Cost Sharing with Dependencies and Fixed Costs
Extended Abstract

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Abstract

Allocating the joint cost of producing a bundle of infinitely divisible consumption goods is a common practical problem with no obvious solution. All the works up to this day have assumed at least one of the following– 1. the lack of individual demand dependencies; namely, it is assumed that the set of all possible aggregate demand vectors is a cube, 2. the differentiability of the cost functions, and 3. the absence of fixed costs. This is obviously not the case in many cost problems of interest. Mirman et al. [9] addressed the third matter but assumed the first two. Samet et al. [13] and Haimanko [5] addressed the second matter, but not the other two. Generally, even dropping the first two assumptions together made the problem unamenable to all known forms of analysis. Recently, we (Edhan [4]) have extended the work of Haimanko [5] to include individual demand dependencies, supplying a characterization of the pricing mechanism. The main hardship in this case is that one can no longer naturally assume demand to be constant so demand aggregation enters the game, while the non-differentiability of the cost functions prevents any type of approximation by cost problems with constant demand. However, Edhan [4] assumes that there are no fixed costs. The current work is dedicated to waving these three assumptions all together. We consider two classes of cost problems exhibiting fixed costs, with generically major non-differentiability of cost functions, and whose sets of aggregated demand vectors may fail to be a cube. The cost functions in the first class are convex exhibiting non-decreasing marginal costs to scale, and those in the second class are piece-wise affine. We show existence and uniqueness of a cost allocation mechanism, satisfying standard axioms, on these classes.

1 Introduction

Allocating the joint cost of producing a bundle of infinitely divisible consumption goods is a common practical problem with no obvious solution. Billera, Heath, and Raanan [2] were the first to apply Aumann and Shapley [1] theory of nonatomic games to set equitable telephone billing rates that share services cost among users. Billera and Heath [3] and Mirman and Tauman [8] offered an axiomatic justification of Aumann-Shapley (A-S) prices using economic terms only. Tauman [14] proved that A-S prices naturally extend the average cost prices from a single product to an arbitrary finite number of products with nonseparable and differentiable cost functions. In the case of non-differentiable cost functions, Haimanko [5] introduced the Mertens price mechanism, which is closely related to the Mertens [7] value and extends the A–S price mechanism to non-differentiable costs. Haimanko [5] offered a characterization of the Mertens mechanism which is similar in nature to those of Billera and Heath [3] and Mirman and Tauman [8].

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A-S and Mertens prices are characterized (see [3], [8], and [5]) by five simple axioms- cost sharing, additivity, rescaling invariance, monotonicity, and consistency, the latter tying the pricing of multiple commodities with that of a single one. Other characterizations of A-S prices were also proposed by Young [15], Hart and Mas-Colell [6], and Monderer and Neyman [10]. Cost sharing problems with fixed costs were introduced by Mirman et al. ([9]), under the assumption of the differentiability of the cost functions. However fixed costs were not addressed in the case of non-differentiable cost functions.

However, all the aforementioned works share a crucial assumption– that consumers' individual demands for each good are independent, namely, that the set of all possible aggregate demand vectors is a cube. However, in many economic applications these assumption may not hold. Lately, Edhan [4] has generalized the work of Haimanko to also incorporate possible dependencies in individual demand with non-differentiable cost functions. This work required dealing with demand aggregation which was redundant in previous cases, as, once the set of production outputs is the cube, individual demand can be essentially assumed to be constant. Nevertheless, Edhan [4] assumes that there are no fixed costs, and treating fixed cost is not a straight forward conclusion from this work, and the ideas of Mirman et al. [9] cannot be applied as they depend on the differentiability of the cost functions.

The purpose of this paper is to lift all the aforementioned assumptions, namely, to study cost sharing with non-differentiable cost functions, fixed costs, and individual demand dependencies. We exhibit a price mechanism on the classes of cost problems whose set of aggregate demand vectors is not a cube, and whose cost functions are convex with fixed costs, and exhibit non-decreasing marginal costs to scale in the first case, or piecewise linear with fixed costs in the second case. The price mechanism is characterized by the axioms of fixed cost sharing, cost separability, rescaling invariance, monotone aggregate cost, and consistency. Our axioms are counterparts of Mirman et al. ([9]) axioms, and in fact generalize them to the case were demand dependencies and fixed costs exist. As in Haimanko [5], our mechanism has an affinity with the Mertens [7] value and we refer to it as the *Mertens price mechanism*. Our main result is Theorem 2.3, which states that this mechanism is uniquely characterized by the five axioms on these classes of cost problems.

