Information Choice as a Correlation Device

Catherine Gendron-Saulnier* Sidartha Gordon†

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Abstract

We study a class of Bayesian games in which players face restrictions on how much information they can obtain on a common payoff relevant state, but have some leeway in choosing the correlation (or similarity) between their signals, before choosing their actions. Using a new dependence stochastic ordering between a player's and other players' signals, we obtain equilibrium necessary conditions that link the complementarity or substitutability in own and other's actions, the monotonicity properties of the second stage action strategies, and the dependence between the chosen signals. We also provide (stronger) sufficient conditions for certain types of equilibria, in particular for public information to arise as an equilibrium outcome. Equilibrium information structures may be inefficient. Making which signals were chosen (but not their realizations) observable by all players may restore efficiency.

Keywords: Dependence ordering; Public information; Stochastic orders; Increasing differences; Weak association; Endogenous information structure; Value of private and public information; Monotone strategies.

JEL Classifications: C72, D81, D83.

*Department of Economics, University of Montreal. Email: catherine.gendron-saulnier@umontreal.ca.
†Department of Economics, Sciences Po. Email: sidartha.gordon@sciencespo.fr

1 Introduction

We study a class of Bayesian games in which players face restrictions on how much information they can obtain on a common payoff relevant state, but have some leeway in choosing the correlation (or similarity) between their signals, before choosing their actions. Using a new dependence stochastic ordering between a player's and others' signals, we obtain equilibrium necessary conditions that link the complementarity or substitutability in own and others' actions, the monotonicity properties of the second stage action strategies, and the dependence between the chosen signals. We also provide (stronger) sufficient conditions for certain types of equilibria, in particular for public information to arise as an equilibrium outcome. Equilibrium information structures may be inefficient. Making information choices (but not the signal realizations) publicly observable may restore efficiency.

2 Related literature

In the classic formulation of a game with incomplete information, a Bayesian game, the information structure is exogenously given.

In recent years, a large literature has developped, whose objective is to understand how the information structure maps to equilibrium outcomes, and how sensitive these outcomes are to the information structure (e.g. Bergemann and Morris, 2013). At the same time, scholars have seeked to endogenize the information structure. Three main forms of endogeneity of the information structure of a game have been considered. First, the player's information can be influenced by what players tell each other via cheap talk communication (Crawford and Sobel, 1982, Myerson, 1986, Forges, 1986), or through some other forms of communication. Second, the information structure can be design by a third party, who has a stake in the game that is being played. For example, one can think of the seller of an object in an auction (Bergemann and Pesendorfer, 2007), the central bank of a macroeconomy (Morris and Shin, 2002), an agent persuading a decision maker (Gentzkow and Kamenica, 2010) or simply the social planner (Taneva, 2014). Third, the players' information can result from the players' information acquisition decentralized choices and effort, as in Li, McKelvey and Page (1987), Vives (1988), Hellwig and Veldkamp (2009), Myatt and Wallace (2011), Szkup and Trevino (2014), Yang (2014), and many others. Our work belongs to this third class of models. We discuss these models in further detail.

2.1 Decentralized information acquisition

Regarding decentralized information acquisition, the most commonly studied framework is a two stage game where players start with a common prior on some unknown common value state that affects all players' payoffs.[1] In the first stage, each player makes an information choice (for example, the precision of the signal she receives) that determines the information on the state that she has when entering the second stage. In the second stage, players simultaneously choose an action. Two different extensive forms have been considered, depending on whether the choices made at the first stage are observed or not. In some models, the acquisition is publicly observed. The game is then an extensive form game where each profile of information acquisition choices defines a subgame, and in each subgame, the information structure is common knowledge: in these games, acquisition is *open.* In other models, the choices of the players in the first period are not observed before actions are taken. Acquisition is then *hidden.* A game where information acquisition is hidden is essentially static, as it is equivalent to one where all players simultaneously choose both their information and a commitment to an action strategy that maps the signal they will observe to the action they choose. The difference between open and hidden information acquisition is in the way a deviation in the first stage is treated: Under open acquisition, a deviation on information choice is commonly observed, and the information structure is common knowledge in the second stage subgame following the deviation; Under hidden acquisition, players form a belief of what the information structure is in the second stage, and this belief is correct in equilibrium. But whenever a player deviates, all other players' belief on the information structure is incorrect. It should be noted that in games with a continuum of players (Hellwig and Veldkamp, 2009; Myatt and Wallace, 2011; Szkup and Trevino, 2014), where players' payoffs only depend on the statistical distribution of the other players' actions, the two forms of acquisition are equivalent. Thus, there is no need to make a distinction in this case. The distinction matters only for games with finitely many players. In this paper, we derive results that apply to games with hidden acquisition and finitely many players, and to games with hidden or open acquisition and a continuum of players. The case where acquisition is open and the number of players is finite is also considered but only in the last section.

2.1.1 The motive inheritance result

The main focus in the literature has been on the player's choice of amount of information (their signal's precision), and on the acquisition of private information. These models assume that signals are independent conditionnaly on the state: The acquired information is therefore

[1] Veldkamp's monograph (2011) and Hellwig, Kohls and Veldkamp (2013) provide excellent surveys on the widely studied special case of the beauty contest games with a continuum of actions and players, quadatic payoffs and a Gaussian information structure, and their applications to macroeconomics and finance. Our paper covers a larger class of models, since we do not rely on specific functional forms and allow for a finite number of players.

private. A central question in this context is whether the players' amount of information acquisition are complements, substitutes, or neither complements nor substitutes. With finitely many players, the question is meaningful only when acquisition is open. With a continuum of players, the question is meaningful for both open and hidden acquisition, since the two are in this case equivalent. Li, McKelvey and Page (1987) study a Cournot market with finitely many firms and open acquisition. The unknown common value state is the demand intercept and the information structure satisfies certain conditions. Actions are substitutes and they find that that the precision levels of the private information acquired in the first stage are substitues as well.[2] Vives (1988) obtains a similar result in the case of a continuum of players. Assuming as well a continuum of players, Hellwig and Veldkamp (2009) obtain a similar result in the context of a beauty contest game, where actions can be either substitutes or complements. They find that when actions are substitutes, acquisition levels are substitutes and when actions are complements, acquisition levels are complements: the strategic motive in actions is *inherited* by the acquisition game. All of these papers assume an unbounded continuum of actions (the real line), quadratic payoffs, and a Gaussian information structure.

In spite of the large number of contexts where the inheritance result is confirmed, it does not generalize to the larger class of all game with strategic complementarities or substitutabilities. In particular, the unbounded continuum of actions, the continuum of players and a Guassian information structure seem to be crucial for the result. Even with unbounded actions, quadratic payoffs and a Gaussian information structure, but only two players (as in the case of a differentiated Bertrand game, which the author uses as an example), Jimenez-Martinez (2013) shows that it only holds for some parameters: when the complementarity in actions is strong, levels of acquired precision may be substitutes. And even with a continuum of players, quadratic payoffs, a Gaussian information structure, but binary actions, i.e. a global game, Szkup and Trevino (2014) present a model where the actions are complements but the acquired precision levels are not.

In contrast with this literature, we do not allow agents to choose how much information they acquire. We hold the amount of information fixed. Instead, we let them choose whether the information they acquire is private or public. More generally, we allow agents to choose the level of conditional dependence between their signals. We obtain another type of inheritance result holds: complementarity in actions implies a preference for positive informational dependence, and substitutability in actions implies a preference for informational independence. But unlike the precision inheritance result, our dependence inheritance results hold for all games where actions are strategic complements or substitutes and do not rely on specific functional forms,

[2]Hwang (1993) exploits this result in a duopoly to derive various comparative statics results. Hwang (1995) studies a similar model but focuses on payoff comparisons between different market structures and different ways in which the levels of information precision of the firms is set. Bergemann, Shi and Välimäki (2009) obtain conditions under which information acquisition levels are substitutes or complements, in a VCG auction with interdependent valuations. Their setting differs from the common value models listed here in several ways.

provided that the second stage strategies are monotonic (which is a possibility in some games, and an implication of Nash-Bayesian equilibrium in a subset of these games).

2.1.2 Public and private information

The issue of the role of public and private information is a central one in the entire literature on endogenous information structures. Morris and Shin (2002) for example, show that in a beauty contest game with a continuum of players, when the planner (the central bank) increases the precision of public information, it can be detrimental to welfare, because agents rely less on their private information.

In the context of information acquisition, Hellwig and Veldkamp (2009) and Myatt and Wallace (2011) let agents choose whether the information is private and potentially public. They provide models where public information is an equilibrium outcome of agents choosing to observe the same potentially public signals. Thus in their models, unlike Morris and Shin (2002), public information is not provided by an external third party, but is the result of the market forces themselves. We follow up on this idea, and go one step further. While their model has private and potentially public signals, and public signals are the potentially public ones that all agents chose to observe, in a version of our model, all signals are potentially public: a private signal is one that only one agent chose to observe, while a public signal is one that all agents chose to observe.

Like Morris and Shin (2002), Hellwig and Veldkamp (2009) are interested in the marginal value of additional public information compared to an initial situation. But because no information is intrinsically public or private, what they really look at is the marginally value of additional potentially public information. They make the important observation that marginal value of acquiring *more* potentially public information is kinked at some profiles that they call symmetric. At a symmetric profile, defined as one where all agents observe the same potentially public signals, if a player deviates and observes one more potentially public signal, she obtains additional information that in effect is private, since nobody else observes it. If she instead drops one of her potential signals, she decreases her own access to public information. This asymetry and discontnuity causes mutiple equilibria that differ in the level of public information. In Myatt and Wallace (2011), public information obtains when all agents pay a substantial amount of attention to the same signal. An implication of this assumption is that agents who hold public information are necessarily well informed agents. In both of these two papers, the problem of the division of information between private and public (and eveything in between) is intrinsically intertwined with the more widely studied isse of the amount of information that the agents acquire. In Hellwig and Veldkamp (2009), it is because of the question they choose to ask, and in Myatt and Wallace (2011), it is in the way they define public information.

In contrast, we choose to completely disentangle the two issues. At the risk of making the model seem less realistic (because in practice, economic agents often face the choice of how much information to acquire), we hold the amount of information fixed by assuming that all signals an agent can choose to observe are equally informative of the unknown state: they all have the same joint marginal distribution with the state. By doing so, we isolate the issue of the partition of the information stucture between public, private and neither private nor public information, from the issue of the amount of information. Doing so enables us to identify a robust force and to obtain general results that hold for a large class of games, not only the Gaussian-quadratic model with a continuum of actions and players. As we argued earlier, no such result holds when the issue of the amount of information is not excluded, even when only private information can be acquired.

Our assumption that agents are restricted in the amount of information they acquire (formally, the joint marginal distribution between their own signal and the state is fixed, no matter what signal they choose). This can be thought of as a form of rational inattention. Agents are limited in how much informationthey can acquire, (Sims, 2003, 2005, 2006), and thus face a choice of what to observe.

2.1.3 Inefficiency of equilibrium under hidden information acquisition

A number of papers are dedicated to the analysis of inefficiencies in the collection of information and in the use of that information when the information structure is exogenously given. Angeletos and Pavan (2007) for instance, study a model with a continuum of agents, quadratic payoff and a Gaussian information structure, where each player observe a private and a public signals. By comparing the equilibrium use of information to an efficiency benchmark (the best society could achieve keeping information decentralized), they show that information use can be inefficient when the incentives to coordinate actions and the social value for coordination are different. The welfare impact depends on the degree of strategic interaction and on its nature (complementarity or substitutability).

