq g_1(0, y_{k+1}) + \beta p(V(k+2) - V(k))$$

$$\therefore (1-\beta^2)V(k) = g_1(0, y_k) + \beta g_1(1, y_{k-1}) + \beta p(V(k+1) - V(k-1))$$

$$\therefore (1-\beta^2)V(k) \geq g_1(1, y_k) + \beta g_1(0, y_{k+1}) + \beta^2 p(V(k+2) - V(k))$$

$$\therefore g_1(0, y_k) + \beta g_1(1, y_{k-1}) + \frac{p}{1-p}(g_1(0, y_k) - g_1(1, y_k))$$

$$\geq g_1(1, y_k) + \beta g_1(0, y_{k+1}) + \beta\frac{p}{1-p}(g_1(0, y_{k+1}) - g_1(1, y_{k+1}))$$

$$\therefore y_{k+1} \leq \frac{1}{\beta}\frac{1-A}{1-Ap}y_k + \frac{A(1-p)}{1-Ap}y_{k-1}$$

$$\therefore y_{k+1} \leq \frac{1}{\beta}(1-A_p)y_k + A_p y_{k-1}$$

(3) $y_{k+1} \leq 1$ and $V(k+2) = g_1(0, y_{k+2}) + \beta(pV(k+3) + (1-p)V(k+1))$.

$$V(k) = g_1(1, y_k) + \beta V(k+1) = g_1(0, y_k) + \beta V(k-1)$$

$$V(k+1) = g_1(1, y_{k+1}) + \beta V(k+2) \geq g_1(0, y_{k+1}) + \beta V(k) + \beta p(V(k+2) - V(k))$$

$$V(k+2) = g_1(0, y_{k+2}) + \beta V(k+1) + \beta p(V(k+3) - V(k+1))$$

$$\therefore (1-\beta^2)V(k+1) = g_1(1, y_{k+1}) + \beta g_1(0, y_{k+2}) + \beta^2 p(V(k+3) - V(k+1))$$

$$\therefore (1-\beta^2)V(k+1) \geq g_1(0, y_{k+1}) + \beta g_1(1, y_k) + \beta p(V(k+2) - V(k))$$

$$\therefore g_1(1, y_{k+1}) + \beta g_1(0, y_{k+2}) + \beta\frac{p}{1-p}(g_1(0, y_{k+2}) - g_1(1, y_{k+2}))$$

$$\geq g_1(0, y_{k+1}) + \beta g_1(1, y_k) + \frac{p}{1-p}(g_1(0, y_{k+1}) - g_1(1, y_{k+1}))$$

$$\therefore y_{k+2} \geq \frac{1}{\beta}\frac{1-A}{1-Ap}y_{k+1} + \frac{A(1-p)}{1-Ap}y_k$$

$$\therefore y_{k+2} \geq \frac{1}{\beta}(1-A_p)y_{k+1} + A_p y_k$$

$\square$

Lemma 1.5: If $K > 3 + \frac{\log \frac{\epsilon-1}{A_p}}{\log \frac{A_p}{\epsilon}}$, then the long-run player strictly prefers to play N in state K, $0 < y_k < 1$ for all $2 \leq k \leq K-1$. $\epsilon = \frac{1}{2\beta}(1 - A_p + \sqrt{(1-A_p)^2 + 4A_p\beta^2})$.

Proof. **Step 1 :** If the long-run player strictly prefers to play NI in state K and $y_{K-1} < 1$, then $0 < y_k < 1$ for all $2 \leq k \leq K-2$.

By lemma 1.4(3), $y_K \geq \frac{1}{\beta}(1-A_p)y_{K-1} + A_p y_{K-2}$. If $y_{K-2} = 1$, then $y_{K-1} \leq \beta$. Since $y_{K-2} = 1$, $y_{K-3} < 1$ by lemma 1.3. By lemma 1.4(2) , $y_{K-2} \leq \frac{1}{\beta}(1-A_p)y_{K-3} + A_p y_{K-4} \leq \frac{1}{\beta}(1-A_p)y_{K-3} + A_p$. Then, $y_{K-3} \geq \beta$. By lemma 1.4(3), $y_{K-1} \geq \frac{1}{\beta}(1-A_p)y_{K-2} + A_p y_{K-3}$, $y_{K-1} > y_{K-3} \geq \beta$, a contradiction. In all, we have shown that $y_{K-2} < 1$.

Show that $0 < y_k < 1$ for all $2 \leq k \leq K-2$ by induction. Assume $y_t < 1$ for all $t \geq k$. Assume $y_{k-1} = 1$, then $y_{k-2} < 1$. By lemma 1.4(2), $y_{k-1} \leq \frac{1}{\beta}(1-A_p)y_{k-2} + A_p y_{k-3} \leq \frac{1}{\beta}(1-A_p)y_{k-2} + A_p$. Then, $y_{k-2} \geq \beta$. By lemma 1.4(3), $y_k \geq \frac{1}{\beta}(1-A_p)y_{k-1} + A_p y_{k-2}$, $y_k > y_{k-2} \geq \beta$. Therefore, $y_{k+1} = \frac{1}{\beta}(1-A_p)y_k + A_p y_{k-1} > 1$, a contradiction. In all, we have show that $0 < y_k < 1$ for all $2 \leq k \leq K-2$.

Step 2 : The long-run player strictly prefers to play NI in state K.

Assume that player 1 weakly prefers to play C in state K. By the same logic of lemma 1.3, player 1 strictly prefers to play N in state $K+1$. By lemma 1.4(3), $y_{K+1} \geq \frac{1}{\beta}(1 - A_p)y_K + A_p y_{K-1}$. Then, $y_{K-1} \leq \frac{1-\frac{1}{\beta}(1-A_p)}{A_p}$.

If $y_{K-2} < 1$, then by the same argument of step 1, we have $0 < y_k < 1$ for all $2 \leq k \leq K-2$. Then, we can estimate y_{K-1}:

$$y_K - \epsilon y_{K-1} \leq (-\frac{A_p}{\epsilon})^{K-2}(y_2 - \epsilon y_1)$$

$$\therefore y_{K-1} \geq \frac{1}{\epsilon} - (\frac{A_p}{\epsilon})^{K-2}$$

If $y_{K-2} = 1$, then by lemma 1.3, we have $y_{K-3} < 1$. By the same argument of step 1, $0 < y_k < 1$ for all $2 \leq k \leq K-3$. By lemma 1.4(3), $y_{K-1} \geq \frac{1}{\beta}(1-A_p)y_{K-2} + A_p y_{K-3}$.

$$\therefore y_{K-1} - \epsilon y_{K-2} \geq (-\frac{A_p}{\epsilon})^{K-3}(y_2 - \epsilon y_1)$$

$$\therefore y_K - \epsilon y_{K-1} \le (-\frac{A_p}{\epsilon})(y_{K-1} - \epsilon y_{K-2}) \le (-\frac{A_p}{\epsilon})^{K-2}(y_2 - \epsilon y_1)$$

$$\therefore y_{K-1} \ge \frac{1}{\epsilon} - (\frac{A_p}{\epsilon})^{K-2}$$

In all, we have $y_{K-1} \ge \frac{1}{\epsilon} - (\frac{A_p}{\epsilon})^{K-2}$.

$$\because y_{K-1} \le \frac{1 - \frac{1}{\beta}(1 - A_p)}{A_p} = \frac{1}{\epsilon} - \frac{\epsilon - 1}{A_p}$$

$$\therefore \frac{\epsilon - 1}{A_p} \le (\frac{A_p}{\epsilon})^{K-2}$$

a contradiction to $K > 3 + \frac{\log \frac{\epsilon - 1}{A_p}}{\log \frac{A_p}{\epsilon}}$. In all, the long-run player strictly prefers NI in state K.

Step 3 : $y_{K-1} < 1$.

Assume that $y_{K-1} = 1$. We have shown in step 2 that player 1 strictly prefer NI in state K. Therefore, $y_K \ge \frac{1}{\beta}(1 - A_p)y_{K-1} + A_p y_{K-2}$. Then, $y_{K-2} \le \frac{1 - \frac{1}{\beta}(1 - A_p)}{A_p}$.

If $y_{K-3} < 1$, then by the same argument of step 1, $0 < y_k < 1$ for all $2 \le k \le K - 3$. We can estimate y_{K-2}:

$$y_{K-1} - \epsilon y_{K-2} \le (-\frac{A_p}{\epsilon})^{K-3}(y_2 - \epsilon y_1)$$

$$\therefore y_{K-2} \ge \frac{1}{\epsilon} - (\frac{A_p}{\epsilon})^{K-3}$$

If $y_{K-3} = 1$, then by lemma 1.3, we have $y_{K-4} < 1$. By the same argument of step 1, $0 < y_k < 1$ for all $2 \le k \le K - 4$.

$$\because y_{K-2} \ge \frac{1}{\beta}(1 - A_p)y_{K-3} + A_p y_{K-4}$$

$$\therefore y_{K-2} - \epsilon y_{K-3} \ge (-\frac{A_p}{\epsilon})^{K-4}(y_2 - \epsilon y_1)$$

$$\therefore y_{K-1} - \epsilon y_{K-2} \le (-\frac{A_p}{\epsilon})(y_{K-2} - \epsilon y_{K-3}) \le (-\frac{A_p}{\epsilon})^{K-3}(y_2 - \epsilon y_1)$$

$$\therefore y_{K-2} \ge \frac{1}{\epsilon} - (\frac{A_p}{\epsilon})^{K-3}$$

In all, $y_{K-2} \ge \frac{1}{\epsilon} - (\frac{A_p}{\epsilon})^{K-3}$.

$$\because y_{K-2} \le \frac{1 - \frac{1}{\beta}(1 - A_p)}{A_p} = \frac{1}{\epsilon} - \frac{\epsilon - 1}{A_p}$$

$$\therefore \frac{\epsilon - 1}{A_p} \leq (\frac{A_p}{\epsilon})^{K-3}$$

a contradiction to $K > 3 + \frac{\log \frac{\epsilon-1}{A_p}}{\log \frac{A_p}{\epsilon}}$. In all, $y_{K-1} < 1$.

Step 4 : Player 1 strictly prefers NI in state $t > K$.

Assume that player 1 weakly prefers C in state $K+i$ where $i \geq 1$. By the same argument of lemma 1.3, player 1 strictly prefers N in state $K+i+1$.

$$\therefore V(K+i+1) = g_1(0,1) + \beta(pV(K+i) + (1-p)V(K+i-2))$$

$$\because V(K+i) = g_1(1,1) + \beta V(K+i+1) \geq g_1(0,1) + \beta(pV(K+i+1) + (1-p)V(K+i-1))$$

$$\therefore (1-\beta^2)V(K+i) = g_1(1,1) + \beta g_1(0,1) + \beta^2 p(V(K+i) - V(K+i-2))$$

$$\because V(K+i-1) \geq g_1(1,1) + \beta V(K+i)$$

$$\therefore (1-\beta^2)V(K+i) \geq g_1(0,1) + \beta g_1(1,1) + \beta p(V(K+i+1) - V(K+i-1))$$

$$\because \beta p(V(K+i+1) - V(K+i-1)) = \frac{p}{1-p}(g_1(0,1) - g_1(1,1))$$

$$\geq \beta p(V(K+i) - V(K+i-2)) \geq \beta^2 p(V(K+i) - V(K+i-2))$$

$$\therefore g_1(1,1) + \beta g_1(0,1) \geq g_1(0,1) + \beta g_1(1,1)$$

a contradiction.

□

Proof of Theorem 1:

Proof. We have shown that the long-run player strictly prefers to play NI in state K, $0 < y_k < 1$ for all $2 \leq k \leq K-1$. There are two cases for us to consider: $y_1 = 1$ and $y_1 < 1$.

Case 1 : $y_1 < 1$.

