Rivalry and Professional Network Formation: The Struggle for Access

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Abstract

We develop a network formation game where principals (e.g., partners at a consulting firm) employ agents (e.g., consultants) from their professional networks to help them complete projects. Since agents only work for one principal at a time, the principal's use of agents is rivalrous. We establish that there's a (pure strategy) equilibrium and we characterize how this rivalry influences equilibrium network structure as well as the principals' welfare. We find, for instance, that the principals always hold "minimally overlapping" networks and that the principals' equilibrium interests are opposed – in an equilibrium where one does best, the other does worst.

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1 Introduction

Professional networks play a key, internal role at Deloitte Consulting:[1] they are the primary means by which partners obtain consultants to help them complete their projects. Deloitte hires consultants into a pool – any partner may request the help of a consultant and each consultant is expected to provide help, unless she is currently working on another project. When a partner gets a project, she must decide which consultants in this pool to employ. Since consulting usually requires close collaboration, a partner usually to employs consultants that she likes and has built a good, productive working relationship with, i.e., a partner usually employs consultants from her professional network.

Once a consultant takes on a partner's project, she's usually unavailable to help other partners complete their projects – projects typically involve daylong meetings and other activities at clients' offices. Thus, the partners' employment of consultants is rivalrous (at least in the short term), implying the partners may exert a negative externality on each other. To illustrate, consider two partners A and B. If A employs a large number of consultants who are part of her network and part of B's network, then B's pool of available, in-network consultants is smaller. This reduces B's ability to complete certain projects, and diminishes her earnings – at many consulting firms, including Deloitte, a partner's pay is partially based on the revenue she brings in from her completed projects and her expenses, including the cost of the consultants she employs on the projects she undertakes. In contrast, A is in a better position to complete her project and do well.

Since partners choose their professional networks, our goal is to understand how this rivalry shapes their network formation decisions and, ultimately, their welfare. To these ends, we build a stylized, two-stage game of network formation and rivalry.[2] In our game, there are two partners, A and B, as well as a finite number of consultants, indexed 1 to N. In the first stage, both partners form their professional networks. It's costly for a partner to include a consultant in her network as she must invest effort (and money) to develop a good, productive working relationship with the consultant. For instance, she often needs to invest effort to mentor the consultant in her production techniques/technology so that the consultant may be a capable assistant.[3]

In the second stage, the partners sequentially receive projects to complete. The projects

[1] My thanks to a West-Coast based Deloitte consultant for conversations about the inner workings of the firm in the fall of 2013.

[2] We wish to emphasize that, while our game is inspired by Deloitte (and other consulting firms), it is not a complete model of how these firms work.

[3] The consultant will also invest costly effort to build this working relationship. For simplicity, we'll assume that the firm pays the consultant enough to cover such costs so that the consultants are happy to learn.

and order in which they are awarded are determined by nature. Both projects are received in a short succession, so the first is not completed before the second is received. Projects are of heterogeneous difficulty and each requires a different number of consultants to complete – some, like an evaluation of a small call-center's effectiveness may only require one or two consultants, while others, like an efficiency review of a complex supply chain, may require a dozen or more consultants.

Once a partner receives her project, she employs consultants in her network to help her complete it. She may employ any subset of in-network consultants she wants (including the empty set), save that she cannot employ consultants who are already working on an ongoing project. If the partner manages to employ enough consultants, then she completes her project and earns a positive reward, from which she pays her labor and networking costs. If a partner does not employ enough consultants, she fails to complete the project and earns nothing, but is still liable for her labor and networking costs, if any.

For simplicity, we don't model consultants' behavior. Instead, we assume that the consultants (i) always agree to be in a partner's network and (ii) always agree to be employed on a partner's project, provided they aren't working on another project. These assumptions reflect the expectations consulting companies have of their employees.

Our solution concept is a pure strategy Nash equilibrium.[4] We first establish that an equilibrium exists (Proposition 1). The argument is non-trivial as we must simultaneously consider selecting the best network and the best employment strategies for both partners. Fortunately, there are "simple strategies." Partner A's (B's) simple strategy of size n tells her to network with the n lowest (highest) indexed consultants and to employ lower (higher) indexed consultants before higher (lower) indexed consultants. When the partners follow simple strategies, their payoffs are submodular in their networks' sizes (i.e., the sizes of their simple strategies) because of the rivalrous use of consultants.

We construct an Auxiliary Game where the partners use simple strategies, A chooses her network's size, and B chooses the "negative" of her network's size. Since the order of B's choice is inverted, the Auxiliary Game is a two-player supermodular game and so has a pure strategy Nash equilibrium. Subsequently, we show that each equilibrium of the Auxiliary Game induces an equilibrium of the full game. We call these induced equilibria "simple equilibria."

In equilibrium, we find that both partners' networks are "minimally overlapping" (Proposition 2). That is, the partners try to have networks that share as few consultants as possible. The intuition is that by networking with different sets of consultants, the partners minimize

[4] We focus on pure strategy equilibrium because typical refinements, like subgame perfection, add little economic insight – see Section 2.

the chance that their access to needed consultants will be blocked. An implication of Proposition 2 is that consultants are often in only one partner's network. This prediction finds qualitative support at Deloitte where consultants usually work with a small group of partners, which implies that the partners' networks don't overlap too much.

Two properties play a key role in our subsequent analysis: "employment lists" and "employment efficiency." An equilibrium has employment lists if the first partner who gets a project always uses a ranked list to determine the identities of the consultants she employs: if she employs l consultants, then she employs the first l consultants on her list. This property captures the intuitive idea that each partner uses an address book or other (mental) list. An equilibrium is employment efficient if the first partner to get a project always employs consultants who are exclusively in her network before she employs consultants who are in both her network and the other partner's network. This ensures that the second partner to get a project has a larger pool of available, in-network consultants than she would if the first partner employed consultants in an arbitrary manner. Thus, the second partner is better able to complete projects and has higher earnings.

We focus on equilibria that have employment lists and are employment efficient because these properties are intuitive, especially as both partners work at the same firm, and these equilibria are "robust" – see Section 4. We call these equilibria "ELEE equilibria." Since simple equilibria are ELEE equilibria (Lemma 5), the existence of ELEE equilibria is assured. We then establish that each ELEE equilibrium is payoff equivalent to a simple equilibrium (Proposition 3). The intuition is that, when employment lists and employment efficiency hold, then the equilibrium behavior of the partners is analogous to their behavior in a permutation of a simple equilibrium.

We next establish that the partners' equilibrium interests are opposed. That is, there's an ELEE equilibrium where A does best and B does worst and vice versa (Proposition 4). The intuition for these results is that rivalry (i) causes the Auxiliary Game to be supermodular and (ii) ensures that a partner's optimal payoff is always *weakly* decreasing in the size of the other partner's network. Since (i), there's a there's a maximal (minimal) equilibrium where A holds her largest (smallest) equilibrium network and B holds her smallest (largest) equilibrium network. Since (ii), the maximal (minimal) equilibrium is most (least) preferred by A and least (most) preferred by B. The desired result then follows from Proposition 3. We also establish that if each partner's optimal payoff is *strictly* decreasing in the size of the other partner's network, then A does best in any equilibrium where B does worst and vice versa (Proposition 4). The insight is that strict monotonicity ensures that the maximal (minimal) equilibrium is *the* best (worst) one for A and *the* worst (best) one for B.

We also develop welfare comparative statics for the ELEE equilibria where the partners' interests are opposed. For concreteness, we suppose that A's reward to completing a project weakly increases and that her cost of networking weakly decreases, while B's reward and cost are held constant. We then compare A and B's payoffs and network sizes before and after the shift in equilibria where their interests are opposed. We show that A does better and holds a larger network, while B does worse and holds a smaller network (Proposition 5). This result follow naturally from Proposition 3 and weak monotonicity as the changes in A's reward and cost cause her best response in the Auxiliary Game to increase.

In light of the opposition of interests, it's natural to wonder if one partner actually does better than the other. We establish that A earns more and has a larger network than B in any ELEE equilibria where she does best and B does worst, provided (i) A and B have the same chance of getting a project, (ii) A has a weakly lower networking cost than B, and (iii) A receives weakly more for completing a project than B (Proposition 6). This result follows naturally from Proposition 5 and the insight that, when A has B's reward and cost functions, then she holds a larger network and does weakly better than B in the maximal equilibrium of the Auxiliary Game.

We next turn our attention to efficiency. Unfortunately, there need not be an (ex-ante) efficient equilibrium; we show this via an example. In the example, we establish that one partner always has a strict incentive to defect from the unique efficient (joint) strategy by over-investing in her network, i.e., by holding a larger network than is socially optimal. This over-investment increases the total networking cost and generates a large negative externality on the other partner. As a result, it decreases social welfare and implies that the efficient strategy isn't an equilibrium. This leads us to investigate why an efficient strategy wouldn't be an equilibrium. We find, at least for efficient simple strategies, that the reason is over-investment (Proposition 7). This result follows from weak monotonicity and the fact that simple strategies are best replies to simple strategies.

We conclude by examining the effects of salesmanship on the partners' equilibrium networks and welfare. One partner, say A, may be more skilled at selling projects and, as a result, may obtain projects more frequently. We model this by allowing A to move first in the second stage with greater probability than B. We find that increases in the probability that A moves first increase A's payoff and network size and decrease B's payoff and network size in ELEE equilibria where the partners' interests are opposed (Proposition 8). The intuition is that it's better to move first in the second stage because then one is not subject to less rivalry. Hence, as the probability A moves first increases, A's and B's best responses in the Auxiliary Game increase. From this fact, one may apply Proposition 3 and monotonicity to deduce Proposition 8.

Related Literature

Our work makes contributions to two literatures. The first literature on trading-on-networks examines how buyers and sellers come together and trade single items – e.g., Kranton and Minehart [11] and Condorelli and Galeotti [4]. We extend this literature by allowing partners to "buy" labor from multiple consultants and examining how the partners' rivalrous use of consultants affects their networks and welfare. The second literature on multiple common pool resources examines how rivalry affects players' consumption of multiple natural resources – e.g., Ilkilic [8]. In this literature, players choose whether to consume a resource to which they have access. We enrich this literature by also allowing players/partners to chooses which resources/consultants they may access.

Kranton and Minehart [11] initiated the trading-on-networks literature with their two-stage game between unit-demand buyers and unit-supply sellers. In the first stage, buyers make costly links/investments with specific sellers that allow them to trade in the second stage. In the second stage, the sellers conduct ascending price auctions with the buyers to whom they're linked. Since players' payoffs are quasi-linear in prices, Kranton and Minehart are able to establish the existence of a pure strategy, efficient equilibrium.[5]

In a related paper, Condorelli and Galeotti [4] develop a two-stage game to examine trade through intermediaries. In the first stage, traders form the links over which trade occurs. In the second stage, one trader is endowed with a single object and traders engage in resell (according to a specified procedure) until the item reaches a trader who would rather keep it than resell it. Each trader has unit demand and quasi-linear preferences. Condorelli and Galeotti find that an equilibrium is generally inefficient because traders from fewer links than is socially optimal because traders fail to internalize the impact an additional link has on the probability the object finds its way to the trader who desires it most.

Our work is complementary to these papers. Like them, we consider a game where players first form links then "transact" over these links. However, we allow partners to tap multiple consultants, we examine the effects of rivalry, and we focus on non-market environments (so there are no "terms of trade" in our game). These differences are economically meaningful. For instance, the papers mentioned above find that equilibria are efficient or are inefficient only due to under-investment; in contrast, we find that equilibria may be inefficient *only* because of over-investment.

The trading-on-networks literature is a subset of the network formation literature. This extensive literature examines the considerations that shape social and economic networks and the effects that these networks have on socio-economic behavior.[6] Our contribution to

[5]In a related work, Corominas-Bosch [6] examines an alternating-offer bargaining game between buyers and sellers and characterizes how the exogenously given topology of the network affects the equilibrium terms of trade.

[6]See Granovetter [7] and Jackson [9] for overviews and discussion of the role of networks and network

this literature is to explore how rivalry shapes networks and welfare. Also falling within this literature is Jackson and Wolinsky's [10] Co-Author Game. In this game, a group of players forms links with each other to co-author papers. Each player benefits from having a link/co-author but is harmed when any of her co-authors have links to other players as these co-authors limited time is divided across more projects. Thus, there's a rivalry for players' time. We differ in the nature of the rivalry we explore and this difference has substantive implications for network structure. For instance, Jackson and Wolinsky find that players form disjoint, fully interconnected clusters, implying players' networks (i.e., neighborhoods) are either entirely overlapping or disjoint. This contrasts with our minimally overlapping result.

A common pool resource is one with rivalrous consumption. In their survey, Ostrom et al. [13] give several examples including ground water (as one city's extraction diminishes the amount of water left for other cities), fisheries, and grazing (i.e., the commons problem).[7] Ilkilic [8] considers a game where cities have different degrees of access to water sources (e.g., cities A and B share an aquifer 1, while cities B, C, and D share aquifer 2) and decide how much water to extract from each source they can access to. He takes the map between cities and water sources as given and characterizes the equilibrium extraction from each water source when cities move simultaneously and have quasi-linear payoffs. He finds that sources that are more "central" are more heavily used and are often over-depleted relative to the social optimum. Our work is complementary because we allow cities to choose the water sources they have access to; though we constrain cities to all or nothing use of their sources so our game doesn't nest Ilkilic's game.

More broadly, our work is related to the team formation literature – e.g., Bolle [3] and Lappas et al. [12]. In these models, a principal seeks to hire heterogeneously skilled agents to help her complete a project. In Bolle's early model, these agents are assumed to work equally well with each other. Lappas et al. enrich Bolle's framework by (i) allowing certain agents to be friends and work well with each other and (ii) allowing other agents to enemies and work poorly with each other. While team formation plays a key role in our environment, we also include a rivalry between principals that is absent in both of these papers.

Our work is also related to the cooperative matching with externalities literature – e.g., Bando [2] and Pycia and Yenmez [14]. In these models, a worker may take jobs with multiple firms and each firm may hire multiple workers. Each player's payoff depends on the identities

formation in economics.

[7] Rivalrous consumption of a resource is also a concern in the club goods literature, where it's known as "congestion" – see Cornes and Sandler [5] for an overview. However, this literature usually leaves the processes behind congestion un-modeled – e.g., Cornes and Sandler frequently regard the "rate of congestion" as exogenous.

of everyone's partners and, thus, allows for many kinds of externalities. Both papers use the pairwise-stable set as a solution concept and give different conditions to guarantee that this set is non-empty and has various welfare properties. We complement this literature by considering an externality these models cannot capture: rivalry. In general, rivalry depends not only on the map between partners and consultants, but also on the partners decisions about which consultants to employ and, thus, on their projects and the order in which they get them. For a given map and given employment strategies, low difficulty projects may lead to no or little rivalry, while very difficult projects may cause significant negative externalities.

2 The Game

This section describes our environment, our solution concept, and gives an example.

ENVIRONMENT

We consider a two-stage game between two partners, A and B. There is a finite set of consultants $\mathcal{C} = \{1, \ldots, N\}$, where $N \geq 1$. Let i denote an arbitrary partner. For simplicity, we don't model consultant behavior. Instead, we assume that all consultants (i) always agree to be in a partner's network and (ii) always agree to be employed on a partner's project, provided they aren't working on another project.

In the first stage, both partners simultaneously form professional networks. That is, they pick subsets of $\mathcal{C}$ with whom to build friendly and productive working relationships. Let $\mathcal{N}_i \subset \mathcal{C}$ denote partner i's selection.[8] It's costly for i to form a network as she must invest effort (and money) to develop a good, productive working relationship with each consultant in her network. Let $c_i : \mathbb{N} \to \mathbb{R}_+$ such that $c_i(0) = 0$. Partner i's cost of holding the network $\mathcal{N} \subset \mathcal{C}$ is given by $c_i(|\mathcal{N}|)$, where $|\cdot|$ denotes the cardinality of a set and $|\emptyset| = 0$.[9]

In the second stage, the partners sequentially get their projects and employ consultants. Initially no consultant is employed by either partner. At the start of the stage, nature picks a partner to get her project first – both partners have a $1/2$ chance of getting their project first. Once the identity of the first partner is realized, nature draws a project for her. Then this partner employs consultants from her network. Subsequently, nature draws a project for the second partner. Then the second partner employs consultants from her network who aren't already employed by the first partner. Both projects are received in short succession

[8]In the language of the network formation literature (e.g., Bala and Goyal [1]), the partners form "directed links" to consultants.

[9]Sometimes, we'll assume that partner i has a constant marginal cost of networking $\kappa_i \geq 0$, so $c_i(|\mathcal{N}|) = \kappa_i |\mathcal{N}|$. This constant marginal cost assumption is common in the literature – e.g., Bala and Goyal [1] and Jackson [9].

(e.g., on the same day), so the first is not completed by the time the second is awarded. The stage ends with both partners receiving their payoffs.

Let X denote the non-empty and finite set of projects that either partner may receive. For, each partner i, let $P_i : X \to [0, 1]$ such that $\sum_{x \in X} P_i(x) = 1$ be the probability that nature draws project $x \in X$ for i. These draws are independent across partners and the order in which they move in the second stage. Let $d : X \to \mathbb{N}_{++}$ give the difficulty of a project, i.e., the minimum number of consultants needed to complete it – some projects are more complicated than other and require more help. Notice that every project requires at least one consultant.

For each partner i, let $r_i : X \to \mathbb{R}_+$ give i's reward to completing the project. Thus, if partner i gets project x, she earns $r_i(x)$ if she employs at least $d(x)$ consultants and she earns 0 if she employs less than $d(x)$ consultants. This production technology reflects two facts. First, clients only pay for completed projects – there is no residual value to a project that's only halfway done.[10] Second, most consultants are of high ability – e.g., Deloitte usually hires college graduates who are in the top of their classes. Hence, in the abstract, in-network consultants are homogenous from a partner's perspective: the partner can ask any consultant with whom she has a good, productive working relationship for help and trust that the consultant will be a capable assistant.

When partner i gets her project first, she observes the network she chose in the first stage $\mathcal{N}_i$ and her project x_i. Subsequently, she decides which in-network consultants to employ, if any. (Note that i may always choose to forgo a project by not employing any consultants.) A behavioral strategy for i is a $\sigma_{1i} : \mathbb{P}(\mathcal{C}) \times X \to \mathbb{P}(\mathcal{C})$ such that $\sigma_{1i}(\mathcal{N}, x) \subset \mathcal{N}$ for all $(\mathcal{N}, x) \in \mathbb{P}(\mathcal{C}) \times X$. Thus, i's strategy σ_{1i} takes her observations $(\mathcal{N}_i, x_i)$ and returns a (possibly empty) list of consultants $\sigma_{1i}(\mathcal{N}_i, x_i)$ in $\mathcal{N}_i$ who she employs. Partner i pays an exogenously fixed and finite amount $w \geq 0$ for each consultant she employs – at many consulting firms a partner's compensation is linked to the cost of the consultants she employs in the projects she attempts.[11] Thus, her total labor cost is $w\,|\sigma_{1i}(\mathcal{N}_i, x_i)|$.[12]

When partner i gets her project second, she observes her network $\mathcal{N}_i$, the set of consultants employed by the other partner $\mathcal{T}$, and her project x_i. Since the consultants in $\mathcal{T}$ are

[10] In the Supplement, we examine an alternative production technology with residual value. Our results are robust to this extension.

[11] We wish to emphasize that w is not actually a consultant's wage, but rather a way of accounting for the cost a partner faces for the use of a consultant. That w is the same for both partners reflects the fact that the firm's cost of a consultant, i.e., her salary, benefits, office space, etc., depend on conditions in the labor market and in other input markets. We relax this assumption in the Supplement and argue that our results are robust.

[12] Assuming a constant marginal cost of labor is without loss; our results continue to hold for any total labor cost function that is increasing in the number of consultants employed.

part of an ongoing project, they are unavailable, i.e., i cannot employ them. Subsequently, i decides which available in-network consultants to employ, if any. A behavioral strategy for i is a $\sigma_{2i} : \mathbb{P}(\mathcal{C})^2 \times X \to \mathbb{P}(\mathcal{C})$ such that $\sigma_{2i}(\mathcal{N}, \mathcal{N}', x) \subset \mathcal{N} \backslash \mathcal{N}'$ for all $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$.[13] Thus, i's strategy σ_{2i} takes her observations $(\mathcal{N}_i, \mathcal{T}, x_i)$ and returns a list of consultants $\sigma_{2i}(\mathcal{N}_i, \mathcal{T}, x_i)$ in $\mathcal{N}_i \backslash \mathcal{T}$ who she employs. As before, i has to pay w for each consultant she employs, so her total labor cost is $w\,|\sigma_{2i}(\mathcal{N}_i, \mathcal{T}, x_i)|$.

A strategy for partner i is $\boldsymbol{s}_i = (\mathcal{N}_i, \sigma_{1i}, \sigma_{2i})$, i.e., is a complete specification of her network selection and her two behavioral strategies. Let $\boldsymbol{S}_i$ denote i's finite set of all possible strategies. Let $\boldsymbol{s} = (\boldsymbol{s}_A, \boldsymbol{s}_B)$ denote a vector of strategies denote the vector of the partners' strategies, and let $\boldsymbol{S} = \boldsymbol{S}_A \times \boldsymbol{S}_B$ be the joint strategy space.

Each partner's (ex-post) payoff is her reward less her labor and networking costs. To fix ideas, let i be a partner and let $-i$ be the other partner. Let $\boldsymbol{s} = (\mathcal{N}_i, \sigma_{1i}, \sigma_{2i}, N_{-i}, \sigma_{1-i}, \sigma_{2-i}) \in \boldsymbol{S}$ and let x_i and x_{-i} be their projects. Suppose both partners follow $\boldsymbol{s}$. Then i's ex-post payoff when she gets her project first is

$$u_{1i}(\boldsymbol{s}, x_i, x_{-i}) = r_i(x_i)\mathbb{I}(|\sigma_{1i}(\mathcal{N}_i, x_i)| \geq d(x_i)) - w\,|\sigma_{1i}(\mathcal{N}_i, x_i)| - c_i(|\mathcal{N}_i|),$$

where $\mathbb{I}(\cdot)$ is an indicator function that's equal to one if $|\sigma_{1i}(\mathcal{N}_i, x_i)| \geq d(x_i)$, i.e., if i employs enough consultants to complete her project, and is equal to zero else. When i gets her project second, $-i$ employs $\sigma_{1-i}(\mathcal{N}_{-i}, x_{-i})$. Thus, i's ex-post payoff is

$$\begin{aligned} u_{2i}(\boldsymbol{s}, x_i, x_{-i}) = {} & r_i(x_i)\mathbb{I}(|\sigma_{2i}(\mathcal{N}_i, \sigma_{1-i}(\mathcal{N}_{-i}, x_{-i}), x_i)| \geq d(x_i)) \\ & - w\,|\sigma_{2i}(\mathcal{N}_i, \sigma_{1-i}(\mathcal{N}_{-i}, x_{-i}), x_i)| - c_i(|\mathcal{N}_i|), \end{aligned}$$

where $\mathbb{I}(\cdot)$ is an indicator function that's equal to one if $|\sigma_{2i}(\mathcal{N}_i, \sigma_{1-i}(\mathcal{N}_{-i}, x_{-i}), x_i)| \geq d(x_i)$, i.e., if i employs enough consultants to complete her project after seeing $-i$ employ $\sigma_{1-i}(\mathcal{N}_{-i}, x_{-i})$, and is equal to zero else.

Two observations are in order. First, once i gets her project her ex-post payoff only depends on the size of her network and the number of consultant that she employs. Second, i may always get a payoff of zero by choosing the empty network.

We assume that players have expected utility, so partner i's ex-ante payoff is

$$U_i(\boldsymbol{s}) = \sum_{(x_i, x_{-i}) \in X^2} (u_{1i}(\boldsymbol{s}, x_i, x_{-i}) + u_{2i}(\boldsymbol{s}, x_i, x_{-i}))P_i(x_i)P_{-i}(x_{-i})/2.$$

We'll occasionally write $U_i(\boldsymbol{s})$ as $U_i(\boldsymbol{s}, r_i, c_i)$ when we wish to emphasize the dependence of

[13] To avoid extensive notation, we define $\emptyset \backslash \emptyset = \emptyset$.

i's ex-ante payoff on her reward and cost functions.

Throughout the rest of the paper, we maintain the following simplifying assumption.

Assumption 1. Project Completion is (Weakly) Good.
For all $x \in X$, we have $r_i(x) - w\, d(x) \geq 0$ for each partner i.

The assumption guarantees that both partners have a non-negative payoff to employing the number of consultants that are needed to complete a project. While this simplifies the statements and proofs of our results, it's not essential to them.

Solution Concept

Our solution concept is a pure strategy Nash equilibrium.