Angeletos and Pavan (2007)'s finding is recurrent in the literature. Morris and Shin (2002) among others also show that an increase in the amount of public information can impair welfare. However, this does not hold necessarily if information is a choice for the players.

Chahrour (2012) proposes a model of endogenous information acquisition where the detrimental effect of public information is still valid. In the model, a central authority chooses both how many signals to divulge and their precisions. He finds that the authority always chooses the highest possible precision and releases a positive but finite number of signals. An important result is that too many signals can cause the players to decrease the amount of information they acquire which in turn decreases welfare.

Colombo and Femminis (2008; 2011) are other examples where endogenizing the information structure makes additional public information beneficial for welfare. By allowing the players to choose the precision of their private signals once the central authority has announced the precision of the public signal, they show that the precisions of private and public signals are strategic substitutes. Moreover, if the cost of public information is lower than the cost of private information, then increasing the precision of the public information increases welfare. While Colombo and Femminis (2008, 2011) investigate the welfare implications of public information provision on incentives to acquire private information, Llosa Gonzalo and Venkateswaran (2012), by considering models different from the beauty-contest type, study how different links and externalities among agents affect the acquisition process of private information.

Existing work allows the players to choose the level of information precision. We take the analysis in an other direction by keeping the amount of information fixed and focusing instead on information correlation. We show that hidden information acquisition sometimes leads to inefficiencies when there are payoff externalities that are not reflected in the players' equilibrium choice. Interestingly, we show that these inefficiencies can sometimes be eliminated when information acquisition is open.

2.2 Statistical dependence

In our main result, we characetrize the precise notion of statistical dependence for which the result that complementarities and monotonicity imply a preference for positively dependent signals. In the case on two players, our result in an application of a result by Tchen (1980) which establishes the equivalence between various concepts of statistical dependence for bivariate random vectors. For the case of more than two players, our result requires a generalization of this result to notions of statistical dependence between a random variable (own signal) and a vector (others' signals). Surprisingly, to the best of our knowledge, such generalizations have not been studied in the probability and statistics literatures.

The statistics and economics literature has up to now taken another view, which was to consider dependence relations in the case $N = 2$ as measures of interdependence between the two components of a bivariate random vector. The natural generalization to the case of a general $N \geq 2$, was then to consider dependence as a measure of interdependence between all univariate components of a multivariate random vector. Partial generalizations of Tchen's (1980) result for interpedepence in the case $N \geq 3$ exist. They are reviewed by Strulovici and Meyer (2012), who also establish new relations. Roughly speaking, one can define a number of "more interdependent than" weak partial orders on the set of all N-variate random vectors with the same fixed marginal distributions: a general covariance interdependence relation, a supermodular interdependence relation, a concordance interdependence relation and many others. One could imagine a way to define a dominance interdependence relation, but to the best of our knowledge, this has not been studied. As Strulovici and Meyer (2012) summarize, the general covariance interdependence relation is strictly stronger than the supermodular interdependence relation (established by Strulovici and Meyer, 2012, generalizing a proof of Christofides and Vaggelatou, 2004, for a special case), which in turn is strictly stronger than the concordance interdependence relation (Müller and Stoyan, 2002). The strictness in these statements is established by examples provided by Hu, Müller and Scarsini (2004) and Strulovici and Meyer (2012).

We believe that the concepts of statistical dependence (as oppose to interdependence) between two random vectors , which we introduce in this work, and the equivalences we establish are of independent interest and could be used in other settings.

3 The Model

Let $I = \{1, ..., N\}$ be a finite set of players. The state of the world is described by a random variable θ, with support Θ, which enters explicitly in the players' payoffs. Each player may observe a signal of θ, that does not explicitly enter his payoff. More precisely, there is an arbitrary set of signals S, and a collection of random variables X_s, with $s \in S$. For each $i \in I$, there is a set of signals $S_i \subseteq S$, that player i has access to. For simplicity, we assume that all the variables X_s have identical finite support $\mathcal{X} = \{1, ..., \overline{k}\}$.[3]

The game unfolds as follows in two stages.

Initially all players start with a common prior, which is a pdf on $\Theta \times \mathcal{X}^S$. In the first stage, each player i chooses a unique signal s_i from the set $S_i \subseteq S$. The players choose their signals simultaneously. Abusing notations, for all i, let X_i denote the signal chosen by player i and let x_i denote its realization. Then each player i privately observes the realization x_i without having observed the other players' signal choices nor their signal realizations. In the second stage, each player then chooses an action a_i from the set $A_i = \mathbb{R}$. Let $a = (a_1, ..., a_N)$ be a profile of actions for all players. Each player then obtains a payoff $u_i(a, \theta)$. In the normal form of this game, a pure strategy for player i is a pair (s_i, α_i) in $S \times A_i^{\mathcal{X} \times S}$, where s_i is the signal chosen by player i and α_i, the player's **action strategy**, is a mapping that determines player i's action choice, given the realization x_i he observed and the source s_i he chose. For the time being, we will restrict attention to pure strategies. We will extend the analysis to mixed strategies in Section 8. With pure strategies, it is without loss of generality to restrict attention to action strategies that do not depend on s_i: holding a strategy profile for the other players (s_{-i}, α_{-i}) fixed, any outcome, i.e. a joint distribution over $\Theta \times A^N$ induced by the profile (s, α)

[3] The finite support assumption is made to simplify the exposition, and to avoid uninteresting technical complications. We conjecture that our results extend to the case where $\mathcal{X}$ is infinite, as in the normal quadratic class of examples in Section 4.2.

such that α_i depends on s_i, is also induced by some other profile $(s, \alpha'_i, \alpha_{-i})$ such that α'_i does not depend on s_i. Thus for simplicity and without loss of generality, we restrict attention to action strategies in $A_i^{\mathcal{X}}$.

A profile $(s, \alpha) = (s_i, \alpha_i)_{i \in I}$ is a **Nash-Bayesian Equilibrium** if for all i all s'_i and all $(s'_i, \alpha'_i) \neq (s_i, \alpha_i)$, we have

$$\mathbb{E}\left(u_i\left(\alpha_i\left(x_i, s_i\right), \theta\right)\right) \geq \mathbb{E}\left(u_i\left(\alpha'\left(x_i, s'_i\right), \theta\right)\right).$$

We are interested in the Nash-Bayesian equilibria of this game. In particular, we are interested in understanding how the monotonicity properties of the equilibrium action strategies in stage 2, together with the complementarity or substitutability between actions is stage 2 affect the players' signal choice in stage 1, which in turn determines the players' higher order beliefs at the beginning of stage 2.

More precisely, for any mapping $\phi : \mathcal{X} \to \mathbb{R}$, we say that ϕ is **increasing** if for all $x_i, x'_i \in \mathcal{X}$, we have $x_i \leq x'_i \implies \phi(x_i) \leq \phi(x'_i)$, and that ϕ is **strictly increasing** if for all $x_i, x'_i \in \mathcal{X}$, we have $x_i < x'_i \implies \phi(x_i) < \phi(x'_i)$. We say that ϕ is **decreasing** if $-\phi$ is increasing and that ϕ is **strictly decreasing** if $-\phi$ is increasing. A profile of action strategies α is **strictly monotonic** if either for all i, the action strategy α_i is strictly increasing, or for all i, α_i is strictly decreasing. A profile of action strategies α is **strictly antimonotonic for** i if for either α_i is strictly increasing and for all $j \neq i$, α_j is strictly decreasing, or if α_i is strictly decreasing and for all $j \neq i$, α_j is strictly increasing. Abusing terminology, we say that a strategy profile is strictly monotonic (antimonotonic for i) if its action profile is strictly monotonic (antimonotonic for i).

In general, an important consideration in a player's choice of a signal could be how informative the different available signals are, on the payoff-relevant state θ, or which aspects of θ the different signals reveal.[4] In order to eliminate these motives, and in order to concentrate on the higher-order motive, we will assume that all signals X_s are equally informative on θ in the sense of Blackwell, namely that the joint marginal distribution of θ and X_s is the same for all signals s.

Intuitively, different signals have higher positive dependence between one another if their distributions are more similar. More precise definitions will be given in Section 5. In one extreme case of positive dependence, when different signals are perfectly positively dependent, their realizations are identical with probability one. In this case, we can say that the signals are **public information on** θ. In an other extreme case, they are independent conditional on θ. In this case, we can say that they are **private information on** θ. Roughly speaking, our main result says that in any strictly monotonic Nash-Bayesian Equilibrium, if the actions

[4]For example, one signal could reveal θ's sign, whereas another could reveal θ's absolute value.

are complements, the players choose signals as positively dependent as possible; if instead the actions are substitutes, the players choose signals that have as little positive dependence as possible. On the contrary, in any Nash-Bayesian Equilibrium that is strictly antimonotonic for i, if the actions are complements, the players choose signals that have as little positive dependence as possible; if instead the actions are substitutes, the players choose signals as positively dependent as possible.

4 Examples

To fix ideas, we start with simple examples

4.1 The binary quadratic model

Suppose that $S = \{X, Y\}$ and $N = 2$. Payoffs are

$$u_i(\theta, a) = -a_i^2 + 2b_{i12}a_i a_j + 2b_{i13}a_i\theta + 2b_{i1}a_i + K(a_j, \theta) \tag{1}$$

where b_{i12}, b_{i13} and b_{i1} are real numbers for $i \in \{1, 2\}$ and $K(\cdot, \cdot)$ is a function that does not affect the set of Nash-Bayesian equilibria, but may have an effect on welfare. The information structure is as follows. The random vector (θ, X, Y) is distributed in $\{0, 1\}^3$ according to a pdf $(f_\theta)_{\theta,x,y\in\{0,1\}}$ such that the probability of the realization (θ, x, y) is $f_{\theta xy}$. The joint marginal distributions of (θ, X) and (θ, Y) are identical, so that

$$\sum_{y'} f_{\theta x y'} = \sum_{x'} f_{\theta x' y}$$

for all triples $(\theta, x, y) \in \{0, 1\}^3$ such that $x = y$.

More specifically assume that $\mathbb{P}(\theta = 1) = p_\theta \in (0, 1)$ and that each signal has the joint distribution with θ such that

	$X = 0$	$X = 1$
$\theta = 0$	q_{00}	q_{01}
$\theta = 1$	q_{10}	q_{11}

where $\mathbb{P}(\theta = \ell, X_s = k) = q_{\ell k}$. Note that these four numbers must add up to 1. Note also that $q_{10} + q_{11} = p_\theta$, so that this matrix can be rewritten as

	$X = 0$	$X = 1$
$\theta = 0$	q_{00}	$1 - p_\theta - q_{00}$
$\theta = 1$	$p_\theta - q_{11}$	q_{11}

Therefore the joint marginal (θ, X) is completely characterized by the three numbers p_θ, q_{00} and q_{11}. Note that

$$\begin{aligned} Cov\left(X,\theta\right) &= q_{11}q_{00} - q_{10}q_{01} \\ &= q_{11}q_{00} - \left(p_\theta - q_{11}\right)\left(1 - p_\theta - q_{00}\right) \\ &= q_{00}p_\theta + q_{11}\left(1 - p_\theta\right) - p_\theta\left(1 - p_\theta\right). \end{aligned}$$

Therefore, the signal is informative if and only if

$$q_{00}p_\theta + q_{11}\left(1 - p_\theta\right) - p_\theta\left(1 - p_\theta\right) \neq 0,$$

which we assume. Without loss of generality, we can assume that

$$q_{00}p_\theta + q_{11}\left(1 - p_\theta\right) - p_\theta\left(1 - p_\theta\right) > 0.$$

If this did not hold, we could simply change the labels 0 and 1 of the realizations of X and it would hold. Note that we will keep these parameters p_θ, q_{00} and q_{11} fixed throughout. They parametrize the level of uncertainty in the economy and the informativeness of each signal.