2 Definitions and The Results

2.1 Cost Problems

We denote by $\mathbb{R}_+$ the nonnegative real numbers and by $\mathbb{R}_{++}$ the strictly positive real numbers. A *cliental* is a standard measurable space (Ω, Σ). A population measure is a nonatomic probability measure λ on the cliental. Denote the space population measures by NA^1. A *demand vector* for a cliental (w.r.t. λ) is an essentially bounded random variable X taking values in $\mathbb{R}^k_+$ for some $k \geq 1$, namely $X \in (L^\infty(\lambda))^n$. The *aggregate demand* $\int X$ of X is the vector measure given by $\int X(A) = \int_A X(\omega) d\lambda(\omega)$ for any $A \in \Sigma$. The range of $\int X$ is the *consumption set generated by the demand vector* X, namely the set $K(X) = \left\{\int X(A) : A \in \Sigma\right\}$, which is a compact and convex set of $\mathbb{R}^k_+$. Denote, for short, $X(\Omega) = \int X(\Omega)$. We interpret $X(\Omega)$ as the maximal aggregate demand a firm may be faced. Denote $\mathcal{X}^k_c = \left\{X : X \in (L^\infty(\lambda))^k, \lambda \in NA^1\right\}$ and let $\mathcal{X}_c = \bigcup_{k=1}^{\infty} \mathcal{X}^k_c$. A *cost problem* is a pair (f, X) with demand vector $X \in \mathcal{X}^k_c$ for some $\lambda \in NA^1$ and $k \geq 1$, and a *cost function* $f : K(X) \to \mathbb{R}$. The cost function f exhibits *fixed costs* iff $f(0_k) > 0$. For any class F of cost problems, denote by F^k its subset of problems (f, X) with $X \in \mathcal{X}^k_c$.

A price mechanism on a class of cost problems F is a function $\phi : F \to \bigcup_{k=1}^{\infty} \mathbb{R}^k$ s.t. the range of $\phi\big|_{F^k}$ is contained in $\mathbb{R}^k$. If $(f, X) \in F^k$ and $1 \leq j \leq k$ we denote by $\phi_j((f, X))$ the jth coordinate of $\phi((f, X))$. If $x, y \in \mathbb{R}^k_+$ denote $x * y = (x_1 y_1, ..., x_k y_k)$, and $(y * f)(x) = f(y * x)$ for a function f on $\mathbb{R}^k_+$. If $x \in \mathbb{R}^k_{++}$ denote $x^{-1} = (x_1^{-1}, ..., x_k^{-1})$.

For $m \geq k \geq 1$ and a partition $(S_1, ..., S_k)$ of $\{1, ..., m\}$ let $\pi^* : \mathbb{R}^m \to \mathbb{R}^k$ be given by $\pi_i^*(x) = \sum_{j \in S_i} x_j$.

Definition 2.1. A price mechanism ϕ on F is:

1. *cost sharing* iff for every $(f, X) \in F$

$$f(X(\Omega)) = X(\Omega) \cdot \phi((f, X)); \tag{2.1}$$

2. *fixed cost separable* iff for every $(f, X), (g_1, X), ..., (g_n, X) \in F^k$ with $f = \sum_{i=1}^{n} g_i$ there is $C \in \mathbb{R}^n$ with $\sum_{i=1}^{n} C_i = 0$ and $g_i(0_k) + C_i \geq 0$ for every $1 \leq i \leq n$, s.t. for every $x \in K(X)$

$$\phi((f, X)) \cdot x = \sum_{i=1}^{n} \phi((g_i + C_i, X)) \cdot x; \tag{2.2}$$

3. *consistent* iff for every $1 \leq k \leq m$, every partition $\pi = \{S_1, ..., S_k\}$ of $\{1, ..., m\}$ s.t. $(f \circ \pi^*, X), (f, \pi^*(X)) \in F$, every $1 \leq i \leq k$, and every $j \in S_i$ we have

$$\phi_j((f \circ \pi^*, X)) = \phi_i((f, \pi^*(X))); \tag{2.3}$$

4. *rescaling invariant* iff for every $(f, X) \in F^k$ and $\alpha \in \mathbb{R}^k_{++}$ with $(\alpha * f, \alpha^{-1} * X) \in F^k$ we have for every $x \in K(X)$

$$\phi((\alpha * f, \alpha^{-1} * X)) \cdot (\alpha^{-1} * x) = \phi((f, X)) \cdot x; \tag{2.4}$$