Definition. An *equilibrium* is a (pure) strategy vector $\boldsymbol{s}^\star = (\boldsymbol{s}^\star_A, \boldsymbol{s}^\star_B)$ such that

$$U_A(\boldsymbol{s}^\star) \geq U_A(\boldsymbol{s}_A, \boldsymbol{s}^\star_B) \text{ for all } \boldsymbol{s}_A \in \boldsymbol{S}_A \text{ and}$$
$$U_B(\boldsymbol{s}^\star) \geq U_B(\boldsymbol{s}^\star_A, \boldsymbol{s}_B) \text{ for all } \boldsymbol{s}_B \in \boldsymbol{S}_B.$$

Let $\boldsymbol{E}$ denote the set of equilibria.

Notice that when $\boldsymbol{E}$ is non-empty, then it's finite and is usually non-singular because (i) the definition of equilibrium places little restriction on what can happen off of the equilibrium path and (ii) the consultants' identities may always be permuted. We focus on (pure strategy) equilibria because typical refinements, like subgame perfection, add no economic insight. The reason is simple: after she selects her network, a partner never learns of the network selected by her rival. Thus, the only subgame is the game as a whole and so the sets of subgame perfect equilibria and pure strategy equilibria coincide. That said, we show in the Supplement that there is an equilibrium where both partners always (i.e., both on-path and off-path) behave optimally in the second stage and so don't make "non-credible threats."

An Example

It's useful to work an example.

Example 1. A Simple Symmetric Example.

Let $\mathcal{C} = \{1, 2\}$ be the set of consultants and let $X = \{x_1, x_2\}$ be the set of projects. Let $d(x_1) = 1$ and $d(x_2) = 2$ be the difficulties of the two projects. Let $r_i(x_1) = 2$ and $r_i(x_2) = 5$ for $i \in \{A, B\}$ be the rewards to both projects. Let $w = 1$ and suppose both partners have a constant marginal networking cost of $1/2$. Also, let both projects be equally likely, i.e., $P_i(x_1) = P_i(x_2) = 1/2$ for $i \in \{A, B\}$.

An equilibrium is $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star, \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star)$, where $\mathcal{N}_A^\star = \mathcal{N}_B^\star = \{1,2\}$,

$$\sigma_{1A}^\star(\mathcal{N}, x) = \begin{cases} \{1\} & \text{if } \mathcal{N} = \mathcal{N}_A^\star \text{ and } x = x_1 \\ \{1,2\} & \text{if } \mathcal{N} = \mathcal{N}_A^\star \text{ and } x = x_2 \\ \emptyset & \text{else} \end{cases} \text{ and } \sigma_{1B}^\star(\mathcal{N}, x) = \begin{cases} \{2\} & \text{if } \mathcal{N} = \mathcal{N}_B^\star \text{ and } x = x_1 \\ \{1,2\} & \text{if } \mathcal{N} = \mathcal{N}_B^\star \text{ and } x = x_2 \\ \emptyset & \text{else,} \end{cases}$$

$$\sigma_{2A}^\star(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{1\} & \text{if } \mathcal{N} = N_A^\star,\ x = x_1, \text{ and } \mathcal{N}' = \{2\} \\ \emptyset & \text{else,} \end{cases}$$

$$\sigma_{2B}^\star(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{2\} & \text{if } \mathcal{N} = N_B^\star,\ x = x_1, \text{ and } \mathcal{N}' = \{1\} \\ \emptyset & \text{else,} \end{cases}$$

where $\mathcal{N}$ and $\mathcal{N}'$ are subsets of $\mathcal{C}$ and x is a project.

Let's verify that $\boldsymbol{s}^\star$ is an equilibrium. We'll first establish that it's best for A to follow $\boldsymbol{s}^\star$ when B does by considering different networks for A. If A holds network $\emptyset$, then her payoff is 0.

If A's network is $\{2\}$, then her (ex-ante) payoff is $-1/4$ when she behaves optimally in the second stage. Let x_A be A's project. If she gets her project first, then it's best for her to employ consultant 2 if $x_A = x_1$ and for her not to employ consultant 2 if $x_A = x_2$. Hence, her expected payoff is $1/2$ when she moves first. If she gets her project x_A second, then A cannot employ consultant 2 as B always employs 2. Thus, her expected payoff is 0 when she moves second. Hence, her (ex-ante) payoff from network $\{2\}$ is $1/2(1/2) - 1/2 = -1/4$.

If A's network is $\{1\}$, her (ex-ante) payoff is $-1/8$ when she behaves optimally in the second stage. If she gets her project first, then it's best for her to employ consultant 1 if $x_A = x_1$ and for her not to employ consultant 1 if $x_A = x_2$. Hence, her expected payoff is $1/2$ when she moves first. If she gets her project second, then it's best for her to employ consultant 1 if $x_A = x_1$ and B employs only consultant 2, otherwise it's best for her to employ no consultants. Since B employs consultant 2 if and only if she gets project x_1, A's expected payoff is $1\,1/2\,1/2 = 1/4$ when she moves second. Thus, her (ex-ante) payoff from network $\{1\}$ is $1/2(1/2 + 1/4) - 1/2 = -1/8$.

If A's network is $\{1,2\}$, her (ex-ante) payoff is $1/4$ when she behaves optimally in the second stage. If she gets her project first, then it's best for her to employ consultant 1 if $x_A = x_1$ and for her to employ both consultants if $x_A = x_2$. Notice that $\sigma_{1A}^\star$ makes exactly this recommendation to A: $\sigma_{1A}^\star(\{1,2\}, x_1) = \{1\}$ and $\sigma_{1A}^\star(\{1,2\}, x_2) = \{1,2\}$. It follows that her expected payoff is $1/2\,1 + 1/2\,3 = 2$ when she moves first. If she gets her project second, then it's best for her to employ consultant 1 if $x_A = x_1$ and B employs consultant 2, otherwise it's best for her to employ no consultants. Notice that $\sigma^\star_{2A}$ makes exactly this recommendation to A: $\sigma^\star_{2A}(\{1,2\},\{2\},x_1) = \{1\}$, $\sigma^\star_{2A}(\{1,2\},\{2\},x_2) = \emptyset$, and $\sigma^\star_{2A}(\{1,2\},\{1,2\},x) = \emptyset$ for all $x \in X$. Hence, her expected payoff is $1\,{}^1\!/_2\,{}^1\!/_2 = {}^1\!/_4$ when she moves second. Thus, her (ex-ante) payoff from network $\{1,2\}$ is ${}^1\!/_2(2 + {}^1\!/_4) - 1 = {}^1\!/_8$.

It follows that it's best for A to hold network $N^\star_A = \{1,2\}$ and follow $\sigma^\star_{1A}$ and $\sigma^\star_{2A}$ when B plays according to equilibrium. Since an analogous argument gives that it's best for B to follow $\boldsymbol{s}^\star$ when A does, we have that $\boldsymbol{s}^\star$ is an equilibrium. $\triangle$

3 Equilibrium Existence

Our goal in this section is to establish the following proposition.

Proposition 1. Existence of an Equilibrium.
The set of equilibria $\boldsymbol{E}$ *is non-empty.*

We'll prove the proposition in three steps. First, we'll introduce "simple strategies." For each partner, a simple strategy takes an integer and returns a network and behavioral strategies. The integer is the size of the network. Second, we'll consider an "Auxiliary Game" where the partners play simple strategies and choose the "sizes" of their networks. We'll show that this is a supermodular game and so has a pure strategy Nash equilibrium. Third, we'll show that each equilibrium of the Auxiliary Game induces an equilibrium of the original game. The key insight behind our approach is that when one partner plays a simple strategy, the other partner does best by also playing a simple strategy.

Let's develop simple strategies. A **simple strategy for $\boldsymbol{A}$** of size $n \in \{0,\ldots,N\}$ is a tuple $\tilde{\boldsymbol{s}}_A(n) = (\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1A}, \tilde{\sigma}_{2A})$ such that, for every $(\mathcal{N},\mathcal{N}',x) \in \mathbb{P}(\mathcal{C})^2 \times X$, we have

$$\tilde{\mathcal{N}}_A = \{1,\ldots,n\}$$

$$\tilde{\sigma}_{1A}(\mathcal{N},x) = \begin{cases} \{1,\ldots,d(x)\} & \text{if } d(x) \le |\mathcal{N}| \text{ and } \mathcal{N} = \tilde{\mathcal{N}}_A \\ \emptyset & \text{else,} \end{cases}$$

$$\tilde{\sigma}_{2A}(\mathcal{N},\mathcal{N}',x) = \begin{cases} \{1,\ldots,d(x)\} & \text{if } d(x) \le |\mathcal{N}\backslash\mathcal{N}'|,\ \mathcal{N}' \subset \{d(x)+1,\ldots,N\}, \\ & \qquad \text{and } \mathcal{N} = \tilde{\mathcal{N}}_A \\ \emptyset & \text{else.} \end{cases}$$

In this strategy,[14] A includes the *first* n consultants in her network and, when she gets a

[14] It's readily verified that $\tilde{\boldsymbol{s}}_A(n) \in \boldsymbol{S}_A$ for every $n \in \{0,\ldots,N\}$.

project x, she employs the *first* $d(x)$ consultants, provided these consultants aren't employed by B and there are $d(x)$ consultants in her network.

A **simple strategy for $\boldsymbol{B}$** of size $n \in \{0, \ldots, N\}$ is a tuple $\tilde{\boldsymbol{s}}_B(n) = (\tilde{\mathcal{N}}_B, \tilde{\sigma}_{1B}, \tilde{\sigma}_{2B})$ such that, for every $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$, we have

$$
\begin{aligned}
\tilde{\mathcal{N}}_B &= \{N+1-n, \ldots, N\} \\
\tilde{\sigma}_{1B}(\mathcal{N}, x) &= \begin{cases} \{N+1-d(x), \ldots, N\} & \text{if } d(x) \leq |\mathcal{N}| \text{ and } \mathcal{N} = \tilde{\mathcal{N}}_B \\ \emptyset & \text{else,} \end{cases} \\
\tilde{\sigma}_{2B}(\mathcal{N}, \mathcal{N}', x) &= \begin{cases} \{N+1-d(x), \ldots, N\} & \begin{array}{l} \text{if } d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|, \mathcal{N}' \subset \{1, \ldots, N-d(x_B)\}, \\ \qquad \text{and } \mathcal{N} = \tilde{\mathcal{N}}_B \end{array} \\ \emptyset & \text{else.} \end{cases}
\end{aligned}
$$

In this strategy, B includes the *last* n consultants in her network and, when she gets a project x, she employs the *last* $d(x)$ consultants, provided these consultants aren't employed by A and there are $d(x)$ consultants in her network.

We need one more piece of notation. For each $(n_A, n_B) \in \{0, \ldots, N\}^2$, let $\tilde{\boldsymbol{s}}(n_A, n_B) = (\tilde{\boldsymbol{s}}_A(n_A), \tilde{\boldsymbol{s}}_B(n_B))$ denote a vector of simple strategies for both players.

In the **Auxiliary Game**, A chooses $z_A \in \{0, \ldots, N\}$ and B chooses $z_B \in \{0, \ldots, N\}$. Then both partners play the original game according to $\tilde{\boldsymbol{s}}(z_A, N - z_B)$ and so get payoffs $U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ and $U_B(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ respectively. Thus, A picks the size of her simple strategy (i.e., the size of her network) and B picks the "negative" of the size of her simple strategy (i.e., the negative of the size of her network). An **equilibrium of the Auxiliary Game** is a $(z_A^\star, z_B^\star) \in \{0, \ldots, N\}^2$ such that (i) $U_A(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B^\star))$ for all $z_A \in \{0, \ldots, N\}$ and $U_B(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_B(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B))$ for all $z_B \in \{0, \ldots, N\}$. Let $\boldsymbol{F}$ be the set of equilibria of this game.

Lemma 1. Supermodularity.
Let $(z_A, z_B) \in \{0, \ldots, N\}^2$, we have that $U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ and $U_B(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ are supermodular in (z_A, z_B).

Proof. The proof is given in the Appendix. □

To see the intuition behind the lemma, focus on A. When z_B decreases, i.e., $N - z_B$ increases, B adds consultants to her network. This allows her to take on more difficult projects and so reduces the probability that A may employ any "shared" consultant, i.e., a consultant who's in both her network and in B's network. Since A has more shared consultants when z_A is larger, the reduction in z_B decreases A's payoff faster when z_A is larger; hence supermodularity as a result of rivalry.

It follows that the Auxiliary Game is a two-player supermodular game. Thus, we have the following result.

Lemma 2. Equilibria of the Auxiliary Game.
The set of equilibria of the Auxiliary Game $\boldsymbol{F}$ is a non-empty, complete lattice.

Proof. Since (i) the joint strategy spaces of the Auxiliary Game $\{0, \ldots, N\}^2$ is trivially a compact and complete (sub-)lattice and (ii) the payoff functions $U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ and $U_B(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ are supermodular and continuous in (z_A, z_B) on $\{0, \ldots, N\}^2$, this result follows from Theorem 4.2.1 of Topkis [15]. □

It remains to show that each of these equilibria induces an equilibrium of the original game. The next lemma does this.

Lemma 3. Inducement of Equilibria.
Let $(z_A^\star, z_B^\star)$ be an equilibrium of the Auxiliary Game, then $\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)$ is an equilibrium of the original game, i.e., $(z_A^\star, z_B^\star) \in \boldsymbol{F}$ implies that $\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star) \in \boldsymbol{E}$.

Proof. The proof is given in the Appendix. □

The intuition behind this result is the intuition behind Proposition 1: when one partner, say B, plays a simple strategy, then A does best by also playing a simple strategy. This is because the (ex-ante) probability that B employs a consultant is monotone increasing in the index of the consultant: B employs consultant N with the highest probability, consultant $N - 1$ with the second highest probability, and so on. Since A wants to network with consultants she'll be able to employ with high probability, she does best by networking with the lowest indexed consultants. Thus, when B follows $\tilde{\boldsymbol{s}}_B(N - z_B^\star)$, then the best A can do is play $\tilde{\boldsymbol{s}}_A(n)$ for some $n \in \{0, \ldots, N\}$. Since $(z_A^\star, z_B^\star)$ is an equilibrium point of the Auxiliary Game, we necessarily have that A does best by setting $n = z_A^\star$. Since an analogous argument holds for B, it follows that $\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star) \in \boldsymbol{E}$.

Proof of Proposition 1. By Lemma 2, there is an equilibrium $(z_A^\star, z_B^\star)$ of the Auxiliary Game. By Lemma 3, $\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star) \in \boldsymbol{E}$. □

Let $\boldsymbol{E}_S$ denote the set of equilibria that are induced by the equilibria of the Auxiliary Game, i.e., let

$$\boldsymbol{E}_S = \{\boldsymbol{s} \in \boldsymbol{S} | \boldsymbol{s} = \tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star) \text{ for some } (z_A^\star, z_B^\star) \in \boldsymbol{F}\}.$$

We refer to E_S as the set of **simple equilibria**. Observe that each element of $\boldsymbol{F}$ corresponds to a unique element in $\boldsymbol{E}_S$: $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star, \ldots) \in \boldsymbol{E}_S$ if and only if $(|\mathcal{N}_A^\star|, N - |\mathcal{N}_B^\star|) \in \boldsymbol{F}$. (We'll make heavy use of this fact in Section 5.)

4 General Properties of Equilibrium

In this section, we establish that the partners' equilibrium networks are minimally overlapping, introduce employment lists and employment efficiency, and establish that employment list and employment efficient equilibria are payoff equivalent to simple equilibria.

Minimally Overlapping Networks

Our goal in this subsection is to establish that the partners' equilibrium networks share as few consultants "as possible." The next definition formalizes this idea.

Definition. Let $\mathcal{N}_A$ and $\mathcal{N}_B$ be networks for A and B. We say that $\mathcal{N}_A$ and $\mathcal{N}_B$ are *minimally overlapping* if there does not exists a $\mathcal{N}'_A \subset \mathcal{C}$ and a $\mathcal{N}'_B \subset \mathcal{C}$, such that $|\mathcal{N}_A| = |\mathcal{N}'_A|$, $|\mathcal{N}_B| = |\mathcal{N}'_B|$, and $|\mathcal{N}'_A \cap \mathcal{N}'_B| < |\mathcal{N}_A \cap \mathcal{N}_B|$.

That is, A and B's networks are minimally overlapping if there does not exist a way to shrink the number of consultants in both networks without changing the size of A's network or B's network. For instance, when $\mathcal{C} = \{1, 2, 3, 4\}$, the networks $\mathcal{N}_A = \{1, 2, 3\}$ and $\mathcal{N}_B = \{2, 3\}$ aren't minimally overlapping, while the networks $\mathcal{N}_A = \{1, 2, 3\}$ and $\mathcal{N}_B = \{3, 4\}$ are minimally overlapping.

We'll prove that all equilibrium networks are minimally overlapping when the following technical assumption holds.

Assumption 2. Strictly Positive Costs and Rewards and Heterogeneous Project Difficulty. We have $w > 0$ and, for each $x \in X$, we have $P_i(x)(r_i(x) - w\, d(x)) > 0$ for each partner i. Additionally, for each $n \in \{1, \ldots, N\}$, there is a project $x \in X$ such that $d(x) = n$.

The first part of the assumption gives that labor costs are strictly positive, that the reward to completing a project is strictly positive after accounting for labor costs, and that all projects occur with strictly positive probability. This ensures that each partners' optimal behavior in the second stage is essentially unique. The second part of the assumption requires that projects are of sufficiently heterogeneous difficulty. This guarantees, for instance, that a partner always finds occasion to employ two-thirds of her network.

Proposition 2. Equilibrium Networks are Minimally Overlapping.
Let Assumption 2 hold and let $\boldsymbol{s}^\star = (\mathcal{N}^\star_A, \ldots, \mathcal{N}^\star_B, \ldots) \in \boldsymbol{E}$*, then* $\mathcal{N}^\star_A$ *and* $\mathcal{N}^\star_B$ *are minimally overlapping.*

The proposition implies, for instance, that if A and B hold small networks, i.e., if $|\mathcal{N}^\star_A| + |\mathcal{N}^\star_B| < |\mathcal{C}|$, then their networks are disjoint. The proposition reflects a general preference of the partners to share as few consultants as possible: all else equal, fewer shared consultants means that a partner can employ more of the consultants in her network more often and so do better. The proof of the proposition makes use of this intuition.

We'll prove Proposition 2 using the following lemma.

Lemma 4. Covering.
Let Assumption 2 hold and let $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \dots, \mathcal{N}_B^\star, \dots) \in \boldsymbol{E}$, *then* $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star \neq \emptyset$ *implies that* $\mathcal{C} \subset \mathcal{N}_A^\star \cup \mathcal{N}_B^\star$.

Proof. The proof is given in the Appendix. □

The intuition is that if $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star \neq \emptyset$ and $\mathcal{C}$ is not in $\mathcal{N}_A^\star \cup \mathcal{N}_B^\star$, then there is at least one consultant who isn't in either partner's network. We show that one partner, say A, can always do strictly better by swapping this consultant for one who she shares with B.[15] Thus, $\boldsymbol{s}^\star$ cannot be an equilibrium, a contradiction. The lemma follows.

Proof of Proposition 2. There are two cases $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star = \emptyset$ and $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star \neq \emptyset$. If $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star = \emptyset$, then $|\mathcal{N}_A^\star \cap \mathcal{N}_B^\star| = 0$ and so it's impossible to find two subsets of $\mathcal{C}$ with a strictly smaller intersection. It follows that $\mathcal{N}_A^\star$ and $\mathcal{N}_B^\star$ are minimally overlapping.

If $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star \neq \emptyset$, we argue by contradiction. Suppose $\mathcal{N}_A^\star$ and $\mathcal{N}_B^\star$ aren't minimally overlapping, then there are $\mathcal{N}_A' \subset \mathcal{C}$ and $\mathcal{N}_B' \subset \mathcal{C}$ with $|\mathcal{N}_A^\star| = |\mathcal{N}_A'|$, $|\mathcal{N}_B^\star| = |\mathcal{N}_B'|$, and $|\mathcal{N}_A' \cap \mathcal{N}_B'| < |\mathcal{N}_A^\star \cap \mathcal{N}_B^\star|$. Since $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star \neq \emptyset$, Lemma 4 gives that $\mathcal{C} \subset \mathcal{N}_A^\star \cup \mathcal{N}_B^\star$, so the sets $\mathcal{N}_A^\star$ and $\mathcal{N}_B^\star \backslash \mathcal{N}_A^\star$ partition $\mathcal{C}$. It follows that $N = |\mathcal{C}| = |\mathcal{N}_A^\star| + |\mathcal{N}_B^\star| - |\mathcal{N}_A^\star \cap \mathcal{N}_B^\star|$, since $|\mathcal{N}_B^\star \backslash \mathcal{N}_A^\star| = |\mathcal{N}_B^\star| - |\mathcal{N}_A^\star \cap \mathcal{N}_B^\star|$. Thus, $N < |\mathcal{N}_A'| + |\mathcal{N}_B'| - |\mathcal{N}_A' \cap \mathcal{N}_B|$. Since $|\mathcal{N}_B' \backslash \mathcal{N}_A'| = |\mathcal{N}_B'| - |\mathcal{N}_A' \cap \mathcal{N}_B'|$, we've $N < |\mathcal{N}_A'| + |\mathcal{N}_B' \backslash \mathcal{N}_A'|$. Since $\mathcal{N}_A'$ and $\mathcal{N}_B' \backslash \mathcal{N}_A'$ are disjoint subsets of $\mathcal{C}$, we necessarily have that $|\mathcal{N}_A'| + |\mathcal{N}_B' \backslash \mathcal{N}_A'| \leq N$. Thus, $N < N$, a contradiction. □

Remark. In the Supplement we show that the conclusion of this proposition (appropriately generalized) holds when there are more than two partners.

Employment List and Employment Efficient Equilibria

In this subsection we introduce the concepts of employment lists and employment efficiency, we argue that these properties are intuitive refinements, and we show that every equilibrium with these properties is payoff equivalent to a simple equilibrium. We begin by formally defining these properties.

Definition. A $\boldsymbol{s} = (\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \mathcal{N}_B, \sigma_{1B}, \sigma_{2B}) \in \boldsymbol{S}$ has an *employment list for partner i* if $\mathcal{N}_i \neq \emptyset$ implies that we can write $\mathcal{N}_i$ as an ordered list $\{j_1, j_2, \dots, j_{|\mathcal{N}_i|}\}$ such that, for each $x \in X$ with $\sigma_{1i}(\mathcal{N}_i, x) \neq \emptyset$, we have $\sigma_{1i}(\mathcal{N}_i, x) = \{j_1, j_2, \dots, j_{|\sigma_{1i}(\mathcal{N}_i, x)|}\}$. We say $\boldsymbol{s}$ has *employment lists* if it has employment lists for both A and B.

If a strategy profile has employment lists, then when i gets project x first, then she always employs the first $|\sigma_{1i}(\mathcal{N}_i, x)|$ consultants on her list when she employs anyone. For instance,

[15] This insight depends critically on the heterogeneity assured by Assumption 2. In the Supplement, we give an example where the conclusion of Lemma 4 doesn't apply because heterogeneity fails.

in the equilibrium in Example 1, B's employment list is $\{2, 1\}$ as she employs consultant 2 to carry out project x_1 and consultants 1 and 2 to carry out project x_2.

Definition. A $\boldsymbol{s} = (\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \mathcal{N}_B, \sigma_{1B}, \sigma_{2B}) \in \boldsymbol{S}$ is *employment efficient for partner i* if, for each $x \in X$, we have:
(i) $\sigma_{1i}(\mathcal{N}_i, x) \subset \mathcal{N}_i \backslash \mathcal{N}_{-i}$ when $|\sigma_{1i}(\mathcal{N}_i, x)| \leq |\mathcal{N}_i \backslash \mathcal{N}_{-i}|$, and
(ii) $\mathcal{N}_i \backslash \mathcal{N}_{-i} \subset \sigma_{1i}(\mathcal{N}_i, x)$ when $|\sigma_{1i}(N_i, x)| > |\mathcal{N}_i \backslash \mathcal{N}_{-i}|$.
We say $\boldsymbol{s}$ is *employment efficient* if it's employment efficient for both A and B.

When partner i moves first, she may employ either "exclusive" consultants in $\mathcal{N}_i \backslash \mathcal{N}_{-i}$ or shared consultants in $\mathcal{N}_i \cap \mathcal{N}_{-i}$. Since every shared consultant i employs is one less consultant that $-i$ may use, i reduces $-i$'s payoff by employing shared consultants. Thus, it's inefficient for i to employ shared consultants before she's employed every one of her exclusive consultants. Employment efficiency requires that this inefficiency not occur. Specifically, it requires (i) that if i wants to employ less than $|\mathcal{N}_i \backslash \mathcal{N}_{-i}|$ consultants, then she chooses behavioral strategies that employ only exclusive consultants, and (ii) that i wants to employ more than $|\mathcal{N}_i \backslash \mathcal{N}_{-i}|$ consultants, she chooses behavioral strategies that employ all of her exclusive consultants. An example of such a strategy is the equilibrium given in Example 1.