Even the agents choices of signal cannot change these values, which hold for any signal that any agent may choose to observe. Let $\mathbb{P}(X = \ell, Y = k) = z_{\ell k}$, such that the conditional probability matrices are

$$\begin{cases} \begin{array}{c|c|c} & Y=0 & Y=1 \\ \hline X=0 & z_{00} & z_{01} \\ \hline X=1 & z_{01} & z_{11} \end{array} & \text{if } \theta = 0 \\ \\ \begin{array}{c|c|c} & Y=0 & Y=1 \\ \hline X=0 & w_{00} & w_{01} \\ \hline X=1 & w_{01} & w_{11} \end{array} & \text{if } \theta = 1. \end{cases} \tag{2}$$

Note that $z_{01} = \frac{1}{2}(1 - z_{00} - z_{11})$ and $w_{01} = \frac{1}{2}(1 - w_{00} - w_{11})$, so that

$$
\begin{cases}
\begin{array}{c|c|c}
 & Y=0 & Y=1 \\
\hline
X=0 & z_{00} & \frac{1}{2}(1 - z_{00} - z_{11}) \\
\hline
X=1 & \frac{1}{2}(1 - z_{00} - z_{11}) & z_{11}
\end{array} & \text{if } \ \theta = 0 \\[2em]
\begin{array}{c|c|c}
 & Y=0 & Y=1 \\
\hline
X=0 & w_{00} & \frac{1}{2}(1 - w_{00} - w_{11}) \\
\hline
X=1 & \frac{1}{2}(1 - w_{00} - w_{11}) & w_{11}
\end{array} & \text{if } \ \theta = 1.
\end{cases}
\tag{3}
$$

Moreover, we have

$$z_{00} + \frac{1}{2}(1 - z_{00} - z_{11}) = \frac{q_{00}}{1 - p_\theta}$$

which implies that

$$
\begin{aligned}
z_{11} &= z_{00} + 1 - \frac{2q_{00}}{1 - p_\theta} \\
\frac{1}{2}(1 - z_{00} - z_{11}) &= \frac{q_{00}}{1 - p_\theta} - z_{00}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
w_{00} &= w_{11} + 1 - \frac{2q_{11}}{p_\theta} \\
\frac{1}{2}(1 - w_{00} - w_{11}) &= \frac{q_{11}}{p_\theta} - w_{11}
\end{aligned}
$$

so that the conditional probability matrices are

$$
\begin{cases}
\begin{array}{c|c|c}
 & Y=0 & Y=1 \\
\hline
X=0 & z_{00} & \frac{q_{00}}{1-p_\theta} - z_{00} \\
\hline
X=1 & \frac{q_{00}}{1-p_\theta} - z_{00} & z_{00} + 1 - \frac{2q_{00}}{1-p_\theta}
\end{array} & \text{if } \ \theta = 0 \\[2em]
\begin{array}{c|c|c}
 & Y=0 & Y=1 \\
\hline
X=0 & w_{11} + 1 - \frac{2q_{11}}{p_\theta} & \frac{q_{11}}{p_\theta} - w_{11} \\
\hline
X=1 & \frac{q_{11}}{p_\theta} - w_{11} & w_{11}
\end{array} & \text{if } \ \theta = 1.
\end{cases}
$$

where the parameters $z_{00} \in \left[0, \frac{q_{00}}{1-p_\theta}\right]$ and $w_{11} \in \left[0, \frac{q_{11}}{p_\theta}\right]$ are determined by the agent's choices of information structure. The cases $z_{00} = 0$ and $w_{11} = 0$ correspond to (imperfect) negative conditional correlation. The cases $z_{00} = \frac{q_{00}}{1-p_\theta}$ and $w_{11} = \frac{q_{11}}{p_\theta}$ correspond to perfect conditional correlation. When both equalities hold, the information is public. Conditional independence correspond to the cases

$$\begin{aligned} z_{00}\left(z_{00}+1-\frac{2q_{00}}{1-p_\theta}\right)-\left(\frac{q_{00}}{1-p_\theta}-z_{00}\right)^2 &= 0 \\ w_{11}\left(w_{11}+1-\frac{2q_{11}}{p_\theta}\right)-\left(\frac{q_{11}}{p_\theta}-w_{11}\right)^2 &= 0 \end{aligned}$$

i.e.

$$\begin{aligned} z_{00} &= \left(\frac{q_{00}}{1-p_\theta}\right)^2 \in \left[0, \frac{q_{00}}{1-p_\theta}\right] \\ w_{11} &= \left(\frac{q_{11}}{p_\theta}\right)^2 \in \left[0, \frac{q_{11}}{p_\theta}\right]. \end{aligned}$$

When both equalities hold, the agent's information is fully private. To summarize, the entire information structure is fully described by the five numbers p_θ, q_{00}, q_{11}, z_{00} and w_{11}. The first three parameters are held fixed throughout. The last two are jointly controlled by the two agents.

We now introduce two elements that will be useful when solving for the equilibrium. First, let k_{ix} be the probability that $\theta = 1$ given that player i has received the signal realization $X_i = x$. As the state is binary, it implies that $\mathbb{E}(\theta|X_i = x) = k_{ix}$. Next, since a player's best response will depend on his expectation of the other player signal realization, let h_{ix} denote the belief of player i on player j's signal realization. More precisely, let $h_{ix} \equiv \mathbb{P}(X_j = x|X_i = x)$.

Note that the assumption that the joint marginals are identical for (X, θ) and (Y, θ) imply that $k_{ix} = k_{jx}$ and that $h_{ix} = h_{jx}$ for all x. Thus we will simply write these conditional probabilities k_x and h_x where

$$k_x = \frac{\mathbb{P}(\theta = 1, X_i = x)}{\mathbb{P}(X_i = x)} = \begin{cases} \frac{p_\theta - q_{11}}{p_\theta - q_{11} + q_{00}} & \text{if } x = 0 \\ \frac{q_{11}}{1-p_\theta-q_{00}+q_{11}} & \text{if } x = 1 \end{cases} \tag{4}$$

$$h_x = \frac{\mathbb{P}(X_j = x, X_i = x)}{\mathbb{P}(X_i = x)} = \begin{cases} \frac{z_{00}(1-p_\theta)+\left(w_{11}+1-\frac{2q_{11}}{p_\theta}\right)p_\theta}{q_{00}+p_\theta-q_{11}} & \text{if } x = 0 \text{ and } X_i \neq X_j \\ \frac{\left(z_{00}+1-\frac{2q_{00}}{1-p\theta}\right)(1-p_\theta)+w_{11}p_\theta}{1-p_\theta-q_{00}+q_{11}} & \text{if } x = 1 \text{ and } X_i \neq X_j \\ 1 & \text{if } X_i = X_j \end{cases} \tag{5}$$

We see that k_x only depends on p_θ, q_{00}, q_{11}, so it is not controlled by the agents. Moreover, we see that both h_0 and h_1 are increasing in z_{00} and w_{11}.

4.1.1 Actions

Fixing signal choices $X_1 \in S$ and $X_2 \in S$, let α_{ix} be the action strategy given that player i has received the realization $X_i = x$. It is convenient to describe agent i's action strategy by the two variables α_i, β_i, such that $\alpha_i := \alpha_{i0}$ and $\beta_i := \alpha_{i1} - \alpha_{i0}$. Let $H = h_0 - (1 - h_1)$ and $K = k_1 - k_0$. We can restrict attention to $1 > K > 0$, which means positive correlation between the signal and θ and $1 > H \geq 0$, which means nonnegative correlation between the signals.

The expected payoff of player i given the profile of signal choice (X_1, X_2) is

$$\begin{aligned}
&\mathbb{P}(X_i = 0)\Big((1 - h_0)\mathbb{E}(u_i(\tilde{\theta}, \alpha_i, \alpha_j + \beta_j)|X_i = 0) + h_0\mathbb{E}(u_i(\tilde{\theta}, \alpha_i, \alpha_j)|X_i = 0)\Big) \\
&+\mathbb{P}(X_i = 1)\Big((1 - h_1)\mathbb{E}(u_i(\tilde{\theta}, \alpha_i + \beta_i, \alpha_j)|X_i = 1) + h_1\mathbb{E}(u_i(\tilde{\theta}, \alpha_i + \beta_i, \alpha_j + \beta_j)|X_i = 1)\Big)
\end{aligned} \tag{6}$$

By taking the first-order condition to (6) with respect to α_i and β_i for $i = 1, 2$, we obtain the following system of linear equations

$$\begin{cases}
\alpha_i &= b_{i12}(1 - h_0)\beta_j + b_{i12}\alpha_j + b_{i13}k_0 + b_{i1} \\
\beta_i &= b_{i12}\beta_j H + b_{i13}K \\
\alpha_j &= b_{j12}(1 - h_0)\beta_i + b_{j12}\alpha_i + b_{j13}k_0 + b_{j1} \\
\beta_j &= b_{j12}\beta_i H + b_{j13}K.
\end{cases}$$

We can first solve for β_i and β_j. We get

$$\begin{cases}
\beta_i &= b_{i12}\beta_j H + b_{i13}K \\
\beta_j &= b_{j12}\beta_i H + b_{j13}K,
\end{cases}$$

so that

$$\beta_i^* = \frac{b_{i13}K + b_{i12}b_{j13}KH}{1 - b_{i12}b_{j12}H^2} \tag{7}$$

$$\beta_j^* = \frac{b_{j13}K + b_{j12}b_{i13}KH}{1 - b_{i12}b_{j12}H^2}. \tag{8}$$

Then we can solve

$$\begin{cases}
\alpha_i &= b_{i12}(1 - h_0)\beta_j^* + b_{i12}\alpha_j + b_{i13}k_0 + b_{i1} \\
\alpha_j &= b_{j12}(1 - h_0)\beta_i^* + b_{j12}\alpha_i + b_{j13}k_0 + b_{j1},
\end{cases}$$

so that

$$\alpha_i^* = \frac{b_{i12}(1-h_0)K(b_{j13}(1+b_{i12}b_{j12}H)+b_{i13}b_{j12}(1+H))}{(1-b_{i12}b_{j12})(1-b_{i12}b_{j12}H^2)} + \frac{b_{i1}+b_{i12}b_{j1}+(b_{i13}+b_{i12}b_{j13})k_0}{1-b_{i12}b_{j12}} \tag{9}$$

$$\alpha_j^* = \frac{b_{j12}(1-h_0)K(b_{i13}(1+b_{j12}b_{i12}H)+b_{j13}b_{i12}(1+H))}{(1-b_{i12}b_{j12})(1-b_{i12}b_{j12}H^2)} + \frac{b_{j1}+b_{j12}b_{i1}+(b_{j13}+b_{j12}b_{i13})k_0}{1-b_{i12}b_{j12}} \tag{10}$$

To sum up, a necessary condition for a Bayesian Nash equilibrium is that player i's action in the second-stage of the game are

$$a_i^* = \begin{cases} \alpha_i^* & \text{if } x_i = 0 \\ \alpha_i^* + \beta_i^* & \text{if } \ x_i = 1. \end{cases} \tag{11}$$

From the expressions for β_i^*, we can conclude on when the equilibrium action strategy is increasing in the signal. In the case of symmetric payoffs, that is with $b_{i12} = b_{j12}$ and $b_{i13} = b_{j13}$, then $\beta_i^* > 0$ if either $b_{13} < 0$ and $b_{12} > 1/H$ or if $b_{13} > 0$ and $b_{12} < 1/H$. Moreover, it is always the case that the actions a_i^* and a_j^* are comonotonic (either strictly increasing or strictly decreasing).