By lemma 1.4(1), there is a system of equations:

$$y_1 = (\frac{1}{\beta}(1-A_p) + 1)y_0$$

$$y_{k+1} = \frac{1}{\beta}(1-A_p)y_k + A_p y_{k-1} \quad \forall 1 \leq k \leq K-1$$

$$y_k = 1 \quad \forall k \geq K$$

The solution is

$$y_0 = \frac{1}{\epsilon^K + (1 - \frac{A_p}{\epsilon})\frac{\epsilon^K - (-\frac{A_p}{\epsilon})^K}{\epsilon + \frac{A_p}{\epsilon}}}$$

$$y_k = \frac{\epsilon^k + (1 - \frac{A_p}{\epsilon})\frac{\epsilon^k - (-\frac{A_p}{\epsilon})^k}{\epsilon + \frac{A_p}{\epsilon}}}{\epsilon^K + (1 - \frac{A_p}{\epsilon})\frac{\epsilon^K - (-\frac{A_p}{\epsilon})^K}{\epsilon + \frac{A_p}{\epsilon}}} \ \forall 1 \leq k \leq K-1$$

In order to satisfy $y_1 < 1$, we need

$$\frac{\epsilon + 1 - \frac{A_p}{\epsilon}}{\epsilon^K + (1 - \frac{A_p}{\epsilon})\frac{\epsilon^K - (-\frac{A_p}{\epsilon})^K}{\epsilon + \frac{A_p}{\epsilon}}} < 1$$

Case 2 : $y_1 = 1$.

If $\frac{\epsilon+1-\frac{A_p}{\epsilon}}{\epsilon^K+(1-\frac{A_p}{\epsilon})\frac{\epsilon^K-(-\frac{A_p}{\epsilon})^K}{\epsilon+\frac{A_p}{\epsilon}}} > 1$, then we can not have solution like in case 1, otherwise $y_1 > 1$, a contradiction. The only choice is that the long-run player strictly prefers I in state 1. Then, we have a system of equations:

$$y_2 = \frac{1 - A_p + \beta + A_p\beta^2}{\beta^2} y_0 - \frac{A_p}{\beta}$$

$$y_1 = 1$$

$$y_{k+1} = \frac{1}{\beta}(1 - A_p)y_k + A_p y_{k-1} \ \forall 2 \leq k \leq K-1$$

The solution is

$$y_0 = \frac{\beta^2 y_2 + A_p\beta}{1 - A_p + A_p\beta^2 + \beta}$$

$$y_1 = 1$$

$$y_k = \epsilon^{k-1} + (1 - \epsilon^{K-1})\frac{\epsilon^{k-1} - (-\frac{A_p}{\epsilon})^{k-1}}{\epsilon^{K-1} - (-\frac{A_p}{\epsilon})^{K-1}} \ \forall 2 \leq k \leq K$$

If $\frac{\epsilon+1-\frac{A_p}{\epsilon}}{\epsilon^K+(1-\frac{A_p}{\epsilon})\frac{\epsilon^K-(-\frac{A_p}{\epsilon})^K}{\epsilon+\frac{A_p}{\epsilon}}} = 1$, player 1 is indifferent between I and NI and $y_1 = 1$. □

Proof of Proposition 1:

Proof. **Step 1:** When $p = 0$, we know that

$$1 = y_K = \frac{1}{\beta}(1 - A_p)y_{K-1} + A_p y_{K-2}$$

The ending condition for the second-order difference equation $y_i = \frac{1}{\beta}(1-A_p)y_{i-1} + A_p y_{i-2}$ is $y_K = 1$.

Step 2: When $p > 0$, we need to figure out y_{K-1} in the limit.

$$\because V(k) = g_1(0,1) + \beta(pV(t+1) + (1-p)V(k-1)) \ \forall k \geq K$$

$$\therefore W(k) = \beta(pW(k+1) + (1-p)W(k-1)) \ \forall k \geq K+1$$

$$\therefore W(k+1) = \frac{1}{\beta p}W(k) - \frac{1-p}{p}W(k-1) \ \forall k \geq K+1$$

$$\therefore W(k) = \lambda_1 x_1^k + \lambda_2 x_2^k \ \forall k \geq K$$

where λ_1, λ_2 are constants, $x_1 < 1$ and $x_2 > 1$ are the roots of the characteristic function:

$$x^2 - \frac{1}{\beta p}x + \frac{1-p}{p} = 0$$

$$\therefore x_1 = \frac{1-\sqrt{1-4\beta^2 p(1-p)}}{2\beta p}, \ x_2 = \frac{1+\sqrt{1-4\beta^2 p(1-p)}}{2\beta p}$$

Because $\{W(t)\}_{t=K}^{\infty}$ is a bounded sequence and $x_2 > 1$, $\lambda_2 = 0$. Therefore, $W(t) = \lambda_1 x_1^t$ for all $k \geq K$.

$$\therefore \frac{W(K+1)}{W(K)} = x_1$$

$$\because V(K-1) = g_1(1, y_{K-1}) + \beta V(K), \ V(K) = g_1(0,1) + \beta(pV(K+1) + (1-p)V(K-1))$$

$$\therefore \beta p(V(K+1) - V(K-1)) = (1+\beta)(V(K) - V(K-1)) - (g_1(0,1) - g_1(1, y_{K-1}))$$

$$1 + \beta(1-p) = \frac{g_1(0,1) - g_1(1,y_{K-1})}{V(K) - V(K-1)} + \beta p \frac{V(K+1) - V(K)}{V(K) - V(K-1)} = \frac{g_1(0,1) - g_1(1,y_{K-1})}{V(K) - V(K-1)} + \beta p x_1$$

$$\therefore V(K) - V(K-1) = \frac{1}{1 + \beta(1-p) - \beta p x_1}(g_1(0,1) - g_1(1, y_{K-1}))$$

$$\therefore \beta p(V(K+1) - V(K-1) = \frac{\beta p + \beta p x_1}{1 + \beta(1-p) - \beta p x_1}(g_1(0,1) - g_1(1, y_{K-1}))$$

In order to satisfy the optimal condition that the long-run player strictly prefers N at state K,

$$\beta p(V(K+1) - V(K-1) < \frac{p}{1-p}(g_1(0,1) - g_1(1,1))$$

$$\therefore \frac{\beta p + \beta p x_1}{1 + \beta(1-p) - \beta p x_1}(1 - Ay_{K-1}) < \frac{p}{1-p}(1-A)$$

$$\therefore y_{K-1} > 1 - \frac{1-A}{A(1-p)} \frac{1-\beta x_1}{\beta(1+x_1)}$$

Because $\lim_{\Delta\to 0}\beta x_1 = 1$, then $\lim_{\Delta\to 0} y_{K-1} = 1$. Therefore, in the limit, the ending condition for the second-order difference equation is $y_{K-1} = 1$.

Step 3: Show that $K > 3 + \frac{\log\frac{\epsilon-1}{A_p}}{\log\frac{A_p}{\epsilon}}$ is always satisfied when $\Delta \to 0$.

Because $\epsilon = \frac{1}{2\beta}(1 - A_p + \sqrt{(1-A_p)^2 + 4A_p\beta^2})$, $\lim_{\Delta\to 0}\epsilon e^{-b\Delta} = 1$.

$$\therefore \lim_{\Delta\to 0}\frac{\log\frac{\epsilon-1}{A_p}}{\log\frac{A_p}{\epsilon}}\Delta = \lim_{\Delta\to 0}\frac{\log\frac{e^{b\Delta}-1}{A_p}}{\log A_p}\Delta = \lim_{\Delta\to 0}\frac{(\log\frac{b}{A_p} + \log\Delta)\Delta}{\log A_p} = 0$$

$$\lim_{\Delta\to 0}(K-3)\Delta = X^* > 0$$

$$\therefore \lim_{\Delta\to 0}(K-3)\Delta > \lim_{\Delta\to 0}\frac{\log\frac{\epsilon-1}{A_p}}{\log\frac{A_p}{\epsilon}}\Delta$$

Step 4: Take the limit $\Delta \to 0$ and use the fact that $\lim_{\Delta\to 0,\ k\Delta\to X}\epsilon^k = e^{b\frac{1-A}{1+(1-2p)A}X}$, we can get the limiting result.

$\square$

Appendix 2

Lemma 2.1: If player 1 weakly prefers NI at state $t \geq K$, then he will strictly prefer NI from t on.

Proof. Assume that $k \geq t+1$ is the smallest state in which player 1 weakly prefers I. Therefore, player 1 plays NI at state $k-1$.

$$V(k-1) = g_1(0,1) + \beta(pV(k) + (1-p)V(k-2)) \geq g_1(1,1) + \beta(qV(k-2) + (1-q)V(k))$$

$$V(k) = g_1(1,1) + \beta(qV(k-1) + (1-q)V(k+1)) \geq g_1(0,1) + \beta(pV(k+1) + (1-p)V(k-1))$$

$$\therefore V(k) - V(k-1) \leq g_1(1,1) + \beta(qV(k-1) + (1-q)V(k+1)) - (g_1(1,1) + \beta(qV(k-2) + (1-q)V(k)))$$

$$= \beta(V(k+1) - V(k)) + \beta q((V(k) - V(k-2)) - (V(k+1) - V(k-1)))$$

$$\because V(k) - V(k-2) \leq \frac{1}{\beta(1-p-q)}(g_1(0,1) - g_1(1,1)) \leq V(k+1) - V(k-1)$$

$$\therefore V(k) - V(k-1) \leq \beta(V(k+1) - V(k))$$

Step 1: The long-run player strictly prefers I at state $k+1$.

Assume that player 1 weakly prefers NI at state $k+1$, then

$$V(k+1) = g_1(0,1) + \beta(pV(k+2) + (1-p)V(k)) \geq g_1(1,1) + \beta(qV(k) + (1-q)V(k+2))$$

$$\therefore V(k+1) - V(k-1) = \beta(p(V(k+2) - V(k)) + (1-p)(V(k) - V(k-2))$$

However, $V(k+1) - V(k-1) \geq \frac{1}{\beta(1-p-q)}(g_1(0,1) - g_1(1,1)) \geq (V(k+2) - V(k))$ and $V(k+1) - V(k-1) \geq \frac{1}{\beta(1-p-q)}(g_1(0,1) - g_1(1,1)) \geq (V(k) - V(k-2))$, a contradiction. Therefore, player 1 strictly prefers I at state $k+1$.

Step 2: $V(k+2) - V(k+1) > V(k+1) - V(k)$.

$$\therefore V(k+1) = g_1(1,1) + \beta(qV(k) + (1-q)V(k+2)) > g_1(0,1) + \beta(pV(k+2) + (1-p)V(k))$$

$$\therefore V(k+2) - V(k+1) = \frac{1}{\beta(1-q)}(V(k+1) - V(k)) - \frac{q}{1-q}(V(k) - V(k-1))$$

$$\because V(k) - V(k-1) \leq \beta(V(k+1) - V(k))$$

$$\therefore V(k+2) - V(k+1) \geq (\frac{1}{\beta(1-q)} - \frac{q\beta}{1-q})(V(k+1) - V(k)) > V(k+1) - V(k)$$

Step 3: The long-run player strictly prefers I at state $k+2$.

Assume that the long-run player weakly prefers NI at period $k+2$, then

$$V(k+2) = g_1(0,1) + \beta(pV(k+3) + (1-p)V(k+1)) \geq g_1(1,1) + \beta(qV(k+1) + (1-q)V(k+3))$$

$$\therefore V(k+3) - V(k+1) \leq \frac{1}{\beta(1-p)}(g_1(0,1) - g_1(1,1)) \leq V(k+2) - V(k)$$

$$\because V(k+1) > g_1(0,1) + \beta(pV(k+2) + (1-p)V(k))$$

$$\therefore V(k+2) = V(k+1) + (V(k+2) - V(k+1)) > g_1(0,1) + \beta(pV(k+2) + (1-p)V(k)) + (V(k+2) - V(k+1))$$

$$= g_1(0,1) + \beta V(k) + (V(k+2) - V(k+1)) + \beta p(V(k+2) - V(k))$$

$$> g_1(0,1) + \beta V(k+1) + \beta p(V(k+3) - V(k+1))$$

where we use the fact that $V(k+2) - V(k+1) > V(k+1) - V(k)$ and $V(k+2) - V(k) \geq V(k+3) - V(k+1)$. Therefore, player 1 strictly prefers I at period $k+2$, a contradiction.

Step 4: Player 1 strictly prefers NI from t on.