The next lemma comments on the relationship between simple strategies, employment list strategies, and employment efficient strategies.

Lemma 5. Simple Strategies Have Employment Lists and are Employment Efficient.
The vector of simple strategies $\tilde{\boldsymbol{s}}(n_A, n_B)$ has employment lists and is employment efficient for all $(n_A, n_B) \in \{0, \ldots, N\}^2$.

Proof. Obvious and omitted. $\square$

We say that a $\boldsymbol{s}^\star \in \boldsymbol{E}$ is an **employment list and employment efficient equilibrium** (abbreviated ELEE equilibrium) if it has employment lists and is employment efficient. We use $\boldsymbol{E}_{LE}$ to denote the set of ELEE equilibria. In light of Lemma 5, we've $\boldsymbol{E}_S \subset \boldsymbol{E}_{LE}$. In fact, the set of ELEE equilibria is substantially larger than the set of simple equilibria. For instance, it contains all equilibria that differ from a simple equilibrium off of the equilibrium path, as well as all permutations of simple equilibria.

We focus on ELEE equilibria for two reasons. First, employment lists and employment efficiency are natural refinements. Employments lists are reasonable because they captures the intuitive idea that each partner uses an address book or other (mental) list to determine which consultants to employ. Employment efficiency is also natural because it's costless to implement (since both exclusive and shared consultants both cost a partner w) and it weakly increases efficiency. Second, ELEE equilibria are "robust." That is, there's an ELEE equilibrium for every parameterization of our game, whereas there are parameterizations for which there are no non-ELEE equilibria. We give an example of this in the Supplement.[16]

The next proposition characterizes the relationship between ELEE and simple equilibria.

Proposition 3. Payoff Equivalency of ELEE Equilibria and Simple Equilibria.
Let Assumption 2 hold. Then, each ELEE equilibrium is payoff equivalent to a simple equilibrium where both partners hold the same sized networks as they do in the ELEE equilibrium. That is, for each $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star, \ldots) \in \boldsymbol{E}_{LE}$*, we have that (i)* $\tilde{\boldsymbol{s}}(|\mathcal{N}_A^\star|, |\mathcal{N}_B^\star|) \in \boldsymbol{E}_S$ *and (ii) that* $U_i(\boldsymbol{s}^\star) = U_i(\tilde{\boldsymbol{s}}(|\mathcal{N}_A^\star|, |\mathcal{N}_B^\star|))$ *for each partner* i.

Proof. The proof is given in the Appendix. □

The key insight behind the proposition is that whenever partner i moves first in the second stage and uses an employment list and employment efficient behavioral strategy, then her behavior in $\boldsymbol{s}^\star$ is a permutation of her behavior in the corresponding simple equilibrium. Thus, the second partner to move $-i$ has the same number of in-network consultants available in both equilibria. It follows that both partners get the same payoffs in these equilibria.

5 Welfare and Network Size

In this section, we establish our welfare results and develop several results on network size in the process.

Opposition of Interests

Our goal in this subsection is to establish (i) that there are equilibria where A does best and B does worst and (ii) that, under a strict monotonicity condition, A does best whenever B does worst.

To understand the monotonicity condition, we first have to understand how rivalry affects the partners' payoffs when they play simple strategies. To these ends, let $b_A(n) = \arg\max_{n' \in \{0,\ldots,N\}} U_A(\tilde{\boldsymbol{s}}(n', n))$ and $b_B(n) = \arg\max_{n' \in \{0,\ldots,N\}} U_B(\tilde{\boldsymbol{s}}(n, n'))$ denote A and B's best responses when playing simple strategies. Let $\overline{b}_i(n) = \max_{\leq} b_i(n)$ denote the maximal selection of partner i's best response. Occasionally, we'll write $\overline{b}_i(n, r_i, c_i)$ when we wish to emphasize the dependence of i's maximal selection on her reward and cost functions.

Lemma 6. Weak Monotonicity.
Each partner's payoff to a simple strategy is weakly decreasing in the size of the other partner's simple strategy when she responds optimally. That is, $U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(n), n))$ *and* $U_B(\tilde{\boldsymbol{s}}(n, \overline{b}_B(n))$

[16] Unfortunately, we cannot appeal to a Pareto dominance argument to justify the focus on ELEE equilibria. In the Supplement, we give an example of a non-ELEE equilibrium where A does strictly better than in any ELEE equilibria, while B does strictly worse than in any ELEE equilibria.

are weakly decreasing in n.

Proof. The proof is given in the Appendix. □

The intuition for this result is that each partner exerts a weakly negative externality on the other by holding a larger network because the use of consultants is rivalrous.

Our monotonicity condition is that the partners' payoffs to simple strategies evaluated at best responses are *strictly* decreasing in the size of the other partner's network. Formally, we make the following assumption.

Assumption 3. Strict Monotonicity and Single-Valued Best Replies.
We have (i) $b_A(n)$ and $b_B(n)$ are single-valued for all $n \in \{0, \ldots, N\}$ and (ii) $U_A(\tilde{\boldsymbol{s}}(b_A(n), n))$ and $U_B(\tilde{\boldsymbol{s}}(n, b_B(n))$ are strictly decreasing in n.

Part (i) is guaranteed if the partners' payoffs to (simple strategies) are strictly quasi-concave or if the partners commit to a selection of their best replies before the game begins. Part (ii) holds whenever both partners find it desirable to hold large networks (e.g., the networking cost is low and the returns to difficult projects are high) as then an increase in the size of her rival's network always increases the number of shared consultants. It's readily verified that the payoffs in Example 1 satisfy both parts of this assumption.

Now we're in the position to give the main result of this subsection. Let $\overline{\boldsymbol{W}}(i) = \{\boldsymbol{s}^\star \in \boldsymbol{E}_{LE} | U_i(\boldsymbol{s}^\star) \geq U_i(\boldsymbol{s}) \text{ for all } \boldsymbol{s} \in \boldsymbol{E}_{LE}\}$ be the set of ELEE equilibria where partner i does best. Also, let $\underline{\boldsymbol{W}}(i) = \{\boldsymbol{s}^\star \in \boldsymbol{E}_{LE} | U_i(\boldsymbol{s}^\star) \leq U_i(\boldsymbol{s}) \text{ for all } \boldsymbol{s} \in \boldsymbol{E}_{LE}\}$ be the set of ELEE equilibria where i does worst. We'll occasionally write $\overline{\boldsymbol{W}}(i, r_i, c_i)$ and $\underline{\boldsymbol{W}}(i, r_i, c_i)$ when we wish to emphasize the dependence of these sets on i's reward and cost functions.

Proposition 4. Opposing Interests.
Let Assumption 2 hold, then there's an ELEE equilibrium where A does best and B does worst and vice-versa, i.e., $\overline{\boldsymbol{W}}(A) \cap \underline{\boldsymbol{W}}(B)$ and $\overline{\boldsymbol{W}}(B) \cap \underline{\boldsymbol{W}}(A)$ are non-empty. When Assumption 3 also holds, then in A does best in any ELEE equilibrium where B does worst and vice versa, i.e., $\overline{\boldsymbol{W}}(A) = \underline{\boldsymbol{W}}(B)$ and $\overline{\boldsymbol{W}}(B) = \underline{\boldsymbol{W}}(A)$.

The proposition gives that there are equilibria where A and B's **interests are opposed**, i.e., where A does best and B does worst and vice versa. It also tells us that A does best precisely when B does worst when our monotonicity condition holds.

We'll prove the proposition by going to the Auxiliary Game. Since this game is supermodular, it has a largest equilibrium, which A weakly most-prefers and B weakly least-prefers (per Lemma 6), and a smallest equilibrium, which B weakly most-prefers and A weakly least-prefers (per Lemma 6). When Assumption 3 holds, the partners' preferences over equilibria become strict. Thus, A does best (worst) in only the largest (smallest) equilibrium and B does worst (worst) only in the largest (smallest) equilibrium. We establish these facts in Lemma 7 below. We then deduce Proposition 4 from this lemma and Proposition 3.

Recall that the set of equilibria of the Auxiliary Game $\boldsymbol{F}$ is a complete lattice. Let $(\overline{z}_A, \overline{z}_B)$ be the maximal element of $\boldsymbol{F}$ and let $(\underline{z}_A, \underline{z}_B)$ be its minimal element. Let $\overline{\boldsymbol{W}}_{\boldsymbol{F}}(i) = \{(z_A^\star, z_B^\star) \in \boldsymbol{F} | U_i(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_i(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ for all $(z_A, z_B) \in \boldsymbol{F}\}$ denote the set of equilibria in the Auxiliary Game where partner i does best. Let $\underline{\boldsymbol{W}}_{\boldsymbol{F}}(i) = \{(z_A^\star, z_B^\star) \in \boldsymbol{F} | U_i(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \leq U_i(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ for all $(z_A, z_B) \in \boldsymbol{F}\}$ denote the set of all equilibria in the Auxiliary Game where i does worst.

Two preliminary facts will be useful. First, $U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B)) = U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - z_B), N - z_B)$ for all $(z_A, z_B) \in \boldsymbol{F}$ as both z_A and $\overline{b}_A(N - z_B)$ were picked to maximize $U_A(\tilde{\boldsymbol{s}}(n, N - z_B))$ by choice of $n \in \{0, \ldots, N\}$, the former by definition of equilibrium and the latter by construction. Second, $U_B(\tilde{\boldsymbol{s}}(z_A, N - z_B)) = U_B(\tilde{\boldsymbol{s}}(z_A, \overline{b}_B(z_A))$ for all $(z_A, z_B) \in \boldsymbol{F}$ as z_B was picked to maximize $U_B(\tilde{\boldsymbol{s}}(z_A, N - n))$ by choice of $n \in \{0, \ldots, N\}$ and $\overline{b}_B(z_A)$ was picked to maximize $U_B(\tilde{\boldsymbol{s}}(z_A, n))$ by choice of $n \in \{0, \ldots, N\}$.

Lemma 7. Equilibria with Opposing Interests in the Auxiliary Game.
We have (i) $(\overline{z}_A, \overline{z}_B) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B)$ *and (ii)* $(\underline{z}_A, \underline{z}_B) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(B) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$. *If Assumption 3 also holds, then we have (iii)* $\overline{\boldsymbol{W}}_{\boldsymbol{F}}(A) = \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B) = \{(\overline{z}_A, \overline{z}_B)\}$ *and (iv)* $\overline{\boldsymbol{W}}_{\boldsymbol{F}}(B) = \underline{\boldsymbol{W}}_{\boldsymbol{F}}(A) = \{(\underline{z}_A, \underline{z}_B)\}$.

Proof. In light of Lemma 6, parts (i) and (ii) are almost obvious. But, we prove part (i) for completeness; the argument for part (ii) is analogous. Subsequently, we prove part (iii) since a similar argument gives part (iv).

We begin by establishing part (i). First, we show that $U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B)) \geq U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ for all $(z_A, z_B) \in \boldsymbol{F}$. Write

$$\begin{aligned} U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B)) &= U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}_B), N - \overline{z}_B)) \\ &\geq U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - z_B), N - z_B)) = U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B)). \end{aligned}$$

The equalities are due to our first fact. The inequality is due to Lemma 6 since $N - \overline{z}_B \leq N - z_B$ by the maximality of $(\overline{z}_A, \overline{z}_B)$. Next we show that $U_B(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B)) \leq U_A(\tilde{\boldsymbol{s}}(z_A, N - z))$ for all $(z_A, z_B) \in \boldsymbol{F}$. Write

$$U_B(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B)) = U_B(\tilde{\boldsymbol{s}}(\overline{z}_A, \overline{b}_B(\overline{z}_A))) \leq U_B(\tilde{\boldsymbol{s}}(z_A, \overline{b}_B(z_A))) = U_B(\tilde{\boldsymbol{s}}(z_A, N - z_B)),$$

where the equalities are due to our second fact and the inequality is due to Lemma 6 as $z_A \leq \overline{z}_A$ by maximality.

Now we establish part (iii) by showing that $(\overline{z}_A, \overline{z}_B)$ is the unique element of $\overline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$

and that it's also the unique element of $\underline{\boldsymbol{W}}_{\boldsymbol{F}}(B)$. To begin, let $(z_A^\star, z_B^\star) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$. Then, for every $(z_A, z_B) \in \boldsymbol{F}$,

$$U_A(\tilde{\boldsymbol{s}}(b_A(N - z_B^\star), N - z_B^\star) = U_A(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star) \geq U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B)) = U_A(\tilde{\boldsymbol{s}}(b_A(N - z_B), N - z_B),$$

where the equalities are due to our first fact and the inequality is due to $(z_A^\star, z_B^\star) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$. Since $U_A(\tilde{\boldsymbol{s}}(b_A(n), n)$ is strictly decreasing by Assumption 3, this implies $z_B \leq z_B^\star$ for all z_B played in some equilibrium. Thus, $z_B^\star = \overline{z}_B$. Since A's best response is single valued by Assumption 3, we have $z_A^\star = b_A(N - z_B^\star) = b_A(N - \overline{z}_B) = \overline{z}_A$. Hence, $(z_A^\star, z_B^\star) = (\overline{z}_A, \overline{z}_B)$, implying that $\overline{\boldsymbol{W}}_{\boldsymbol{F}}(A) = \{(\overline{z}_A, \overline{z}_B)\}$.

Now consider $\underline{\boldsymbol{W}}_{\boldsymbol{F}}(B)$. Let $(z_A^\star, z_B^\star) \in \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B)$. Then, for every $(z_A, z_B) \in \boldsymbol{F}$,

$$U_B(\tilde{\boldsymbol{s}}(z_A^\star, b_B(z_A^\star)) = U_B(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star) \leq U_B(\tilde{\boldsymbol{s}}(z_A, N - z_B)) = U_B(\tilde{\boldsymbol{s}}(z_A, b_B(z_A)).$$

Since $U_A(\tilde{\boldsymbol{s}}(n, b_B(n))$ is strictly decreasing, we have $z_A^\star = \overline{z}_A$. Thus, $z_B^\star = \overline{z}_B$ since B's best reply is unique, implying $(z_A^\star, z_B^\star) = (\overline{z}_A, \overline{z}_B)$. Hence, $\underline{\boldsymbol{W}}_{\boldsymbol{F}}(B) = \{(\overline{z}_A, \overline{z}_B)\}$. $\square$

Two facts will be useful in the proof of Proposition 4. First, if $\boldsymbol{s} = (\mathcal{N}_A, \ldots, \mathcal{N}_B, \ldots) \in \overline{\boldsymbol{W}}(A)$, then $(|\mathcal{N}_A|, N - |\mathcal{N}_B|) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$. (To see this, suppose that $(|\mathcal{N}_A|, N - |\mathcal{N}_B|) \notin \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$. Then, $U_A(\tilde{\boldsymbol{s}}(|\mathcal{N}_A|, |\mathcal{N}_B|)) < U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B))$ by Lemma 7, so Proposition 3 gives $U_A(\boldsymbol{s}^\star) < U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B))$. Since $\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B) \in \boldsymbol{E}_S \subset \boldsymbol{E}_{LE}$ by Lemmas 3 and 5, $U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B)) \leq U_A(\boldsymbol{s})$. Thus, $U_A(\boldsymbol{s}) < U_A(\boldsymbol{s})$, an impossibility.) Second, if $\boldsymbol{s}^\star = (\mathcal{N}_A, \ldots, \mathcal{N}_B, \ldots) \in \underline{\boldsymbol{W}}(B)$, then $(|\mathcal{N}_A|, N - |\mathcal{N}_B|) \in \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B)$. (The argument for this fact is analogous to the argument for the first fact.)

Proof of Proposition 4. We begin by showing that $\overline{\boldsymbol{W}}(A) \cap \underline{\boldsymbol{W}}(B)$ is non-empty. Subsequently, we'll show that $\overline{\boldsymbol{W}}(A) = \underline{\boldsymbol{W}}(B)$ when Assumption 3 holds. The arguments that (i) $\overline{\boldsymbol{W}}(B) \cap \underline{\boldsymbol{W}}(A) \neq \emptyset$ and (ii) that $\overline{\boldsymbol{W}}(B) = \underline{\boldsymbol{W}}(A)$ when Assumption 3 holds are analogous.

We first establish that $\overline{\boldsymbol{W}}(A) \cap \underline{\boldsymbol{W}}(B)$ is non-empty. Consider $(\overline{z}_A, \overline{z}_B)$. Lemma 7 and the fact that $\boldsymbol{s}' = (\mathcal{N}'_A, \ldots, \mathcal{N}'_B, \ldots) \in \boldsymbol{E}_S$ if and only if $(|\mathcal{N}'_A|, N - |\mathcal{N}'_B|) \in \boldsymbol{F}$ (per the construction of simple equilibria) imply

$$U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B)) \geq U_A(\boldsymbol{s}') \text{ and } U_B(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B)) \leq U_B(\boldsymbol{s}') \text{ for all } \boldsymbol{s}' \in \boldsymbol{E}_S.$$

Thus, Proposition 3 implies that $\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B) \in \overline{\boldsymbol{W}}(A) \cap \underline{\boldsymbol{W}}(B)$. (To see this, let $\boldsymbol{s}^\star \in \boldsymbol{E}_{LE}$, then Proposition 3 gives that $U_A(\boldsymbol{s}^\star) = U_A(\boldsymbol{s}')$ and $U_B(\boldsymbol{s}^\star) = U_B(\boldsymbol{s}')$ for an $\boldsymbol{s}' \in \boldsymbol{E}_S$. Thus, $U_A(\boldsymbol{s}^\star) \leq U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B))$ and $U_B(\boldsymbol{s}^\star) \geq U_B(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B))$. Since this holds for all elements of $\boldsymbol{E}_{LE}$, we've $\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B) \in \overline{\boldsymbol{W}}(A)$ and $\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B) \in \underline{\boldsymbol{W}}(B)$.)

Now we establish that $\overline{\boldsymbol{W}}(A) = \underline{\boldsymbol{W}}(B)$ when Assumption 3 holds. First, we show that $\overline{\boldsymbol{W}}(A) \subset \underline{\boldsymbol{W}}(B)$. Let $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star, \ldots) \in \overline{\boldsymbol{W}}(A)$, the our first fact gives $(|\mathcal{N}_A^\star|, N - |\mathcal{N}_B^\star|) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$. Thus, Lemma 7 gives that $(|\mathcal{N}_A^\star|, N - |\mathcal{N}_B^\star|) \in \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B)$. The construction of simple equilibria then implies that $U_B(\tilde{\boldsymbol{s}}(|\mathcal{N}_A^\star|, |\mathcal{N}_B^\star|)) \leq U_B(\boldsymbol{s}')$ for all $\boldsymbol{s}' \in \boldsymbol{E}_S$. Thus, Proposition 3 gives that $\boldsymbol{s}^\star \in \underline{\boldsymbol{W}}(B)$ since every other ELEE equilibrium maps to a simple equilibrium with a (weakly) higher payoff. Next, we establish that $\underline{\boldsymbol{W}}(B) \subset \overline{\boldsymbol{W}}(A)$. Let $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star, \ldots) \in \underline{\boldsymbol{W}}(B)$. Our second fact gives $(|\mathcal{N}_A^\star|, N - |\mathcal{N}_B^\star|) \in \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B)$. Lemma 7 then implies that $(|\mathcal{N}_A^\star|, N - |\mathcal{N}_B^\star|) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A)$. Then the construction of simple equilibria implies $U_A(\tilde{\boldsymbol{s}}(|\mathcal{N}_A^\star|, |\mathcal{N}_B^\star|)) \geq U_A(\boldsymbol{s}')$ for all $\boldsymbol{s}' \in \boldsymbol{E}_S$. Proposition 3 then gives that $\boldsymbol{s}^\star \in \overline{\boldsymbol{W}}(A)$. It follows that that $\overline{\boldsymbol{W}}(A) = \underline{\boldsymbol{W}}(B)$. □

Comparative Statics

In this subsection, we characterize how the partners' payoffs and network sizes shift in ELEE equilibria where their interests are opposed as their reward and cost functions shift. We consider what happens in the event of the following shift.

Assumption 4. Changes in Rewards and Costs.
The cost and reward functions change as follows:
(i) A's reward function r_A increases to r'_A, i.e., $r_A(x) \leq r'_A(x)$ for each project x.
(ii) A's cost function c_A decreases to c'_A, i.e., $c'_A(n) \leq c_A(n)$ for all n.
(iii) B's cost and reward functions don't change.
Also, $c_A(n) - c'_A(n)$ is weakly increasing in n.

The last part of the assumption is a technical requirement. It's satisfied, for instance, when A has constant marginal costs of networking.[17]

Proposition 5. Comparative Statics in Rewards and Costs.
Let Assumptions 2 and 4 hold. As A's reward and cost functions shift, then A's payoff increases and B's payoff decreases in the ELEE equilibria that are best for A and worst for B. That is, for $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star) \in \overline{\boldsymbol{W}}(A, r_A, c_A) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$ *and* $\boldsymbol{s}' = (\mathcal{N}'_A, \ldots, \mathcal{N}'_B) \in \overline{\boldsymbol{W}}(A, r'_A, c'_A) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$*, we have that*

$$U_A(\boldsymbol{s}', r'_A, c'_A) \geq U_A(\boldsymbol{s}^\star, r_A, c_A) \text{ and } U_B(\boldsymbol{s}', r_B, c_B) \leq U_B(\boldsymbol{s}^\star, r_B, c_B).$$

If Assumption 3 also holds both before and after the shifts in rewards and costs, then the size of A's network increases and the size of B's network decreases, i.e., $|\mathcal{N}'_A| \geq |\mathcal{N}_A^\star|$ *and*

[17] If A's cost decreases from κ_A to κ'_A, then $c_A(n) - c'_A(n) = (\kappa_A - \kappa'_A)n$, which is trivially increasing in n.

$|\mathcal{N}'_B| \leq |\mathcal{N}^\star_B|$. *An analogous result holds for the ELEE equilibria that are best for* B *and worst for* A.

We'll prove this proposition by going to the Auxiliary Game and showing that as A's reward and cost functions shift, her best responses increase – see Lemma 8 below. This increases the size of A's network and decreases the size of B's network in the maximal and minimal equilibria – see Lemma 9 below. Then, Lemma 6 and the shift in rewards and costs imply that A's payoffs in these equilibria increases, while B's payoffs in decreases, while Lemma 7 implies that the size of A's network increases and the size of B's network decreases – see Lemma 10 below. We'll use this last result and Proposition 3 to deduce Proposition 5.

Let $\mathcal{S}$ and $\mathcal{S}'$ be subsets of $\{0, \ldots, N\}^n$, with $n \geq 1$. Recall that $\mathcal{S}$ is less than $\mathcal{S}'$ in the **strong set order**, denoted $\preceq$, if $(\min(n_1, n'_1), \ldots, \min(n_n, n'_n)) \in \mathcal{S}$ and $(\max(n_1, n'_1), \ldots, \max(n_n, n'_n)) \in \mathcal{S}'$ for all $(n_1, \ldots, n_n) \in \mathcal{S}$ and $(n'_1, \ldots, n'_n) \in \mathcal{S}'$. For each $z \in \{0, \ldots, N\}$, let $\phi_A(z) = \arg\max_{z' \in \{0,\ldots,N\}} U_A(\tilde{\boldsymbol{s}}(z', N-z))$ and $\phi_B(z) = \arg\max_{z' \in \{0,\ldots,N\}} U_A(\tilde{\boldsymbol{s}}(z, N-z'))$ denote A and B's best responses in the Auxiliary Game.[18] We'll occasionally write $\phi_A(z, r_A, c_A)$ and $\phi_B(z, r_B, c_B)$ when we wish to emphasize the dependence of these best replies on the partners' reward and cost functions.

Lemma 8. Shifts in Best Responses.
Let Assumption 4 hold, then for each $z \in \{0, \ldots, N\}$, *we have* $\phi_A(z, r_A, c_A) \preceq \phi_A(z, r'_A, c'_A)$.

Proof. The proof is given in the Appendix. □

The key insight of this proof is that when r_A increases and c_A decreases, then A never does worse by expanding the size of her network. Thus, her best response shifts out. Consequently, A winds up holding a larger network in equilibrium, while B winds up holding a smaller network.

Lemma 9. Comparisons of Extremal Elements.
Let Assumption 4 hold. Let $\boldsymbol{F}$ *denote the set of equilibria in the Auxiliary Game before the parameter shift and let* $\boldsymbol{F}'$ *denote the set of equilibria after the parameter shift. Let* $(\overline{z}_A, \overline{z}_B)$ *and* $(\underline{z}_A, \underline{z}_B)$ *be the maximal and minimal elements of* $\boldsymbol{F}$ *and let* $(\overline{z}'_A, \overline{z}'_B)$ *and* $(\underline{z}'_A, \underline{z}'_B)$ *be the maximal and minimal elements of* $\boldsymbol{F}'$. *Then* $(\overline{z}_A, \overline{z}_B) \leq (\overline{z}'_A, \overline{z}'_B)$ *and* $(\underline{z}_A, \underline{z}_B) \leq (\underline{z}'_A, \underline{z}'_B)$.