The assumption of identical joint marginal implies that at the profiles of signal choice (Y, X) and (X, Y) the players will have identical belief on the other's signal and on the state. The same claim remains true at the profiles of signal choice (Y, Y) and (X, X). Therefore, without loss of generality, we only need to compare the optimal profile of actions a^* at the profiles of signal choice (Y, X) and (X, X).

Let $a^*(X, X)$ be the equilibrium action profile of the second-stage given that the profile of signal choice is (X, X). Define similarly, $a^*(Y, X)$. In particular, we have

$$a_i^*(X,X) = \begin{cases} \frac{b_{i1}+b_{i13}+b_{i12}(b_{j1}+b_{j13}) - \frac{(b_{i13}+b_{i12}b_{j13})q_{00}}{p_\theta+q_{00}-q_{11}}}{1-b_{i12}b_{j12}} & \text{if } x_i = 0 \\ \\ \frac{(b_{i1}+b_{i12}b_{j1})(1-p_\theta-q_{00})+(b_{i1}+b_{i13}+b_{i12}(b_{j1}+b_{j13}))q_{11}}{(1-b_{i12}b_{j12})(1-p\theta-q_{00}+q_{11})} & \text{if } \ x_i = 1, \end{cases} \tag{12}$$

and

$$a_i^*(Y,X) = \begin{cases} \frac{b_{i12}(1-h_0)K(b_{j13}(1+b_{i12}b_{j12}H)+b_{i13}b_{j12}(1+H))}{(1-b_{i12}b_{j12})(1-b_{i12}b_{j12}H^2)} + \frac{b_{i1}+b_{i12}b_{j1}+(b_{i13}+b_{i12}b_{j13})k_0}{1-b_{i12}b_{j12}} & \text{if } x_i = 0 \\ \\ \Big(\frac{b_{i12}(1-h_0)K(b_{j13}(1+b_{i12}b_{j12}H)+b_{i13}b_{j12}(1+H))}{(1-b_{i12}b_{j12})(1-b_{i12}b_{j12}H^2)} + \frac{b_{i1}+b_{i12}b_{j1}+(b_{i13}+b_{i12}b_{j13})k_0}{1-b_{i12}b_{j12}} \\ \quad + \frac{b_{i13}K+b_{i12}b_{j13}KH}{1-b_{i12}b_{j12}H^2}\Big) & \text{if } \ x_i = 1. \end{cases} \tag{13}$$

4.1.2 Signal Choice

Suppose that the profile of signal choice is (Y, X) so that it is the less dependent profile given the available information structure. Then, player i has a profitable deviation if the profile (X, X) gives him a strictly higher payoff than (Y, X) keeping the action profile $a^*(Y, X)$ fixed. This provides a necessary but non sufficient condition for an equilibrium. When deviating from (Y, X) to (X, X) we say that player i increases the dependence. We address now the conditions under which increasing the dependence constitutes a profitable deviation.

The type of deviation that we study implies that when increasing the dependence, a player has an effect on his ex-ante expected payoff only by changing the probability that the realization of his signal is the same as the other player. Hence, the ex-ante expected payoff change only because h_1 and h_0 do. To analyze whether increasing the dependence is profitable, we evaluate the ex-ante expected payoff given in (6) at the action profile $a^*(Y, X)$ and then compare it to the ex-ante expected payoff where nothing changes except the probability h_1 and h_0.

At the profile (Y, X), ex-ante expected utility of player i is

$$\begin{aligned}&\mathbb{P}(Y=0)\left(\alpha_i^*(2b_{i1}+2b_{i13}k_0-\alpha_i^*+2b_{i12}(\alpha_j^*+(1-h_0)\beta_j^*))\right)\\&+\mathbb{P}(Y=1)\left(-(\alpha_i^*+\beta_i^*)(-2b_{i1}-2b_{i13}k1+\alpha_i^*+\beta_i^*-2b_{i12}(\alpha_j^*+h_1\beta_j^*))\right).\end{aligned} \tag{14}$$

Consider a deviation from player i to (X, X), such deviation increases the dependence and the payoff is now

$$\begin{aligned}&\mathbb{P}(X=0)\left(\alpha_i^*(2b_{i1}+2b_{i13}k_0-\alpha_i^*+2b_{i12}(\alpha_j^*))\right)\\&+\mathbb{P}(X=1)\left(-(\alpha_i^*+\beta_i^*)(-2b_{i1}-2b_{i13}k1+\alpha_i^*+\beta_i^*-2b_{i12}(\alpha_j^*+\beta_j^*))\right).\end{aligned} \tag{15}$$

The difference between (14) and (15) is

$$-2b_{i12}\mathbb{P}(Y=1)\mathbb{P}(X=0|Y=1)\beta_i^*\beta_j^* \tag{16}$$

with $\mathbb{P}(Y=1)\mathbb{P}(X=0|Y=1)=q_{00}+q_{11}-(1-p_\theta)z_{00}-p_\theta w_{11}$. If Equation (16) is negative, then increasing the dependence by deviating to the profile of signal choice (X, X) is profitable for the player i. From (16) it follows that whenever the actions are monotonic, i.e. $\beta_i^*\beta_j^*>0$, then the deviation will be profitable only if $b_{i12}>0$. Similarly, under an antimonotonic equilibrium, i.e. $\beta_i^*\beta_j^*<0$, then player i does not benefit from increasing the dependence if there are complementarities in actions.

Finally, we investigate under which conditions do the players benefit from decreasing the dependence, that is going from a profile of signal choice (X, X) to (Y, X). In this case, the difference between the ex-ante expected payoff at the profile (X, X) and the ex-ante expected payoff at the profile (Y, X) keeping the actions fixed to $a^*(X, X)$ is

$$2b_{i12}\mathbb{P}(Y = 1)\mathbb{P}(X = 0|Y = 1)\beta_i^*\beta_j^*. \tag{17}$$

The case where (17) is negative implies that decreasing the dependence is profitable. Hence, with complementarity in actions, decreasing the dependence is profitable only if the actions are strictly antimonotonic.

4.2 The normal quadratic model

The second example is not strictly speaking a special case of our model, because the support of the signals is infinite. We include it because of the important role it plays in the literature and because our finiteness assumption is made to keep the exposition simple, not for more fundamental reasons.

Suppose that $S = \{X, Y\}$ and $N = 2$. Payoffs are

$$u_i(\theta, a) = -a_i^2 + 2b_{i12}a_ia_j + 2b_{i13}a_i\theta + 2b_{i1}a_i + K(a_j, \theta) \tag{18}$$

where b_{i12}, b_{i13} and b_{i1} are real numbers for $i \in \{1, 2\}$ and $K(\cdot, \cdot)$ is a function that does not affect the set of Nash-Bayesian equilibria, but may have an effect on welfare. The information structure is as follows. The random vector (θ, X, Y) is a Gaussian vector with expectation $(0, 0, 0)$ and covariance matrix

$$\sigma = \begin{pmatrix} \sigma_{\theta\theta} & \sigma_{\theta X} & \sigma_{\theta Y} \\ \sigma_{\theta X} & \sigma_{XX} & \sigma_{XY} \\ \sigma_{\theta Y} & \sigma_{XY} & \sigma_{YY} \end{pmatrix}$$

with $\sigma_{XX} = \sigma_{YY}$ and $\sigma_{\theta X} = \sigma_{\theta Y}$, so that (θ, X) and (θ, Y) have identical joint marginal distributions.

In this setting, a strategy for player i has two components. First, a source choice $X_i \in S$ and a reaction function $\alpha_i : \mathbb{R}^{S_i} \longrightarrow \mathbb{R}$, which maps a signal x_s to an action a. Let $\mathbb{A}$ be the class of such functions. Given a profile $(X_1, X_2, \alpha_1, \alpha_2)$, the expected payoff of player i is

$$U_i(X_1, X_2, \alpha_1, \alpha_2) = \mathbb{E}[u_i(\widetilde{\theta}, \alpha_i(\widetilde{x}_i), \alpha_j(\widetilde{x}_j))], \tag{19}$$

where $u_i(\theta, \alpha_i, \alpha_j)$ is given by (18).

A necessary condition for the profile $(X_1, X_2, \alpha_1, \alpha_2)$ to be a Nash-Bayesian equilibrium is that no player has an incentive to change its reaction function, i.e., (α_1, α_2) form a Nash-equilibrium in the game where (X_1, X_2) is already fixed. Hence, fixing signal choices $X_1 \in S$ and $X_2 \in S$, one can without loss of generality[5] restrict attention to affine action strategies of the form

$$\alpha_i\left(x_i\right) = w_i x_i + \kappa_i.$$

Given $X_1 \in S$ and $X_2 \in S$, we assume that (θ, X_1, X_2) is a Gaussian vector with expectation $(0,0,0)$ and covariance matrix

$$\sigma = \begin{pmatrix} \sigma_{\theta\theta} & \sigma_{\theta 1} & \sigma_{\theta 2} \\ \sigma_{\theta 1} & \sigma_{11} & \sigma_{12} \\ \sigma_{\theta 2} & \sigma_{12} & \sigma_{22} \end{pmatrix} \tag{20}$$

Taking the first-order condition to (19) with respect to α_i conditional on $X_i = x_i$ yields the best-response

$$\alpha_i = b_{i12}\mathbb{E}(\alpha_j | X_i = x_i) + b_{i13}\mathbb{E}(\theta | X_i = x_i) + b_{i1} \tag{21}$$

Conjecturing that $\alpha_j = w_j x_j + \kappa_j$, and using the covariance matrix given in (20), we get that

$$\mathbb{E}(\alpha_j | X_i = x_i) = w_j \left(\frac{\sigma_{ij} x_i}{\sigma_{ii}}\right) + \kappa_j \tag{22}$$

and

$$\mathbb{E}(\theta | X_i = x_i) = \frac{\sigma_{\theta i} x_i}{\sigma_{ii}}. \tag{23}$$

The best-response in (21) becomes

$$\alpha_i = \left(\frac{b_{i12} w_j \sigma_{ij}}{\sigma_{ii}} + \frac{b_{i13}\sigma_{\theta i}}{\sigma_{ii}}\right) x_i + b_{i12} k_j + b_{i1} \tag{24}$$

An equation similar to (24) also holds for player j, that is

$$\alpha_j = \left(\frac{b_{j12} w_i \sigma_{ij}}{\sigma_{jj}} + \frac{b_{j13}\sigma_{\theta j}}{\sigma_{jj}}\right) x_j + b_{j12} k_i + b_{j1}. \tag{25}$$

[5]It is known that in a game with quadratic payoffs and Gaussian information, there exists a unique equilibrium and the equilibrium reaction functions α_1 and α_2 are affine functions of the signals.Jiménez-Martinez (2013) and Calvo-Armengol & de Marti Beltran (2009) prove uniqueness of the reaction functions for a game with quadratic payoffs and an exogenous level of information using the theory of potential games and Radner (1962)'s team theory.