Keep using the argument of Step 3, player 1 strictly prefers I at all state $t \geq k+1$.

$$\therefore V(t) = g_1(1,1)+\beta(qV(t-1)+(1-q)V(t+1)) > g_1(0,1)+\beta(pV(t+1)+(1-p)V(t-1))\ \forall t \geq k+1$$

Since $\{V(t)\}_{t\geq k}$ is a strictly increasing and bounded sequence, there is a limit V^* such that $V^* = g_1(1,1) + \beta(qV^* + (1-q)V^*)$. Therefore, $V^* = \frac{g_1(1,1)}{1-\beta}$. However, $V(t+1) - V(t-1) \geq \frac{1}{\beta(1-p-q)}(g_1(0,1) - g_1(1,1)$ implies that $0 = \lim_{t\to+\infty} V(t+1) - V(t-1) \geq \frac{1}{\beta(1-p-q)}(g_1(0,1) - g_1(1,1)$, a contradiction.

Therefore, the long-run player strictly prefers NI at state $t+1$. By induction, player 1 strictly prefers NI from t on. □

Proof of Theorem 2:

Proof. Prove by contradiction. Assume k as the smallest i to satisfy $M \leq i \leq K$ and $a(i) = y_i = 1$. Therefore, $a_k = y_k = 1$, $a_i = \pi(i)$ and $0 < y_i < 1$ for all $M \leq i \leq k-1$.

Step 1: Figure out the upper bound of y_{k-1}.

(1) $0 < y_{k+1} < 1$.

By lemma 1.4(3), $y_{k+1} \geq \frac{1}{\beta}(1 - A_{pq})y_k + A_{pq}y_{k-1}$.

$$\therefore y_{k-1} \leq \frac{1 - \frac{1}{\beta}(1 - A_{pq})}{A_{pq}}$$

(2) Player 2 strictly prefers B and player 1 weakly prefers I until state $k+i+2$: for some $i \geq 0$, $a_t > 0$, $y_t = 1$ for $k+1 \leq t \leq k+i+1$. Moreover, $a_{k+i+2} = 0$, $y_{k+i+2} = 1$ or $a_{k+i+2} = \pi(k+i+2)$, $0 < y_{k+i+2} < 1$.

$$\therefore V(k+1) - V(k-1) - \beta(1-q)(V(k+2) - V(k)) = \frac{q}{1-p-q}(1-A)y_{k-1} + A(1-y_{k-1})$$

$$\therefore V(k+i+2)-V(k+i)-\beta(1-p)(V(k+i+1)-V(k-i+1)) \leq \beta p(V(k+i+3)-V(k+i+1)) < \frac{p(1-A)}{1-p-q}$$

We know that

$$V(k+i+1) - V(k+i-1) = \beta q(V(k+i) - V(k+i-2)) + \beta(1-q)(V(k+i+2) - V(k+i))$$

It is true that $V(k+i) - V(k+i-2) > V(k+i+2) - V(k+i)$.

$$\therefore V(k+i+1)-V(k+i-1) < \frac{\beta(1-q)}{1+\beta(1-q)}(V(k+i+2)-V(k+i))+\frac{1}{1+\beta(1-q)}(V(k+i)-V(k+i-2))$$

$$\therefore V(k+i+1) - V(k+i-1) - \beta(1-q)(V(k+i+2) - V(k+i))$$

$$< V(k+i) - V(k+i-2) - \beta(1-q)(V(k+i+1) - V(k+i-1))$$

By induction,

$$\therefore V(k+i+1) - V(k+i-1) - \beta(1-q)(V(k+i+2) - V(k+i))$$

$$< V(k+1) - V(k-1) - \beta(1-q)(V(k+2) - V(k))$$

$$= \frac{q}{1-p-q}(1-A)y_{k-1} + A(1-y_{k-1})$$

$$\therefore (1-\beta(1-q))(V(k+i+2) - V(k+i)) + (1-\beta(1-p))(V(k+i+1) - V(k+i-1))$$

$$< \frac{p(1-A)}{1-p-q} + \frac{q(1-A)}{1-p-q}y_{k-1} + A(1-y_{k-1})$$

$$\therefore (1-\beta(1-q) + 1 - \beta(1-p))\frac{1-A}{\beta(1-p-q)}$$

$$< \frac{p(1-A)}{1-p-q} + \frac{q(1-A)}{1-p-q}y_{k-1} + A(1-y_{k-1})$$

$$\therefore y_{k-1} \leq 1 - \frac{2(1-\beta)(1-A)}{\beta(A(1-p)-q)}$$

Step 2: If $y_{k-2} \leq y_{k-1}$, then we reach a contradiction.

By the same argument as lemma 4(2), $1 \leq \frac{1}{\beta}(1-A_{pq})y_{k-1} + A_{pq}y_{k-2}$, where $A_{pq} = \frac{A(1-p)-q}{1-q-pA}$.

$$\therefore y_{k-1} \geq \frac{1}{\frac{1}{\beta}(1-A_{pq}) + A_{pq}} = 1 - \frac{(1-\beta)(1-A_{pq})}{1-A_{pq}+A_{pq}\beta} = 1 - \frac{(1-\beta)(1-A)}{1-A+(A(1-p)-q)\beta}$$

If $0 < y_{k+1} < 1$, then $y_{k-1} \leq \frac{1-\frac{1}{\beta}(1-A_{pq})}{A_{pq}}$. Therefore, $\frac{1}{\frac{1}{\beta}(1-A_{pq})+A_{pq}} < \frac{1-\frac{1}{\beta}(1-A_{pq})}{A_{pq}}$, a contradiction.

Player 2 strictly prefers B and player 1 weakly prefers I until state $k+i+2$, then $y_{k-1} \leq 1 - \frac{2(1-\beta)(1-A)}{\beta(A(1-p)-q)}$ by step 1.

$$\therefore \frac{(1-\beta)(1-A)}{1-A+(A(1-p)-q)\beta} > \frac{2(1-\beta)(1-A)}{\beta(A(1-p)-q)}$$

$$\therefore 2(1-A) + \beta(A(1-p)-q) < 0$$

a contradiction.

Step 3: If $0 < y_i < 1$ for all $i \leq k-1$, then we can show by solving the second-order difference equation that $y_{k-2} \leq y_{k-1}$ by the definition of M.

Step 4: For any $M+1 \le i \le K-2$, if $y_{i-1}=1$, then it is impossible that $0<y_{i+1}<1$, $0<y_i<1$.

(1)$0<y_{i-2}<1$

If $y_i<1$ and $y_{i+1}<1$, then

$$y_i+\beta A y_{i-1}+\frac{p(1-A)}{1-p-q}y_k-\beta^2 q(V(i)-V(i-2))$$

$$=Ay_i+\beta y_{i+1}+\beta\frac{p(1-A)}{1-p-q}y_{k+1}-\frac{q(1-A)}{1-p-q}y_i$$

$$\therefore y_{i+1}=\frac{1}{\beta}\frac{1-A}{1-q-Ap}y_i+\frac{A(1-p-q)}{1-q-Ap}-\frac{\beta q(1-p-q)}{1-q-Ap}(V(i)-V(i-2))$$

$$\because V(i)=g_1(0,y_i)+\beta(pV(i+1)+(1-p)V(i-1))$$

$$\because V(i-2)=g_1(1,y_{i-2})+\beta(qV(k-3)+(1-q)V(k-1))$$

$$\therefore V(i)-V(i-2)=y_i-Ay_{i-2}+\beta p(V(i+1)-V(i-1))+\beta q(V(i-1)-V(i-3))$$

$$=(1+\frac{p(1-A)}{1-p-q})y_i-(A-\frac{q(1-A)}{1-p-q})y_{i-2}$$

$$\therefore y_{i+1}=(\frac{1}{\beta}\frac{1-A}{1-q-Ap}-\beta q)y_i+\frac{A(1-p-q)}{1-q-Ap}+\frac{\beta q(A(1-p)-q)}{1-q-Ap}y_{i-2}$$

If $\frac{1}{\beta}\frac{1-A}{1-q-Ap}-\beta q>0$, then $y_i>y_{i-2}\ge\beta$ implies that

$$\therefore y_{i+1}>(\frac{1}{\beta}\frac{1-A}{1-q-Ap}-\beta q)\beta+\frac{A(1-p-q)}{1-q-Ap}+\frac{\beta q(A(1-p)-q)}{1-q-Ap}\beta$$

$$=\frac{(1-\beta^2 q)(1-A)}{1-q-Ap}+\frac{A(1-p-q)}{1-q-Ap}>1$$

a contradiction.

(2) $y_{i-2}=1$.

$$y_i+\beta A y_{i-1}+\frac{p(1-A)}{1-p-q}y_i-\beta^2 q(V(i)-V(i-2))$$

$$=Ay_i+\beta y_{i+1}+\beta\frac{p(1-A)}{1-p-q}y_{i+1}-\frac{q(1-A)}{1-p-q}y_i$$

Assume that $y_t=1$ for $j\le t\le i-2$ and $0<y_{j-1}<1$. We can show that

$$V(i)-V(i-2)-\beta q(V(i-1)-V(i-3))=\frac{1-q}{1-p-q}(1-A)y_i-A(1-y_i)$$

$$V(j+1)-V(j-1)-\beta(1-q)(V(j+2)-V(j))=\frac{q}{1-p-q}(1-A)y_{j-1}+A(1-y_{j-1})$$

$$\because V(i-1)-V(i-3)-\beta(1-q)(V(i)-V(i-2)) < V(j+1)-V(j-1)-\beta(1-q)(V(j+2)-V(j))$$

$$\therefore (1-\beta(1-q))(V(i)-V(i-2)) + (1-\beta q)(V(i-1)-V(i-3))$$

$$< \frac{1-A}{1-p-q} y_i + (\frac{q(1-A)}{1-p-q} - A)(y_{j-1} - y_i)$$

$$\because V(i) - V(i-2) < V(i-1) - V(i-3)$$

$$\therefore V(i) - V(i-2) < \frac{1}{2-\beta}(\frac{1-A}{1-p-q} y_i + (\frac{q(1-A)}{1-p-q} - A)(y_{i-3} - y_i))$$

$$\therefore y_{i+1} > \frac{1}{\beta}\frac{1-A}{1-q-Ap} y_i + \frac{A(1-p-q)}{1-q-Ap}$$

$$-\frac{\beta q(1-p-q))}{(2-\beta)(1-q-Ap)}(\frac{1-A}{1-p-q} y_i + (\frac{q(1-A)}{1-p-q} - A)(y_{i-3} - y_i))$$

$$\therefore y_{i+1} > (\frac{1}{\beta}\frac{1-A}{1-q-Ap} - \frac{\beta q}{2-\beta}) y_i + \frac{A(1-p-q)}{1-q-Ap} + \frac{\beta q}{2-\beta}\frac{A(1-p)-q}{1-q-Ap} y_{i-3}$$

If $\frac{1}{\beta}\frac{1-A}{1-q-Ap} - \frac{\beta q}{2-\beta} > 0$, then $y_i > y_{i-3} \geq \beta$ implies that

$$y_{i+1} > (\frac{1-A}{1-q-Ap} - \frac{\beta^2 q}{2-\beta}) + \frac{A(1-p-q)}{1-q-Ap} + \frac{\beta^2 q}{2-\beta}\frac{A(1-p)-q}{1-q-Ap}$$

$$= \frac{1-Ap-Aq}{1-q-Ap} - \frac{\beta^2}{2-\beta}\frac{q(1-A)}{1-q-Ap} > \frac{1-Ap-Aq}{1-q-Ap} - \frac{q(1-A)}{1-q-Ap} = 1$$

a contradiction.

Step 5: Show that $y_i = 1$ for all $M \leq i \leq k-2$.

We know $0 < y_{k-1} < 1$. Assume that $0 < y_{k-2} < 1$. By step 3, there exists $i \leq k-2$ such that $y_i = 1$. Let i^* be the largest one to satisfy the above condition. Then, $0 < y_{i^*+1} < 1$, $0 < y_{i^*+2} < 1$ and $y_{i^*} = 1$, a contradiction to step 4. Therefore, $y_{k-2} = 1$. By the definition of k, $y_i = 1$ for all $M \leq i \leq k-2$.

Step 6: There is a sequence $k-1 = k_0 < k_1 < \ldots < k_N \leq K-1$ such that $0 < y_{k_i} < 1$ and $y_j = 1$ for any $M \leq j \leq K-1$ and $j \notin \{k_i\}_{i=0}^N$.