Proof. The proof is given in the Appendix. □

Lemma 10. Comparative Statics in the Auxiliary Game.
Let Assumption 4 hold. For all $(z^\star_A, z^\star_B) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A, r_A, c_A) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B, r_B, c_B)$ *and all* $(z'_A, z'_B) \in$

[18] Notice that $\phi_A(z) = b_A(N-z)$ and that $\phi_B(z) = \{z' \in \{0, \ldots, N\} | N - z' \in b_B(z)\}$.

$\overline{\boldsymbol{W}}_{\boldsymbol{F}}(A, r'_A, c'_A) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B, r_B, c_B)$, *we have that*

$$U_A(\tilde{\boldsymbol{s}}(z'_A, N - z'_B), r'_A, c'_A) \geq U_A(\tilde{\boldsymbol{s}}(z^\star_A, N - z^\star_B), r_A, c_A) \text{ and} \tag{5.1}$$

$$U_B(\tilde{\boldsymbol{s}}(z'_A, N - z'_B), r_B, c_B) \leq U_B(\tilde{\boldsymbol{s}}(z^\star_A, N - z^\star_B), r_B, c_B). \tag{5.2}$$

If Assumption 3 also holds both before and after the shifts in rewards and costs, then $(z^\star_A, z^\star_B) \leq (z'_A, z'_B)$. *Analogous result hold for equilibria were B does best and A does worst.*

Proof. We'll only establish the lemma for the equilibria of the Auxiliary Game where A does best and B does worst. The argument for all equilibria where B does best and A does worst is analogous. We begin by proving that equation (5.1) is true. Subsequently, we'll prove the result for network sizes is true.

Let $(\overline{z}_A, \overline{z}_B)$ and $(\overline{z}'_A, \overline{z}'_B)$ be as in the statement of Lemma 9. To simplify notation, for all $(n_A, n_B) \in \{0, \dots, N\}^2$, let $U_i(n_A, n_B)$ denote $U_i(\tilde{\boldsymbol{s}}(n_A, N - n_B), r_i, c_i)$ for each partner i and let $U'_A(n_A, n_B)$ denote $U_A(\tilde{\boldsymbol{s}}(n_A, N - n_B), r'_A, c'_A)$. Lemma 7 implies that $U_A(z^\star_A, z^\star_B) = U_A(\overline{z}_A, \overline{z}_B)$, that $U_B(z^\star_A, z^\star_B) = U_B(\overline{z}_A, \overline{z}_B)$, that $U'_A(z'_A, z'_B) = U'_A(\overline{z}'_A, \overline{z}'_B)$, that $U_B(z'_A, z'_B) = U_B(\overline{z}'_A, \overline{z}'_B)$. Thus, we only need to show that

$$U'_A(\overline{z}'_A \overline{z}'_B) \geq U_A(\overline{z}_A, \overline{z}_B) \text{ and } U_B(\overline{z}'_A \overline{z}'_B) \leq U_B(\overline{z}_A, \overline{z}_B) \tag{5.3}$$

to establish equations (5.1) and (5.2).

Let's prove (5.3) for A. Let $\overline{b}_A(n)$ denote $\overline{b}_A(n, r_A, c_A)$ and let $\overline{b}'_A(n)$ denote $\overline{b}_A(n, r'_A, c'_A)$. Write

$$\begin{aligned} U_A(\overline{z}_A, \overline{z}_B) &= U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}_B), N - \overline{z}_B), r_A, c_A) \\ &\leq U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}'_B), N - \overline{z}'_B), r_A, c_A) \\ &\leq U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}'_B), N - \overline{z}'_B), r'_A, c'_A) \\ &\leq U_A(\tilde{\boldsymbol{s}}(\overline{b}'_A(N - \overline{z}'_B), N - \overline{z}'_B), r'_A, c'_A) \\ &= U'_A(\overline{z}'_A, \overline{z}'_B) \end{aligned}$$

The first line is standard – see the facts before Lemma 7. Since $(\overline{z}_A, \overline{z}_B) \leq (\overline{z}'_A, \overline{z}'_B)$ by Lemma 9, the second line follows from Lemma 6 as $(\overline{z}_A, \overline{z}_B) \leq (\overline{z}'_A, \overline{z}'_B)$ implies that $N - \overline{z}'_B \leq N - \overline{z}_B$. The third line follows from the fact A's costs fall and her rewards increase. The fourth line follows from the optimality of $\overline{b}'_A(\cdot)$. The fifth line is standard. Since the argument for B is analogous, equation (5.3) holds.

It remains to show that the size of A's network increases and that the size of B's network decreases. Since Assumption 3 holds, Lemma 7 implies $(z^\star_A, z^\star_B) = (\overline{z}_A, \overline{z}_B)$ and that

$(z'_A, z'_B) = (\overline{z}'_A, \overline{z}'_B)$. Thus, the desired result follows directly from Lemma 9. $\square$

Proof of Proposition 5. We'll only establish the result for ELEE equilibria where A does best and B does worst. The argument for ELEE equilibria where B does best and A does worst is analogous.

Let $\boldsymbol{s}^\star = (\mathcal{N}^\star_A, \ldots, \mathcal{N}^\star_B, \ldots) \in \overline{\boldsymbol{W}}(A, r_A, c_A) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$ and let $\boldsymbol{s}' = (\mathcal{N}'_A, \ldots, \mathcal{N}'_B, \ldots) \in \overline{\boldsymbol{W}}(A, r'_A, c'_A) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$. The two facts before the Proof of Proposition 4 imply that $(|\mathcal{N}^\star_A|, N - |\mathcal{N}^\star_B|) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A, r_A, c_A) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B, r_B, c_B)$ and that $(|\mathcal{N}'_A|, N - |\mathcal{N}'_B|) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A, r'_A, c'_A) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B, r_B, c_B)$. Thus, Lemma 10 gives

$$U_A(\tilde{\boldsymbol{s}}(|\mathcal{N}'_A|, N - |\mathcal{N}'_B|), r'_A, c'_A) \geq U_A(\tilde{\boldsymbol{s}}(|\mathcal{N}^\star_A|, N - |\mathcal{N}^\star_B|), r_A, c_A) \text{ and}$$
$$U_B(\tilde{\boldsymbol{s}}(|\mathcal{N}'_A|, N - |\mathcal{N}'_B|), r_B, c_B) \leq U_B(\tilde{\boldsymbol{s}}(|\mathcal{N}^\star_A|, N - |\mathcal{N}^\star_B|), r_B, c_B).$$

Since $U_i(\boldsymbol{s}^\star) = U_i(\tilde{\boldsymbol{s}}(|\mathcal{N}^\star_A|, N - |\mathcal{N}^\star_B|))$ and $U_i(\boldsymbol{s}') = U_i(\tilde{\boldsymbol{s}}(|\mathcal{N}'_A|, N - |\mathcal{N}'_B|))$ for each partner i by Proposition 3, $U_A(\boldsymbol{s}', r'_A, c'_A) \geq U_A(\boldsymbol{s}^\star, r_A, c_A)$ and $U_B(\boldsymbol{s}', r_B, c_B) \leq U_B(\boldsymbol{s}^\star, r_B, c_B)$.

If Assumption 3 also holds, then we have $(|\mathcal{N}^\star_A|, N - |\mathcal{N}^\star_B|) \leq (|\mathcal{N}'_A|, N - |\mathcal{N}'_B|)$ by Lemma 11. Thus, $|\mathcal{N}^\star_A| \leq |\mathcal{N}'_A|$ and $|\mathcal{N}^\star_B| \geq |\mathcal{N}'_B|$. $\square$

Comparison of Payoffs and Network Sizes

In this subsection, we show that A has a larger network and payoff than B in any ELEE equilibrium that's best for her and worst for B whenever she has a reward or cost advantage, i.e., when the next assumption holds.

Assumption 5. Partners' Costs and Rewards.
For each $x \in X$, we have that $P_A(x) = P_B(x)$ and $r_A(x) \geq r_B(x)$. For all n, we also have (i) $c_A(n) \leq c_B(n)$ and (ii) $c_B(n) - c_A(n)$ is weakly increasing.

Proposition 6. Comparison of Payoffs and Network Sizes.
Let Assumptions 2 and 5 hold, then A earns more than B in any ELEE equilibrium where she does best and B does worst, i.e., $U_A(\boldsymbol{s}^\star) \geq U_B(\boldsymbol{s}^\star)$ for all $\boldsymbol{s}^\star = (\mathcal{N}^\star_A, \ldots, \mathcal{N}^\star_B, \ldots) \in \overline{\boldsymbol{W}}(A) \cap \underline{\boldsymbol{W}}(B)$. If Assumption 3 also holds, then A has a larger network than B, i.e., $|\mathcal{N}^\star_A| \geq |\mathcal{N}^\star_B|$.

Outside of these equilibria, however, A needn't have a higher payoff or a larger network than B – we give an example of this in the Supplement. We'll establish the proposition by first showing that when A has B's reward and cost functions, then A does weakly better than B and holds a weakly larger network than B in the maximal equilibrium of the Auxiliary Game – see the next lemma. Then we'll prove Proposition 6 by applying Proposition 5.

Lemma 11. An Intermediate Result.

Let Assumptions 2 and 5 hold, then, $U_A(\boldsymbol{s}^\star, r_B, r_B) \geq U_B(\boldsymbol{s}^\star, r_B, r_B)$ *for all* $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star, \ldots) \in \overline{\boldsymbol{W}}(A, r_B, c_B) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$*. If Assumption 3 also holds, then* $|\mathcal{N}_A^\star| \geq |\mathcal{N}_B^\star|$*.*

The intuition is that, under the hypothesis of the lemma, A holds a larger network than B in the largest equilibrium of the Auxiliary Game because of supermodularity/rivalry. Thus, Lemma 6 implies that A makes more than B. The lemma then follows from Proposition 3 when coupled with Lemma 7.

Proof. We'll first establish that A does better than B. Subsequently, we'll establish that A's network is weakly larger than B's network. Let $\boldsymbol{F}$ denote the equilibrium set of the Auxiliary Game when A has B's reward and cost function and let $(\overline{z}_A, \overline{z}_B)$ be it's maximal element.

We need a preliminary fact, that $N - \overline{z}_B \leq \overline{z}_A$. To see this, recall that $\phi_A(z) = b_A(N - z)$ and that $\phi_B(z) = \{z' \in \{0, \ldots, N\} | N - z' \in b_B(z)\}$. Since $\overline{z}_A \in \phi_A(\overline{z}_B)$ and $\overline{z}_B \in \phi_B(\overline{z}_A)$, we have $\overline{z}_A \in b_A(N - \overline{z}_B)$ and $N - \overline{z}_B \in b_B(\overline{z}_A)$. Since $b_A = b_B$, it follows that $N - \overline{z}_B \in b_A(\overline{z}_A)$ and $\overline{z}_A \in b_B(N - \overline{z}_B)$. Hence, $N - \overline{z}_B \in \phi_A(N - \overline{z}_A)$ and that $N - \overline{z}_A \in \phi_B(N - \overline{z}_B)$. So $(N - \overline{z}_B, N - \overline{z}_A) \in \boldsymbol{F}$, implying $(N - \overline{z}_B, N - \overline{z}_A) \leq (\overline{z}_A, \overline{z}_B)$, which gives $N - \overline{z}_B \leq \overline{z}_A$.

We first establish that A does better than B. Since both partners' have the same costs and rewards, $U_A(\tilde{\boldsymbol{s}}(n_A, n_B), r_B, c_B) = U_B(\tilde{\boldsymbol{s}}(n_B, n_A), r_B, c_B)$, i.e., the game is symmetric; implying that both partners have the same best response, i.e., that $b_A(n) = b_B(n)$. Consequently,

$$\begin{aligned}
U_A(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B), r_B, c_B) &= U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}_B), N - \overline{z}_B), r_B, c_B) \\
&\geq U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(\overline{z}_A), \overline{z}_A), r_B, c_B) \\
&= U_B(\tilde{\boldsymbol{s}}(\overline{z}_A, \overline{b}_B(\overline{z}_A)), r_B, c_B) \\
&= U_B(\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B), r_B, c_B).
\end{aligned}$$

The first line is standard – see the facts before Lemma 7. The second line is due to Lemma 6 and the fact that $N - \overline{z}_B \leq \overline{z}_A$. The third line is due to the symmetry of the game. The last line is standard. Since $\tilde{\boldsymbol{s}}(\overline{z}_A, N - \overline{z}_B) \in \overline{\boldsymbol{W}}(A, r_B, c_B) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$ by Proposition 3 (the argument is analogous to the Proof of Proposition 4), we have the desired result.

It remains to show that $|\mathcal{N}_A^\star| \geq |\mathcal{N}_B^\star|$. By the preliminary fact of the Proof of Proposition 6, $(|\mathcal{N}_A^\star|, N - |\mathcal{N}_B^\star|) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A, r_B, c_B) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B, r_B, c_B)$. Thus, Lemma 7 gives $(|\mathcal{N}_A^\star|, N - |\mathcal{N}_B^\star|) = (\overline{z}_A, \overline{z}_B)$ as Assumption 3 holds. The desired result now follows from the preliminary fact. □

Proof of Proposition 6. This is almost obvious. Since $r_A \geq r_B$, $c_A \leq c_B$, and $c_B - c_A$ is weakly increasing, Proposition 5 gives that $U_A(\boldsymbol{s}', r_A, c_A) \geq U_A(\boldsymbol{s}^\star, r_B, c_B)$ and $U_B(\boldsymbol{s}', r_B, c_B) \leq U_B(\boldsymbol{s}^\star, r_B, c_B)$ and that $|\mathcal{N}_A'| \geq |\mathcal{N}_A^\star|$ and $|\mathcal{N}_B'| \leq |\mathcal{N}_B^\star|$ for all $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star, \ldots) \in$

$\overline{\boldsymbol{W}}(A, r_B, c_B) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$ and $\boldsymbol{s}' = (\mathcal{N}'_A, \ldots, \mathcal{N}'_B, \ldots) \in \overline{\boldsymbol{W}}(A, r_A, c_A) \cap \underline{\boldsymbol{W}}(B, r_B, c_B)$.

We've been a bit loose in the use of Assumption 3; strictly speaking, we haven't verified that it holds when A has B's reward and cost function. To see that it does hold, recall that $U_A(\tilde{\boldsymbol{s}}(b_A(n, r_A, c_A), n), r_A, c_A)$ and $U_B(\tilde{\boldsymbol{s}}(b_B(n, r_B, c_B), n), r_B, c_B)$ are strictly decreasing in n by hypothesis. Thus,

$$U_A(\tilde{\boldsymbol{s}}(b_A(n, r_B, c_B), n), r_B, c_B) = U_B(\tilde{\boldsymbol{s}}(b_B(n, r_B, c_B), n), r_B, c_B)$$

is also strictly decreasing in n, so Assumption 3 holds when A has B's reward and cost function. □

6 Efficiency

In this section we show, via an example, that equilibria may be ex-ante inefficient. We also discuss why this is so and prove that over-investment is the "usual culprit." The next definition is useful.

Definition. We say that an $\boldsymbol{s} \in \boldsymbol{S}$ is *efficient* if it maximizes ex-ante surplus, i.e., if $\boldsymbol{s}$ solves

$$\max_{\boldsymbol{s}' \in \boldsymbol{S}} U_A(\boldsymbol{s}') + U_B(\boldsymbol{s}').$$

Since $\boldsymbol{S}$ is finite, there's always an efficient strategy. However, this efficient strategy need not be an equilibrium. The next example illustrates.

Example 2. Inefficient Equilibria.

Let $\mathcal{C} = \{1, 2\}$ be the set of consultants and let $X = \{x_1\}$ be the set of projects. Let $d(x_1) = 2$ be the difficulty of the project, let $r_A(x_1) = 25$, and let $r_B(x_1) = 5$. Let $w = 1$ and suppose both partners have a constant marginal networking cost of $1/2$. Also, let $P_i(x_1) = 1$ for each partner i.

In the unique equilibrium path outcome, A and B to hold network $\{1, 2\}$, employ both consultants to complete the project when they move first in the second stage, and employ no consultants when they move second. Thus, ex-post, the second partner wastes resources networking with consultants she can't employ. This is undesirable from a social standpoint. Instead, because of A's high reward function, it's uniquely efficient to have A hold the network $\{1, 2\}$ and employ both consultants whenever she moves, and to have B hold the empty network. It follows that no equilibrium is efficient. △

To see why, exactly, an efficient strategy can't be an equilibrium in the example, suppose both players initially follow an efficient strategy. Then A earns 22 and B earns 0. Suppose

B adds consultants 1 and 2 to her network. She incurs a cost of 1, but now can complete project x_1 whenever she moves before A, so her expected payoff is $3/2 - 1 = 1/2$. Thus, B has incentive to defect from $\boldsymbol{s}'$. Specifically, she has an incentive to **over-invest** in her network, i.e., form a larger network that is socially efficient. In over-investing, B increases the total cost of networking from 1 to 2 and so diminishes social welfare. Additionally, B also exerts a negative externality on A: she reduces the probability that A completes her high value project from 1 to $1/2$. This causes A's contribution to welfare to fall to $\frac{23}{2} - 1 = 10.5$, which further reduces welfare.

In light of this, it's natural to wonder if over-investment is the reason efficient strategies fail to be equilibria. The next result establishes that, at least for simple strategies, it is. Let $\boldsymbol{S}_S$ be the set of simple strategies, i.e.,

$$\boldsymbol{S}_S = \{\boldsymbol{s} \in \boldsymbol{S} | \boldsymbol{s} = \tilde{\boldsymbol{s}}(n_A, n_B) \text{ for some } (n_A, n_B) \in \{0, \dots, N\}^2\}.$$

Proposition 7. Efficient Simple Strategies and Over-Investment.
Let $\boldsymbol{s} = (\boldsymbol{s}_A, \boldsymbol{s}_B) = (\mathcal{N}_A, \dots, \mathcal{N}_B, \dots) \in \boldsymbol{S}_S$ be efficient. If $\boldsymbol{s}$ is not an equilibrium, then one of the two partners strictly benefits from over-investing in her network. That is, $\boldsymbol{s} \notin \boldsymbol{E}$ implies that either (i) a $n \in \{|\mathcal{N}_A| + 1, \dots, N\}$ such that $U_A(\tilde{\boldsymbol{s}}_A(n), \boldsymbol{s}_B) > U_A(\boldsymbol{s})$ or (ii) a $n \in \{|\mathcal{N}_B| + 1, \dots, N\}$ such that $U_B(\boldsymbol{s}_A, \tilde{\boldsymbol{s}}_B(n)) > U_B(\boldsymbol{s})$.

Proof. The proof is given in the Appendix as a corollary to a more general result which we omit for simplicity since it's statement requires several intermediate concepts that are introduced in the Appendix. □

The key insights of the proof are (i) that simple strategies are best replies to simple strategies and (ii) that, when both partners play simple strategies, then one partner does better when the other decreases the size of her network (per Lemma 6). Thus, if one partner, say A, defects from an efficient simple strategy, she does best by defecting to another simple strategy. If A plays a simple strategy where her network is smaller than in the efficient simple strategy, then B does better and social welfare improves. Since this is impossible as the original strategy is efficient, we necessarily have that A defects to a simple strategy where she plays a larger network.

7 Extension: Asymmetric Probabilities of Moving First

So far, we've assumed that both partners have an equal chance of moving first in the second stage. However, it's sometimes the case that one partner is a better salesperson than the other and, thus, is capable of obtaining projects more quickly. Naturally, this partner naturally has a greater chance of moving first in the second stage. In this section, we allow one partner, say A, to move first more frequently than B and we characterize how changes in this probability effect the partners' payoffs and network sizes.[19]

Environment

Let $\alpha \in [0,1]$ be the probability that A gets her project first in the second stage, so $1-\alpha$ is the probability B gets her project first. Thus, for each strategy vector $\boldsymbol{s}$, A's and B's ex-ante payoffs are

$$U_A(\boldsymbol{s},\alpha) = \sum_{(x_A,x_B)\in X^2} (\alpha\, u_{1A}(\boldsymbol{s},x_A,x_B) + (1-\alpha)u_{2A}(\boldsymbol{s},x_A,x_B))P_A(x_A)P_B(x_B)$$
$$U_B(\boldsymbol{s},\alpha) = \sum_{(x_A,x_B)\in X^2} ((1-\alpha)u_{1B}(\boldsymbol{s},x_B,x_A) + \alpha\, u_{2B}(\boldsymbol{s},x_B,x_A))P_A(x_A)P_B(x_B).$$

It's readily verified that, for each value of α, our all of our existing results hold – an equilibrium exists, all equilibria are minimally overlapping, A's and B' equilibrium interests are opposed, and so on.

Results

We write $\overline{\boldsymbol{W}}(i,\alpha)$ and $\underline{\boldsymbol{W}}(i,\alpha)$ to emphasize the dependence of the sets of best and worst equilibria for partner i on the probability that A moves first in the second stage.

Proposition 8. Comparative Statics in α.
Let Assumption 2 hold. As the probability that A moves first increases from α to α', A's payoff increases and B payoff decreases in the ELEE equilibria that are best for A and worst for B. That is, for $\boldsymbol{s}^\star = (\mathcal{N}_A^\star,\ldots,\mathcal{N}_B^\star) \in \overline{\boldsymbol{W}}(A,\alpha)\cap\underline{\boldsymbol{W}}(B,\alpha)$ and $\boldsymbol{s}' = (\mathcal{N}_A',\ldots,\mathcal{N}_B') \in \overline{\boldsymbol{W}}(A,\alpha')\cap\underline{\boldsymbol{W}}(B,\alpha')$, we have

$$U_A(\boldsymbol{s}',\alpha') \geq U_A(\boldsymbol{s}^\star,\alpha) \text{ and } U_B(\boldsymbol{s}',\alpha') \leq U_B(\boldsymbol{s}^\star,\alpha).$$

If Assumption 3 also holds both before and after the shifts in α, then the size of A's network increases and the size of B's network decreases, i.e., $|\mathcal{N}_A'| \geq |\mathcal{N}_A^\star|$ and $|\mathcal{N}_B'| \leq |\mathcal{N}_B^\star|$. An analogous result holds for the ELEE equilibria that are best for B and worst for A.

The intuition for this result is that it's better to move first in the second stage because then one is not subject to less rivalry. Thus, an increase in α increases A and B's best responses in the Auxiliary Game – see Lemma 12 below. This, in turn, increases the size of A's network and decreases the size of B's network in the maximal and minimal equilibria – see

[19] As previously mentioned, we consider three additional extensions in the Supplement that examine (i) multiple partners, (ii) a production technology with residual value, and (iii) heterogeneous labor costs.

Lemma 13 below. Then, Lemma 6 implies that A's payoffs in these equilibria increases and that B's payoffs in decreases, while Lemma 7 implies that the size of A's network increases and the size of B's network decreases – see Lemma 14 below. This last result along with Proposition 3 imply Proposition 8. We defer the proof to give the following corollary.

Corollary 1. Comparison of Payoffs and Network Sizes.
Let Assumptions 2 and 5 hold, and let $\alpha' \geq 1/2$, then A earns more than B in any ELEE equilibrium where she does best and B does worst, i.e., $U_A(\boldsymbol{s}^\star, \alpha') \geq U_B(\boldsymbol{s}^\star, \alpha')$ for all $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star) \in \overline{\boldsymbol{W}}(A, \alpha') \cap \underline{\boldsymbol{W}}(B, \alpha')$. If Assumption 3 also holds when A moves first with probability 1/2 and with probability α', then A holds a larger network than B, i.e., $|\mathcal{N}_A^\star| \geq |\mathcal{N}_B^\star|$.

Proof. Set $\alpha = 1/2$, then Proposition 6 gives that $U_A(\boldsymbol{s}^\star, 1/2) \geq U_B(\boldsymbol{s}^\star, 1/2)$ for all $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \ldots, \mathcal{N}_B^\star) \in \overline{\boldsymbol{W}}(A, 1/2) \cap \underline{\boldsymbol{W}}(B, 1/2)$ and that $|\mathcal{N}_A^\star| \geq |\mathcal{N}_B^\star|$. Thus, Proposition 8 gives that $U_A(\boldsymbol{s}', \alpha') \geq U_B(\boldsymbol{s}', \alpha')$ and that $|\mathcal{N}_A'| \geq |\mathcal{N}_B'|$ for all $\boldsymbol{s}' = (\mathcal{N}_A', \ldots, \mathcal{N}_B') \in \overline{\boldsymbol{W}}(A, \alpha') \cap \underline{\boldsymbol{W}}(B, \alpha')$. □

When $\alpha' < 1/2$, however, this result need not be true. For instance, if $\alpha = 0$, then A may choose the null network when (i) B frequently uses most of the consultants and (ii) has a high networking cost that can't be covered by the low value, simple projects she could complete.

We'll prove Proposition 8 by formalizing the intuition we sketched above. We write $\phi_A(z, \alpha)$ and $\phi_B(z, \alpha)$ to emphasize the dependence of A's and B's best replies in the Auxiliary Game on the probability A moves first.