Then (24) and (25) define the following system of linear equations

$$\begin{pmatrix} 1 & -\frac{b_{i12}\sigma_{ij}}{\sigma_{ii}} & 0 & 0 \\ -\frac{b_{j12}\sigma_{ij}}{\sigma_{jj}} & 1 & 0 & 0 \\ 0 & 0 & 1 & -b_{i12} \\ 0 & 0 & -b_{j12} & 1 \end{pmatrix} \times \begin{pmatrix} w_i \\ w_j \\ \kappa_i \\ \kappa_j \end{pmatrix} = \begin{pmatrix} \frac{b_{i13}\sigma_{\theta i}}{\sigma_{ii}} \\ \frac{b_{j13}\sigma_{\theta j}}{\sigma_{jj}} \\ b_{i1} \\ b_{j1} \end{pmatrix} \tag{26}$$

Solving (26) with respect to $(w_i, w_j, \kappa_i, \kappa_j)$ yields the solution $(w_1^*, w_2^*, \kappa_1^*, \kappa_2^*)$ with

$$w_i^* = \frac{b_{i13}\sigma_{jj}\sigma_{\theta i} + b_{i12}b_{j13}\sigma_{ij}\sigma_{\theta j}}{\sigma_{ii}\sigma_{jj} - b_{i12}b_{j12}\sigma_{ij}^2} \tag{27}$$

$$\kappa_i^* = \frac{b_{i1} + b_{i12}b_{j1}}{1 - b_{i12}b_{j12}}. \tag{28}$$

In a Nash Bayesian equilibrium, given a signal profile (X_1, X_2), we must have $\alpha^* = (\alpha_1^*, \alpha_2^*)$ where

$$\alpha_i^* = w_i^* x_i + \kappa_i^*. \tag{29}$$

We proceed next with the first-stage of the game as an equilibrium profile must also specify a choice of signal for each player. The expected payoff for player i in the first-stage at a profile (X_1, X_2) is obtained by evaluating (19) at α^*

$$\mathbb{E}(u_i(\widetilde{\theta}, \alpha_i^*(\tilde{x}_i), \alpha^*(\tilde{x}_j)) = -w_i^{2*}\sigma_{ii} + 2b_{i12}w_i^*w_j^*\sigma_{ij} + 2b_{i13}w_i^*\sigma_{\theta i} - \kappa_i^* + 2b_{i12}\kappa_i^*\kappa_j^* + 2b_{i1}\kappa_i^* + \mathbb{E}(K(\alpha_j^*, \theta)). \tag{30}$$

Once again, we are interested in necessary conditions for a Bayesian Nash equilibrium. In particular, we would like to obtain conditions under which a deviation that increases (decreases) the dependence is profitable.

Suppose that the profile of signal choice is (Y, X), then the ex-ante expected payoff is

$$\begin{aligned} & -w_i^{2*}(Y,X)\sigma_{YY} + 2b_{i12}w_i^*(Y,X)w_j^*(Y,X)\sigma_{XY} + 2b_{i13}w_i^*(Y,X)\sigma_{\theta Y} \\ & \quad -\kappa_i^*(Y,X) + 2b_{i12}\kappa_i^*(Y,X)\kappa_j^*(Y,X) + 2b_{i1}\kappa_i^*(Y,X) + \mathbb{E}(K(\alpha_j^*(Y,X), \theta)). \end{aligned} \tag{31}$$

Suppose next that player i considers increasing the dependence with the deviation $X_i = X$. Then as the signal choice are not observable, this means that player j will not modify his action to take into account the new dependence between X_i and X_j. Note that player i's action is also fixed. Thus, the effect of the deviation is to change σ_{12} the covariance between the players' signals. In particular, this means that we can restrict attention to the term $2b_{i12}w_i^*(Y,X)w_j^*(Y,X)\sigma_{ij}$ in the payoff function to conclude on the effect of the deviation.

In this respect, notice that we must have $\sigma_{XX} \geq \sigma_{YX}$ in order for the covariance matrix σ to be positive definite. This means that going from (Y, X) to (X, X) necessarily increases the covariance between the signals. The deviation will be profitable if the term $b_{i12} w_i^*(Y, X) w_j^*(Y, X)$ is positive.

In the case of symmetric payoffs, that is with $b_{i12} = b_{j12}$ and $b_{i13} = b_{j13}$, then it is always the case that the actions α_i^* and α_j^* are comonotonic (either strictly increasing or strictly decreasing). Therefore, the profitability of increasing the dependence only depends on the sign of b_{12}. If actions are substitutes (or equivalently if the payoff is submodular in actions), then increasing the dependence is not a profitable deviation. However, if actions are complements (or equivalently if the payoff is supermodular in actions), then increasing the dependence is indeed a profitable deviation for player i. Hence the profile of signal choice (Y, X) cannot be an equilibrium if there is complementarity.

Turning next to a deviation that consists of decreasing the dependence, as a result of the assumption that marginals are identical, we can restrict the analysis to the study of the deviation (X, X) to (Y, X). In this change, since $\sigma_{XX} \geq \sigma_{YX}$, we necessarily decreases the covariance between the signals. Thus, the deviation will be profitable if the term $b_{i12} w_i^*(X, X) w_j^*(X, X)$ is negative. The fact that actions are comonotonic in the symmetric case allows us to conclude that it is the sign of b_{12} that drives whether decreasing the dependence is profitable. If actions are complements (or equivalently if the payoff is supermodular in actions), then decreasing the dependence is not a profitable deviation. However, if actions are substitutes (or equivalently if the payoff is submodular in actions), then decreasing the dependence is indeed a profitable deviation for player i. Hence the profile of signal choice (X, X) cannot be an equilibrium if there is substitutability.

In the rest of the paper, we will generalize some of the results obtained in these examples to a larger class of games. In the next section, we introduce a notion of dependence of a random variable on a random vector that will play an important role in this analysis.

5 Dependence of a random variable on a multivariate vector

We start by introducing various concepts of dependence between a random variable and a fixed random vector and establish some relationships between them. We relate this results to a few known results.

Let h be a pdf distribution on $\mathcal{X}$. Let $\mathbb{X}$ be the set of random variables X_j on $\mathcal{X}$ whose pdf is h. Consider a fixed random vector X_{-i} with support in $\mathcal{X}^{N-1}$, with $N \geq 2$, such that each of the univariate components X_j is an element of $\mathbb{X}$. We will define various weak partial orders on the set $\mathbb{X}$ that compare the similarity of the elements of this set with the components of the vector X_{-i}.

Definition 1 (General covariance dependence)**.** *For all Y_i and Z_i in $\mathbb{X}$, the variable Y_i has a weakly greater general covariance dependence on X_{-i} than Z_i, denoted $Y_i \succeq_G^{X_{-i}} Z_i$ if for all increasing functions $r : \mathcal{X} \to \mathbb{R}$ and $s : \mathcal{X}^{N-1} \to \mathbb{R}$,*

$$Cov\left(r\left(Y_i\right), s\left(X_{-i}\right)\right) \geq Cov\left(r\left(Z_i\right), s\left(X_{-i}\right)\right).$$

We say that the function $\phi : \mathcal{X}^N \to \mathbb{R}$ has **increasing differences in x_i and x_{-i}** if for all $x = (x_i, x_{-i}) \in \mathcal{X}^N$ and $y = (y_i, y_{-i}) \in \mathcal{X}^N$, such that $x \leq y$, we have

$$\phi\left(y_i, x_{-i}\right) - \phi\left(x_i, x_{-i}\right) \leq \phi\left(y_i, y_{-i}\right) - \phi\left(x_i, y_{-i}\right).$$

We say that the function ϕ has **strictly increasing differences in x_i and x_{-i}** if for all $x = (x_i, x_{-i}) \in \mathcal{X}^N$ and $y = (y_i, y_{-i}) \in \mathcal{X}^N$, such that $x_i < y_i$ and $x_{-i} < y_{-i}$ we have

$$\phi\left(y_i, x_{-i}\right) - \phi\left(x_i, x_{-i}\right) < \phi\left(y_i, y_{-i}\right) - \phi\left(x_i, y_{-i}\right).$$

We say that ϕ has **decreasing differences in x_i and x_{-i}** if $-\phi$ has increasing differences in x_i and x_{-i} and that ϕ has **strictly decreasing differences in x_i and x_{-i}** if $-\phi$ has strictly increasing differences in x_i and x_{-i}.

Definition 2 (Increasing differences dependence)**.** *The random variable Y_i has a weakly greater increasing-differences dependence on X_{-i} than the random variable Z_i, denoted $Y_i \succeq_\Delta^{X_{-i}} Z_i$, if for all functions $\phi : \mathcal{X}^N \to \mathbb{R}$ that have increasing differences in x_i and x_{-i}, $\mathbb{E}\left(\phi\left(Y_i, X_{-i}\right)\right) \geq \mathbb{E}\left(\phi\left(Z_i, X_{-i}\right)\right)$.*

Definition 3 (Concordance dependence)**.** *The random variable Y_i has weakly greater concordance dependence on X_{-i} than the random variable Z_i, denoted $Y_i \succeq_C^{X_{-i}} Z_i$, if the cdfs F and G of the random vectors (Y_i, X_{-i}) and (Z_i, X_{-i}) satisfy for all $x \in \mathcal{X}^N$, $G(x) \leq F(x)$ and in addition if the survival function $\overline{F}(x) = \mathbb{P}_F\left(\left(Y_i, X_{-i}\right) \geq x\right)$ and $\overline{G}(x) = \mathbb{P}_G\left(\left(Z_i, X_{-i}\right) \geq x\right)$ satisfy $\overline{G}(x) \leq \overline{F}(x)$.*

Given that the random vectors (Y_i, X_{-i}) and (Z_i, X_{-i}) have the same univariate marginal distributions, an equivalent definition is that $Y_i \succeq_C^{X_{-i}} Z_i$, if for all $x \in \mathcal{X}^I$, $\Pr\left(\left(Y_i, X_{-i}\right) \geq x\right) \geq \Pr\left(\left(Z_i, X_{-i}\right) \geq x\right)$.

Our next dependence weak partial order relies on the notion of stochastic dominance, which we need to introduce first.

The following result due to Lehman (1955) and Levhari, Paroush and Peleg (1975) establishes the equivalence between four different possible ways to define the stochastic dominance ordering for multivariate vectors, which generalizes the familiar first-order stochastic dominance ordering from the univariate case. A set $L \subseteq \mathcal{X}^k$ is called comprehensive if for all $x \in L$, $x' \leq x \implies x' \in L$. For any vector $x \in \mathcal{X}^k$, let 1_x be the indicator function of the singleton $\{x\}$.

Theorem 1. *Let X and Y be random vectors with respective pdfs f and g on the support $\mathcal{X}^k$. The following conditions are equivalent:*

i. For all comprehensive L, we have

$$\sum_{x \in L} f(x) \geq \sum_{x \in L} g(x).$$

ii. For all nondecreasing mapping $W : \mathcal{X}^k \to \mathbb{R}$, $\mathbb{E}(W(Y)) \geq \mathbb{E}(W(X))$.

iii. There exist random vectors X' and Y' with respective pdfs f and g such that $X' \leq Y'$.

iv. There exist a finite list of vector pairs $(x_t, y_t)_{t=1,\ldots,T}$ with $x_t \leq y_t$ and a list of reals $(\Delta_t)_{t=1,\ldots,m}$, with $\Delta_t \in [0,1]$ such that

$$g(x) - f(x) = \sum_t \Delta_t \left(1_{y_t}(x) - 1_{x_t}(x)\right).$$

This result allows us to provide the following definition.

Definition 4. *A random variable Z with pdf g on the support $\mathcal{X}^k$ stochastically dominates another random vector Y with pdf f on $\mathcal{X}^k$ if they satisfy the equivalent conditions (i) to (iv) in Theorem 1.*

The notion of multivariate stochastic dominance enables us to define our next dependence weak partial order.