Define $k_0 = k-1$. Construct a sequence $\{k_i\}$ as below. For each $i \geq 0$, let k_{i+1} be the smallest $t \geq k_i + 1$ such that $0 < y_t < 1$, then by step 4, $y_{k_{i+1}+1} = 1$. Therefore, we have shown that there is a sequence $k-1 = k_0 < k_1 < \ldots < k_N \leq K-1$ such that $0 < y_{k_i} < 1$ and $y_j = 1$ for any $k-1 \leq j \leq K-1$ and $j \notin \{k_i\}_{i=0}^N$. Combined with step 5, we get the result.

Step 7: Show that it is impossible to have more than $\frac{\log(D)}{\log(x_1)} + 2$ consecutive complete trust and player 1 weakly prefer I.

Prove by contradiction. Define $n = i_1 - i_0$. For all $i_0 \le i \le i_1$,

$$V(i) = g_1(1,1) + \beta(qV(i-1) + (1-q)V(i+1)) \ge g_1(1,1) + \beta(qV(i-1) + (1-q)V(i+1))$$

Define $W(i) = V(i) - V(i-2)$, then $W(i) > W(i+1)$ for all $i_0 + 1 \le i \le i_1 - 1$.

$$W(i+1) = \frac{1}{\beta(1-q)}W(i) - \frac{q}{1-q}W(i-1)$$

$$\therefore W(i) = \lambda_1 x_1^{i-i_0} + \lambda_2 x_2^{i-i_0}$$

where $x_1 = \frac{1-\sqrt{1-4\beta^2 q(1-q)}}{2\beta(1-q)} < 1$ and $x_2 = \frac{1+\sqrt{1-4\beta^2 q(1-q)}}{2\beta(1-q)} > 1$. We know that $\lambda_1 x_1^{n-1} + \lambda_2 x_2^{n-1} > \lambda_1 x_1^n + \lambda_2 x_2^n$. Therefore,

$$x_1^{n-1} > \frac{(\lambda_1 x_1^{n-1} + \lambda_2 x_2^{n-1})(x_2 - 1)}{\lambda_1(x_2 - x_1)}$$

Next, figure out the upper bound of λ_1. Assume that $0 < y_{i_0-1} < 1$, then $y_{i_0-1} \ge \beta$.

$$\because W(i_0) = \beta(1-q)W(i_0+1) + \frac{q(1-A)}{1-p-q}y_{i_0-1} + A(1-y_{i_0-1})$$

$$\therefore \lambda_1 + \lambda_2 \le \beta(1-q)(\lambda_1 x_1 + \lambda_2 x_2) + \frac{q(1-A)\beta}{1-p-q} + A(1-\beta)$$

Because $\lambda > 0$ and $\beta(1-q)x_2 - 1 < 0$, then

$$\lambda_1 < \frac{\frac{q(1-A)\beta}{1-p-q} + A(1-\beta)}{1-\beta(1-q)x_1}$$

Because $\lambda_1 x_1^{n-1} + \lambda_2 x_2^{n-1} > \frac{1-A}{\beta(1-p-q)}$, then

$$x_1^{n-1} > \frac{(\frac{1-A}{\beta(1-p-q)})(x_2-1)}{\frac{\frac{q(1-A)\beta}{1-p-q}+A(1-\beta)}{1-\beta(1-q)x_1}(x_2 - x_1)} \equiv D$$

$$\therefore i_1 - i_0 + 1 = n + 1 < \frac{\log(D)}{\log(x_1)} + 2$$

Step 8: Show that $N > \frac{\log(\frac{2}{2+\beta})}{\log(A_{pq})}$ as $\Delta \to 0$.

Assume that $N \le \frac{\log(\frac{2}{2+\beta})}{\log(A_{pq})}$. Therefore, there are $K - M - N - 1$ complete trust $y_i = 1$ for $M \le i \le K-1$. Because there are $N+1$ incomplete trust, then there exists a sequence of consecutive complete trust with the number at least $\frac{K-M-N-1}{N+2}$. By step 7, $\frac{K-M-N-1}{N+2} \le \frac{\log(D)}{\log(x_1)} + 2$.

$$\therefore K - M < (N+2)(\frac{\log(D)}{\log(x_1)} + 3) < (\frac{\log(\frac{2}{2+\beta})}{\log(A_{pq})} + 2)(\frac{\log(D)}{\log(x_1)} + 3)$$

a contradiction when $\Delta \to 0$.

Step 9: $y_{k_{i+1}} > \frac{2-\beta}{\beta}(1-A_{pq}) + A_{pq}y_{k_i}$ and $y_{k_0} \geq \beta$. Then, it is trivial to show that $y_{k_N} > 1$, a contradiction.

$$\frac{2-\beta}{\beta} - y_{k_N} < A_{pq}^N(\frac{2-\beta}{\beta} - y_{k_0}) \leq A_{pq}^N(\frac{2-\beta}{\beta} - \beta)$$

$$\therefore y_{k_N} > \frac{2-\beta}{\beta} - A_{pq}^N(\frac{2-\beta}{\beta} - \beta) > 1$$

The last inequality holds since $N > \frac{\log(\frac{2}{2+\beta})}{\log(A_{pq})}$ □

Appendix 3

Lemma 3.1: If player 1 weakly prefers action I in state $k \geq K$ and $V(k) \geq V(k+1)$, then in each state $i = 0, 1, \ldots, k-1$, we have (1) player 1 weakly prefers action I; (2) $y(i) = y(k) = 1$; and (3) $V(i-1) \geq V(i)$.

Proof. We know that $y(k) = 1$ for each $t \geq K$. Assume, for induction, that, for $i = k+1, \ldots, t$, the three properties hold. Consider $i = k$. Prove (2) by contradiction, assume that $y(k) < y(k+1) = 1$.

Case 1: $a(k) > \pi(k)$.

In this case, it is optimal for player 2 to choose B, so $y(k) = 1$, a contradiction.

Case 2: $a(k) \leq \pi(k)$.

$$\therefore V(k) = g_1(0, y(k)) + \beta((1-p)V(0) + pV(k+1)) \geq g_1(1, y(k)) + \beta((1-q)V(k+1) + qV(0))$$

By submodularity,

$$g_1(0, y(k)) - g_1(1, y(k)) < g_1(0, 1) - g_1(1, 1)$$

$$\therefore g_1(0, 1) + \beta((1-p)V(0) + pV(k+1)) > g_1(1, 1) + \beta((1-q)V(k+1) + qV(0))$$

$$\therefore V(k+1) = g_1(0, 1) + \beta((1-p)V(0) + pV(k+2))$$

$$= g_1(0, 1) + \beta((1-p)V(0) + pV(k+1)) + \beta p(V(k+2) - V(k+1))$$

$$> g_1(1, 1) + \beta((1-q)V(k+1) + qV(0)) + \beta p(V(k+2) - V(k+1))$$

$$\geq g_1(1, 1) + \beta((1-q)V(k+2) + qV(0))$$

The last inequality uses the fact that $(1-p-q)(V(k+1)-V(k+2)) \geq 0$. Therefore, player 1 strictly prefers NI in state $k+1$, a contradiction. Therefore, we have proved (2). Then, (1) and (3) holds trivially,

□

Corollary 3.1: If $V(t) \geq V(t+1)$, then player 1 strictly prefers action NI in state $t \geq K$ and $a(t) = 0$.

Proof. If player 1 weakly prefers action I in state $t \geq K$ and $V(t) \geq V(t+1)$, then by Lemma 3.1, $y(i) = 1$ for $i = 1, 2, \ldots, t-1$. It is obvious that $y(i) = 1$ for $i \geq t$, because $i \geq t \geq K$. In all, $y(i) = 1$ for all state i. Therefore, player 2's strategy does not depend on the history of the game. As a result, player 1 would strictly prefer action NI, a contradiction. □

Lemma 3.2: If $V(t) < V(t+1)$, then player 1 strictly prefers action NI in state $t \geq K$ and $a(t) = 0$.

Proof. Assume that player 1 weakly prefer I at $t \geq K$ and $V(t) < V(t+1)$.
Case 1: $V(i) < V(i+1)$ for all $i \geq t$.

Then, $\{V(i)\}_{i=t}^{+\infty}$ is a strictly increasing and bounded sequence and assume the limit is V^*. Furthermore,

$$V(i) = g_1(1,1) + \beta((1-q)V(i+1) + qV(0)) \quad \forall i \geq t$$

$$\therefore V^* = g_1(1,1) + \beta((1-q)V^* + qV(0))$$

$$\therefore V^* = \frac{g_1(1,1) + \beta V(0)}{1-\beta+\beta q}\ ,\ V(i) < V(i+1) < \frac{g_1(1,1) + \beta V(0)}{1-\beta+\beta q}\ \forall i \geq t$$

$$\because V(i) = g_1(1,1) + \beta((1-q)V(i+1) + qV(0)) > g_1(1,1) + \beta((1-q)V(i) + qV(0))$$

$$\therefore V(i) > \frac{g_1(1,1) + \beta V(0)}{1-\beta+\beta q}$$

This is a contradiction.
Case 2: $V(i) \geq V(i+1)$ for some $i > t$.

Assume i^* is the smallest $i > t$ such that $V(i) \geq V(i+1)$. Therefore, $V(t) < V(t+1) < \ldots < V(i^*)$.

If player 1 weakly prefer I at i^* and $V(i^*) \geq V(i^*+1)$, by lemma 3.1, we know that $V(i) \geq V(i+1)$ for all $i \leq i^*$, a contradiction to $V(t) < V(t+1)$.

If player 1 strictly prefer NI at i^*, then

$$V(i^*) = g_1(0,1) + \beta((1-p)V(0) + pV(i^*+1)) > g_1(1,1) + \beta((1-q)V(i^*+1) + qV(0))$$

Because player 1 weakly prefer I at $t \geq K$,

$$V(t) = g_1(1,1) + \beta((1-q)V(t+1) + qV(0)) \geq g_1(0,1) + \beta((1-p)V(0) + pV(t+1))$$

$$\therefore V(i^*+1) < V(t+1)$$

Since $V(t) < V(i^*)$, then $V(t+1) < V(i^*+1)$, a contradiction.

In all, we have shown that if $V(t) < V(t+1)$, then player 1 strictly prefers action NI in state $t \geq K$ and $a(t) = 0$.

$\square$

Corollary 3.2: Player 1 strictly prefers action NI in state $t \geq K$ and $V(t) = V(K)$ for all $t \geq K$.

Proof. By Corollary 3.1 and lemma 3.2, Player 1 strictly prefers action NI in state $t \geq K$. Furthermore, $V(t) = V(K) = g_1(0,1) + \beta((1-p)V(0) + pV(K))$. $\square$

Lemma 3.3: If for some $j < K$, $y(j+1) > 0$, then they are strictly increasing for all i such that $j \leq i \leq K$.

Proof. Firstly, show that $y(K-1) < y(K)$.

If $y(K-1) = 0$, then $y(K-1) < y(K)$ holds. Furthermore, $V(K-1) = g_1(0,0) + \beta((1-p)V(0) + pV(K))$ and $V(K) = g_1(0,1) + \beta((1-p)V(0) + pV(K))$. Therefore,$V(K-1) < V(K)$.

If $y(K-1) > 0$, then $a(K-1) \geq \pi(K-1) > 0$.

$$\therefore g_1(1, y(K-1)) + \beta((1-q)V(K) + qV(0)) \geq g_1(0, y(K-1)) + \beta((1-p)V(0) + pV(K))$$

Since player 1 strictly prefers NI in state K,

$$V(K) = g_1(0, y(K)) + \beta((1-p)V(0) + pV(K)) > g_1(1, y(K)) + \beta((1-q)V(K) + qV(0))$$

$$\therefore g_1(0, y(K)) - g_1(1, y(K)) > g_1(0, y(K-1)) - g_1(1, y(K-1))$$

By submodularity, $y(K-1) < y(K)$.