Lemma 12. Shifts in Best Responses.
Suppose α increase to α', then for each $z \in \{0, \ldots, N\}$, we've $\phi_A(z, \alpha) \preceq \phi_A(z, \alpha')$ and $\phi_B(z, \alpha) \preceq \phi_B(z, \alpha')$.

Proof. The proof is given in the Appendix. □

Lemma 13. Comparisons of Extremal Elements.
Let $\boldsymbol{F}$ denote the set of equilibria in the Auxiliary Game at α and let $\boldsymbol{F}'$ denote the set of equilibria at α', where $\alpha \leq \alpha'$. Let $(\overline{z}_A, \overline{z}_B)$ and $(\underline{z}_A, \underline{z}_B)$ be the maximal and minimal elements of $\boldsymbol{F}$ and let $(\overline{z}'_A, \overline{z}'_B)$ and $(\underline{z}'_A, \underline{z}'_B)$ be the maximal and minimal elements of $\boldsymbol{F}'$. Then $(\overline{z}_A, \overline{z}_B) \leq (\overline{z}'_A, \overline{z}'_B)$ and $(\underline{z}_A, \underline{z}_B) \leq (\underline{z}'_A, \underline{z}'_B)$.

Proof. Analogous to the Proof of Lemma 9 and omitted. □

Lemma 14. Welfare Comparative Statics in the Auxiliary Game.

Let $\alpha \leq \alpha'$. For all $(z_A^\star, z_B^\star) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A,\alpha) \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B,\alpha)$ and all $(z'_A, z'_B) \in \overline{\boldsymbol{W}}_{\boldsymbol{F}}(A,\alpha') \cap \underline{\boldsymbol{W}}_{\boldsymbol{F}}(B,\alpha')$, we have that

$$U_A(\tilde{\boldsymbol{s}}(z'_A, N - z'_B), \alpha') \geq U_A(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star), \alpha) \text{ and}$$
$$U_B(\tilde{\boldsymbol{s}}(z'_A, N - z'_B), \alpha') \leq U_B(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star), \alpha).$$

If Assumption 3 also holds both before and after the shift in α, then $(z_A^\star, z_B^\star) \leq (z'_A, z'_B)$. Analogous result hold for equilibria were B does best and A does worst.

Proof. The proof is given in the Appendix. □

Proof of Proposition 8. In light of Lemma 14, the proof is analogous to the Proof of Proposition 5 and, thus, is omitted. □

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A Proofs Appendix

In this section we give the proofs that were omitted from the main text.

PROOF OF LEMMA 1

To establish Lemma 1, we need the following result.

Lemma A1. Payoffs when Partners Play Simple Strategies.
Let $i \in \{A, B\}$ *and let* $(n_i, n_{-i}) \in \{0, \ldots, N\}^2$. *Then,*

$$U_i(\tilde{\mathbf{s}}(n_i, n_{-i})) = \sum_{\{x|d(x) \le n_i\}} (r_i(x) - w\, d(x)) \frac{P_i(x)}{2} + \sum_{l=1}^{n_i} \sum_{\{x|d(x)=l\}} (r_i(x) - w\, d(x)) g_i(l, n_{-i}) \frac{P_i(x)}{2} - c_i(n_i)$$

where, for $l \in \{1, \ldots, N\}$,

$$g_i(l, n_{-i}) = 1 - \sum_{\{x|N+1-l \le d(x) \le n_{-i}\}} P_{-i}(x).$$

Proof. This result is purely computational. We perform the computation for A as the it's analogous for B. The claim is true if $n_A = 0$. Thus, we take $n_A \ge 1$.

We begin by computing A's payoff when she gets her project first. It's useful to think about the difficulty of A's project x_A. If $d(x_A) \le n_A$, then A employs $\tilde{\sigma}_{1A} = \{1, \ldots, d(x_A)\}$, completes her project, and earns $r_A(x_A) - w\, d(x_A)$ (before networking costs). If $d(x_A) > n_A$, then A employs $\tilde{\sigma}_{1A} = \emptyset$, does not complete her project, and earns 0. Thus, before accounting for network costs, A's expected payoff from getting her project first are

$$\sum_{\{x|d(x) \le n_A\}} (r_A(x) - w\, d(x)) P_A(x).$$

Next, we compute A's payoff when she gets her project second. Let x_A and x_B be A and B's projects. We need to consider the possibility that A can't complete her project. Let $l = d(x_A)$. Since B follows her simple strategy there are four sub-cases: (i) $d(x_B) > n_B$ and $l > n_A$, (ii) $d(x_B) \le n_B$ and $l > n_A$, (iii) $d(x_B) > n_B$ and $l \le n_A$, and (iv) $d(x_B) \le n_B$ and $l \le n_A$. If cases (i) or (ii), then $\tilde{\sigma}_{2A} = \emptyset$ and so A earns nothing.

If case (iii), then $\tilde{\sigma}_{2A} = \{1, \ldots, l\}$, A completes her project, and earns $r_A(x_A) - w\, d(x_A)$ (before networking costs). Since A follows her simple strategy, she completes her project if and only if B doesn't employ a consultant with an index at or below l.[20] Since B employs

[20] To see this, suppose B employs a set of consultants $\mathcal{T}$ and recall that A's network $\tilde{\mathcal{N}}_A = \{1, \ldots, n_A\}$

no consultants, the desired result follows.

If case (iv), then A might not complete her project because B might employ consultants with indices at or below l. Since B follows her simple strategy, she'll employ a consultant with an index at or below l if an only if she gets a project x_B with $d(x_B) \geq N+1-l$.[21] Thus, if $d(x_B) \geq N+1-l$, A does not complete her project and earns 0. However, if $d(x_B) < N+1-l$ or $d(x_B) > n_B$, A completes her project and earns $r_A(x_A) - w\,d(x_A)$.

It follows that, A's expected payoff from getting project x_A second are:
(1) 0 if $d(x_A) > n_A$, and
(2) $(r_A(x_A) - w\,d(x_A))(1 - \sum_{\{x|N+1-l \leq d(x) \leq n_B\}} P_B(x))$ if $d(x_A) \leq n_A$.
The first follows directly from sub-cases (i) and (ii). The second follows from sub-cases (iii) and (iv), where A gets $r_A(x_A) - w\,d(x_A)$ if either $d(x_B) < N+1-l$ or $d(x_B) > n_B$, an event of probability $1 - \sum_{\{x|N+1-l \leq d(x) \leq n_B\}} P_B(x) = g_A(l, n_B)$. Thus, before networking costs, A's expected payoff from getting her project second is

$$\sum_{\{x|d(x) \leq n_A\}} (r_A(x) - w\,d(x)) g_A(d(x), n_B) P_A(x).$$

Since A has a 1/2 chance of getting her project first and a 1/2 chance of getting her project second, her expected payoff inclusive of networking costs are

$$\sum_{\{x|d(x) \leq n_A\}} (r_A(x) - w\,d(x)) \frac{P_A(x)}{2} + \sum_{\{x|d(x) \leq n_A\}} (r_A(x) - w\,d(x)) g_A(d(x_A), n_B) \frac{P_A(x)}{2} - c_A(n_A).$$

Reordering the second sum shows that this expression is equivalent to the one given in the statement of the lemma. □

Proof of Lemma 1. This result is largely computational. We prove it for A as the argument for B is analogous. Let n_A, n_B, and n'_B be in $\{0, \ldots, N\}$ with $n'_B \geq n_B$. We first establish that the difference $U_A(\tilde{\boldsymbol{s}}(n_A, n_B)) - U_A(\tilde{\boldsymbol{s}}(n_A, n'_B))$ is weakly increasing in n_A, for all $n_A \in \{0, \ldots, N\}$. Given this, for $n'_A \geq n_A$ with $n'_A \in \{0, \ldots, N\}$,

$$U_A(\tilde{\boldsymbol{s}}(n'_A, n_B)) - U_A(\tilde{\boldsymbol{s}}(n'_A, n'_B)) \geq U_A(\tilde{\boldsymbol{s}}(n_A, n_B)) - U_A(\tilde{\boldsymbol{s}}(n_A, n'_B))$$
$$U_A(\tilde{\boldsymbol{s}}(n'_A, n_B)) + U_A(\tilde{\boldsymbol{s}}(n_A, n'_B)) \geq U_A(\tilde{\boldsymbol{s}}(n'_A, n'_B)) + U_A(\tilde{\boldsymbol{s}}(n_A, n_B)).$$

That is, $U_A(\tilde{\boldsymbol{s}}(n_A, n_B))$ is submodular in (n_A, n_B). It follows that $U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B))$ is

under $\tilde{\mathbf{s}}(n_A, n_B)$. If $\mathcal{T} \not\subset \{l+1, \ldots, N\}$, then $\tilde{\sigma}_{2A} = \emptyset$ (by construction) and A fails to complete her project. If, however, $\mathcal{T} \subset \{l+1, \ldots, N\}$, then $|\tilde{\mathcal{N}}_A \backslash \mathcal{T}| \geq l$, and so $\tilde{\sigma}_{2A} = \{1, \ldots, l\}$, implying A completes her project.

[21] Recall that B employs $\{N+1-d(x_B), \ldots, N\}$. If $d(x_B) < N+1-l$, then B doesn't employ a with an index at or below l as $d(x_B) < N+1-l \implies N+1-d(x_B) > l$. If, however, $d(x_B) \geq N+1-l$, then B employs a consultant with an index at or below l as $d(x_B) \geq N+1-l \implies N+1-d(x_B) \leq l$.

supermodular in $(z_A, z_B) \in \{0, \ldots, N\}^2$.

It remains to establish that $U_A(\tilde{\boldsymbol{s}}(n_A, n_B)) - U_A(\tilde{\boldsymbol{s}}(n_A, n_B'))$ is weakly increasing in n_A. To these ends, use Lemma A1 to write

$$U_A(\tilde{\boldsymbol{s}}(n_A, n_B)) - U_A(\tilde{\boldsymbol{s}}(n_A, n_B')) = \sum_{l=1}^{n_A} \sum_{\{x|d(x)=l\}} (r_A(x) - w\, d(x))(g_A(l, n_B) - g_A(l, n_B')) \frac{P_A(x)}{2}.$$

Since $g_A(l, n)$ is weakly decreasing in n and Assumption 1 holds, the sum on the right hand side is weakly increasing in n_A. □

Proof of Lemma 3

To prove Lemma 3, we need to show that a partner always does best by playing her simple strategy whenever the other plays a simple strategy. We'll make this argument via three lemmas. The first lemma describes an optimal way for a partner to employ her consultants, given her network and given that the other partner plays a simple strategy. The second lemma calculates a partner's payoff to a network under these optimal employment strategies. The third lemma uses the first two lemmas to show that a simple strategy is a best response to a simple strategy. This lets prove Lemma 3.

Remark A1. Optimal Behavior in the Second Stage.

It's useful to characterize an optimal behavior for partner i in the second stage. Suppose i gets project x and has non-empty network $\mathcal{N}$. (If $\mathcal{N}$ is empty, i cannot employ any consultants and so gets 0.) If i gets her project first, then it's optimal for her to employ exactly $d(x)$ consultants when $d(x) \leq |\mathcal{N}|$ and to employ zero consultants otherwise. If i gets her project second, then it's optimal for her to employ exactly $d(x)$ consultants when $d(x) \leq |\mathcal{N} \backslash \mathcal{T}|$ and to employ zero consultants otherwise, where $\mathcal{T}$ is the set of consultants initially employed by $-i$.

Simply, i may employ up to (i) $|\mathcal{N}|$ consultants when she gets her project first or (ii) $|\mathcal{N}_i \backslash \mathcal{T}|$ consultants when she gets her project second. Let n be the number of consultants i employs. If $n < d(x)$, then i fails to complete her project and earns $-w\, n$ (before networking costs). If, however, $n \geq d(x)$, then i earns $r_i(x) - w\, n$ (before networking costs). Since $w \geq 0$ and $r_i(x) - w\, d(x) \geq 0$, it follows that it's best for i to set $n = d(x)$ if she has at that many consultants available and to set $n = 0$ otherwise.

Next we develop pair of second stage behavioral strategies that implements a partner's optimal behavior, given her network and given the other partner plays her simple strategy. To these ends, we introduce the following notation.

Let $\mathcal{N}$ be a non-empty network of consultants, and let $n = |\mathcal{N}|$. The **standard form of $\boldsymbol{\mathcal{N}}$ for $\boldsymbol{A}$** is a secondary labeling of the consultants in $\mathcal{N}$ such that the *lowest* indexed consultant receives the label m_1, the second lowest indexed consultant receives the label m_2, and so on until the highest indexed consultant gets the label m_n. For instance, if $\mathcal{N} = \{1, 5, 2, 4\}$, then 1 is labeled m_1, 2 is labeled m_2, 4 is labeled m_3, and 5 is labeled m_4, so we write $\mathcal{N} = \{m_1, m_2, m_3, m_4\}$. The **standard form of $\boldsymbol{\mathcal{N}}$ for $\boldsymbol{B}$** is a secondary labeling of the consultants in $\mathcal{N}$ such that the *highest* indexed consultant receives the label m_1, the second highest indexed consultant gets the label m_2, and so on until the lowest indexed consultant gets the label m_n.

We now develop a candidate pair of second stage behavioral strategies for A. Let $\mathcal{N}_A$ be a network for A. If $\mathcal{N}_A$ is empty, let

$$\hat{\sigma}_{1A}^{\mathcal{N}_A}(\mathcal{N}, x) = \hat{\sigma}_{2A}^{\mathcal{N}_A}(\mathcal{N}, \mathcal{N}', x) = \emptyset \text{ for all } (\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X.$$

If $\mathcal{N}_A$ is non-empty, let $\{m_1, \ldots, m_n\}$ be the standard form of $\mathcal{N}_A$ for A, where $n = |\mathcal{N}_A|$. For each $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$, let

$$\hat{\sigma}_{1A}^{\mathcal{N}_A}(\mathcal{N}, x) = \begin{cases} \{m_1, m_2 \ldots, m_{d(x)}\} & \text{if } d(x) \leq |\mathcal{N}| \text{ and } \mathcal{N} = \mathcal{N}_A \\ \emptyset & \text{else,} \end{cases}$$

$$\hat{\sigma}_{2A}^{\mathcal{N}_A}(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{m_1, m_2, \ldots, m_{d(x)}\} & \begin{array}{c} \text{if } d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|,\ \mathcal{N}' \subset \{m_{d(x)} + 1, \ldots, N\}, \\ \text{and } \mathcal{N} = \mathcal{N}_A \end{array} \\ \emptyset & \text{else.} \end{cases}$$

We refer to $\hat{\sigma}_{1A}^{\mathcal{N}_A}$ and $\hat{\sigma}_{2A}^{\mathcal{N}_A}$ as ***A*'s hat strategies given $\boldsymbol{\mathcal{N}_A}$**.[22] Under these strategies, when A gets project x, she employs the $d(x)$ consultants in her network with the *lowest* indices, when these consultants are available and there are $d(x)$ consultants in her network. Observe that, when $\mathcal{N}_A = \{1, \ldots, n\}$ for some $n \in \{0, \ldots, N\}$, then $m_1 = 1$, $m_2 = 2$, ..., and $m_n = n$, so the hat strategies are equivalent to A's second stage strategies under her simple strategy.

Consider B. Let $\mathcal{N}_B$ be a network for B. If $\mathcal{N}_B$ is empty, let

$$\hat{\sigma}_{1B}^{\mathcal{N}_B}(\mathcal{N}, x) = \hat{\sigma}_{2A}^{\mathcal{N}_B}(\mathcal{N}, \mathcal{N}', x) = \emptyset \text{ for all } (\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X.$$

If $\mathcal{N}_B$ is non-empty, let $\{m_1, \ldots, m_n\}$ be the standard form of $\mathcal{N}_B$ for B, where $n = |\mathcal{N}_B|$.

[22] It is readily verified that $(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}) \in \boldsymbol{S}_A$ for every $\mathcal{N}_A \subset \mathcal{C}$.

For each $(\mathcal{N}, \mathcal{N}', x) \in \mathbb{P}(\mathcal{C})^2 \times X$, let

$$\hat{\sigma}_{1B}^{\mathcal{N}_B}(\mathcal{N}, x) = \begin{cases} \{m_1, m_2 \ldots, m_{d(x)}\} & \text{if } d(x) \leq |\mathcal{N}| \text{ and } \mathcal{N} = \mathcal{N}_B \\ \emptyset & \text{else,} \end{cases}$$

$$\hat{\sigma}_{2B}^{\mathcal{N}_B}(\mathcal{N}, \mathcal{N}', x) = \begin{cases} \{m_1, m_2, \ldots, m_{d(x)}\} & \text{if } d(x) \leq |\mathcal{N} \backslash \mathcal{N}'|,\ \mathcal{N}' \subset \{1, \ldots, m_{d(x)} - 1\}, \\ & \text{and } \mathcal{N} = \mathcal{N}_B \\ \emptyset & \text{else.} \end{cases}$$

We refer to $\hat{\sigma}_{1B}^{\mathcal{N}_B}$ and $\hat{\sigma}_{2B}^{\mathcal{N}_B}$ as ***B*'s hat strategies given $\boldsymbol{\mathcal{N}_B}$**. Under these strategies, when B gets project x, she employs the $d(x)$ consultants in her network with the *highest* indices, when these consultants are available there are $d(x)$ consultants in her network. Observe that, when $\mathcal{N}_B = \{N + 1 - n, \ldots, N\}$ for some $n \in \{0, \ldots, N\}$, then $m_1 = N$, $m_2 = N - 1$, ..., and $m_n = N + 1 - n$, so the hat strategies are equivalent to A's second stage strategies under her simple strategy.

The next lemma shows that the hat strategies function as desired, i.e., given partner i's network, they implement i's optimal behavior when $-i$ plays a simple strategy.

Lemma A2. Optimality of Hat Strategies.
*LA*2.1 : *Let* $\mathcal{N}_A \subset \mathcal{C}$, *then, for all* $(\sigma_{1A}, \sigma_{2A})$ *such that* $(\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}) \in \boldsymbol{S}_A$ *and all* $n_B \in \{0, \ldots, N\}$,

$$U_A(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}, \tilde{s}_B(n_B)) \geq U_A(\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \tilde{s}_B(n_B)).$$

*LA*2.2 : *Let* $\mathcal{N}_B \subset \mathcal{C}$, *then, for all* $(\sigma_{1B}, \sigma_{2B})$ *such that* $(\mathcal{N}_B, \sigma_{1B}, \sigma_{2B}) \in \boldsymbol{S}_B$ *and all* $n_A \in \{0, \ldots, N\}$,

$$U_B(\tilde{s}_A(n_A), \mathcal{N}_B, \hat{\sigma}_{1B}^{\mathcal{N}_B}, \hat{\sigma}_{2B}^{\mathcal{N}_B}) \geq U_B(\tilde{s}_A(n_A), \mathcal{N}_B, \sigma_{1B}, \sigma_{2B}).$$

Proof. We'll prove $LA2.1$ as the argument for $LA2.2$ is analogous. Since A has expected utility, it suffices to show that

$$u_{1A}(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}, \tilde{s}_B(n_B), x_A, x_B) \geq u_{1A}(\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \tilde{s}_B(n_B), x_A, x_B) \tag{A.1}$$

$$u_{2A}(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}, \tilde{s}_B(n_B), x_A, x_B)) \geq u_{2A}(\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \tilde{s}_B(n_B), x_A, x_B)), \tag{A.2}$$

for each $(x_A, x_B) \in X^2$. Since this argument is trivial if $\mathcal{N}_A = \emptyset$ because A always gets 0 when she gets her project first or second, we take $\mathcal{N}_A$ to be non-empty. Let $\{m_1, \ldots, m_{n_A}\}$ be the standard form of $\mathcal{N}_A$, where $n_A = |\mathcal{N}_A|$.

Let $(x_A, x_B) \in X^2$. Suppose A gets her project first. If $d(x_A) \le |\mathcal{N}_A|$, then $\hat{\sigma}_{1A}^{\mathcal{N}_A} = \{m_1, \ldots, m_{d(x)}\}$ and so A completes her project. If, however, $d(x_A) > |\mathcal{N}_A|$, then $\hat{\sigma}_{1A}^{\mathcal{N}_A} = \emptyset$ and A doesn't complete her project. This behavior is optimal per Remark A1, so we necessarily have that equation (A.1) holds.

Now, suppose A gets her project second. Let $l = d(x_A)$. Since B follows her simple strategy, there are four cases: (i) $d(x_B) > n_B$ and $l > n_A$, (ii) $d(x_B) > n_B$ and $l \le n_A$, (iii) $d(x_B) \le n_B$ and $l > n_A$, and (iv) $d(x_B) \le n_B$ and $l \le n_A$. If (i) or (iii), then $\hat{\sigma}_{2A}^{\mathcal{N}_A} = \emptyset$ as $l > |\mathcal{N}_A|$ and A does not complete her project. This is optimal per Remark A1. Thus, we necessarily have that equation (A.2) holds.

If (ii), then B employs no consultants, i.e., $\tilde{\sigma}_{1B} = \emptyset$. Since $d(x_A) \le n_A = |\mathcal{N}_A \backslash \tilde{\sigma}_{1B}|$, we've $\hat{\sigma}_{2A}^{\mathcal{N}_A} = \{m_1, \ldots, m_l\}$, so A completes her project. This is also optimal per Remark A1. Thus, we necessarily have that equation (A.2) holds.

If (iv), then A may or may not complete her project. Since A follows her hat strategy, she completes her project if and only if B doesn't employ a consultant with an index at or below m_l.[23] Since B follows her simple strategy, she'll employ a consultant with an index at or below m_l if and only if $d(x_B) \ge N + 1 - m_l$.[24]

If $d(x_B) \ge N + 1 - m_l$, then $\hat{\sigma}_{2A}^{\mathcal{N}_A} = \emptyset$ and A does *not* complete her project. This is optimal by Remark A1. Specifically, B employs all consultants with indices of $N + 1 - d(x_B)$ and above, let $\mathcal{T}$ denote this set. Since $m_l \ge N + 1 - d(x_B)$, $\mathcal{N}_A$ less $\mathcal{T}$ is a *subset* of $\{m_1, \ldots, m_{l-1}\}$. Thus, $l > |\mathcal{N}_A \backslash \mathcal{T}|$ and so Remark A1 gives that it's best for A not to complete her project. Thus, we necessarily have that equation (A.2) holds.

If $d(x_B) < N + 1 - m_l$, then $\hat{\sigma}_{2A}^{\mathcal{N}_A} = \{m_1, \ldots, m_l\}$ and A completes her project. This is optimal by Remark A1. Simply, B employs all consultants with indices of $N + 1 - d(x_B)$ or above, let $\mathcal{T}$ denote this set. Since $N + 1 - d(x_B) > m_l$, $\mathcal{N}_A$ less $\mathcal{T}$ *contains* $\{m_1, \ldots, m_l\}$. It follows that $|\mathcal{N}_A \backslash \mathcal{T}| \ge l$, so Remark A1 gives that it's best for A to complete her project. Thus, equation (A.2) necessarily holds. □

The next lemma calculates the partner's expected payoffs to different networks when they use their hat strategies in the second stage and the other partner plays a simple strategy.

Lemma A3. Payoffs to Different Networks.
Let $i \in \{A, B\}$, let $n_{-i} \in \{0, \ldots, N\}$, and let $\mathcal{N}_i \subset \mathcal{C}$.

[23] To see this, suppose B employs a set of consultants $\mathcal{T}$. If $\mathcal{T} \not\subset \{m_l + 1, \ldots, N\}$, then $\hat{\sigma}_{2A}^{\mathcal{N}_A} = \emptyset$ (by construction) and A fails to complete her project. If, however, $\mathcal{T} \subset \{m_l + 1, \ldots, N\}$, then $|\mathcal{N}_A \backslash \mathcal{T}| \ge l$ and so $\tilde{\sigma}_{2A}^{\mathcal{N}_A} = \{m_1, \ldots, m_l\}$, implying A completes her project.

[24] Recall that B employs $\{N+1-d(x_B), \ldots, N\}$. If $d(x_B) < N+1-m_l$, then B doesn't employ a consultant with an index at or below m_l as $d(x_B) < N+1-m_l \implies N+1-d(x_B) > m_l$. If, however, $d(x_B) \ge N+1-m_l$, then B employs a consultant with an index at or below m_l as $d(x_B) \ge N+1-m_l \implies N+1-d(x_B) \le m_l$.