Definition 5 (Dominance dependence)**.** *The random variable Y_i has a weakly greater dominance dependence on X_{-i} than the random variable Z_i, denoted $Y_i \succeq_D^{X_{-i}} Z_i$, if for all x_i, the distribution of X_{-i} conditional on $Z_i \leq x_i$ stochastically dominates the distribution of X_{-i} conditional on $Y_i \leq x_i$.*

For each of these weak partial orderings, we can define the associated strict orderings as follows: for each $O \in \{G, \Delta, C, D\}$, we let $Y \succ_O X$ if $Y \succeq_O X$ and not $X \succeq_O Y$.

We are now ready to present a result that describes the relation between the different dependence weak partial orders.

Theorem 2. *The covariance dependence, increasing differences dependence and dominance dependence weak partial orders are equivalent. If $N = 2$, these weak partial orders are equivalent to the concordance weak partial order. If $N \geq 3$, they are strictly stronger than the concordance weak partial order.*

Proof. In the case $N = 2$, the equivalence of the four weak partial orders are by now classic results in probability and statistics. The equivalence between the increasing differences and concordance orderings was established by Tchen (1980). The other equivalence is well known (e.g. Müller and Stoyan, 2002, Theorem 3.8.2). In the continuation, we prove the result in the case where $N = 3$.

Step 1: The general covariance dependence weak partial order is at least as strong as the increasing differences dependence weak partial order.

This proof is adapted from the proof of Strulovici and Meyer (2012) of the analogous implication for interdependence weak partial orders, which builds on a proof by Christofides and Vaggelatou (2004) that establishes this implication in the special case of the comparison between a random vector and its independent counterpart. Let X_{-i} be a fixed random vector. Let Y_i and Z_i in $\mathbb{X}$, such that $Y_i \succeq_G^{X_{-i}} Z_i$. We will show that $Y_i \succeq_\Delta^{X_{-i}} Z_i$.

First, let $W = (W_1, ..., W_N)$ be a random vector that has the same distribution as (Z_i, X_{-i}) but is independent of (Y_i, X_{-i}). This implies that the vectors (Y_i, W_{-i}) and (W_i, X_{-i}) have the same distribution. Thus it follows that for any function $\phi : \mathcal{X}^N \to \mathbb{R}$, we have

$$\mathbb{E}\phi\left(W_i, X_{-i}\right) - \mathbb{E}\phi\left(Y_i, W_{-i}\right) = 0. \tag{32}$$

Now, let ϕ be any function that has increasing differences in x_i and x_{-i}. We will show that

$$\mathbb{E}\phi\left(Y_i, X_{-i}\right) - \mathbb{E}\phi\left(W_i, W_{-i}\right) \geq \mathbb{E}\phi\left(W_i, X_{-i}\right) - \mathbb{E}\phi\left(Y_i, W_{-i}\right)$$

Together with (32) and the equality $\mathbb{E}\left(\phi\left(W_i, W_{-i}\right)\right) = \mathbb{E}\left(\phi\left(Z_i, X_{-i}\right)\right)$, this will imply that $\mathbb{E}\left(\phi\left(Y_i, X_{-i}\right)\right) - \mathbb{E}\left(\phi\left(Z_i, X_{-i}\right)\right) \geq 0$.

Label $k = 1, ..., \overline{k}$ the elements of $\mathcal{X}$. We can write

$$\phi\left(Y_i, X_{-i}\right) - \phi\left(W_i, X_{-i}\right) = \sum_{k=0}^{\overline{k}-1} \left(I_{Y_i > k} - I_{W_i > k}\right)\left(\phi\left(k+1, X_{-i}\right) - \phi\left(k, X_{-i}\right)\right).$$

Therefore

$$\begin{aligned}
& \mathbb{E}\phi\left(Y_i, X_{-i}\right)-\mathbb{E}\phi\left(W_i, X_{-i}\right) \\
= & \sum_{k=0}^{\bar{k}-1} \mathbb{E}\left[I_{Y_i>k}\left(\phi\left(k+1, X_{-i}\right)-\phi\left(k, X_{-i}\right)\right)\right]-\sum_{k=0}^{\bar{k}-1} \mathbb{E}\left[I_{W_i>k}\left(\phi\left(k+1, X_{-i}\right)-\phi\left(k, X_{-i}\right)\right)\right] \\
= & \sum_{k=0}^{\bar{k}-1} \mathbb{E}\left[I_{Y_i>k}\left(\phi\left(k+1, X_{-i}\right)-\phi\left(k, X_{-i}\right)\right)\right]-\sum_{k=0}^{\bar{k}-1} \mathbb{E}\left[I_{W_i>k}\right] \mathbb{E}\left(\phi\left(k+1, X_{-i}\right)-\phi\left(k, X_{-i}\right)\right) \\
= & \sum_{k=0}^{\bar{k}-1} \mathbb{E}\left[I_{Y_i>k}\left(\phi\left(k+1, X_{-i}\right)-\phi\left(k, X_{-i}\right)\right)\right]-\sum_{k=0}^{\bar{k}-1} \mathbb{E}\left[I_{Y_i>k}\right] \mathbb{E}\left(\phi\left(k+1, X_{-i}\right)-\phi\left(k, X_{-i}\right)\right) \\
= & \sum_{k=0}^{\bar{k}-1} Cov\left(I_{Y_i>k},\left(\phi\left(k+1, X_{-i}\right)-\phi\left(k, X_{-i}\right)\right)\right) \\
\geq & \sum_{k=0}^{\bar{k}-1} Cov\left(I_{Z_i>k},\left(\phi\left(k+1, X_{-i}\right)-\phi\left(k, X_{-i}\right)\right)\right) \\
= & \sum_{k=0}^{\bar{k}-1} Cov\left(I_{W_i>k},\left(\phi\left(k+1, W_{-i}\right)-\phi\left(k, W_{-i}\right)\right)\right) \\
= & \mathbb{E}\phi\left(W_i, W_{-i}\right)-\mathbb{E}\phi\left(Y_i, W_{-i}\right) .
\end{aligned}$$

where the second equality holds because W_i is independent from X_{-i}, the third equality holds because W_i and Y_i have the same marginal distribution, the inequality holds because $Y_i \succeq_G^{X_{-i}} Z_i$, the function $r(x_i) = I_{x_i>k}$ is increasing in x_i, and the function $s(x_{-i}) = \phi(k+1, x_{-i}) - \phi(k, x_{-i})$, is increasing in x_{-i} since ϕ has increasing differences, the fifth equality holds because (Z_i, X_{-i}) has the same distribution as W, and the last equality holds by the same arguments invoked in the first four equalities. We thus obtain $\mathbb{E}(\phi(Y_i, X_{-i})) - \mathbb{E}(\phi(Z_i, X_{-i})) \geq 0$. Since this is true for any function ϕ that has increasing differences, it follows that $Y_i \succeq_\Delta^{X_{-i}} Z_i$, the desired conclusion.

Step 2: The increasing differences dependence weak partial order is at least as strong as the general covariance dependence weak partial order.

Let X_{-i} be a fixed random vector. Let Y_i and Z_i in $\mathbb{X}$, such that $Y_i \succeq_\Delta^{X_{-i}} Z_i$. We will show that $Y_i \succeq_G^{X_{-i}} Z_i$.

Let $r : \mathcal{X} \to \mathbb{R}$ and $s : \mathcal{X}^{N-1} \to \mathbb{R}$ be two increasing functions. Let $\phi : \mathcal{X}^N \to \mathbb{R}$ be such that $\phi(x_i, x_{-i}) = r(x_i) s(x_{-i})$. Let $x \in \mathcal{X}^N$ and $y \in \mathcal{X}^N$ be such that $x \leq y$. Then

$$\begin{aligned}
\phi\left(y_i, x_{-i}\right)-\phi\left(x_i, x_{-i}\right) & = \left(r\left(y_i\right)-r\left(x_i\right)\right) s\left(x_{-i}\right) \\
& \leq \left(r\left(y_i\right)-r\left(x_i\right)\right) s\left(y_{-i}\right) \\
& = \phi\left(y_i, y_{-i}\right)-\phi\left(x_i, y_{-i}\right)
\end{aligned}$$

where the inequality holds because both r and s are increasing functions. Thus ϕ has increasing differences in x_i and x_{-i}. It follows that

$$\begin{aligned}\mathbb{E}\left[r\left(Y_i\right)s\left(X_{-i}\right)\right] &= \mathbb{E}\phi\left(Y_i, X_{-i}\right)\\ &\geq \mathbb{E}\phi\left(Z_i, X_{-i}\right)\\ &= \mathbb{E}\left[r\left(Z_i\right)s\left(X_{-i}\right)\right]\end{aligned}$$

where the inequality holds because $Y_i \succeq_{\Delta}^{X_{-i}} Z_i$ and ϕ has increasing differences in x_i and x_{-i}. Because Y_i and Z_i have the same marginal distribution, it also holds that $\mathbb{E}\left[r\left(Y_i\right)\right] = \mathbb{E}\left[r\left(Z_i\right)\right]$. Together with $\mathbb{E}\left[r\left(Y_i\right)s\left(X_{-i}\right)\right] \geq \mathbb{E}\left[r\left(Z_i\right)s\left(X_{-i}\right)\right]$, it implies that

$$Cov\left(r\left(Y_i\right), s\left(X_{-i}\right)\right) \geq Cov\left(r\left(Z_i\right), s\left(X_{-i}\right)\right).$$

Since this holds for any increasing functions r and s, we obtain that $Y_i \succeq_{G}^{X_{-i}} Z_i$, the desired conclusion.

Step 3: The general covariance dependence weak partial order is at least as strong as the concordance dependence weak partial order.

This can be shown by using $r\left(x_i\right) = I_{x_i \geq k_i}$ for all $k_i = 1, ..., \overline{k}$ and $s\left(x_{-i}\right) = I_{x_{-i} \geq k_{-i}}$ for all $k_{-i} \in \left\{1, ..., \overline{k}\right\}^{N-1}$.

The following example, which we adapt from Hu, Müller and Scarsini (2004) shows that the second implication is strict.

Let $\left(X_2, X_3\right)$ be uniformly distributed on the set $\left\{(0,0), (1,1), (2,1), (0,2), (1,2), (2,2)\right\}$. Next let $Y_1 = \phi\left(X_2, X_3\right)$ and $Z_1 = \psi\left(X_2, X_3\right)$, where ϕ and ψ are defined by $\phi(0,0) = 0$; $\phi(1,1) = 2$; $\phi(2,1) = 1$; $\phi(0,2) = 2$; $\phi(1,2) = 1$; $\phi(2,2) = 2$ and $\psi(0,0) = 2$; $\psi(1,1) = 1$; $\psi(2,1) = 2$; $\psi(0,2) = 0$; $\psi(1,2) = 2$; $\psi(2,2) = 1$. Both Y_1 and Z_1 have the marginal distribution $h(0) = 1/6$, $h(1) = 1/3$ and $h(2) = 1/2$. One can verify that $Y_1 \succeq_{C}^{X_{-1}} Z_1$ holds but that $Y_1 \succeq_{D}^{X_{-1}} Z_1$ does not hold.

Step 4: The general covariance dependence weak partial order is equivalent to the dominance dependence weak partial order.