$\therefore V(K-1) = g_1(1, y(K-1)) + \beta((1-p)V(K) + pV(0)) \leq g_1(1, y(K)) + \beta((1-q)V(K) + qV(0)) < V(K)$

Prove by contradiction. Suppose that $y(i) > 0$ and $y(i) \leq y(i-1)$. Let i^* be the largest state such that $0 < y(i^*) \leq y(i^*-1)$. Since $y(i^*) < y(i^*+1)$, $y(i^*) < 1$. Therefore, $a(i^*) = \pi(i^*)$ and $a(i^*-1) \geq \pi(i^*-1)$. Furthermore, $y(i) > 0$ for any $i \geq i^*$ means that $a(i) \geq \pi(i)$ for any $i \geq i^*$. Therefore, for any $i \geq i^*$, we have

$$V(i) = (g_1(1, y(i)) + \beta q V(0)) + \ldots + (\beta(1-q))^{K-i-1}(g_1(1, y(K-2)) + \beta q V(0)) + (\beta(1-q))^{K-i} V(K-1)$$

$$V(i+1) = (g_1(1, y(i+1)) + \beta q V(0)) + \ldots + (\beta(1-q))^{K-i-1}(g_1(1, y(K-1)) + \beta q V(0)) + (\beta(1-q))^{K-i} V(K)$$

$$\therefore V(i) < V(i+1) \ \forall i \geq i^*$$

$$V(i^*-1) = g_1(1, y(i^*-1)) + \beta((1-q)V(i^*) + qV(0)) \geq g_1(0, y(i^*-1)) + \beta((1-p)V(0) + pV(i^*))$$

$$V(i^*) = g_1(1, y(i^*)) + \beta((1-q)V(i^*+1) + qV(0)) = g_1(0, y(i^*)) + \beta((1-p)V(0) + pV(i^*+1))$$

$$\therefore g_1(0, y(i^*-1)) - g_1(1, y(i^*-1)) \leq \beta(1-p-q)(V(i^*) - V(0))$$

$$< \beta(1-p-q)(V(i^*+1) - V(0)) = g_1(0, y(i^*)) - g_1(1, y(i^*))$$

By submodularity, $y(i^*-1) < y(i^*)$, a contradiction.

$\square$

Lemma 3.4: If $\beta > \frac{1-A+\gamma}{1-q-Ap}$, then $0 < y(X_i) < 1$ and $a(i) = \pi(i)$ for each $i \leq K-1$ and $\{y(i)\}_{i=0}^{K}$ is strictly increasing in i.

Proof. Assume, by contradiction, that $y(0) = 0$, then $a(0) \leq \pi(0) < 1$

$$\therefore V(0) = g_1(0, 0) + \beta((1-p)V(0) + pV(1)) \geq g_1(1, 0) + \beta((1-q)V(1) + qV(0))$$

$$\therefore V(1) \leq V(0) + \frac{g_1(0,0) - g_1(1,0)}{\beta(1-p-q)}$$

$$\therefore V(0) \leq \frac{g_1(0,0) + \beta p \frac{g_1(0,0) - g_1(1,0)}{\beta(1-p-q)} + \beta p V(0)}{1 - \beta + \beta p}$$

$$\therefore V(0) \leq \frac{g_1(0,0)}{1-\beta} + \frac{p}{1-p-q} \frac{g_1(0,0) - g_1(1,0)}{1-\beta}$$

Because $\beta > \frac{1-A+\gamma}{1-q-Ap}$ and $y(K) = 1$, we can show that

$$g_1(0,1) + \beta((1-p)V(0) + p\frac{g_1(1,1) + \beta V(0)}{1-\beta(1-q)}) < \frac{g_1(1,1) + \beta V(0)}{1-\beta(1-q)}$$

Therefore, $V(K) = \frac{g_1(1,1)+\beta V(0)}{1-\beta(1-q)}$ which means that $V(K) = g_1(1,1)+\beta((1-q)V(K)+qV(0))$, a contradiction to the fact that player 1 will strictly prefer NI in state K.

Next, assume, by contradiction, that $y(1) = 0$. Then, $a(1) \leq \pi(1) < 1$, so NI is an optimal choice for player 1 in state 1.

$$\therefore V(1) = g_1(0,0) + \beta((1-p)V(0) + pV(2)) \geq g_1(1,0) + \beta((1-q)V(2) + qV(0))$$

$$\therefore V(2) - V(0) \leq \frac{g_1(0,0) - g_1(1,0)}{(1-p-q)\beta}$$

$y(0) > 0$ implies $a(0) \geq \pi(0) > 0$, so I is an optimal choice for player 1 in state 0,

$$\therefore V(0) = g_1(1,y(0)) + \beta((1-q)V(1) + qV(0)) \geq g_1(0,y(0)) + \beta((1-p)V(0) + pV(1))$$

$$\therefore V(1) - V(0) \geq \frac{g_1(0,y(0)) - g_1(1,y(0))}{(1-p-q)\beta} \geq \frac{g_1(0,0) - g_1(1,0)}{(1-p-q)\beta}$$

$$\therefore V(2) \leq V(1)$$

$$\therefore V(0) = g_1(1,y(0)) + \beta((1-q)V(1) + qV(0)) \leq g_1(1,y(0)) + \beta V(1)$$

$$= g_1(1,y(0)) + \beta g_1(0,0) + \beta^2((1-p)V(0) + pV(2))$$

$$\leq g_1(1,y(0)) + \beta g_1(0,0) + \beta^2((1-p)V(0) + pV(1))$$

$$< g_1(0,y(0)) + \beta g_1(0,0) + \beta^2((1-p)V(0) + pV(1))$$

$$< g_1(0,y(0)) + \beta g_1(0,y(0)) + \beta^2((1-p)V(0) + pV(1))$$

$$= g_1(0,y(0)) + \beta g_1(0,y(0)) + \beta^2((1-p)V(0) + pV(1))$$

$$\leq g_1(0,y(0)) + \beta V(0) \leq V(0)$$

This is a contradiction.

By Lemma 3.3, $y(1) > 0$ implies that $\{y(i)\}_{i=0}^{K}$ is strictly increasing in i. Therefore, $y(i) > 0$ for each $i < K$. Because $y(K) = 1$ and $\{y(i)\}_{i=0}^{K}$ is strictly increasing in i, $y(i) < 1$ for each $i < K$. Therefore, $0 < y(i) < 1$ for each $i < K$ implies that $a(i) = \pi(i)$ for each $i < K$.

□

Proof of Theorem 3:

Proof. It is obvious that $y(t) = 1$ for each $t \geq K$. By Corollary 3.2, player 1 strictly prefers action NI in state $t \geq K$. Then, we have proved (2). Lemma 3.4 proved (1). Then, let's characterize $y(k)$ for $0 \leq k \leq K-1$.

$$\because V(k) = g_1(0, y(k)) + \beta((1-p)V(0) + pV(k+1)) = g_1(1, y(k)) + \beta((1-q)V(k+1) + qV(0))$$

$$\therefore V(k+1) - V(0) = \frac{g_1(0, y(k)) - g_1(1, y(i))}{\beta(1-p-q)}$$

$$\therefore V(k) - V(0) = g_1(0, y(k)) - (1-\beta)V(0) + \beta p(V(k+1) - V(0))$$

$$\therefore \frac{g_1(0, y(k-1)) - g_1(1, y(k-1))}{\beta(1-p-q)} = g_1(0, y(k)) - g_1(0, y(0)) + \beta p \frac{g_1(0, y(i)) - g_1(1, y(k))}{\beta(1-p-q)}$$

$$\therefore y(k) = \eta_1 y(k-1) + \eta_3 y(0) + \eta_2$$

Then, all the results follow if we let $y(k) = 1$ for all $k \geq K$. $\square$

Proof of Proposition 3:

Proof. It is trivial to show that

$$y(0) = \frac{1 - \eta_1 - \eta_2}{\eta_3}$$

$$y(k) = 1 - \eta_1^k (1 - y(0)) \ \forall k \geq 0$$

$$\because y(0) = \frac{1 - \eta_1 - \eta_2}{\eta_3} = \frac{(1-\beta p)(g_1(1,1) - g_1(0,0)) - (1 - \beta(1-q))(g_1(0,1) - g_1(0,0))}{\beta(1-p-q)(g_1(0,1) - g_1(0,0))}$$

$$\therefore \frac{\partial y(0)}{\partial p} = \frac{-(1-\beta(1-q))(g_1(0,1) - g_1(1,1))}{\beta(1-p-q)^2(g_1(0,1) - g_1(0,0))} < 0$$

$$\therefore \frac{\partial y(0)}{\partial q} = \frac{-(1-\beta p)(g_1(0,1) - g_1(1,1))}{\beta(1-p-q)^2(g_1(0,1) - g_1(0,0))} < 0$$

Furthermore, it is true that $\frac{\partial \eta_1}{\partial p} > 0$ and $\frac{\partial \eta_1}{\partial q} > 0$.

$$\because y(k) = 1 - \eta_1^t (1 - y(0)) \ \forall k \geq 0$$

$$\therefore \frac{\partial y(k)}{\partial p} < 0, \ \frac{\partial y(k)}{\partial q} < 0$$

$\square$

Proof of Claim 1:

Proof. Assume that $y(0) > 0$, then we have shown that in the limit case,

$$y(0) = \frac{(1-\beta p)(g_1(1,1) - g_1(0,0)) - (1-\beta(1-q))(g_1(0,1) - g_1(0,0))}{\beta(1-p-q)(g_1(0,1) - g_1(0,0))}$$

$$= \frac{g_1(1,1) - g_1(0,0) - ((g_1(1,1) - g_1(1,0))p + (g_1(0,1) - g_1(0,0))q)}{(1-p-q)(g_1(0,1) - g_1(0,0))}$$

If assumption 3.1 is violated, then $y(0) \le 0$, a contradiction. Therefore, $y(0) = 0$ in the limit case. $\square$

Appendix 4

Lemma 4.1 : Show that the short run player strictly prefer NI at all $t \ge K+1$.

Proof. There are no two consecutive states $t \ge K$ and $t+1 \ge K$ such that the long-run player weakly prefers I. Otherwise,

$$V(t) = g_1(1,1) + \beta V(t+1) \ge g_1(0,1) + \beta V(t-1)$$

$$V(t+1) = g_1(1,1) + \beta V(t+2) \ge g_1(0,1) + \beta V(t)$$

$$\therefore V(t+2) = V(t+1) + (V(t+2) - V(t+1)) > g_1(0,1) + \beta V(t) + \beta(V(t+1) - V(t)) = g_1(0,1) + \beta V(t+1)$$

where we use the fact that $V(t+2) - V(t+1) = \frac{1}{\beta}(V(t+1) - V(t)) > \beta(V(t+1) - V(t))$.

$$V(t+2) = g_1(1,1) + \beta V(t+3) > g_1(0,1) + \beta V(t+1)$$

By induction,

$$V(i) = g_1(1,1) + \beta V(i+1) \ge g_1(0,1) + \beta V(i-1) \ \forall i \ge t$$

a contradiction. Therefore, $V(t+1) = g_1(1,1) + \beta V(t+2) \ge g_1(0,1) + \beta V(t)$ implies that

$$V(t) = g_1(0,1) + \beta V(t-1) > g_1(1,1) + \beta V(t+1)$$

$$V(t+2) = g_1(0,1) + \beta V(t+1) > g_1(1,1) + \beta V(t+3)$$

$$\therefore \frac{1}{\beta}(g_1(0,1) - g_1(1,1)) < V(t+2) - V(t) = \beta(V(t+1) - V(t-1)) < g_1(0,1) - g_1(1,1)$$

a contradiction. $\square$

Lemma 4.2: If $K \geq \hat{K}$, then there is a unique absorbing equilibrium. Furthermore, the necessary condition for the existence of absorbing equilibrium is $K \geq \hat{K}$.

Proof. **Step 1:** If $0 < y_i < 1$ for $1 \leq i \leq K-1$, then we need $T^* = \hat{T}$.

By lemma 4.1, $a(i) = 0$ for all $i \geq K+1$. Firstly, assume that $a(K) = 1$ and check the equilibrium.

$$y_0 = V_0 = 0$$

$$g_1(0,1) + \beta V_0 > g_1(1,1) + \beta V_1$$

$$V_i = (1-y_i)\beta V_i + y_i(g_1(1,1) + \beta V_{i+1}) = (1-y_i)\beta V_i + y_i(g_1(0,1) + \beta V_{i-1}) \; \forall 1 \leq i \leq K-1$$

$$V_i = g_1(0,1) + \beta V_{i-1} > g_1(1,1) + \beta V_{i+1} \; \forall i \geq K$$

Because $g_1(0,1) + \beta V_0 = g_1(1,1) + \beta V_2$, $V_2 = \frac{1}{\beta}(g_1(0,1) - g_1(1,1))$.