*LA*3.1 : *If* $\mathcal{N}_i = \emptyset$, *then*

$$U_i(\mathcal{N}_i, \hat{\sigma}_{1i}^{\mathcal{N}_i}, \hat{\sigma}_{2i}^{\mathcal{N}_i}, \tilde{\boldsymbol{s}}_{-i}(n_{-i})) = 0.$$

*LA*3.2 : *If* $\mathcal{N}_i \neq \emptyset$, *let* $\{m_1, \ldots, m_n\}$ *be the standard form of* $\mathcal{N}_i$ *for partner* i, *where* $n = |\mathcal{N}_i|$. *Then,*

$$U_i(\mathcal{N}_i, \hat{\sigma}_{1i}^{\mathcal{N}_i}, \hat{\sigma}_{2i}^{\mathcal{N}_i}, \tilde{\boldsymbol{s}}_{-i}(n_{-i})) = \sum_{\{x|d(x)\leq n\}} (r_i(x) - w\, d(x))\frac{P_i(x)}{2},$$
$$+ \sum_{l=1}^{n} \sum_{\{x|d(x)=l\}} (r_i(x) - w\, d(x))\frac{P_i(x)}{2} h_i^{\mathcal{N}_i}(l, n_{-i}) - c_i(n)$$

where, for $l \in \{1, \ldots, N\}$,

$$h_i^{\mathcal{N}_i}(l, n_{-i}) = \begin{cases} 1 - \sum_{\{x|N+1-m_l \leq d(x) \leq n_{-i}\}} P_{-i}(x) & \text{if } i = A \\ 1 - \sum_{\{x|m_l \leq d(x) \leq n_{-i}\}} P_{-i}(x) & \text{if } i = B. \end{cases}$$

Proof. The argument is analogous to the proof of Lemma 1. The only difference is when a partner moves second. We document this difference and omit the balance of the argument. We focus on $i = A$ as the argument when $i = B$ is analogous. As usual we take $\mathcal{N}_A \neq \emptyset$ to avoid trivialities.

Let x_A and x_B be A and B's projects. Let $l = d(x_A)$ and let $n_A = |\mathcal{N}_A|$. There are four cases: (i) $d(x_B) > n_B$ and $l > n_A$, (ii) $d(x_B) \leq n_B$ and $l > n_A$, (iii) $d(x_B) > n_B$ and $l \leq n_A$, and (iv) $d(x_B) \leq n_B$ and $l \leq n_A$. If (i) or (ii), then $\hat{\sigma}_{2A}^{\mathcal{N}_A} = \emptyset$ and A earns nothing, while if (iii) then B employs no consultants, so $\hat{\sigma}_{2A}^{\mathcal{N}_A} = \{m_1, \ldots, m_l\}$ and A earns $r_A(x_A) - w\, l$.

If (iv), then A may or may not complete her project. Since A follows $\hat{\sigma}_{2A}^{\mathcal{N}_A}$, she'll complete her project if and only if B employs no consultant with an index at or below m_l. Since B follows her simple strategy, she'll employ a consultant with an index at or below m_l if and only if $d(x_B) \geq N + 1 - m_l$. Thus, A earns $r_A(x_A) - w\, l$ if $d(x_B) < N + 1 - m_l$ and earns 0 if $d(x_B) \geq N + 1 - m_l$.

It follows that, A's expected payoff from getting project x_A second are:
(i) 0 if $d(x_A) > n_A$, and
(ii) $(r_A(x_A) - w\, d(x_A))(1 - \sum_{\{x|N+1-m_l \leq d(x) \leq n_B\}} P_B(x))$ if $d(x_A) \leq n_A$.
The first is follows directly from cases (i) and (ii). The second follows from cases (iii) and (iv), where A gets $r_A(x_A) - w\, d(x_A)$ if $d(x_B) < N + 1 - m_l$ or $d(x_B) > n_B$, an event of probability $1 - \sum_{\{x|N+1-m_l \leq d(x) \leq n_B\}} P_B(x)$. Thus, before networking costs, A's expected

payoff from getting her project second is

$$\sum_{\{x|d(x)\leq n_A\}} (r_A(x) - w\, d(x)) h_A^{\mathcal{N}_A}(d(x_A), n_B) P_A(x).$$

The lemma follows. □

The next lemma shows that a partner always does best by playing a simple strategy when the other partner plays a simple strategy.

Lemma A4. Optimality of Simple Strategies.
$LA4.1$: *Let* $\mathcal{N}_A \subset \mathcal{C}$*, let* $n_A = |\mathcal{N}_A|$*, and let* $n_B \in \{0, \ldots, N\}$*. Then,*

$$U_A(\tilde{\boldsymbol{s}}_A(n_A), \tilde{\boldsymbol{s}}_B(n_B)) \geq U_A(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}, \tilde{\boldsymbol{s}}_B(n_B)).$$

$LA4.2$: *Let* $\mathcal{N}_B \subset \mathcal{C}$*, let* $n_B = |\mathcal{N}_B|$*, and let* $n_A \in \{0, \ldots, N\}$*. Then,*

$$U_B(\tilde{\boldsymbol{s}}_A(n_A), \tilde{\boldsymbol{s}}_B(n_B)) \geq U_B(\tilde{\boldsymbol{s}}_A(n_A), \mathcal{N}_B, \hat{\sigma}_{1B}^{\mathcal{N}_B}, \hat{\sigma}_{2B}^{\mathcal{N}_B}).$$

The key insight of $LA4.1$ is that the (ex-ante) probability B employs a consultant is monotone increasing in the index of the consultant – B employs consultant N with the highest probability, consultant $N-1$ with the second highest probability, and so on. Since A wants to network with consultants she'll be able to employ with high probability, she does best by networking with the lowest indexed consultants.

Proof. We prove $LA4.1$ as the argument for $LA4.2$ is analogous. If $\mathcal{N}_A$ is empty, the result is true as A makes zero. Thus, we take $\mathcal{N}_A$ to be non-empty.

Write $\tilde{\boldsymbol{s}}_A(n_A) = (\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1A}, \tilde{\sigma}_{2A})$ and recall that $\hat{\sigma}_{1A}^{\tilde{\mathcal{N}}_A} = \tilde{\sigma}_{1A}$ and $\hat{\sigma}_{2A}^{\tilde{\mathcal{N}}_A} = \tilde{\sigma}_{2A}$ since $\tilde{\mathcal{N}}_A = \{1, \ldots, n_A\}$, i.e., A's hat strategies are the same as her second stage strategies under her simple strategy of size n_A. Thus, $U_A(\tilde{\boldsymbol{s}}_A(n_A), \tilde{\boldsymbol{s}}_B(n_B)) = U_A(\tilde{\mathcal{N}}_A, \hat{\sigma}_{1A}^{\tilde{\mathcal{N}}_A}, \hat{\sigma}_{2A}^{\tilde{\mathcal{N}}_A}, \tilde{\boldsymbol{s}}_B(n_B))$. So we only need to show that

$$U_A(\tilde{\mathcal{N}}_A, \hat{\sigma}_{1A}^{\tilde{\mathcal{N}}_A}, \hat{\sigma}_{2A}^{\tilde{\mathcal{N}}_A}, \tilde{\boldsymbol{s}}_B(n_B)) - U_A(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}, \tilde{\boldsymbol{s}}_B(n_B)) \geq 0.$$

Since both of A's networks are of size n_A, Lemma A3 gives

$$U_A(\tilde{\mathcal{N}}_A, \hat{\sigma}_{1A}^{\tilde{\mathcal{N}}_A}, \hat{\sigma}_{2A}^{\tilde{\mathcal{N}}_A}, \tilde{\boldsymbol{s}}_B(n_B)) - U_A(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}, \tilde{\boldsymbol{s}}_B(n_B))$$
$$= \sum_{l=1}^{n_A} \sum_{\{x|d(x)=l\}} (r_A(x) - w\, d(x)) \frac{P_A(x)}{2} (h_A^{\tilde{\mathcal{N}}_A}(l, n_B) - h_A^{\mathcal{N}_A}(l, n_B)).$$

Since Assumption 1 holds and $P_A \geq 0$, we only need to show that $h_A^{\tilde{\mathcal{N}}_A}(l, n_B) - h_A^{\mathcal{N}_A}(l, n_B) \geq 0$.

Let $\{\tilde{m}_1, \ldots, \tilde{m}_{n_A}\}$ be the standard from of $\tilde{\mathcal{N}}_A$ for A and let $\{m_1, \ldots, m_{n_A}\}$ be the standard form of $\mathcal{N}_A$ for A. Thus,

$$h_A^{\tilde{\mathcal{N}}_A}(l, n_B) - h_A^{\mathcal{N}_A}(l, n_B) = \sum_{\{x | N+1-m_l \leq d(x) \leq n_B\}} P_B(x) - \sum_{\{x | N+1-\tilde{m}_l \leq d(x) \leq n_B\}} P_B(x).$$

Observe that, for every $k \in \{1, \ldots, n_A\}$, we've $\tilde{m}_k \leq m_k$.[25] Thus, $N + 1 - m_l \leq N + 1 - \tilde{m}_l$. It follows that

$$h_A^{\tilde{\mathcal{N}}_A}(l, n_B) - h_A^{\mathcal{N}_A}(l, n_B) = \sum_{\{x | N+1-m_l \leq d(x) < N+1-\tilde{m}_l\}} P_B(x) \geq 0,$$

where the inequality follows from the fact that $P_B \geq 0$. □

Proof of Lemma 3. Let $(z_A^\star, z_B^\star)$ be an equilibrium of the Auxiliary Game. We'll prove that $\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star) = (\tilde{\boldsymbol{s}}_A(z_A^\star), \tilde{\boldsymbol{s}}_B(N - z_B^\star))$ is an equilibrium by showing that

$$U_A(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_A(\boldsymbol{s}_A, \tilde{\boldsymbol{s}}_B(N - z_B^\star)) \text{ for all } \boldsymbol{s}_A \in \boldsymbol{S}_A \text{ and}$$
$$U_B(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_B(\tilde{\boldsymbol{s}}_A(z_A^\star), \boldsymbol{s}_B) \text{ for all } \boldsymbol{s}_B \in \boldsymbol{S}_B.$$

We'll establish this for A since the argument for B is analogous.

Since $(z_A^\star, z_B^\star)$ is an equilibrium of the Auxiliary Game,

$$U_A(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_A(\tilde{\boldsymbol{s}}(n_A, N - z_B^\star)) \text{ for all } n_A \in \{0, \ldots, N\}.$$

Thus, Lemma A4 implies that

$$U_A(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_A(\mathcal{N}_A, \hat{\sigma}_{1A}^{\mathcal{N}_A}, \hat{\sigma}_{2A}^{\mathcal{N}_A}, \tilde{\boldsymbol{s}}_B(N - z_B^\star)) \text{ for all } \mathcal{N}_A \subset \mathcal{C}.$$

Hence, Lemma A2 gives

$$U_A(\tilde{\boldsymbol{s}}(z_A^\star, N - z_B^\star)) \geq U_A(\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \tilde{\boldsymbol{s}}_B(N - z_B^\star)) \text{ for all } (\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}) \in \boldsymbol{S}_A.$$

The desired result follows. □

Proof of Lemma 4

To prove Lemma 4, we first need to refine our understanding of the partners' optimal

[25]To see this, note that for every $j \in \{1, \ldots, n_A\}$, we have $j \leq m_j$. Simply, $m_1 \geq 1$. Since $m_2 > m_1$, we necessarily have $m_2 \geq m_1 + 1 \geq 2$. Continuing establishes the desired result. Since $\tilde{m}_1 = 1$, $\tilde{m}_2 = 2$, ..., and $\tilde{m}_{n_A} = n_A$, we have that $\tilde{m}_k \leq m_k$ for every $k \in \{1, \ldots, n_A\}$.

behavior in the second stage. Once we do this, we'll prove a preliminary lemma: that partners always behave according to Remark A1 in any equilibrium when Assumption 2 holds. Subsequently, we'll prove Lemma 4.

Remark A2. Unique Optimal Behavior in the Second Stage.
Under Assumption A2, the optimal behavior described in Remark A1 is the *unique* optimal behavior in the second stage. This follows from the facts $w > 0$ and $r_i(x) - w\,d(x) > 0$ for each partner i and each $x \in X$. (Since $r_i(x) - w\,d(x) > 0$, it's always best for i to complete project x by employing at least $d(x)$ consultants. Since $w > 0$, it's best for i to never employ more than $d(x)$ consultants.) Consequently, if partner i behaves in any other manner, she can do *strictly* better by switching to the behavior described in Remark A1.

With this in mind, we introduce a notion inspired by sequential rationality. Let $\boldsymbol{s} = (\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \mathcal{N}_B, \sigma_{1B}, \sigma_{2B}) \in \boldsymbol{S}$ and let $\mathcal{E}_i^{\boldsymbol{s}}$ denote the set of sets of consultants partner i employs under $\boldsymbol{s}$ when she gets her project first, i.e.,

$$\mathcal{E}_i^{\boldsymbol{s}} = \{\mathcal{T} \in \mathbb{P}(\mathcal{C}) | \mathcal{T} = \sigma_{1i}(\mathcal{N}_i, x) \text{ for some } x \in X\}.$$

Definition. We say that a $\boldsymbol{s} = (\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \mathcal{N}_B, \sigma_{1B}, \sigma_{2B}) \in \boldsymbol{S}$ is *weakly rational for partner i* if she behaves optimally in the second stage when the other partner follows $\boldsymbol{s}$. That is, if, for each $(\mathcal{T}, x) \in \mathcal{E}_{-i}^{\boldsymbol{s}} \times X$, we have:
(i) $|\sigma_{1i}(\mathcal{N}_i, x)| = d(x)$ when $d(x) \leq |\mathcal{N}_i|$ and $|\sigma_{1i}(\mathcal{N}_i, x)| = 0$ when $d(x) > |\mathcal{N}_i|$; and
(ii) $|\sigma_{2i}(\mathcal{N}_i, \mathcal{T}, x)| = d(x)$ when $d(x) \leq |\mathcal{N}_i \backslash \mathcal{T}|$ and $|\sigma_{2i}(\mathcal{N}_i, \mathcal{T}, x)| = 0$ when $d(x) > |\mathcal{N}_i \backslash \mathcal{T}|$.
We say that $\boldsymbol{s}$ is *weakly rational* if it is weakly rational for both partners A and B.

Notice that simple strategies of any size are always weakly rational. The next lemma describes another type of weakly rational strategy vector.

Lemma A5. Every Equilibrium is Weak Rationality.
Let Assumption 2 hold, then each $\boldsymbol{s}^\star \in \boldsymbol{E}$ is weakly rational.

The lemma follows from the fact that every project occurs with positive probability. Thus, if a partner follows a strategy that isn't weakly rational, she can do strictly better in expectation by switching to a strategy that is weakly rational, violating the conjecture of equilibrium.

Proof. This is almost obvious, we only give the proof for completeness. We argue by contradiction. Let $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star, \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star) \in \boldsymbol{E}$. We suppose, without loss, that A's strategy $(\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star)$ isn't weakly rational for her; the argument is analogous for B. We'll establish that $(\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star)$ isn't a best response for A when B plays according to $\boldsymbol{s}^\star$, a contradiction.

There are three ways in which $(\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star)$ may not be weakly rational:

1. There is an $x' \in X$ such that $|\sigma_{1A}^\star(\mathcal{N}_A^\star, x')| \neq d(x')$ when $d(x) \leq |\mathcal{N}_A^\star|$ or $|\sigma_{1A}^\star(\mathcal{N}_A^\star, x)| \neq 0$ when $d(x) > |\mathcal{N}_A^\star|$.

   In this case, upon getting project x first, A does strictly better by employing exactly $d(x)$ consultants when $d(x) \leq |\mathcal{N}_A^\star|$ or employing no consultants when $d(x) > |\mathcal{N}_A^\star|$.

2. There is an $(\mathcal{T}', x') \in \mathcal{E}_B^{\boldsymbol{s}} \times X$, such that $|\sigma_{2A}^\star(\mathcal{N}_A^\star, \mathcal{T}', x')| \neq d(x')$ when $d(x') \leq |\mathcal{N}_A^\star \backslash \mathcal{T}'|$ or $|\sigma_{2A}^\star(\mathcal{N}_A^\star, \mathcal{T}', x')| \neq 0$ when $d(x') > |\mathcal{N}_A^\star \backslash \mathcal{T}'|$.

   In this case, upon getting project x' second and observing B employ consultants in $\mathcal{T}'$, A does strictly better by employing exactly $d(x')$ consultants when $d(x') \leq |\mathcal{N}_A^\star \backslash \mathcal{T}'|$ or employing no consultants when $d(x') > |\mathcal{N}_A^\star \backslash \mathcal{T}'|$.

3. Both 1 and 2.

We'll only consider the second case, since it's the hardest and the other two are analogous.

Suppose case 2 occurs. Let σ_{2A}' be a new strategy for A such that $\sigma_{2A}'(\mathcal{N}, \mathcal{N}', x) = \sigma_{2A}^\star(\mathcal{N}, \mathcal{N}', x)$ for all $(\mathcal{N}, \mathcal{N}', x) \in (\mathbb{P}(\mathcal{C})^2 \times X) \backslash \{(\mathcal{N}_A^\star, \mathcal{T}', x')\}$ and such that $\sigma_{2A}'(\mathcal{N}_A^\star, \mathcal{T}', x')$ selects $d(x')$ consultants from $\mathcal{N}_A^\star \backslash \mathcal{T}'$ when $d(x') \leq |\mathcal{N}_A^\star \backslash \mathcal{T}'|$ or select no consultants when $d(x') > |\mathcal{N}_A^\star \backslash \mathcal{T}'|$. Let $\boldsymbol{s}' = (\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}', \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star)$.

Consider the difference of A's payoff under $\boldsymbol{s}'$ and $\boldsymbol{s}^\star$. Since (i) A has the same network in both strategies, (ii) follows the same behavioral strategy when she gets her project first, and (iii) follows the same behavioral strategy when she gets her project second, unless B gets a project that causes her to employ $\mathcal{T}'$ and A gets project x',

$$U_A(\boldsymbol{s}') - U_A(\boldsymbol{s}^\star) = \sum_{\{x_B | \sigma_{1B}^\star(\mathcal{N}_B^\star, x_B) = \mathcal{T}'\}} (u_{2A}(\boldsymbol{s}', x', x_B) - u_{2A}(\boldsymbol{s}^\star, x', x_B)) \frac{P_A(x') P_B(x_B)}{2}.$$

Since $P_A > 0$ and $P_B > 0$, this difference is strictly positive if $u_{2A}(\boldsymbol{s}', x', x_B) > u_{2A}(\boldsymbol{s}^\star, x', x_B)$, which is exactly the case since A's behavior under $\boldsymbol{s}'$ is optimal and her behavior under $\boldsymbol{s}^\star$ is sub-optimal. Thus, $(\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star)$ is not a best response for A when B plays according to $\boldsymbol{s}^\star$. □

Proof of Lemma 4. We argue by contradiction. Suppose that $\mathcal{N}_A^\star \cap \mathcal{N}_B^\star \neq \emptyset$ and that $\mathcal{C} \not\subset \mathcal{N}_A^\star \cup \mathcal{N}_B^\star$. Then there is a consultant $j \in \mathcal{C} \backslash (\mathcal{N}_A^\star \cup \mathcal{N}_B^\star)$. We'll show that A can do strictly better by swapping a consultant $k \in \mathcal{N}_A^\star \cap \mathcal{N}_B^\star$ for consultant j, when B follows $\boldsymbol{s}^\star$. It follows that $\boldsymbol{s}^\star$ cannot be an equilibrium.

We proceed by constructing a "post-swap" strategy for A and showing that this strategy leads to a strictly higher payoff than A gets in equilibrium. Write $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star, \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star)$ for the equilibrium given in the statement of this lemma.

Let $X_A \subset X$ such that $x \in X_A$ implies $d(x) = |\mathcal{N}_A^\star \backslash \mathcal{N}_B^\star| + 1$. Also, let $X_B \subset X$ such that $x \in X_B$ implies $d(x) = |\mathcal{N}_B^\star|$.[26] By Lemma A5, $x \in X_B$ if and only if $\sigma_{1B}^\star(\mathcal{N}_B^\star, x_B) = \mathcal{N}_B^\star$. (To see this, let $x \in X_B$. Since every equilibrium is weakly rational, we must have $|\sigma_{1B}^\star(\mathcal{N}_B^\star, x)| = d(x) = |\mathcal{N}_B^\star|$, which implies $\sigma_{1B}^\star(\mathcal{N}_B^\star, x) = \mathcal{N}_B^\star$. Conversely, let $x \notin X_B$. Then either $d(x) < |\mathcal{N}_B^\star|$, implying $|\sigma_{1B}^\star(\mathcal{N}_B^\star, x)| < |\mathcal{N}_B^\star|$, or $d(x) > |\mathcal{N}_B^\star|$, implying $|\sigma_{1B}^\star(\mathcal{N}_B^\star, x')| = 0$. In both cases, $\sigma_{1B}^\star(\mathcal{N}_B^\star, x')$ is a strict subset of $\mathcal{N}_B^\star$.)

We now write down a post-swap strategy for A, which we denote $(\mathcal{N}_A', \sigma_{1A}', \sigma_{2A}')$. Since A swaps k for j, we have $\mathcal{N}_A' = \{j\} \cup \mathcal{N}_A^\star \backslash \{k\}$. We'll choose σ_{1A}' so that, when A observes her network is $\mathcal{N}_A'$ and observes her project is x, she employs exactly the same consultants as she would under $\sigma_{1A}^\star$, when she observes her network is $\mathcal{N}_A^\star$ and observes her project is x, save she swaps k for j. Formally, for every $x \in X$, let

$$\sigma_{1A}'(\mathcal{N}_A', x) = \begin{cases} \sigma_{1A}^\star(\mathcal{N}_A^\star, x) & \text{if } k \notin \sigma_{1A}^\star(\mathcal{N}_A^\star, x) \\ \{j\} \cup \sigma_{1A}^\star(\mathcal{N}_A^\star, x) \backslash \{k\} & \text{if } k \in \sigma_{1A}^\star(\mathcal{N}_A^\star, x). \end{cases}$$

And for every $(\mathcal{N}, x) \in (\mathbb{P}(\mathcal{C}) \backslash \mathcal{N}_A') \times X$, let $\sigma_{1A}'(\mathcal{N}, x) = \emptyset$.

We'll choose σ_{2A}' so that, when A observes her network is $\mathcal{N}_A'$, observes B employ $\mathcal{T} \in \mathcal{E}_B^{\boldsymbol{s}^\star}$, and observes her project is x, she employs exactly the same consultants as she would under $\sigma_{2A}^\star$, when she observes her network is $\mathcal{N}_A^\star$, observes B employ $\mathcal{T}$, and observes her project is x, save she swaps k for j. Formally, for every $(\mathcal{T}, x) \in \mathcal{E}_B^{\boldsymbol{s}^\star} \times X$, let

$$\sigma_{2A}'(\mathcal{N}_A', \mathcal{T}, x) = \begin{cases} \sigma_{2A}^\star(\mathcal{N}_A^\star, \mathcal{T}, x) & \text{if } k \notin \sigma_{2A}^\star(\mathcal{N}_A^\star, \mathcal{T}, x) \\ \{j\} \cup \sigma_{2A}^\star(\mathcal{N}_A^\star, \mathcal{T}, x) \backslash \{k\} & \text{if } k \in \sigma_{2A}^\star(\mathcal{N}_A^\star, \mathcal{T}, x). \end{cases}$$

And for every $(\mathcal{N}, \mathcal{N}', x) \in (\mathbb{P}(\mathcal{C})^2 \times X) \setminus \big(\{\mathcal{N}_A'\} \times \mathcal{E}_B^{\boldsymbol{s}^\star} \times X\big)$, let $\sigma_{2A}'(\mathcal{N}, \mathcal{N}', x) = \emptyset$.

We make one modification to σ_{2A}' before proceeding: if A observes B employ $\mathcal{N}_B^\star$ and then gets a project in X_A, she employs all of $\mathcal{N}_A' \backslash \mathcal{N}_B^\star$. Formally, for every $x \in X_A$, let $\sigma_{2A}'(\mathcal{N}_A', \mathcal{N}_B^\star, x) = \mathcal{N}_A' \backslash \mathcal{N}_B^\star$. It is readily verified that $(\mathcal{N}_A', \sigma_{1A}', \sigma_{2A}') \in \boldsymbol{S}_A$. Let $\boldsymbol{s}' = (\mathcal{N}_A', \sigma_{1A}', \sigma_{2A}', \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star)$.