This equivalence is easy to prove. We leave it to the reader.■

In the case $N = 2$, Theorem 2 is well knows and was established by Tchen (1980) and other authors (see Müller and Stoyan, 2002, Theorem 3.8.2 for references). When $N = 2$, dependence relations can be viewed as comparisons on how different random variables depend on a fixed random vector that happens to be univariate. This is the view that we take and that we generalize to the case where the fixed vector is not necessarily univariate. The statistics and economics literature has up to now taken another view, which is to consider dependence relations in the case $N = 2$ as measures of interdependence between the two components of a bivariate random vector. The natural generalization to the case of a general $N \geq 2$, is to consider dependence as a measure of interdependence between all univariate components of a multivariate random vector. Partial generalizations for $N \geq 3$ along this line exist. They are reviewed by Strulovici and Meyer (2012), who also establish new relations. Roughly speaking, one can define a number "more interdependent than" weak partial orders on the set of all N-variate random vectors with the same fixed marginal distributions: a general covariance interdependence relation, a supermodular interdependence relation, a concordance interdependence relation and many others. One could imagine a way to define a dominance interdependence relation, but to the best of our knowledge, this has not been studied. As Strulovici and Meyer (2012) summarize, the general covariance interdependence relation is strictly stronger than the supermodular interdependence relation (established by Strulovici and Meyer, 2012, generalizing a proof of Christofides and Vaggelatou, 2004, for a special case), which in turn is strictly stronger than the concordance interdependence relation (Müller and Stoyan, 2002). The strictness in these statements is established by examples provided by Hu, Müller and Scarsini (2004) and Strulovici and Meyer (2012).

Our result suggests that the fact that these relations are not equivalent when $N \geq 3$, unlike what occurs in the case $N = 2$, is not due to the multivariate nature of the objects being compared, but rather to the multilateral comparisons between many pairs of variables, and the fact that they all can change, as opposed to our comparisons that only involve two vectors, one of them held fixed, and the other one univariate.

We now return to the study of the equilibria of games with endogenous information structures.

6 Equilibrium information structures: necessary conditions

Using the result in Theorem 2, we will first obtain necessary conditions for equilibrium information structures.

For any random variable X, let X^θ denote the random variable X conditional on the state being θ. For all $i \in I$, and all s_i, s_i' and s_{-i}, we say that (s_i', s_{-i}) has **weakly greater (smaller) dependence than** (s_i, s_{-i}) if for all θ, conditional on θ, the vector $\left(X^\theta_{s_i'}, X^\theta_{s_{-i}}\right)$ has weakly greater (smaller) dependence than the variable $\left(X^\theta_{s_i}, X^\theta_{s_{-i}}\right)$. We say that (s_i', s_{-i}) **has strictly greater (smaller) dependence than** (s_i, s_{-i}) if (s_i', s_{-i}) has weakly greater (smaller) dependence than (s_i, s_{-i}) but (s_i, s_{-i}) does not have weakly greater (smaller) dependence than

(s_i', s_{-i}) . At any signal profile s, and for all $i \in I$, we say that player i can **strictly increase (decrease) the dependence at** s of his own signal on the other players' signals, if there exists s_i' such that (s_i', s_{-i}) has strictly greater (smaller) dependence than (s_i, s_{-i}) .

We say that u_i has **strictly increasing differences in own and other's actions** if for all θ, the function $u_i(\theta, \cdot)$ has strictly increasing differences in a_i and a_{-i}, and that u_i has **strictly decreasing differences in actions** if $-u_i$ has strictly increasing differences in actions.

Strictly increasing (decreasing) differences implies that in the complete information game (where θ is commonly known), player i has an increasing (decreasing) best reply function (Topkis, 1998; Milgrom and Roberts, 1994).

We are now ready to state the necessary condition that must hold in equilibrium in a game with endogenous information structures. It relates the signal choice of a player, the action strategy monotonicity properties and the payoff function of this player.

Theorem 3. *Let i be a player such that u_i has strictly increasing (decreasing) differences in own and others' actions. Let (s, α) be a pure Nash-Bayesian equilibrium of the game.*

1. *If α is strictly monotonic, then player i cannot strictly increase (decrease) the dependence of his own signal on the other players' signals at s.*
2. *If α is strictly antimonotonic for i, then player i cannot strictly decrease (increase) the dependence of his own signal on the other players' signals at s.*

Proof. We only prove point 1, and only in the case where α is strictly monotonic and player i has strictly increasing differences in actions. The proofs of the other three cases are similar. First, suppose that u_i has strictly increasing differences in own and other's actions and suppose that player i can strictly increase the dependence at s, so that there exists a source $s_i' \neq s_i$ such that (s_i', s_{-i}) has strictly greater dependence than s. We will show that the strategy (s_i', α_i) is a profitable deviation for player i.

Since the action strategy profile is kept fixed and is strictly monotonic, and since u_i has strictly increasing differences in own and other's actions, then for each θ, the function that associates the payoff $u_i\left(\alpha_i\left(x_{s_i}\right), \alpha_{-i}\left(x_{s_{-i}}\right), \theta\right)$ to the signals realization profile $(x_s, x_{s'})$ has strictly increasing differences in x_i and x_{-i}. Thus by Theorem 2, the inequalities

$$\mathbb{E}^{\theta}\left[u_i\left(\alpha_i\left(x_{s_i}\right), \alpha_{-i}\left(x_{s_{-i}}\right), \theta\right)\right] \leq \mathbb{E}^{\theta}\left[u_i\left(\alpha_i\left(x_{s_i'}\right), \alpha_{-i}\left(x_{s_{-i}}\right), \theta\right)\right]$$

hold for all realizations of θ, and at least strictly for some realization of θ that has positive probability. Thus we obtain that

$$\mathbb{E}\left[u_i\left(\alpha_i\left(x_{s_i}\right), \alpha_{-i}\left(x_{s_{-i}}\right), \theta\right)\right] < \mathbb{E}\left[u_i\left(\alpha_i\left(x_{s_i'}\right), \alpha_{-i}\left(x_{s_{-i}}\right), \theta\right)\right],$$

the desired conclusion.■

Theorem 3 says that when a player's payoff function has strictly increasing differences in actions, and expects a strictly monotonic action strategy profile, he chooses to acquire information as dependent of the other player's information as possible. In particular, if $N = 2$ and both players have access to the same signals, in any strictly monotonic equilibrium, they choose to acquire essentially the same information. The next definition formalizes this idea. For any signal profile $s \in S$, let $F^{\theta}_{s_i}$ be the marginal distribution of X_{s_i}, conditional on θ. We say that s is **public information** if all i and j, the event $X_{s_i} = X_{s_j}$ has probability one.

Corollary 1. *Suppose that $N = 2$, that $S_1 = S_2$ and let i be a player such that the payoff function u_i has strictly increasing (decreasing) differences in actions. Let $(s_i, s_{-i}, \alpha_i, \alpha_{-i})$ be a pure Nash-Bayesian equilibrium of the game. If α is strictly monotonic (antimonotonic for i), the signals profile is public information.*

Proof. For any signal profile s, and any player i, we can construct the profile (s'_i, s_{-i}) such that $s'_i := s_{-i}$ and $X_{s'_i} \succeq_C^{X_{s_{-i}}} X_{s_i}$. By Theorem 3, it must be that $X_{s_i} \succeq_C^{X_{s_{-i}}} X_{s'_i}$. Since $X_{s'_i} = X_{s_{-i}}$, we obtain $X_{s_i} \succeq_C^{X_{s_{-i}}} X_{s_{-i}}$, which implies that the profile s is public information.■

It is worth noting that Corollary 1 does not generalize to the case of three players or more, as can be seen in the following example.

Example 1. *Let $N = 3$ and $S = S_1 = S_2 = S_3 = \{a, b, c\}$ and let f be a cdf on $\{0, 1\}^4$ such that the joint marginal of (θ, X_s, X_t) is identical for all $s, t \in \{a, b, c\}$, with $s \neq t$. Let*

$$u_i(\theta, a) = -a_i^2 + b_{12} \sum_{j \neq i} a_i a_j - b_{13} a_i \theta + b_1 a_i$$

be player i's payoff, with $b_{12} > 0$. Then there exists a Nash-Bayesian equilibrium where each player plays the same action strategy $\alpha_i = (\alpha_{i0}, \alpha_{i1})$ and where $s = (a, b, c)$.

Even when $S_i \neq S_{-i}$, Theorem 3 leads to sharp predictions when the sets of signals S_i and S_{-i} have a certain structure. Each signal profile s_{-i} induces a weak partial order on the set S_i, namely the "weakly more dependent on s_{-i}" relation. If this weak partial order has a greatest element, let us say that this signal s_i is **most dependent on** s_{-i} in S_i. We say that the tuple $(S_1, ..., S_N, F)$ is **positively simple for** i if every signal in S_{-i} has a most dependent signal in S_i. For example, if $N = 2$ and $S_1 = S_2$, the triple (S_1, S_2, F) is positively simple for both players, and each signal is its own most dependent signal in S.[6]

Corollary 2. *Suppose that $(S_1, ..., S_N, F)$ is positively simple. Let $(s_1, ..., s_N, \alpha_1, ..., \alpha_N)$ be a Nash-Bayesian equilibrium of the game. Suppose that either (i) u_i has strictly increasing*

[6]Is there a nontrivial example of a positively simple tuple $(S_1, ..., S_N, F)$ when $N \geq 2$? It is not clear. For example $S_1 = ... = S_N = S$ will not be sufficient in general, except in trivial cases, such as if S is a singleton or if $N = 2$.

differences and α is strictly monotonic or (ii) u_i has strictly decreasing differences and α is strictly antimonotonic for i. Then the signal s_i is most dependent on s_{-i} in S_i.

Proof. This is a direct implication of Theorem 3.■

Similarly, for each i, and each signal profile s, we say that signal s_i is **least dependent on s_{-i} in S_i** if it is a least element in S_i for the "weakly more dependent on s_{-i}" relation. We say that the tuple $(S_1, ..., S_N, F)$ is **negatively simple for i** if every signal in S_{-i} has a least dependent signal in S_i. For example, if for every i and every signal s_{-i}, the set S_i contains a signal that is independent from s_{-i} and all signals in S are pairwise positively dependent, the tuple $(S_1, ..., S_N, F)$ is negatively simple for all players, and each signal s_i independent from s_{-i} is least dependent on s_{-i} in S_i and vice-versa.

Corollary 3. *Suppose that $(S_1, ..., S_N, F)$ is negatively simple. Let $(s_1, ..., s_N, \alpha_1, ..., \alpha_N)$ be a Nash-Bayesian equilibrium of the game. Suppose that for some player i, either (i) u_i has strictly decreasing differences in actions and α is strictly monotonic; or (ii) u_i has strictly increasing differences in actions and α is strictly antimonotonic for i. Then the signal s_i is least dependent on s_{-i} in S_i.*

Proof. This is a direct implication of Theorem 3.■

7 Equilibrium information structures: sufficient conditions

In this section we provide sufficient conditions for any information structure s at which all agents maximize the dependence of their own signal on the other signals to be part of a Nash-Bayesian equilibrium of the game with an endogenous information structure.

Theorem 4. *Let $N \geq 2$. Suppose that for each i,*

i. u_i has increasing differences in own and others' actions.

ii. u_i has increasing differences in a_i and θ.

iii. For all $x_i < x_i'$, the distribution of θ conditional on $X_i = x_i'$ first order stochastically dominates the distribution of θ conditional on $X_i = x_i$.

iv. For every profile s, all i and all $x_i < x_i'$, the distribution of X_{-i} conditional on $X_i = x_i'$ stochastically dominates the distribution of X_{-i} conditional on $X_i = x_i$.

Then for any profile s such that for each i, the signal s_i is most dependent on s_{-i} in S_i, there exists an increasing action strategy profile α such that (s, α) is a Nash-Bayesian equilibrium for the game.