Find the restriction on y_1. Because $V_1 = (1-y_1)\beta V_1 + y_1(g_1(0,1) + \beta V_0) = (1-y_1)\beta V_1 + g_1(0,1)y_1$, $\frac{g_1(0,1)\beta y_1}{1-\beta+\beta y_1} = \beta V_1 < g_1(0,1) - g_1(1,1)$, then $y_1 < \frac{1-\beta}{\beta}\frac{1-A}{A}$.

Solve for y_i. Firstly, solve for $\frac{V_1}{y_1}$ and $\frac{V_2}{y_2}$. We have shown that

$$V_1 = \frac{g_1(0,1)y_1}{1-\beta+\beta y_1}, \; \frac{V_1}{y_1} = \frac{g_1(0,1)}{1-\beta+\beta y_1}$$

$$\because V_2 = (1-y_2)\beta V_2 + y_2(g_1(0,1) + \beta V_1)$$

$$\therefore y_2 = \frac{1-\beta}{\beta}\frac{1-A}{A}\frac{1-\beta+\beta y_1}{1-\beta+(1+1/A)\beta y_1}, \; \frac{V_2}{y_2} = \frac{(A + \frac{(1+A)\beta y_1}{1-\beta})g_1(0,1)}{1-\beta+\beta y_1}$$

Then, show that $\frac{V_{i+2}}{y_{i+2}} = \frac{V_i}{y_i}$ for any $1 \leq i \leq K-1$. $V_i = (1-y_i)\beta V_i + y_i(g_1(1,1) + \beta V_{i+1}))$ and $V_{i+2} = (1-y_{i+2})\beta V_{i+2} + y_{i+2}(g_1(0,1) + \beta V_{i+1})$ implies that $(1-\beta)(\frac{V_{i+2}}{y_{i+2}} - \frac{V_i}{y_i}) + \beta(V_{i+2} - V_i) = g_1(0,1) - g_1(1,1)$. We know that $\beta(V_{i+2} - V_i) = g_1(0,1) - g_1(1,1)$ for any $1 \leq i \leq K-1$. Therefore, $\frac{V_{i+2}}{y_{i+2}} = \frac{V_i}{y_i}$.

$$\therefore y_{2i+1} = y_{2i-1} + \frac{(1-\beta+\beta y_1)(1-A)}{\beta}, \; y_{2i+2} = y_{2i} + \frac{(1-\beta+\beta y_1)(1-A)}{\beta(A + \frac{(1+A)\beta y_1}{1-\beta})}$$

Define $c_1 = (1-\beta+\beta y_1)(1-A)$ and $c_2 = \frac{(1-\beta+\beta y_1)(1-A)}{A+\frac{(1+A)\beta y_1}{1-\beta}}$

$$\therefore y_{2i+1} = y_1 + \frac{c_1}{\beta}i, \; y_{2i} = \frac{c_2}{\beta}i$$

We have the boundary end condition $y_K = 1$.

Case 1: K is an odd number.

Then we can solve y_1 by backward induction.

$$1 = y_1 + \frac{c_1}{\beta}\frac{K-1}{2} = y_1 + \frac{(1-\beta+\beta y_1)(1-A)}{\beta}\frac{K-1}{2}$$

$$\therefore y_1 = \frac{\frac{2\beta}{1-A} - (1-\beta)(K-1)}{\beta(K+\frac{2}{1-A}-1)}$$

$$\therefore c_1 = \frac{2}{K+\frac{2}{1-A}-1},\ c_2 = \frac{2(1-\beta)}{\frac{2(1+A)\beta}{1-A} - (1-\beta)(K-\frac{1+A}{1-A})}$$

In order for $y_{K-1} \leq 1$, we need $\frac{c_2}{\beta}\frac{K-1}{2} \leq 1$.

$$\therefore (1-\beta)K \leq \frac{1+A}{1-A}\beta + \frac{1-\beta}{1+\beta}$$

Check the optimality at state K. We need $\beta(V(K+1) - V(K-1)) < g_1(0,1) - g_1(1,1)$.

$$\therefore \beta(g_1(0,1) + g_1(0,1)\beta - (1-\beta^2)V(K-1)) = \beta(V(K+1) - V(K-1)) < g_1(0,1) - g_1(1,1)$$

$$\therefore V(K-1) > \frac{g_1(0,1)}{1-\beta} - \frac{g_1(0,1)-g_1(1,1)}{\beta(1-\beta^2)}$$

Because $V(K-1) = \frac{(K-1)(1-A)}{2\beta}g_1(0,1)$, then

$$\frac{(K-1)(1-A)}{2\beta} > \frac{1}{1-\beta} - \frac{1-A}{\beta(1-\beta^2)}$$

$$\therefore (1-\beta)K \leq \frac{1+A}{1-A}\beta - \frac{1-\beta}{1+\beta}$$

In all,

$$\frac{1+A}{1-A}\beta - \frac{1-\beta}{1+\beta} < K < \frac{1+A}{1-A}\beta + \frac{1-\beta}{1+\beta}$$

$$\therefore K = [\frac{1+A}{1-A}\frac{\beta}{1-\beta} - \frac{1}{1+\beta}] + 1$$

Case 2: K is an even number.

Then, we can solve y_1 by backward induction.

$$1 = \frac{K}{2}\frac{c_2}{\beta}$$

$$y_1 = \frac{(1-\beta)(K(1-\beta) - \frac{2A}{1-A}\beta)}{\frac{2(1+A)}{1-A}\beta^2 - K\beta(1-\beta)}$$

$$c_1 = \frac{2\beta(1-\beta)}{\frac{2(1+A)}{1-A}\beta - K(1-\beta)},\ c_2 = \frac{2\beta}{K}$$

In order for $y_{T^*-1} \leq 1$, we need $y_1 + \frac{c_1}{\beta}\frac{K-2}{2} \leq 1$.

$$\therefore (1-\beta)K \leq \frac{1+A}{1-A}\beta + \beta\frac{1-\beta}{1+\beta}$$

Check the optimality at state K. We need $\beta(V(K+1) - V(K-1)) < g_1(0,1) - g_1(1,1)$.

$$\therefore \beta(g_1(0,1) + g_1(0,1)\beta - (1-\beta^2)V(K-1)) = \beta(V(K+1) - V(K-1)) < 1$$

$$\therefore V(K-1) > \frac{g_1(0,1)}{1-\beta} - \frac{g_1(0,1) - g_1(1,1)}{\beta(1-\beta^2)}$$

Because $V(K-1) = \frac{y_1 g_1(0,1)}{1-\beta+\beta y_1} + \frac{(K-2)(1-A)}{2\beta}g_1(0,1)$, then

$$\frac{y_1}{1-\beta+\beta y_1} + \frac{(K-2)(1-A)}{2\beta} > \frac{1}{1-\beta} - \frac{1-A}{\beta(1-\beta^2)}$$

$$\therefore (1-\beta)K > \frac{1+A}{1-A}\beta - \beta\frac{1-\beta}{1+\beta}$$

In all,

$$\therefore \frac{1+A}{1-A}\beta - \beta\frac{1-\beta}{1+\beta} < (1-\beta)K < \frac{1+A}{1-A}\beta + \beta\frac{1-\beta}{1+\beta}$$

$$\therefore K = [\frac{1+A}{1-A}\frac{\beta}{1-\beta} - \frac{\beta}{1+\beta}] + 1$$

We define that

$$\hat{K} \equiv [\frac{1+A}{1-A}\frac{\beta}{1-\beta} - \frac{\beta}{1+\beta}] + 1 = [\frac{1+A}{1-A}\frac{\beta}{1-\beta} - \frac{1}{1+\beta}] + 1$$

Define $\hat{X} \equiv \lim_{\Delta\to 0} \hat{K}\Delta$. Then, $b\hat{X} = \frac{1+A}{1-A}$.

In the limit, if $X^* = \hat{X}$, then $\lim_{\Delta\to 0}\frac{c_1}{\Delta} = \lim_{\Delta\to 0}\frac{c_2}{\Delta} = \frac{2b(1-A)}{1+A}$.

$$\therefore y_X = \frac{b(1-A)}{1+A}X,\ a(X) = \pi(X)\ \forall 0 \leq X \leq X^*$$

$$y_X = 1,\ a(X) = 0\ \ \forall X \geq X^*$$

Next, show that it is impossible that $V_K = g_1(1,1) + \beta V_{T^*+1} \geq g_1(0,1) + \beta V_{K-1}$. Prove by contradiction. Therefore, $V(K+1) = \frac{g_1(0,1)+\beta g_1(1,1)}{1-\beta^2}$ and $V(K) = \frac{g_1(1,1)+\beta g_1(0,1)}{1-\beta^2}$. Then, there exists $1 \leq m \leq K-1$ such that $y_m = 1$ and $0 < y(i) < 1$ for all $1 \leq i \leq m-1$.

(1) K is even.

$$\frac{g_1(1,1)+\beta g_1(0,1)}{1-\beta^2} = V(K) > V(2) + \frac{(K-2)(g_1(0,1)-g_1(1,1))}{2\beta} = \frac{(g_1(0,1)-g_1(1,1))K}{2\beta}$$

$$\therefore (1-\beta)K < \frac{1+A}{1-A}\beta - \beta\frac{1-\beta}{1+\beta}$$

a contradiction to $K = \hat{K}$.

(2) K is odd.

$$\frac{g_1(1,1)+\beta g_1(0,1)}{1-\beta^2} = V(K) = V(1) + \frac{(K-1)(g_1(0,1)-g_1(1,1))}{2\beta} < \frac{(K+1)(g_1(0,1)-g_1(1,1))}{2\beta}$$

$$\therefore (1-\beta)(K+1) > \frac{1+A}{1-A}\beta - \beta\frac{1-\beta}{1+\beta}$$

$$\frac{g_1(0,1)+\beta g_1(1,1)}{1-\beta^2} = V(K+1) > V(2) + \frac{(K-1)(g_1(0,1)-g_1(1,1))}{2\beta} = \frac{(K+1)(g_1(0,1)-g_1(1,1))}{2\beta}$$

$$\therefore (1-\beta)(K+1) < \frac{1+A}{1-A}\beta + \beta\frac{1-\beta}{1+\beta}$$

$$\therefore K+1 = \hat{K}$$

a contradiction to $K = \hat{K}$.

Step 2: $K < \hat{K}$.

Show that there is no absorbing equilibrium. Otherwise, there exists $k \geq 0$ such that $y_i = 0$ for all $0 \leq i \leq k$ and $y_j > 0$ for all $j \geq k+1$. If we assume that $0 < y_j < 1$ for all $k+1 \leq j \leq K-1$, then by step 1, we need $K = k + \hat{K}$, a contradiction to $K < \hat{K}$. Therefore, there is $k+1 \leq m \leq K-1$ such that $y_m = 1$ and $0 < y(i) < 1$ for all $k+1 \leq i \leq m-1$. Step 4 proves that there is no absorbing equilibrium.

Step 3: $K > \hat{K}$.

Then, state $K - \hat{K}$ plays the same role as state 0 in step 1:

$$y_i = V_i = 0 \ \ \forall 0 \leq i \leq K - \hat{K}$$

$$V_i = (1-y_i)\beta V_i + y_i(g_1(1,1)+\beta V_{i+1}) = (1-y_i)\beta V_i + y_i(g_1(0,1)+\beta V_{i-1}) \ \forall K-\hat{K}+1 \leq i \leq K-1$$

$$V_i = g_1(0,1) + \beta V_{i-1} > g_1(1,1) + \beta V_{i+1} \ \forall i \geq K$$

In the limit,

$$y(X) = 0 \ \ \forall 0 \leq X \leq X^* - \hat{X}$$

$$y(X) = 1 + \frac{b(1-A)}{1+A}(X - X^*) \; \forall \; X^* - \hat{X} \leq X \leq X^*$$

$$y(X) = 1, \; a(X) = 0 \;\; \forall X \geq X^*$$

Step 4 shows that this is the only absorbing equilibrium.