Before we establish that A makes strictly under $\boldsymbol{s}'$ than $\boldsymbol{s}^\star$, we need two preliminary facts. First, $|\sigma_{1A}'(\mathcal{N}_A', x)| = |\sigma_{1A}^\star(\mathcal{N}_A^\star, x)|$ for all $x \in X$, i.e., A uses the same number of consultants to complete project x under $\boldsymbol{s}'$ and $\boldsymbol{s}^\star$ when she moves first in the second stage. This is a

[26] Observe that X_A and X_B are non-empty. Since j isn't in both partner's networks, we have $|\mathcal{N}_A| \leq N - 1$ and $|\mathcal{N}_B| \leq N - 1$. Thus, we have $0 \leq |\mathcal{N}_A^\star \backslash \mathcal{N}_B^\star| \leq N - 1$. Hence, Assumption 2 gives that there's a x' and x'' in X such that $d(x') = |\mathcal{N}_A^\star \backslash \mathcal{N}_B^\star| + 1$ and $d(x'') = |\mathcal{N}_B^\star|$.

direct consequence the construction of σ'_{1A}. Second, for all $(x_A, x_B) \in X^2 \backslash (X_A \times X_B)$,

$$|\sigma'_{2A}(\mathcal{N}'_A, \sigma^\star_{1B}(\mathcal{N}^\star_B, x_B), x_A)| = |\sigma^\star_{2A}(\mathcal{N}^\star_A, \sigma^\star_{1B}(\mathcal{N}^\star_B, x_B), x_A)|.$$

That is, given A and B's projects are in $X^2 \backslash (X_A \times X_B)$, then A uses the same number of consultants to complete x_A under $\boldsymbol{s}'$ and $\boldsymbol{s}^\star$ when she moves second. Let's establish this. Since $\sigma^\star_{1B}(\mathcal{N}^\star_B, x_B) = \mathcal{N}^\star_B$ if and only if $x_B \in X_B$, we have $(\sigma^\star_{1B}(\mathcal{N}^\star_B, x_B), x_A) \in \{\mathcal{N}^\star_B\} \times X_A$ if and only if $(x_A, x_B) \in X_A \times X_B$. Thus, $(\sigma^\star_{1B}(\mathcal{N}^\star_B, x_B), x_A) \in \left(\mathcal{E}^{\boldsymbol{s}^\star}_B \times X\right) \setminus (\{\mathcal{N}^\star_B\} \times X_A)$ if and only if $(x_A, x_B) \in X^2 \setminus (X_A \times X_B)$. Since $|\sigma'_{2A}(\mathcal{N}'_A, \mathcal{T}, x_A)| = |\sigma^\star_{2A}(\mathcal{N}^\star_A, \mathcal{T}, x_A)|$ for all $(\mathcal{T}, x_A) \in \left(\mathcal{E}^{\boldsymbol{s}^\star}_B \times X\right) \setminus (\{\mathcal{N}^\star_B\} \times X_A)$ by construction,[27] we have the secondary preliminary fact.

Recall that A's ex-post payoff depends only on the size of her network and the number of consultants she employs. Since A holds the same sized network under $\boldsymbol{s}'$ and $\boldsymbol{s}^\star$, the first preliminary fact implies $u_{1A}(\boldsymbol{s}', x_A, x_B) = u_{1A}(\boldsymbol{s}^\star, x_A, x_B)$ for all $(x_A, x_B) \in X^2$, while the second preliminary fact implies that $u_{2A}(\boldsymbol{s}', x_A, x_B) = u_{2A}(\boldsymbol{s}^\star, x_A, x_B)$ for all $(x_A, x_B) \in X^2 \backslash (X_A \times X_B)$. Thus,

$$U_A(\boldsymbol{s}') - U_A(\boldsymbol{s}) = \sum_{(x_A, x_B) \in X_A \times X_B} (u_{2A}(\boldsymbol{s}', x_A, x_B) - u_{2A}(\boldsymbol{s}^\star, x_A, x_B)) \frac{P_A(x_A) P_B(x_B)}{2}.$$

Under $\boldsymbol{s}^\star$, when A moves second, she employs no consultants when her project x_A is in X_A and B's project x_B is in X_B. Since B employs her entire network, A is left with $|\mathcal{N}^\star_A \backslash \mathcal{N}^\star_B|$ consultants to possibly employ. Since $x \in X_A$ implies $d(x) > |\mathcal{N}^\star_A \backslash \mathcal{N}^\star_B|$, it's optimal for A to employ no consultants per Remarks A1 and A2. Since $\boldsymbol{s}^\star$ is an equilibrium, Lemma A5 tells us that this is exactly what A does. Hence, $u_{2A}(\boldsymbol{s}^\star, x_A, x_B) = -c_A(|\mathcal{N}^\star_A|)$.

Under $\boldsymbol{s}'$, when A moves second, she employs $\mathcal{N}'_A \backslash \mathcal{N}^\star_B$ if her project x_A is in X_A and B's project x_B is in X_B by construction of σ'_{2A}. Since $|\mathcal{N}'_A \backslash \mathcal{N}^\star_B| = |\{j\} \cup \mathcal{N}^\star_A \backslash \mathcal{N}^\star_B| = 1 + |\mathcal{N}^\star_A \backslash \mathcal{N}^\star_B|$ (as $j \notin \mathcal{N}^\star_B$ and $k \in \mathcal{N}^\star_B$) and $d(x_A) = 1 + |\mathcal{N}^\star_A \backslash \mathcal{N}^\star_B|$, A completes her project and gets a payoff of $u_{2A}(\boldsymbol{s}^\star, x_A, x_B) = r_A(x_A) - w\, d(x_A) - c_A(\mathcal{N}^\star_A)$.

It follows that

$$U_A(\boldsymbol{s}') - U_A(\boldsymbol{s}) = \sum_{(x_A, x_B) \in X_A \times X_B} (r_A(x_A) - w\, d(x_A)) P_A(x_A) P_B(x_B) > 0,$$

where the strict inequality follows from Assumption 2. □

Proof of Proposition 3

[27] This equality does not hold on $\{\mathcal{N}^\star_B\} \times X_A$ due to our modification.

To prove Proposition 3, we first establish a useful technical lemma concerning the payoffs of a partner in a certain type of strategy. We then leverage this lemma to prove the proposition. We say that a $\boldsymbol{s} = (\mathcal{N}_A, \ldots, \mathcal{N}_B, \ldots) \in \boldsymbol{S}$ has the **covering property** if $\mathcal{N}_A \cap \mathcal{N}_B \neq \emptyset$ implies $\mathcal{C} \subset \mathcal{N}_A \cup \mathcal{N}_B$. Lemma 4 gives that all equilibria have the covering property under Assumption 2.

Lemma A6. Payoff Equivalent Strategies.
LA6.1 : Let $\boldsymbol{s} = (\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \mathcal{N}_B, \sigma_{1B}, \sigma_{2B}) \in \boldsymbol{S}$ *such that: (i)* $\boldsymbol{s}$ *has the covering property, (ii)* $\boldsymbol{s}$ *is employment efficient for* B*, (iii)* $\boldsymbol{s}$ *is weakly rational for* A*, and (iv)* σ_{1B} *fulfills part (i) of the definition of weak rationality for* B*. Then,* $U_A(\boldsymbol{s}) = U_A(\tilde{\boldsymbol{s}}(|\mathcal{N}_A|, |\mathcal{N}_B|))$.
LA6.2 : Let $\boldsymbol{s} = (\mathcal{N}_A, \sigma_{1A}, \sigma_{2A}, \mathcal{N}_B, \sigma_{1B}, \sigma_{2B}) \in \boldsymbol{S}$ *such that: (i)* $\boldsymbol{s}$ *has the covering property, (ii)* $\boldsymbol{s}$ *is employment efficient for* A*, (iii)* $\boldsymbol{s}$ *is weakly rational for* B*, and (iv)* σ_{1A} *fulfills part (i) of the definition of weak rationality for* A*. Then,* $U_B(\boldsymbol{s}) = U_B(\tilde{\boldsymbol{s}}(|\mathcal{N}_A|, |\mathcal{N}_B|))$.

The intuition behind this lemma is exactly the same as the intuition behind Proposition 3: whenever partner i moves first in the second stage, her behavior is a permutation of her behavior in the corresponding simple equilibrium.

Proof. We prove $LA6.1$ as the argument for $LA6.2$ is analogous. Let $\tilde{\boldsymbol{s}}$ denote $\tilde{\boldsymbol{s}}(|\mathcal{N}_A|, |\mathcal{N}_B|) = (\tilde{\mathcal{N}_A}, \tilde{\sigma}_{1A}, \tilde{\sigma}_{2A}, \tilde{\mathcal{N}_B}, \tilde{\sigma}_{1B}, \tilde{\sigma}_{2B})$. We prove the lemma by establishing that (i) $u_{1A}(\boldsymbol{s}, x_A, x_B) = u_{1A}(\tilde{\boldsymbol{s}}, x_A, x_B)$ and (ii) $u_{2A}(\boldsymbol{s}, x_A, x_B) = u_{2A}(\tilde{\boldsymbol{s}}, x_A, x_B)$ for every $(x_A, x_B) \in X^2$. It follows that $U_A(\boldsymbol{s}) = U_A(\tilde{\boldsymbol{s}})$. Let $n_A = |\mathcal{N}_A|$ and let $n_B = |\mathcal{N}_A|$.

We need to establish a preliminary result: $|\mathcal{N}_A \cap \mathcal{N}_B| = |\tilde{\mathcal{N}_A} \cap \tilde{\mathcal{N}_B}|$. There are two cases, $\mathcal{N}_A \cap \mathcal{N}_B = \emptyset$ and $\mathcal{N}_A \cap \mathcal{N}_B \neq \emptyset$. If $N_A \cap N_B = \emptyset$, then $\mathcal{N}_A$, $\mathcal{N}_B$, and $\mathcal{S} = \mathcal{C} \setminus (\mathcal{N}_A \cup \mathcal{N}_B)$ partition $\mathcal{C}$. Thus, $N = n_A + n_B + |\mathcal{S}|$, implying $N \geq n_A + n_B$. Since

$$|\tilde{\mathcal{N}_A} \cap \tilde{\mathcal{N}_B}| = |\{N+1-n_B, \ldots, n_A\}| = \begin{cases} 0 & \text{if } n_A + n_B \leq N \\ n_A + n_B - N & \text{if } n_A + n_B > N, \end{cases}$$

(by definition of simple strategies) we have $|\tilde{\mathcal{N}_A} \cap \tilde{\mathcal{N}_B}| = 0$. Thus, $|N_A \cap N_B| = |\tilde{N}_A \cap \tilde{N}_B|$.

If $\mathcal{N}_A \cap \mathcal{N}_B \neq \emptyset$, then the covering property implies $\mathcal{C} \subset \mathcal{N}_A \cup \mathcal{N}_B$. Thus, we've that $\mathcal{N}_A \setminus \mathcal{N}_B$, $\mathcal{N}_B \setminus \mathcal{N}_A$, and $\mathcal{N}_A \cap \mathcal{N}_B$ partition $\mathcal{C}$, so $N = |\mathcal{N}_A \setminus \mathcal{N}_B| + |\mathcal{N}_B \setminus \mathcal{N}_A| + |\mathcal{N}_A \cap \mathcal{N}_B|$. Since $|\mathcal{N}_i \setminus \mathcal{N}_{-i}| = n_i - |\mathcal{N}_i \cap \mathcal{N}_i|$ for $i \in \{A, B\}$, we have $|\mathcal{N}_A \cap \mathcal{N}_B| = n_A + n_B - N$. Since $\mathcal{N}_A \cap \mathcal{N}_B \neq \emptyset$, we have $|\mathcal{N}_A \cap \mathcal{N}_B| > 0$, implying $n_A + n_B - N > 0$. Thus, $|\tilde{\mathcal{N}_A} \cap \tilde{\mathcal{N}_B}| = n_A + n_B - N$ and so $|\mathcal{N}_A \cap \mathcal{N}_B| = |\tilde{\mathcal{N}_A} \cap \tilde{\mathcal{N}_B}|$.

Suppose A gets project first. Let x_A and x_B be A and B's projects. Since $\boldsymbol{s}$ is weakly rational for A, $|\sigma_{1A}(\mathcal{N}_A, x_A)| = d(x_A)$ when $d(x_A) \leq n_A$ and $|\sigma_{1A}(\mathcal{N}_A, x_A)| = 0$ when $d(x_A) > n_A$. Likewise, $|\tilde{\sigma}_{1A}(\mathcal{N}_A, x_A)| = d(x_A)$ when $d(x_A) \leq n_A$ and $|\tilde{\sigma}_{1A}(\mathcal{N}_A, x_A)| = 0$ when $d(x) >$

n_A by definition of a simple strategy. Thus, A employs the same number of consultants under both strategies, i.e., $|\sigma_{1A}(\mathcal{N}_A, x_A)| = |\tilde{\sigma}_{1A}(\mathcal{N}_A, x_A)|$. Since A holds the same sized network under $\boldsymbol{s}$ and $\tilde{\boldsymbol{s}}$ and since A's ex-post earnings are determined by the number of consultants she employs and the size of her network, we have $u_{1A}(\boldsymbol{s}, x_A, x_B) = u_{1A}(\tilde{\boldsymbol{s}}, x_A, x_B)$. Since (x_A, x_B) were arbitrary, the desired result follows.

Suppose A gets her project second. Let x_A and x_B be A and B's projects. There are four cases: (i) $d(x_B) > n_B$ and $d(x_A) > n_A$, (ii) $d(x_B) > n_B$ and $d(x_A) \leq n_A$, (iii) $d(x_B) \leq n_B$ and $d(x_A) > n_A$, and (iv) $d(x_B) \leq n_B$ and $d(x_A) \leq n_A$. In each case, we establish that A's behavioral strategies σ_{2A} and $\tilde{\sigma}_{2A}$ select the same number of consultants when B follows σ_{1B} and $\tilde{\sigma}_{1B}$ respectively, i.e., that $|\sigma_{2A}(\mathcal{N}_A, \sigma_{1B}(\mathcal{N}_B, x_B), x_A)| = |\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)|$. Since A's ex-post payoff is determined entirely by the size of her network, which is the same under both strategies, and the number of consultants that she employs, we have $u_{2A}(\boldsymbol{s}, x_A, x_B) = u_{2A}(\tilde{\boldsymbol{s}}, x_A, x_B)$. Since (x_A, x_B) were arbitrary, the desired result follows.

If (i), then $|\sigma_{1B}(\mathcal{N}_B, x_B)| = 0$ by the weak rationality of σ_{1B} and $|\sigma_{2A}(\mathcal{N}_A, \emptyset, x_A)| = 0$ by the weak rationality of $\boldsymbol{s}$ for A. Likewise, $|\tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)| = 0$ and $|\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \emptyset, x_A)| = 0$ by the construction of the simple strategies. Thus, $|\sigma_{2A}(\mathcal{N}_A, \emptyset, x_A)| = |\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \emptyset, x_A)|$.

If case (ii), then $|\sigma_{1B}(\mathcal{N}_B, x_B)| = 0$ by the weak rationality of σ_{1B} and $|\sigma_{2A}(\mathcal{N}_A, \emptyset, x_A)| = d(x_A)$ by the weak rationality of $\boldsymbol{s}$ for A. Likewise, $|\tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)| = 0$ and $|\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \emptyset, x_A)| = d(x_A)$ by the construction of simple strategies. Thus, $|\sigma_{2A}(\mathcal{N}_A, \emptyset, x_A)| = |\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \emptyset, x_A)|$.

If case (iii), then $|\sigma_{1B}(\mathcal{N}_B, x_B)| = d(x_B)$ by the weak rationality of σ_{1B} and $|\sigma_{2A}(\mathcal{N}_A, \sigma_{1B}(\mathcal{N}_B, x_B), x_A)| = 0$ by the weak rationality of $\boldsymbol{s}$ for A. Likewise, $|\tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)| = d(x_B)$ and $|\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)| = 0$ by the construction of simple strategies. Thus, $|\sigma_{2A}(\mathcal{N}_A, \sigma_{1B}(\mathcal{N}_B, x_B), x_A)| = |\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)|$.

If case (iv), then $|\sigma_{1B}(\mathcal{N}_B, x_B)| = d(x_B)$ by the weak rationality of σ_{1B}. It's useful to think about the number of consultants left for A after B moves. Since $\boldsymbol{s}$ is employment efficient for B, $\sigma_{1B}(\mathcal{N}_B, x_B) \subset \mathcal{N}_B \backslash \mathcal{N}_A$ when $d(x_B) \leq |\mathcal{N}_B \backslash \mathcal{N}_A|$, leaving A with n_A consultants in $\mathcal{N}_A$. If $d(x_B) > |\mathcal{N}_B \backslash \mathcal{N}_A|$, then employment efficiency implies B employs all consultants in $\mathcal{N}_B \backslash \mathcal{N}_A$ and so employs $d(x_B) - |\mathcal{N}_B \backslash \mathcal{N}_A|$ consultants from $\mathcal{N}_A \cap \mathcal{N}_B$. This leaves A with $|\mathcal{N}_A \cap \mathcal{N}_B| - (d(x_B) - |\mathcal{N}_B \backslash \mathcal{N}_A|)$ consultants in $\mathcal{N}_A \cap \mathcal{N}_B$ and with $|\mathcal{N}_A \backslash \mathcal{N}_B|$ consultants in $\mathcal{N}_A \backslash \mathcal{N}_B$. Thus, there are

$$|\mathcal{N}_A \backslash \sigma_{1B}(\mathcal{N}_B, x_B)| = \begin{cases} n_A & \text{if } d(x_B) \leq |\mathcal{N}_B \backslash \mathcal{N}_A| \\ N - d(x_B) & \text{if } d(x_B) > |\mathcal{N}_B \backslash \mathcal{N}_A| \end{cases}$$

consultants left for A after B moves under $\boldsymbol{s}$. Since $\tilde{\boldsymbol{s}}$ is also employment efficient and $\tilde{\sigma}_{1B}$ satisfies part (i) of the definition of weak rationality for B, an analogous argument gives that there are

$$|\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)| = \begin{cases} n_A & \text{if } d(x_B) \leq |\tilde{\mathcal{N}}_B \backslash \tilde{\mathcal{N}}_A| \\ N - d(x_B) & \text{if } d(x_B) > |\tilde{\mathcal{N}}_B \backslash \tilde{\mathcal{N}}_A| \end{cases}$$

consultants left for A after B moves. Since (i) $|\mathcal{N}_B \backslash \mathcal{N}_A| = n_B - |\mathcal{N}_A \cap \mathcal{N}_B|$, (ii) $|\tilde{\mathcal{N}}_B \backslash \tilde{\mathcal{N}}_A| = n_B - |\tilde{\mathcal{N}}_B \cap \tilde{\mathcal{N}}_A|$, and (iii) $|\mathcal{N}_A \cap \mathcal{N}_B| = |\tilde{\mathcal{N}}_A \cap \tilde{\mathcal{N}}_B|$, we have

$$|\mathcal{N}_A \backslash \sigma_{1B}(\mathcal{N}_B, x_B)| = |\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)|. \tag{A.3}$$

That is, A has the same number of consultants left after B moves in both $\boldsymbol{s}$ and $\tilde{\boldsymbol{s}}$.

Since $\boldsymbol{s}$ is weakly rational for A, we've $|\sigma_{2A}(\mathcal{N}_A, \sigma_{1B}(\mathcal{N}_B, x_B), x_A)| = d(x_A)$ when $d(x_A) \leq |\mathcal{N}_A \backslash \sigma_{1B}(\mathcal{N}_B, x_B)|$ and $|\sigma_{2A}(\mathcal{N}_A, \sigma_{1B}(\mathcal{N}_B, x_B), x_A)| = 0$ when $d(x_A) > |\mathcal{N}_A \backslash \sigma_{1B}(\mathcal{N}_B, x_B)|$. Likewise, $|\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)| = d(x_A)$ when $d(x_A) \leq |\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)|$ and $|\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)| = 0$ when $d(x_A) > |\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)|$ by construction of simple strategies.[28] Hence, equation (A.3) gives

$$|\sigma_{2A}(\mathcal{N}_A, \sigma_{1B}(\mathcal{N}_B, x_B), x_A)| = |\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)|.$$

The desired result follows. □

Corollary A1. Employment Efficient Equilibria and Simple Strategies.
Let Assumption 2 hold and let $\boldsymbol{s}^\star = (\mathcal{N}_A^\star, \dots, \mathcal{N}_B^\star, \dots) \in \boldsymbol{E}$ *be employment efficient, then* $U_i(\boldsymbol{s}^\star) = U_i(\tilde{\boldsymbol{s}}(|\mathcal{N}_A^\star|, |\mathcal{N}_B^\star|))$ *for each partner* i.

Proof. Since every equilibrium has the covering property by Lemma 4 and is weakly rational by Lemma A5, the antecedents of Lemma 6 are satisfied. The corollary follows. □

This corollary, by itself, does not imply Proposition 3 because *both* partners' strategies are different in $\boldsymbol{s}^\star$ and $\tilde{\boldsymbol{s}}(|\mathcal{N}_A^\star|, |\mathcal{N}_B^\star|)$, even when $\boldsymbol{s}^\star$ is an equilibrium. Thus, one partner may have a profitable defection available under the latter strategy. As we show in the next proof, employment lists mitigate this concern.

Proof of Proposition 3. Let $\boldsymbol{s}^\star = (\boldsymbol{s}_A^\star, \boldsymbol{s}_B^\star) = (\mathcal{N}_A^\star, \sigma_{1A}^\star, \sigma_{2A}^\star, \mathcal{N}_B^\star, \sigma_{1B}^\star, \sigma_{2B}^\star) \in \boldsymbol{E}_{LE}$, let $n_A^\star = |\mathcal{N}_A^\star|$ and $n_B^\star = |\mathcal{N}_B^\star|$. Let $\tilde{\boldsymbol{s}}$ denote $\tilde{\boldsymbol{s}}(n_A^\star, n_B^\star)$. We'll prove the proposition by showing

[28] Let's establish these facts. Suppose that $d(x_A) > |\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)|$, then the the definition of $\tilde{\sigma}_{2A}$ gives that $|\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)| = 0$. Now suppose that $d(x_A) \leq |\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)|$, then the definition of $\tilde{\sigma}_{2A}$ gives that $|\tilde{\sigma}_{2A}(\tilde{\mathcal{N}}_A, \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B), x_A)| = d(x_A)$ if $\tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B) \subset \{d(x_A)+1, \dots, N\}$. We'll establish that $\tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B) \subset \{d(x_A)+1, \dots, N\}$. Recall that $\tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B) = \{N+1-d(x_B), \dots, N\}$ and $\tilde{\mathcal{N}}_A = \{1, \dots, n_A\}$. Thus, $\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B) = \{1, \dots, \min\{n_A, N - d(x_B)\}\}$. Hence, $d(x_A) \leq |\tilde{\mathcal{N}}_A \backslash \tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B)|$ implies that $d(x_A) \leq N - d(x_B)$. It follows that $N + 1 - d(x_B) \geq d(x_A) + 1$ and so $\tilde{\sigma}_{1B}(\tilde{\mathcal{N}}_B, x_B) \subset \{d(x_A)+1, \dots, N\}$.

that $(n_A^\star, N - n_B^\star) \in \boldsymbol{F}$, i.e., is an equilibrium of the Auxiliary Game, as then Lemma 3 implies $\tilde{\boldsymbol{s}}(n_A^\star, n_B^\star) \in \boldsymbol{E}_S$.

We argue that $(n_A^\star, N - n_B^\star) \in \boldsymbol{F}$ by contradiction. Suppose that $(n_A^\star, N - n_B^\star) \notin \boldsymbol{F}$, then at least one partner, say A, does strictly better in the Auxiliary Game by picking a new network size n', with $n' \in \{0, \ldots, N\}$ and $n' \neq n_A^\star$, i.e.,

$$U_A(\tilde{\boldsymbol{s}}(n', n_B^\star))) > U_A(\tilde{\boldsymbol{s}}(n_A^\star, n_B^\star)). \tag{A.4}$$

The argument is analogous if B is the partner who does better.

Construct a network $\mathcal{N}_A'$ of size n' for A as follows. If $n' \leq |\mathcal{C}\backslash\mathcal{N}_B^\star|$, let $\mathcal{N}_A'$ consist of n' elements of $\mathcal{C}\backslash\mathcal{N}_B^\star$. If $n' > |\mathcal{C}\backslash\mathcal{N}_B^\star|$, then $\mathcal{N}_B^\star \neq 0$ as $n' \leq N$ and so B has an employment list $\{j_1, \ldots, j_{n_B^\star}\}$ for $\boldsymbol{s}^\star$. Let $\mathcal{N}_A'$ consist of $\mathcal{C}\backslash\mathcal{N}_B^\star$ and the last $n' - |\mathcal{C}\backslash\mathcal{N}_B^\star|$ elements of B's employment list. Thus,

$$\mathcal{N}_A' = \mathcal{C}\backslash\mathcal{N}_B^\star \cup \{j_{n_B^\star}, j_{n_B^\star - 1}, \ldots, j_{\psi+1}, j_\psi\},$$

where $\psi = n_B^\star + 1 - (n' - |\mathcal{C}\backslash\mathcal{N}_B^\star|)$. Also, construct weakly rational strategies σ_{1A}' and σ_{2A}' for A when her network is $\mathcal{N}_A'$ and B follows $\boldsymbol{s}^\star$. For all $x \in X$, let $\sigma_{1A}'(\mathcal{N}_A', x)$ select $d(x)$ consultants from $\mathcal{N}_A'$ when $d(x) \leq n'$ and let $\sigma_{1A}'(\mathcal{N}_A', x)$ be empty when $d(x) > n'$. For all other $(\mathcal{N}, x) \in \mathbb{P}(\mathcal{C}) \times X$, let $\sigma_{1A}'(\mathcal{N}, x) = \emptyset$. For all $(\mathcal{T}, x) \in \mathcal{E}_B^{\boldsymbol{s}^\star} \times X$, let $\sigma_{2A}'(\mathcal{N}_A', \mathcal{T}, x)$ select $d(x)$ consultants from $\mathcal{N}_A'\backslash\mathcal{T}$ when $d(x) \leq |\mathcal{N}_A'\backslash\mathcal{T}|$ and select no consultants when $d(x) > |\mathcal{N}_A'\backslash\mathcal{T}|$. For all other $(\mathcal{N}, \mathcal{T}, x) \in \mathcal{E}_B^{\boldsymbol{s}^\star} \times X$, let $\sigma_{2A}'(\mathcal{N}, \mathcal{T}, x) = \emptyset$.