Proof. Let Γ_s denote the game with exogenous information structure s, and Γ the game with endogenous information structure. The main result in Van Zandt and Vives (2007) imply that there exists an increasing action strategy profile α such that in the game Γ_s, the profile α is a Nash-Bayesian equilibrium of Γ_s. Let α be such a profile. We will now show that the profile (s, α) is a Nash-Bayesian equilibrium of the game with endogenous information structure Γ.

Suppose by contradiction that (s'_i, α'_i) is a profitable deviation for player i. Let α''_i be a player i's best response to α_{-i} under the information structure (s'_i, s_{-i}) Then by Proposition 11 in Van Zandt and Vives (2007), the action strategy α''_i is increasing. Since (s'_i, α'_i) is a profitable deviation for player i from profile (s, α), therefore (s'_i, α''_i) is also a profitable deviation for player i from profile (s, α). But, because s_i has weakly greater dependence on s_i than s'_i, the same argument used in Theorem 3 implies that (s_i, α''_i) is an even (weakly) better profitable deviation, which contradicts the definition of profile (s, α), as a Nash-Bayesian equilibrium of the game Γ_s with exogenous information structure s. Therefore no player has any profitable deviation, the desired conclusion.■

The following result is a direct implication of Theorem 4.

Corollary 4. *Let $N \geq 2$. Suppose that conditions (i) to (iv) of Theorem 4 hold. In addition suppose that the set $\bigcap_{i \in I} S_i$ is nonempty, i.e. that public information is feasible. Then for any public information signal profile, there exists an increasing action strategy profile α such that (s, α) is a Nash-Bayesian equilibrium for the game Γ.*

8 Mixed strategies

The results obtained in Theorem 3 generalize to mixed strategies, but they imply very few restrictions for Nash-Bayesian equilibria where players play non degenerate mixed strategies. For example, consider a game with two players and two signals, with a fixed information structure such that both players observe each of the two signals with equal probabilities (independent draws). Suppose that this game admits a pure Nash-Bayesian equilibrium in action strategies (they could be pure or not).

Then it is easy to see that the game with an endogenous information structure admits a Nash-Bayesian equilibrium, where both players randomize with equal probabilities between the two signals. To see this, suppose that player 2 uses this strategy. From the point of view of the player 1, the two signals are then equally informative in a Blackwell sense on the vector (θ, a_2), which is all he cares about. It is then a best response for him to play this half half mixed strategy and the same argument holds for player 2. This phenomenon is more general. A symmetric fully mixed equilibrium exists, for any number of players, if and only if the Bayesian game where this structure is fixed admits a Nash-Bayesian equilibrium. What is important for the result is that there are only two signals in S. A more general result can be obtained for a larger number of signals in S, provided that some symmetry condition, which automatically holds in the case of two signals, is imposed on the signal structure.

Theorem 5. *Let $N \geq 2$ and $S = \{a, b\}$ Consider the game with an exogenous information structure, where each player observes a or b with probability $1/2$ (independent draws across players). Suppose that this game admits a pure Nash-Bayesian equilibrium in action strategies (pure or not). Then this action profile and this information structure form a Nash-Bayesian equilibrium of the game Γ where the information structure is endogenous.*

9 Ex ante Pareto-inefficiency of equilibrium information structures

In this Section, we show that the players' equilibrium signal choices sometimes differ from what a planner would design. We suppose that the planner may choose the information structure (within the same set available to the players), but the players still choose their actions non-cooperatively in the second stage. We show that in the absence of a planner, pure Nash equilibria need not be ex ante Pareto-efficient. More precisely, we provide an example with two players that has the property that, in all pure Nash-Bayesian equilibria, the players choose an ex ante Pareto-dominated information structure.

Consider a duopoly where the firms produce a differentiated product and compete in Bertrand.[7] Each of the two firms $i = 1, 2$ sets a price a_i for its product and faces the linear demand $Q_i = b_0 + b_1\theta - \gamma(a_i - a_j)$. Each firm i has convex costs $C_i = c_0 Q_i + c_1 Q_i^2$. The intercept θ of the demand function is the unknown state.

Firm i's profit is then

$$\begin{aligned} u_i(a, \theta) &= Q_i[a_i - C_i] \\ &= [b_0 + b_1\theta - \gamma(a_i - a_j)][a_i - c_0 - c_1(b_0 + b_1\theta - \gamma(a_i - a_j))]. \end{aligned} \tag{33}$$

The payoff in (33) is equivalent to our general payoff, that is

$$u_i(\theta, a) = -a_i^2 + 2b_{12}a_i a_j + 2b_{13}a_i\theta + K(a_j, \theta) \tag{34}$$

where $K(a_j, \theta) = -b_{22}a_j^2 + 2b_{23}a_j\theta - b_{33}\theta^2$ and

$$b_{12} = \tfrac{1+2c_1\gamma}{2+2c_1\gamma}, \quad b_{22} = \tfrac{c_1\gamma}{1+c_1\gamma}, \quad b_{13} = \tfrac{b_1(1+2c_1\gamma)}{2\gamma(1+c_1\gamma)}, \quad b_{23} = -\tfrac{c_1 b_1}{1+c_1\gamma}, \quad b_{33} = \tfrac{c_1 b_1^2}{\gamma(1+c_1\gamma)}.$$

[7] This example comes from Jimenez (2013).

Note that b_{22}, b_{23} and b_{33} do not affect the Nash Bayesian equilibrium in the second-stage. As pricing decisions are complements, the payoff function in (34) is supermodular ($b_{12} > 0$) in actions.

Suppose that 2 signals are available, that is $S = \{X, Y\}$, and that the random vector (θ, X, Y) is a Gaussian vector with expectation $(0, 0, 0)$ and covariance matrix

$$\sigma = \begin{pmatrix} \sigma_{\theta\theta} & \sigma_{\theta X} & \sigma_{\theta Y} \\ \sigma_{\theta X} & \sigma_{XX} & \sigma_{XY} \\ \sigma_{\theta Y} & \sigma_{XY} & \sigma_{YY} \end{pmatrix}$$

with $\sigma_{XX} = \sigma_{YY}$ and $\sigma_{\theta X} = \sigma_{\theta Y}$, so that (θ, X) and (θ, Y) have identical joint marginal distributions. This information structure is identical to the one in Section 4.2.

We now show that when the firms do not observe the information structure, then $(X_1^*, X_2^*) \in \{(X, X), (Y, Y)\}$, but this might no longer be true if the firm can observe the information structure.

First, from Section 4.2 we know that in the case of a symmetric payoff function, the equilibrium in the second-stage is of the form $w_i^* x_i + \kappa_i^*$ and that it is always strictly monotonic, i.e. $w_i^* w_j^* > 0$. From Section 4.2 we know also that deviating from the profile of signal choice affect a player's payoff only up to the covariance between X_1 and X_2. Moreover, we show that, when $b_{12} > 0$ increasing the dependence is a profitable deviation, but that decreasing the dependence is not. Therefore, in equilibrium, in the first-stage, we will have $(X_1^*, X_2^*) \in \{(X, X), (Y, Y)\}$.

Suppose now we assume the players can observe the information structure and in particular, that they see when a player deviates. Then, if player i changes the dependence between the signals, then player j detects the deviation, acknowledges the resulting change in the dependence and changes his action consequently through a change in w_j^*. We say that a deviation by player i triggers a reaction and hence induces a strategic effect.

For instance suppose that the profile of signal choice is (X, X), then

$$w_i^*(X, X) = w_j^*(X, X) = \frac{b_{13}\sigma_{\theta X}}{\sigma_{XX}(1 - b_{12})}, \tag{35}$$

and player i's ex-ante expected payoff is

$$\frac{b_{13}(b_{13}(1 - b_{22}) + 2(1 - b_{12})b_{23})\sigma_{\theta X}^2}{(1 - b_{12})^2 \sigma_{XX}}. \tag{36}$$

For a deviation by player i to the profile (Y, X), then

$$w_i^*(Y, X) = w_j^*(Y, X) = \frac{b_{13}\sigma_{\theta X}}{\sigma_{XY}(1 - b_{12})}, \tag{37}$$

and player i's ex-ante expected payoff is ex-ante expected payoff is

$$\frac{b_{13}(b_{13}(1-b_{22})\sigma_{XX}+2(\sigma_{XX}-b_{12}\sigma_{XY})b_{23})\sigma_{\theta X}^2}{(\sigma_{XX}-b_{12}\sigma_{XY})^2}. \tag{38}$$

The difference between (36) and (38) is

$$\frac{b_{12}b_{13}(\sigma_{XX}-\sigma_{XY})(2(1-b_{12})b_{23}(\sigma_{XX}-b_{12}\sigma_{XY})-b_{13}(1-b_{22})((b_{12}-2)\sigma_{XX}+b_{12}\sigma_{XY}))\sigma_{\theta X}^2}{(1-b_{12})^2\sigma_{XX}(\sigma_{XX}-b_{12}\sigma_{XY})^2}. \tag{39}$$

Whenever, (39) is negative, then decreasing the dependence will be a profitable deviation for player i. For instance, if $b_{12}>0, b_{13}>0$ and $b_{23}<0$, as it is the case in the duopoly example, then it is possible that (39) be negative. Table 1 shows the sign for Equation (39) for different parameters of the initial duopoly game γ, c_1 and b_1.

	b_{12}	b_{13}	b_{23}	b_{22}	Sign of (39)
i. $\gamma=3$, $c_1=0.1$, $b_1=0.8$	0.62	0.08	-0.06	0.23	+
ii. $\gamma=3$, $c_1=0.5$, $b_1=0.8$	0.8	0.11	-0.16	0.6	+
iii. $\gamma=3$, $c_1=0.1$, $b_1=2$	0.62	0.21	-0.15	0.23	+
iv. $\gamma=5$, $c_1=0.5$, $b_1=0.8$	0.86	0.07	-0.11	0.71	-

Table 1: Bertrand Competition, $\sigma_{XX}=1, \sigma_{XY}=0.75, \sigma_{\theta X}=0.5$

Hence, we can conclude that in a Bertrand environment, the equilibrium is sensitive to the parameters and to the assumption on the observability of information structure. Our example proves the following result.

Theorem 6. *In general, the Nash-Bayesian equilibria of* Γ *need not be ex ante Pareto-efficient. There exist examples were all Nash-Bayesian equilibria of* Γ *are ex ante Pareto-inefficient.*

This result suggests that policy intervention is sometimes desirable in markets for information. In a decentralized system, agents may choose either too similar or too dissimilar information, and policy intervention can help mitigating this type of inefficiency.

10 Observable signal choices

In the main model, we made the assumption that signal choices of the first stage are not observed by the players. This is important, since it implies that a deviation from equilibrium play does not affect the other player's action choices in the second stage: the choice of signal and actions are strategically simultaneous. One can imagine situations where signal choices are observable. For example a company may sign a contract with a market research firms and this may be observable by all other companies.

Interestingly, this difference can have important effects. To see this, consider the case of two symmetric players. In this case, both players in stage 1 face the problem of the planner, which we analyze in Section 9. As we showed, the planner's solution may be disjoint from the set of Nash equilibria of the game where signal choices are unobservable. Thus, we can deduce the following result.

Theorem 7. *In general, the set of Nash-Bayesian equilibria of the game* Γ^* *where signals are observable is not equal to the set of Nash-Bayesian equilibria of the game* Γ *where signals are not observable. There exist example where the two sets are disjoint. If there are two players and the game is symmetric, the second set coincides with the set of ex ante Pareto efficient profiles of both games.*

This result suggests that an intervention that mandates players to publicly disclose their sources of information may sometimes be desirable, in that it could help to mitigate excessive information similarity or dissimilarity that may result from a decentralized market for information.

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