Step 4: Show that the following is impossible: there exists $1 \leq k+1 \leq m \leq K-1$ where $K - k \neq \hat{K}$ or $K < \hat{T}$ such that (1) $y_m = 1$, (2) $y_i = 0$ for all $0 \leq i \leq k-1$, (3) $0 < y_i < 1$ for all $k \leq i \leq m-1$.

We can show that $y_{m+2i} = 1$ and $0 < y_{m+2i+1} < 1$, where $m+2i, m+2i+1 < K$. Furthermore, $V(k+1) < \frac{1}{\beta}$ and $V(k+2) = \frac{1}{\beta}$.

Case 1: $V(K) = g_1(1,1) + \beta V(K+1) \geq g_1(0,1) + \beta V(K-1)$.

Therefore, $V(K+1) = \frac{g_1(0,1)+\beta g_1(1,1)}{1-\beta^2}$ and $V(K) = \frac{g_1(1,1)+\beta g_1(0,1)}{1-\beta^2}$.

(1) $K - k$ is even.

$$V(K) = V(k+2) + \frac{(K-k-2)(g_1(0,1) - g_1(1,1))}{2\beta} = \frac{(K-k)(g_1(0,1) - g_1(1,1))}{2\beta}$$

$$\therefore (1-\beta)(K-k) = \frac{1+A}{1-A}\beta - \beta\frac{1-\beta}{1+\beta}$$

However, this is not generically true since $K - k$ is an integer number.

(2) $K - k$ is odd.

$$V(K) = V(k+1) + \frac{(K-k-1)(g_1(0,1) - g_1(1,1))}{2\beta} < \frac{(K-k+1)(g_1(0,1) - g_1(1,1))}{2\beta}$$

$$\therefore (1-\beta)(K-k+1) > \frac{1+A}{1-A}\beta - \beta\frac{1-\beta}{1+\beta}$$

$$V(K+1) > V(k+2) + \frac{(K-k-1)(g_1(0,1) - g_1(1,1))}{2\beta} = \frac{(K-k+1)(g_1(0,1) - g_1(1,1))}{2\beta}$$

$$\therefore (1-\beta)(K-k+1) < \frac{1+A}{1-A}\beta + \beta\frac{1-\beta}{1+\beta}$$

$$\therefore K - k + 1 = \hat{K}$$

We need $\hat{K}$ is even. It is easy to show that $K - m$ is even since $y_K = 1$ and $y_m = 1$. Furthermore, $K - k$ is odd implies that $m - k$ is odd.

$$\therefore V(m+1) = V(k+2) + \frac{(m-k-1)(g_1(0,1) - g_1(1,1))}{2\beta} = \frac{(m-k+1)(g_1(0,1) - g_1(1,1))}{2\beta}$$

Because $V(i+1) - V(i-1) = \frac{g_1(0,1)-g_1(1,1)}{\beta^2}$ for $m+3 \leq i \leq K$

$$V(K+1) = V(m+1) + \frac{(K-m)(g_1(0,1)-g_1(1,1))}{2\beta^2}$$

$$= \frac{(m-k+1)(g_1(0,1)-g_1(1,1))}{2\beta} + \frac{(K-m)(g_1(0,1)-g_1(1,1))}{2\beta^2}$$

However, this is not generically true since $K-m$ and $m-k+1$ are integer numbers.

Case 2: $V(K) = g_1(0,1) + \beta V(K-1) > g_1(1,1) + \beta V(K+1)$.

Therefore, $V(K) = \frac{g_1(0,1)+\beta g_1(1,1)}{1-\beta^2}$ and $V(K-1) = \frac{g_1(1,1)+\beta g_1(0,1)}{1-\beta^2}$.

(1) $K-k$ is even.

$$V(K) > V(k+2) + \frac{(K-k-2)(g_1(0,1)-g_1(1,1))}{2\beta} = \frac{(K-k)(g_1(0,1)-g_1(1,1))}{2\beta}$$

$$\therefore (1-\beta)(K-k) < \frac{1+A}{1-A}\beta + \beta\frac{1-\beta}{1+\beta}$$

$$V(K-1) = V(k+1) + \frac{(K-k-2)(g_1(0,1)-g_1(1,1))}{2\beta} < \frac{(K-k)(g_1(0,1)-g_1(1,1))}{2\beta}$$

$$\therefore (1-\beta)(K-k) > \frac{1+A}{1-A}\beta - \beta\frac{1-\beta}{1+\beta}$$

$$\therefore K-k = \hat{K}$$

a contradiction to $K-k \neq \hat{K}$ and $K < \hat{K}$.

(2) $K-k$ is odd.

$$V(K-1) = V(k+2) + \frac{(K-k-3)(g_1(0,1)-g_1(1,1))}{2\beta} = \frac{(K-k-1)(g_1(0,1)-g_1(1,1))}{2\beta}$$

$$\therefore (1-\beta)(K-k-1) = \frac{1+A}{1-A}\beta - \beta\frac{1-\beta}{1+\beta}$$

However, this is not generically true since $K-k-1$ is an integer number.

$\square$

Lemma 4.3 There is a unique non-absorbing equilibrium and the necessary condition for the existence of non-absorbing equilibrium is $K \leq \hat{K} - 1$.

Assume that K is even.

Step 1: It is impossible that $V(K) = g_1(1,1) + \beta V(K+1) \geq g_1(0,1) + \beta V(K-1)$.

Prove by contradiction, then $0 < y_{K-1} < 1$ and $V(K+1) = g_1(0,1) + \beta V(K) > g_1(1,1) + \beta V(K+1)$ by Lemma 3.1. If $0 < y_{K-2} < 1$, then $0 < y_i < 1$ for all $0 \leq i \leq K-2$ and $\frac{V(K-2)}{y_{K-2}} = \frac{V(0)}{y_0} = \frac{g_1(0,1)}{1-\beta}$. However,

$$(1-\beta)(V(K) - \frac{V(K-2)}{y_{K-2}}) \geq \beta(V(K-1) - V(K-3)) - \beta(V(K) - V(K-2)) = 0$$

$$\therefore V(K) > \frac{V(K-2)}{y_{K-2}} = \frac{g_1(0,1)}{1-\beta}$$

a contradiction. Therefore, $y_{K-2} = 1$, thus $0 < y_{K-3} < 1$. If $0 < y_{K-4} < 1$, then, $0 < y_i < 1$ for all $0 \leq i \leq K-4$ and $\frac{V(K-4)}{y_{K-4}} = \frac{V(0)}{y_0} = \frac{g_1(0,1)}{1-\beta}$. However,

$$(1-\beta)(V(K-2) - \frac{V(K-4)}{y_{K-4}}) > \beta(V(K-3) - V(K-5)) - \beta(V(K-2) - V(K-4)) = 0$$

$$\therefore V(K-2) > \frac{V(K-4)}{y_{K-4}} = \frac{g_1(0,1)}{1-\beta}$$

a contradiction. Therefore, $y_{K-4} = 1$, thus $0 < y_{K-5} < 1$. Use this argument repeatedly, we reach a contradiction.

Step 2: $K \leq 2[\frac{\beta^2}{1-\beta^2}]$.

Show that $y_{K-1} = 1$. If $0 < y_{K-1} < 1$, then $\frac{V(K-2)}{y_{K-2}} = V(K)$ and $0 < y_{K-2} < 1$. By induction, $\frac{V(t-2)}{y_{t-2}} = V(t)$ and $0 < y_{t-2} < 1$ for all $t \leq K$. However, we know that $\frac{V(0)}{y_0} = \frac{g_1(0,1)}{1-\beta}$. Therefore, $V(K) = \frac{V(0)}{y_0} = \frac{g_1(0,1)}{1-\beta}$, a contradiction to the fact that $\frac{g_1(0,1)}{1-\beta}$ is the highest continuation payoff. In all, $y_{K-1} = 1$.

$$\therefore V(K) = \frac{g_1(0,1) + \beta g_1(1,1)}{1-\beta^2}$$

$$\frac{V(K-2)}{y_{K-2}} = V(K) + \frac{g_1(0,1) - g_1(1,1)}{\beta} = \frac{g_1(0,1) + \beta g_1(1,1)}{1-\beta^2} + \frac{g_1(0,1) - g_1(1,1)}{\beta}$$

Since we can not have two consecutive complete trust: $y_t = y_{t+1} = 1$ for $t \leq K-1$. Then, $0 < y_{K-2} < 1$. If $0 < y_{K-3} < 1$, then we can show that $\frac{V(t-2)}{y_{t-2}} = V(t)$ and $0 < y_{t-2} < 1$ for all $t \leq K-1$. Therefore, $\frac{V(K-2)}{y_{K-2}} = \frac{V(0)}{y_0} = \frac{g_1(0,1)}{1-\beta}$. As long as $\frac{g_1(0,1)+\beta g_1(1,1)}{1-\beta^2} + \frac{g_1(0,1)-g_1(1,1)}{\beta} < \frac{g_1(0,1)}{1-\beta}$, a contradiction. Therefore, $y_{K-3} = 1$ and

$$\frac{V(K-4)}{y_{K-4}} = \frac{V(K-2)}{y_{K-2}} + \frac{1}{\beta} = \frac{g_1(0,1) + \beta g_1(1,1)}{1-\beta^2} + \frac{2(g_1(0,1) - g_1(1,1))}{\beta}$$

Because $K \geq 2[\frac{\beta^2}{1-\beta^2}] + 2$, then for all $k \leq \frac{K}{2}$, $\frac{g_1(0,1)+\beta g_1(1,1)}{1-\beta^2} + \frac{k(g_1(0,1)-g_1(1,1))}{\beta} < \frac{g_1(0,1)}{1-\beta}$.

By induction, we get that for all $1 \le k \le \frac{K}{2}$, $y_{K-2k+1} = 1$ and $0 < y_{K-2k} < 1$. Specifically, for all $k \le \frac{K}{2}$,

$$\frac{V(K-2k+2)}{y_{K-2k+2}} = \frac{g_1(0,1)+\beta g_1(1,1)}{1-\beta^2} + \frac{(k-1)(g_1(0,1)-g_1(1,1))}{\beta}$$

$$V(K-2k+2) = \frac{g_1(0,1)+\beta g_1(1,1)}{1-\beta^2} - \frac{(k-1)(g_1(0,1)-g_1(1,1))}{\beta^2}$$

$$\therefore y_{K-2k+2} = \frac{\beta^2(1+A\beta)-(1-\beta^2)(k-1)(1-A)}{\beta^2(1+A\beta)+(1-\beta^2)\beta(k-1)(1-A)}$$

Next, figure out y_0:

$$\because \frac{V_0}{y_0} = \frac{g_1(0,1)}{1-\beta}$$

$$\because (1-\beta)(\frac{V_2}{y_2} - \frac{V_0}{y_0}) = (g_1(0,1)-g_1(1,1)) - \beta(V_2 - V_0)$$

$$\therefore y_0 = (\frac{1+A\beta}{1+\beta} - \frac{(1-A)(K-2)(1-\beta)}{2\beta^2}) + \frac{1-\beta}{\beta}(\frac{(1-A)(K-2)(1-\beta)}{2\beta} - \frac{(1-A)(1+2\beta)}{1+\beta})$$

$$\therefore \lim_{\Delta\to 0} y_0 = \frac{1+A-b(1-A)X^*}{2}$$

For all $0 \le X \le X^*$ and $i \ge 1$,

$$\lim_{\Delta\to 0,\ 2i\Delta\to X} y_{2i\Delta} = \frac{1+A-b(1-A)(X^*-X)}{1+A+b(1-A)(X^*-X)}$$

$$\lim_{\Delta\to 0,\ (2i+1)\Delta\to X} y_{(2i+1)\Delta} = 1$$

Step 3: $K \ge 2[\frac{\beta^2}{1-\beta^2}] + 2$.