5. *monotone aggregate cost* iff for every $(f, X), (g, Y) \in F^k$ satisfying[1] $f\left(\int X(A)\right) - g\left(\int Y(A)\right) \leq f\left(\int X(A')\right) - g\left(\int Y(A')\right)$ for any $A \subseteq A'$, we have for every $A \in \Sigma$

$$\phi((f, X)) \cdot \int X(A) \geq \phi((g, Y)) \cdot \int Y(A). \tag{2.5}$$

2.2 The Classes F_c and F_l

Given a cost problem (f, X) with $X \in \mathcal{X}_c^k$, a point x in the relative interior of $K(X)$, and $y \in AF(X)$, the affine space generated by $K(X)$, the *directional derivative* of f at x in the direction y is

$$df(x, y) = \lim_{\varepsilon \searrow 0} \frac{f(x + \varepsilon y) - f(x)}{\varepsilon}. \tag{2.6}$$

The limit exists for every piecewise convex function f. For $k \geq 1$ let $\widehat{F}_c^k$ be the set of cost problems (f, X) with f being a non-decreasing Lipschitz continuous convex function with $f(0_k) \geq 0$, exhibiting *non-decreasing marginal cost to scale*, namely, for x in the relative interior of $K(X)$, $y \in AF(K(X))$, and $t \geq 1$ s.t. tx is in the relative interior of $K(X)$ it holds that

$$df(x, y) \leq df(tx, y). \tag{2.7}$$

[1] The underlying population measures of X and Y may be different!

Let $\widehat{F}_l^k$ be the set of cost problems (f, K) with f being non-decreasing, continuous, piecewise affine function with $f(0_k) \geq 0$. For $* \in \{c, l\}$ let $\widehat{F}_* = \bigcup_{k=1}^{\infty} \widehat{F}_*^k$.

2.3 The Mertens Mechanism

Given a cost problem $(f, X) \in \widehat{F}_c^k \cup \widehat{F}_\ell^k$, and x in the relative interior of $K(X)$, the function $df(x, \cdot)$ is convex or piecewise affine (respectively) and finite on $AF(X)$ by [12, Theorem 23.1]. Hence its directional derivative exist at any point $y \in AF(X)$ in any direction $z \in AF(X)$. We shall denote it by

$$df(x, y, z) = \lim_{\varepsilon \searrow 0} \frac{df(x, y + \varepsilon z) - df(x, y)}{\varepsilon}. \tag{2.8}$$

For any cost problem $(f, X) \in \widehat{F}_c^k \cup \widehat{F}_\ell^k$ with $f(X(\Omega)) > f(0_k)$ consider the following cost mechanism. For $y \in AF(X)$ let $\|y\|_X = \max\{\langle x, y\rangle : x \in 2X(\Omega) - X(\Omega)\}$. Then $\|\cdot\|_X$ is a norm on $AF(X)$ and there is a probability distribution P_X on $AF(X)$ whose Fourier transform is $\mathcal{F}[P_X](y) = \exp(-\|y\|_X)$ ([11, Lemma 1]). For any $1 \leq j \leq k$ define

$$(\phi_M)_j((f, X)) = \frac{f(X(\Omega))}{f(X(\Omega)) - f(0_k)} \int_{AF(X)} \left(\int_0^1 df(tX(\Omega), y, e_j) dt \right) dP_X(y), \tag{2.9}$$

with e_j the jth unit vector. We call ϕ_M the *Mertens mechanism*, due to its affinity with the Mertens [7] value.

Lemma 2.2. *The Mertens price mechanism satisfies the axioms (1)-(5)*

Our main result in this paper is:

Theorem 2.3. *If ϕ is a cost sharing, fixed cost separable, rescaling invariant, monotone aggregate cost, and consistent price mechanism on $\widehat{F}_c$ ($\widehat{F}_l$) then ϕ is the Mertens price mechanism.*

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