Let $\boldsymbol{s}_A' = (N_A', \sigma_{1A}', \sigma_{2A}')$. Because of the construction, we have that $\boldsymbol{s}_A' \in \boldsymbol{S}_A$ and that $(\boldsymbol{s}_A', \boldsymbol{s}_B^\star)$ is weakly rational for A.

We'll prove that $U_A(\boldsymbol{s}_A', \boldsymbol{s}_B^\star) = U_A(\tilde{\boldsymbol{s}}(n', n_B^\star))$. Given this, we have

$$U_A(\boldsymbol{s}_A^\star, \boldsymbol{s}_B^\star) \geq U_A(\boldsymbol{s}_A', \boldsymbol{s}_B^\star) = U_A(\tilde{\boldsymbol{s}}(n', n_B^\star)),$$

where the weak inequality follows from the fact $(\boldsymbol{s}_A^\star, \boldsymbol{s}_B^\star)$ is an equilibrium. Corollary A1 gives that $U_A(\boldsymbol{s}_A^\star, \boldsymbol{s}_B^\star) = U_A(\tilde{\boldsymbol{s}}(n_A^\star, n_B^\star))$, since $\boldsymbol{s}^\star$ is employment efficient. Thus, we have

$$U_A(\tilde{\boldsymbol{s}}(n_A^\star, n_B^\star)) \geq U_A(\tilde{\boldsymbol{s}}(n', n_B^\star)),$$

a contradiction of our initial supposition (A.4).

It remains to establish that $U_A(\boldsymbol{s}_A', \boldsymbol{s}_B^\star) = U_A(\tilde{\boldsymbol{s}}(n', n_B^\star))$. This follows from Lemma A6, we just need to verify that the antecedents hold. To these ends, recall that $(\boldsymbol{s}_A', \boldsymbol{s}_B^\star)$ is weakly rational for A by construction. Since $\boldsymbol{s}^\star$ is an equilibrium, Lemma A5 gives that $\sigma_{1B}^\star$ satisfies part (i) of the definition of weak rationality for B. Additionally, $(\boldsymbol{s}'_A, \boldsymbol{s}^\star_B)$ has the covering property: if $\mathcal{N}'_A \cap \mathcal{N}^\star_B \neq \emptyset$, then $\mathcal{C}\backslash\mathcal{N}^\star_B \subset \mathcal{N}'_A$ by construction and so $\mathcal{C}$ is in $\mathcal{N}'_A \cup \mathcal{N}^\star_B$. Finally, $(\boldsymbol{s}'_A, \boldsymbol{s}^\star_B)$ is employment efficient for B. If $\mathcal{N}'_A \cap \mathcal{N}^\star_B = \emptyset$, this is trivially the case. If $\mathcal{N}'_A \cap \mathcal{N}^\star_B \neq \emptyset$, observe that $\mathcal{N}^\star_B\backslash\mathcal{N}'_A = \{j_1, \ldots, j_{\psi-1}\}$ since we constructed $\mathcal{N}'_A$ to contain the last $n' - |\mathcal{C}\backslash\mathcal{N}^\star_B|$ elements of B's employment list. Since B sticks to her employment list when she gets a project x, we've $\sigma^\star_{1B}(\mathcal{N}^\star_B, x) = \{j_1, \ldots, j_{d(x)}\}$ when B employs consultants. Thus, if $d(x) \leq |\mathcal{N}^\star_B\backslash\mathcal{N}'_A| = \psi - 1$, we have $\sigma^\star_{1B} \subset \mathcal{N}^\star_B\backslash\mathcal{N}'_A$. If, however, $d(x) > \psi - 1$, then $\mathcal{N}^\star_B\backslash\mathcal{N}'_A \subset \sigma^\star_{1B}$. Thus, Lemma A6 applies. □

Proof of Lemma 6

Proof of Lemma 6. We'll prove the lemma for A as the argument for B is analogous. Let $n < n'$. Since Assumption 1 holds, Lemma A1 gives that $U_A(\tilde{\boldsymbol{s}}(n_A, n_B))$ is weakly decreasing in n_B for all n_A. Thus,

$$U_A(\tilde{\boldsymbol{s}}(\bar{b}_A(n), n)) \geq U_A(\tilde{\boldsymbol{s}}(\bar{b}_A(n'), n)) \geq U_A(\tilde{\boldsymbol{s}}(\bar{b}_A(n'), n')).$$

The first inequality is due to the optimality of $\bar{b}_A(n)$ and the second inequality is due to the fact $U_A(\cdot)$ is weakly decreasing in n. The lemma follows. □

Proof of Lemma 8

We prove Lemma 8 by applying Topkis' Monotonicity Theorem.

Proof of Lemma 8. We'll establish that $\phi_A(z, r_A, c_A) \preceq \phi_A(z, r'_A, c'_A)$. Let $\theta \in \{0, 1\}$ and let

$$f(z_A, z_B, \theta) = \begin{cases} U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B), r_A, c_A) & \text{if } \theta = 0 \\ U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B), r'_A, c'_A) & \text{if } \theta = 1. \end{cases}$$

We'll show that $f(z_A, z_B, \theta)$ is supermodular in (z_A, θ) for each $z_B \in \{0, \ldots, N\}$. Given this, Theorem 2.8.1 in Topkis [15] implies that $\rho(z_B, \theta) = \arg\max_{z_A \in \{0,\ldots,N\}} f(z_A, z_B, \theta)$ is weakly increasing in θ, i.e., that $\rho(z_B, 0) \preceq \rho(z_B, 1)$ for each $z_B \in \{0, \ldots, N\}$. Since $\phi_A(z, r_A, c_A) = \rho(z, 0)$ and $\phi_A(z, r'_A, c'_A) = \rho(z, 1)$, it follows that $\phi_A(z, r_A, c_A) \preceq \phi_A(z, r'_A, c'_A)$ for all $z \in \{0, \ldots N\}$.

To show that $f(z_A, z_B, \theta)$ is supermodular in (z_A, θ), let $\theta' = 1$ and $\theta = 0$. Lemma A1 gives

$$f(z_A, z_B, \theta') - f(z_A, z_B, \theta) = U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B), r'_A, c'_A) - U_A(\tilde{\boldsymbol{s}}(z_A, N - z_B), r_A, c_A)$$

$$f(z_A, z_B, \theta') - f(z_A, z_B, \theta) = \sum_{\{x|d(x)\leq z_A\}} (r'_A(x) - r_A(x))\frac{P_A(x)}{2}$$
$$+ \sum_{l=1}^{z_A} \sum_{\{x|d(x)=l\}} (r'_A(x) - r_A(x))g_A(l, N - z_B)\frac{P_A(x)}{2} + (c_A(z_A) - c'_A(z_A)).$$

Since $r'_A \geq r_A$ and $c_A \geq c'_A$ (and $g_A \geq 0$ and $P_A \geq 0$), this sum is positive. Additionally, the sum is increasing in z_A – the first two terms are trivially increasing in z_A and the last term is increasing in z_A by Assumption 4. Thus, for $z'_A \geq z_A$,

$$f(z'_A, z_B, \theta') - f(z'_A, z_B, \theta) \geq f(z_A, z_B, \theta') - f(z_A, z_B, \theta)$$
$$f(z'_A, z_B, \theta') + f(z_A, z_B, \theta) \geq f(z_A, z_B, \theta') + f(z'_A, z_B, \theta),$$

that is, f is supermodular in (z_A, θ). □

Proof of Lemma 9

Proof of Lemma 9. This is almost obvious. We'll establish that $(\overline{z}_A, \overline{z}_B) \leq (\overline{z}'_A, \overline{z}'_B)$ as the case for the minimal elements is analogous. We'll work with the maximal selections of ϕ_A and ϕ_B for simplicity, which we denote $\overline{\phi}_A$ and $\overline{\phi}_B$. Let $\overline{\Phi} = (\overline{\phi}_A(z_B, r_A, c_A), \overline{\phi}_B(z_A, r_B, c_B))$ and let $\overline{\Phi}' = (\overline{\phi}_A(z_B, r'_A, c'_A), \overline{\phi}_B(z_A, r_B, c_B))$. Since $\overline{\phi}_A$ and $\overline{\phi}_B$ are increasing functions by Lemma 1 and Theorem 2.8.1 of Topkis [15], $\overline{\Phi}$ and $\overline{\Phi}'$ are increasing functions that map $\{0, \ldots, N\}^2$ into itself. Thus, Tarski's Fixed Point Theorem gives that $\overline{\boldsymbol{F}}$ the set of fixed points of $\overline{\Phi}$ and $\overline{\boldsymbol{F}}'$ the set of fixed points of $\overline{\Phi}'$ are non-empty complete lattices. It follows that $(\overline{z}_A, \overline{z}_B)$ is the maximal element of $\overline{\boldsymbol{F}}$ and $(\overline{z}'_A, \overline{z}'_B)$ is the maximal element of $\overline{\boldsymbol{F}}'$.

Lemma 8 implies that $\overline{\phi}_A(z_B, r_A, c_A) \leq \overline{\phi}_A(z_B, r'_A, c'_A)$. Thus, $\overline{\Phi} \leq \overline{\Phi}'$. We can now establish the desired result. Let $D = \{\overline{z}_A, \ldots, N\} \times \{\overline{z}_B, \ldots, N\}$. Since $\overline{\Phi}'(\overline{z}_A, \overline{z}_B) \geq \overline{\Phi}(\overline{z}_A, \overline{z}_B) = (\overline{z}_A, \overline{z}_B)$ and $\overline{\Phi}'$ is increasing, we've that $\overline{\Phi}'$ takes D into itself. Hence, Tarski's Fixed Point Theorem gives that there's a $(z'_A, z'_B) \in D \cap \overline{\boldsymbol{F}}'$. It follows that $(\overline{z}_A, \overline{z}_B) \leq (z'_A, z'_B) \leq (\overline{z}'_A, \overline{z}'_B)$. □

Proof of Proposition 7

We prove Proposition 7 by proving a more general result and then deriving the proposition as a corollary. Let $\boldsymbol{S}_E$ be the set of employment efficient strategies, let $\boldsymbol{S}_R$ be the set of weakly rational strategies, and let $\boldsymbol{S}_C$ be the set of strategies with the covering property. Let $\boldsymbol{S}_{CER} = \boldsymbol{S}_E \cap \boldsymbol{S}_C \cap \boldsymbol{S}_R$.

Proposition A1. Efficiency and Over-Investment.
Let Assumption 2 hold and let $\boldsymbol{s} = (\boldsymbol{s}_A, \boldsymbol{s}_B) = (\mathcal{N}_A, \ldots, \mathcal{N}_B, \ldots) \in \boldsymbol{S}_{CER}$ *be efficient. If* $\boldsymbol{s}$ *is not an equilibrium, then one of the two partners strictly benefits from over-investing in*

her network. That is, $\boldsymbol{s} \notin \boldsymbol{E}$ implies that either (i) there is a $\boldsymbol{s}'_A = (\mathcal{N}'_A, \sigma'_{1A}, \sigma'_{2A})$ such that $U_A(\boldsymbol{s}'_A, \boldsymbol{s}_B) > U_A(\boldsymbol{s})$ and $|\mathcal{N}'_A| > |\mathcal{N}_A|$, or (ii) there is a $\boldsymbol{s}'_B = (\mathcal{N}'_B, \sigma'_{1B}, \sigma'_{2B})$ such that $U_B(\boldsymbol{s}_A, \boldsymbol{s}'_B) > U_B(\boldsymbol{s})$ and $|\mathcal{N}'_B| > |\mathcal{N}_B|$.

Proof. Since $\boldsymbol{s} \notin \boldsymbol{E}$, at least one partner benefits by unilaterally defecting. Without loss, suppose this partner is A and let $\boldsymbol{s}'_A$ be A's new strategy, so $U_A(\boldsymbol{s}'_A, \boldsymbol{s}_B) > U_A(\boldsymbol{s})$.

We first establish that there is a strategy $\boldsymbol{s}''_A$ such that (i) $(\boldsymbol{s}''_A, \boldsymbol{s}_B) \in \boldsymbol{S}_{CER}$ and (ii) $U_A(\boldsymbol{s}''_A, \boldsymbol{s}_B) \geq U_A(\boldsymbol{s}'_A, \boldsymbol{s}_B)$. We establish this fact in three steps. First, observe that we may construct an (interim) strategy $\boldsymbol{s}^R_A$ such that $(\boldsymbol{s}^R_A, \boldsymbol{s}_B) \in \boldsymbol{S}_R$ and $U_A(\boldsymbol{s}^R_A, \boldsymbol{s}_B) \geq U_A(\boldsymbol{s}'_A, \boldsymbol{s}_B)$. To do this, let $\boldsymbol{s}^R_A$ specify the same network as $\boldsymbol{s}'_A$, while specifying weakly rational behavioral strategies for A under the supposition that B follows $\boldsymbol{s}_B$. (We can construct such behavioral strategies because our game is finite.) Since Assumption 2 holds, Remarks A1 and A2 give that A gets a higher payoff under $\boldsymbol{s}^R_A$ than under $\boldsymbol{s}'_A$. (The weak inequality follows from the fact $\boldsymbol{s}'_A$ may be in $\boldsymbol{S}_R$.)

Second, observe that we may construct a strategy $\boldsymbol{s}^{RC}_A$, such that (i) $(\boldsymbol{s}^{RC}_A, \boldsymbol{s}_B) \in \boldsymbol{S}_{CR}$ and (ii) $U_A(\boldsymbol{s}^{RC}_A, \boldsymbol{s}_B) \geq U_A(\boldsymbol{s}^R_A, \boldsymbol{s}_B)$. We do this in the same manner as in the Proof of Lemma 4: swap each of A's shared consultants for a consultant who aren't in either partners' network (until all consultants are in a partner's network) and, given A's post swap network, choose weakly rational strategies for A under the hypothesis B follows $\boldsymbol{s}_B$. Since Assumption 2 holds and $\boldsymbol{s}_B$ satisfies part (i) of the definition of weak rationality for B (as $(\boldsymbol{s}^R_A, \boldsymbol{s}_B)$ is weakly rational), an argument analogous to the Proof of Lemma 4 gives that A does weakly better under $\boldsymbol{s}^{RC}_A$ than under $\boldsymbol{s}^R_A$, given B plays $\boldsymbol{s}_B$. (The weak inequality follows from the fact $\boldsymbol{s}^R_A$ may be in $\boldsymbol{S}_C$.)

Third, observe that we may construct a strategy $\boldsymbol{s}^{ARC}_A$ such that $(\boldsymbol{s}^{ARC}_A, \boldsymbol{s}_B) \in \boldsymbol{S}_{CER}$ and (ii) $U_A(\boldsymbol{s}^{ARC}_A, \boldsymbol{s}_B) = U_A(\boldsymbol{s}^{RC}_A, \boldsymbol{s}_B)$. We do this by simply re-ordering A's behavioral strategy to be employment efficient, i.e., to recommend that A employ exclusive consultants before shared consultants. Such a re-ordering doesn't change A's payoff as she employs exactly the same number of workers before and after the re-ordering. We take $\boldsymbol{s}''_A = \boldsymbol{s}^{ACR}_A$ to complete the argument.

Since $(\boldsymbol{s}_A, \boldsymbol{s}_B) \in \boldsymbol{S}_{CER}$ and $(\boldsymbol{s}''_A, \boldsymbol{s}_B) \in \boldsymbol{S}_{CER}$, Lemma A6 gives that $U_i(\boldsymbol{s}) = U_i(\tilde{\boldsymbol{s}}(n_A, n_B))$ and $U_i(\boldsymbol{s}''_A, \boldsymbol{s}_B) = U_i(\tilde{\boldsymbol{s}}(n''_A, n_B))$ for each partner i, where n_i is the size of i's network under $\boldsymbol{s}$ and n''_A is the size of i's network in $\boldsymbol{s}''_A$. By hypothesis,

$$U_A(\boldsymbol{s}''_A, \boldsymbol{s}_B) = U_A(\tilde{\boldsymbol{s}}(n''_A, n_B)) > U_A(\boldsymbol{s}). \tag{A.5}$$

Since $\boldsymbol{s}$ is efficient on $\boldsymbol{S}$,

$$\begin{aligned} U_A(\tilde{\boldsymbol{s}}(n''_A, n_B)) &\leq U_A(\boldsymbol{s}) + U_B(\boldsymbol{s}) - U_B(\tilde{\boldsymbol{s}}(n''_A, n_B)) \\ &= U_A(\boldsymbol{s}) + U_B(\tilde{\boldsymbol{s}}(n_A, n_B)) - U_B(\tilde{\boldsymbol{s}}(n''_A, n_B)). \end{aligned}$$

Since Lemma A1 gives

$$U_B(\tilde{\boldsymbol{s}}(n_A, n_B)) - U_B(\tilde{\boldsymbol{s}}(n''_A, n_B)) = \sum_{l=1}^{n_B} \sum_{\{x|d(x)=l\}} (r_B(x) - w\, d(x)) \frac{P_B(x)}{2} (g_B(l, n_A) - g_B(l, n''_A)),$$

we have

$$U_A(\tilde{\boldsymbol{s}}(n''_A, n_B)) \leq U_A(\boldsymbol{s}) + \sum_{l=1}^{n_B} \sum_{\{x|d(x)=l\}} (r_B(x) - w\, d(x)) \frac{P_B(x)}{2} (g_B(l, n_A) - g_B(l, n''_A)). \quad \text{(A.6)}$$

To complete the proof, we argue by contradiction. Suppose $n''_A \leq n_A$, i.e., A holds a smaller network after defecting. Then, $g_B(l, n_A) - g_B(l, n''_A) \leq 0$ as $g_B(l, n)$ is weakly decreasing in n. Thus, equation (A.6) implies that $U_A(\boldsymbol{s}) \geq U_A(\tilde{\boldsymbol{s}}(n''_A, n_B)) = U_A(\boldsymbol{s}''_A, \boldsymbol{s}_B)$. This is a contradiction of equation (A.5). It follows that $n''_A > n_A$. □

Remark. We cannot extend Proposition A1 to all efficient strategy vectors. The reason is that efficiency may require behavior that isn't weakly rational. The intuition is that it may be in society's interest for the first partner to pass on a difficult and low value project so as to allow the second partner a better chance of completing a complete a difficult project of very high value.

Proof of Proposition 7. Since $\boldsymbol{S}_S \subset \boldsymbol{S}_{CER}$ by Lemma 5 and the fact every simple strategy is weakly rational, Proposition 7 is an immediate corollary of Proposition A1. □

Proof of Lemma 12

Proof of Lemma 12. The proof is analogous to the proof of Lemma 8. We'll prove that $\phi_A(z, \alpha) \preceq \phi_A(z, \alpha')$ by showing that A's payoff in the Auxiliary Game is supermodular in (z, α). The argument that $\phi_B(z, \alpha) \preceq \phi_B(z, \alpha')$ is analogous.

Let $\alpha' \geq \alpha$. Lemma A1 (appropriately modified), gives that

$$U_A(\tilde{\boldsymbol{s}}(z_A, z_B), \alpha') - U_A(\tilde{\boldsymbol{s}}(z_A, z_B), \alpha) = \sum_{l=1}^{z_A} \sum_{\{x|d(x)=l\}} (r_A(x) - w\, l) P_A(x)(1 - g_A(l, z_B))\, (\alpha' - \alpha).$$

Since $g_A \in [0, 1]$, we have that $(1 - g_A(l, z_B)) \geq 0$. Thus, the summand on the right-hand-side is positive as Assumption 1 and $P_A \geq 0$. Hence, the sum is increasing in z_A. It follows for

$z'_A \geq z_A$ that

$$U_A(\tilde{\boldsymbol{s}}(z'_A, z_B), \alpha') - U_A(\tilde{\boldsymbol{s}}(z'_A, z_B), \alpha) \geq U_A(\tilde{\boldsymbol{s}}(z_A, z_B), \alpha') - U_A(\tilde{\boldsymbol{s}}(z_A, z_B), \alpha)$$
$$U_A(\tilde{\boldsymbol{s}}(z'_A, z_B), \alpha') + U_A(\tilde{\boldsymbol{s}}(z_A, z_B), \alpha) \geq U_A(\tilde{\boldsymbol{s}}(z_A, z_B), \alpha') + U_A(\tilde{\boldsymbol{s}}(z'_A, z_B), \alpha).$$

That is, $U_A(\tilde{\boldsymbol{s}}(z_A, z_B), \alpha)$ is supermodular in (z_A, α) for each $z_B \in \{0, \ldots, N\}$. It follows from Topkis' Monotonicity Theorem (Theorem 2.8.1 [15]) that $\phi_A(z, \alpha) \preceq \phi_A(z, \alpha')$ for all $z \in \{0, \ldots N\}$. □

Proof of Lemma 14

Proof of Lemma 14. Analogous to the proof of Lemma 10. We'll establish the result for the equilibria that are best for A and worst for B. An analogous argument applies at the equilibria that are best for B and worst for A. Let $(\overline{z}_A, \overline{z}_B)$ and $(\overline{z}'_A, \overline{z}'_B)$ be as in the statement of Lemma 13. To simplify notation, for all $(n_A, n_B) \in \{0, \ldots, N\}^2$, let $U_i(n_A, n_B)$ denote $U_i(\tilde{\boldsymbol{s}}(n_A, N - n_B), \alpha)$ and let $U'_i(n_A, n_B)$ denote $U_i(\tilde{\boldsymbol{s}}(n_A, N - n_B), \alpha')$ for each partner i. Lemma 7 implies that $U_A(z^\star_A, z^\star_B) = U_A(\overline{z}_A, \overline{z}_B)$, that $U_B(z^\star_A, z^\star_B) = U_B(\overline{z}_A, \overline{z}_B)$, that $U'_A(z'_A, z'_B) = U'_A(\overline{z}'_A, \overline{z}'_B)$, that $U'_B(z'_A, z'_B) = U_B(\overline{z}'_A, \overline{z}'_B)$. Thus, we only need to show that

$$U'_A(\overline{z}'_A \overline{z}'_B) \geq U_A(\overline{z}_A, \overline{z}_B) \text{ and } U'_B(\overline{z}'_A \overline{z}'_B) \leq U_A(\overline{z}_A, \overline{z}_B) \tag{A.7}$$

to establish the display equation of the lemma. Let's prove (A.7) for A. Let $\overline{b}_A(n)$ denote $\overline{b}_A(n, \alpha)$ and let $\overline{b}'_A(n)$ denote $\overline{b}_A(n, \alpha')$. Write

$$\begin{aligned} U_A(\overline{z}_A, \overline{z}_B) &= U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}_B), N - \overline{z}_B), \alpha) \\ &\leq U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}'_B), N - \overline{z}'_B), \alpha) \\ &\leq U_A(\tilde{\boldsymbol{s}}(\overline{b}_A(N - \overline{z}'_B), N - \overline{z}'_B), \alpha') \\ &\leq U_A(\tilde{\boldsymbol{s}}(\overline{b}'_A(N - \overline{z}'_B), N - \overline{z}'_B), \alpha') \\ &= U'_A(\overline{z}'_A, \overline{z}'_B) \end{aligned}$$

The and fifth lines are standard. Since $(\overline{z}_A, \overline{z}_B) \leq (\overline{z}'_A, \overline{z}'_B)$ by Lemma 13, the second line follows from Lemma 6 as $(\overline{z}_A, \overline{z}_B) \leq (\overline{z}'_A, \overline{z}'_B)$ implies that $N - \overline{z}'_B \leq N - \overline{z}_B$. The third line follows from the Proof of Lemma 12, where we showed $U_A(\tilde{\boldsymbol{s}}(n_A, n_B), \alpha') - U_A(\tilde{\boldsymbol{s}}(n_A, n_B), \alpha) \geq 0$ for all (n_A, n_B). The fourth line follows from optimality. Since the argument for B is analogous, we've (A.7).

It remains to show that the size of A's network increases and that the size of B's network decreases. Since Assumption 3 holds, Lemma 7 implies $(z^\star_A, z^\star_B) = (\overline{z}_A, \overline{z}_B)$ and that $(z'_A, z'_B) = (\overline{z}'_A, \overline{z}'_B)$. Thus, the desired result follows directly from Lemma 13. □