Therefore, $\frac{g_1(0,1)+\beta g_1(1,1)}{1-\beta^2} + \frac{K(g_1(0,1)-g_1(1,1))}{2\beta} < \frac{g_1(0,1)}{1-\beta}$. Denote $k^* < \frac{K}{2}$ as the largest integer k such that $\frac{g_1(0,1)+\beta g_1(1,1)}{1-\beta^2} + \frac{k(g_1(0,1)-g_1(1,1))}{\beta} < \frac{g_1(0,1)}{1-\beta}$. By the same argument as before, we get that for all $1 \le k \le k^*$, $y_{K-2k+1} = 1$ and $0 < y_{K-2k} < 1$.

$$\therefore y_{K-2k+2} = \frac{\beta^2(1+A\beta)-(1-\beta^2)(k-1)(1-A)}{\beta^2(1+A\beta)+(1-\beta^2)\beta(k-1)(1-A)}$$

Denote $\tilde{K} = K - 2k^* - 2$. Because $k^* = [\frac{\beta^2}{1-\beta^2}]$, then $\tilde{K} = K - 2[\frac{\beta^2}{1-\beta^2}] - 2$.

(1) Show that $y_{\tilde{K}+1} = 1$. Prove by contradiction by assuming $0 < y_{\tilde{K}+1} < 1$, then $0 < y_i < 1$ and $\frac{V(i-1)}{y_{i-1}} = \frac{V(i+1)}{y_{i+1}}$ for all $0 \le i \le \tilde{K}+1$. Specifically, $\frac{V(\tilde{K}+2)}{y_{\tilde{K}+2}} = \frac{V(0)}{y_0} = \frac{g_1(0,1)}{1-\beta}$.

$$\because (1-\beta)(\frac{V(\tilde{K}+4)}{y_{\tilde{K}+4}} - \frac{V(\tilde{K}+2)}{y_{\tilde{K}+2}}) = g_1(0,1) - g_1(1,1) - \beta(V(\tilde{K}+4) - V(\tilde{K}+2))$$

$$\because V(\tilde{K}+1) < g_1(1,1) + \beta V(\tilde{K}+2),\ V(\tilde{K}+3) = g_1(1,1) + \beta V(\tilde{K}+4)$$

$$\therefore \beta(V(\tilde{K}+4) - V(\tilde{K}+2)) < V(\tilde{K}+3) - V(\tilde{K}+1) = \frac{g_1(0,1) - g_1(1,1)}{\beta}$$

$$\therefore (1-\beta)(\frac{V(\tilde{K}+4)}{y_{\tilde{K}+4}} - \frac{V(\tilde{K}+2)}{y_{\tilde{K}+2}}) > (1 - \frac{1}{\beta})(g_1(0,1) - g_1(1,1))$$

$$\therefore \frac{g_1(0,1)}{1-\beta} = \frac{V(\tilde{K}+2)}{y_{\tilde{K}+2}} < \frac{V(\tilde{K}+4)}{y_{\tilde{K}+4}} + \frac{g_1(0,1) - g_1(1,1)}{\beta} = \frac{g_1(0,1) + \beta g_1(1,1)}{1-\beta^2} + \frac{k^*(g_1(0,1) - g_1(1,1))}{\beta}$$

a contradiction to the definition of k^*.

(2) Show that $0 < y_i < 1$ for all $i \leq \tilde{K} - 1$.

Because $y_{\tilde{K}+1} = 1$, $0 < y_{\tilde{K}} < 1$. If we assume $y_{\tilde{K}-1} = 1$, then

$$\frac{V(\tilde{K})}{y_{\tilde{K}}} = \frac{g_1(0,1) + \beta g_1(1,1)}{1-\beta^2} + \frac{(k^*+1)(g_1(0,1) - g_1(1,1))}{\beta} > \frac{g_1(0,1)}{1-\beta}$$

a contradiction. Therefore, $0 < y_{\tilde{K}-1} < 1$ and $0 < y_{\tilde{K}} < 1$. This implies that $0 < y_i < 1$ for all $0 \leq i \leq \tilde{K} - 1$.

(3) Show the analytic solution of $\{y_i\}_{i=0}^{\tilde{K}-1}$.

Because $V(0) = \beta(1-y_0)V(0) + y_0(g_1(0,1) + \beta V(0))$, $\frac{V(0)}{y_0} = \frac{g_1(0,1)}{1-\beta}$. Since $V(1) = \beta(1-y_1)V(1) + y_1(g_1(1,1) + \beta V(1))$ and $V(2) = V(1)$, $\frac{V(1)}{y_1} = \frac{g_1(1,1)}{1-\beta}$.

$$\therefore \frac{V(2i)}{y_{2t}} = \frac{g_1(0,1)}{1-\beta} \quad \forall\ 0 \leq 2i \leq \tilde{K}$$

$$\therefore \frac{V(2i+1)}{y_{2t+1}} = \frac{g_1(1,1)}{1-\beta} \quad \forall\ 0 \leq 2i+1 \leq \tilde{K}+1$$

$$\therefore y_{2i} = y_0 + \frac{(1-\beta)(1-A)}{\beta} i,\ \ y_{2i+1} = y_1 + \frac{(1-\beta)(1-A)}{\beta A} i$$

Figure out the boundary condition $y_{\tilde{K}}$. We know that $\frac{V(\tilde{K})}{y_{\tilde{K}}} = \frac{g_1(0,1)}{1-\beta}$ and

$$(1-\beta)(\frac{V(\tilde{K}+2)}{y_{\tilde{K}+2}} - \frac{V(\tilde{K})}{y_{\tilde{K}}}) = g_1(0,1) - g_1(1,1) - \beta(V(\tilde{K}+2) - V(\tilde{K}))$$

Furthermore, we know $V(\tilde{K}+2)$ and $y_{\tilde{K}+2}$,

$$\therefore y_{\tilde{K}} = (\frac{1+A\beta}{1+\beta} - \frac{(1-A)(K-\tilde{K}-2)(1-\beta)}{2\beta^2}) + \frac{1-\beta}{\beta}(\frac{(1-A)(K-\tilde{K}-2)(1-\beta)}{2\beta} - \frac{(1-A)(1+2\beta)}{1+\beta})$$

Denote $\tilde{X} = \lim_{\Delta \to 0} \tilde{K}\Delta$. Then, $b(X^* - \tilde{X}) = 1$, thus $\tilde{X} = X^* - \frac{1}{b}$.

In the limit, $y_{\tilde{K}} \to 1$. Because $\frac{V(1)}{Ay_1} = \frac{V(0)}{y_0} = \frac{g_1(0,1)}{1-\beta}$ and $\beta(V(1)-V(0)) = g_1(0,1)-g_1(1,1)$, then $y_1 = \frac{1}{A}y_0 + \frac{1-\beta}{\beta}\frac{1-A}{A}$.

$$\therefore y_{\tilde{K}-1} = \frac{1}{A}y_0 + \frac{(1-\beta)(1-A)}{\beta A}(\tilde{K}-1) = \frac{y_{\tilde{K}}}{A}$$

Therefore, $y_{\tilde{K}-1} \to A$.

If $0 \leq X \leq X^* - \frac{1}{b}$, then

$$(a(X), y(X)) = \begin{cases} (\pi(X), \frac{(1+A)-b(1-A)(X^*-X)}{2A}) & X = \lim_{\Delta\to 0}(2k+1)\Delta \\ (\pi(X), \frac{(1+A)-b(1-A)(X^*-X)}{2}) & X = \lim_{\Delta\to 0} 2k\Delta \end{cases}$$

If $X^* - \frac{1}{b} < X \leq X^*$, then

$$(a(X), y(X)) = \begin{cases} (1,1) & X = \lim_{\Delta\to 0}(2k+1)\Delta \\ (\pi(X), \frac{1+A-b(1-A)(X^*-X)}{1+A+b(1-A)(X^*-X)}) & X = \lim_{\Delta\to 0} 2k\Delta \end{cases}$$

Step 4: Show that $K \leq \hat{K} + 1$.

Because $k^* < \frac{\beta^2}{1-\beta^2}$ and $K - \tilde{K} - 2 = 2k^*$,

$$\therefore \frac{1-\beta}{\beta}(\frac{(1-A)(K-\tilde{K}-2)(1-\beta)}{2\beta} - \frac{(1-A)(1+2\beta)}{1+\beta}) \leq -\frac{(1-\beta)(1-A)}{\beta}$$

$$\therefore y_{\tilde{K}} \leq (\frac{1+A\beta}{1+\beta} - \frac{(1-A)(K-\tilde{K}-2)(1-\beta)}{2\beta^2}) - \frac{(1-\beta)(1-A)}{\beta}$$

Furthermore, $0 \leq y_0 = y_{\tilde{K}} - \frac{(1-\beta)(1-A)}{\beta}\frac{\tilde{K}}{2}$ implies that $y_{\tilde{K}} \geq \frac{(1-\beta)(1-A)}{\beta}\frac{\tilde{K}}{2}$.

$$\therefore \frac{(1-\beta)(1-A)}{\beta}\frac{\tilde{K}+2}{2} \leq \frac{1+A\beta}{1+\beta} - \frac{(1-A)(K-\tilde{K}-2)(1-\beta)}{2\beta^2}$$

$$\therefore \frac{(1-\beta)(1-A)}{\beta}\frac{K}{2} \leq \frac{1+A\beta}{1+\beta} - \frac{k^*(1-A)(1-\beta)^2}{\beta^2}$$

Eventually, we can show that

$$\therefore (1-\beta)K < \frac{1+A}{1-A}\beta - \beta\frac{1-\beta}{1+\beta}$$

$$\therefore K \leq \hat{K} - 1$$

Step 5: If K is odd, then denote $K^* = K + 1$. It can be show that K^* plays the same role as K in previous steps in which K is even. In all, all the results in the previous steps hold for K^*, if we denote $K^* = K + 1$ if K is odd and $K^* = K$ if K is even.

Proof of Theorem 4 and Proposition 4:

Proof. If $K \geq \hat{K}$, then by lemma 4.3, there is no non-absorbing equilibrium. By lemma 4.2, there is a unique absorbing equilibrium. In all, there is a unique stationary Markov equilibrium and it is an absorbing equilibrium.

If $K \leq \hat{K} - 1$, then by lemma 4.2, there is no absorbing equilibrium. Therefore, the equilibrium needs to a non-absorbing equilibrium. By lemma 4.3, there is only a unique non-absorbing equilibrium. In all, there is only a unique stationary Markov equilibrium and it is a non-absorbing equilibrium.

The limiting result where $\Delta \to 0$ is characterized in lemma 4.2 and lemma 4.3.

□

Appendix 5

Proof of Proposition 5:

Proof. We have shown in section 2.2 that if player 1 only has binary choices I_{i^*} and I_0, then the stationary equilibrium can be characterized by a reputation building phase $X < X^*$ and a reputation exploitation phase $X \geq X^*$.

Look at the equilibrium behavior of player 1 in state $X < X^*$ when player 2 plays mixed strategy $0 < y(X) < 1$. We show that it is an equilibrium that player 1 only puts positive probability on I_{i^*} and I_0.

$$\because g_1(I_{i^*}, B)y(X)+\beta((1-q_{i^*})V(X+1)+q_{i^*}V(X-1)) = g_1(I_0, B)y(X)+\beta((1-q_0)V(X+1)+q_0V(X-1))$$

Furthermore, by the definition of i^*: $i^* = \arg\min_i\{\frac{c_i}{q_0 - q_i}\}$, it is easy to show that

$$g_1(I_i, B)y(X)+\beta((1-q_i)V(X+1)+q_iV(X-1)) < g_1(I_0, B)y(X)+\beta((1-q_0)V(X+1)+q_0V(X-1))$$

Therefore, player 1 puts zero probability on investment other than I_{i^*}

Look at the equilibrium of player in state $X \geq X^*$ where play 2 buys the product for sure: $y(X) = 1$. We can show that it is an equilibrium that player 1 exploits the reputation by playing I_0 for sure.

$$\because g_1(I_{i^*}, B)+\beta((1-q_i)V(X+1)+q_iV(X-1)) < g_1(I_0, B)y(X)+\beta((1-q_0)V(X+1)+q_0V(X-1))$$

By the definition of i^*: $i^* = \arg\min_i\{\frac{c_i}{q_0-q_i}\}$, it is easy to show that for any $i \geq 1$,

$\because g_1(I_i, B)+\beta((1-q_i)V(X+1)+q_iV(X-1)) < g_1(I_0, B)y(X)+\beta((1-q_0)V(X+1)+q_0V(X-1))$

Therefore, player 1 plays I_0 for sure at $X \geq X^*$. $\square$