Auctions without a common prior

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Abstract

In this paper we propose an auction model where players are fully rational but may not share a common prior. Assumptions are made on players beliefs and ensure that players do not gain rank information from their valuation: that is to say a player's valuation does not give him information about whether his valuation is likely to be higher than that of his opponent. It is shown that bidding a constant fraction of one's valuation is an equilibrium. An explanation for the Bertrand entry paradox is provided and a simple auction which extracts the full surplus is outlined.

1 Introduction

Standard auction theory is based on a common prior assumption that requires the entire distribution of valuations to be common knowledge among players. Moreover in order to calculate the bidding function that players use in equilibrium, it is often necessary to perform complex calculations. For instance in a symmetric two-player affiliated value model, the equilibrium bidding function - using t to denote a player's valuation - looks as follows:

$$\sigma(t) = t - \int_{\underline{t}}^{t} t dL(x|t)$$

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where

$$L(x|t) = exp\Big(- \int_x^t \frac{\gamma(y|y)}{\Gamma(y|y)} dy \Big)$$

In this paper we propose an auction model, which is less cognitively demanding and where agents do not have to perform such complex calculations. Furthermore this theory ensures that the bid function of agents is easy to estimate and hence makes the model easily testable. Unlike other simple models that consider beliefs to be primitives, we preserve the assumption that players are fully rational and require that agents play according to Nash equilibrium strategies. We do this by proposing a model where the valuations of players are highly correlated and players have no rank information: that is to say players do not know how their valuation compares to that of others.

The first part of the paper considers traditional auctions where buyers compete over a single object. We assume that the probability of one player having a valuation very close to that of another player is common knowledge. This parameter γ^* captures how competitive the auction is and how closely the valuation of players are clustered. In the framework considered we prove that the following is a symmetric Nash equilibrium:

$$\sigma(t) = \frac{2\gamma^*}{2\gamma^* + 1} t$$

Note that this bid function is particularly easy to estimate since it requires estimating the single parameter γ^*, rather than an entire distribution function. Indeed, this property holds for all the results presented here hence making the model both easy to apply to data and easy to falsify. Having studied the first price auction as a benchmark, we proceed to study the all pay and fractional all pay auctions. We find that - as long as a participation constraint is satisfied - the higher the all pay component auction the higher players bid. This means that the revenue a seller collects increases for two reasons: first higher bids from buyers leads to higher revenue; and secondly a higher share of the loser's bid leads to higher revenue.

This effect can be exploited to design a simple full surplus extraction mechanism, which is a fractional all-pay auction with the fraction chosen so the all players' participation constraint binds. This is possible because players - unlike in the standard case of independent types - do not demand information rents, since they do not have information about how their valuation compares to that of others.

The second part of the paper presents results related to procurement auctions, where firms are competing for the right to deliver a single contract. It is shown how the model presented here can resolve a paradox which standard auction theory cannot easily explain. It is shown that in a setting where i) no contract is allocated when only one firm submits a bid, ii) submitting a bid is costly and iii) marginal costs are drawn from a distribution with finite support from $[\underline{c}, \overline{c}]$, then no firm will enter the market. This version of the Bertrand paradox we refer to as the entry paradox.

This paradox stems from the fact that those firms with marginal costs between $[\overline{c} - \epsilon, \overline{c}]$ know they have little chance of recouping their costs. These firms do not enter, but this means that firms with marginal costs of $[\overline{c} - 2\epsilon, \overline{c} - \epsilon]$ do not enter for similar reasons. Hence by an inductive argument no firms enter the procurement auction, and the entry paradox binds. The model presented here resolves the entry paradox since a distribution with finite support is not assumed; no firm knows that they have the highest - or almost the highest - marginal cost.

Modeling assumptions

In a departure from most existing literature, all assumptions are made directly on players' beliefs and a common prior is not explicitly considered. The motivation for making assumptions on the beliefs of players directly is that - from the point of view of the player - the belief formation process starts *after* the players are aware of their valuations. In these games players form beliefs about other players and their valuations after becoming aware of their own valuation, rather than having a prior in mind when the game starts and updating the prior upon learning their type. As an example consider the auction of a house. It seems likely that a potential buyer first goes to see the house and decides his private value. This information derived from seeing the house is then used in a second step where he forms beliefs about other potential bidders' valuations. The model presented here reflects this natural ordering, where first valuations are observed and secondly beliefs are formed.

The common prior assumption is discussed in detail in Morris (1995). The standard objection to discarding a common prior is that it allows for too wide a range of predictions from a model. In order to somewhat counter this criticism we show in the appendix that all results could be derived by using a common prior. The theoretical advantage of placing assumptions on beliefs is that agents do not necessarily need to share a common prior. Moreover in a one-shot auction environment it seems natural for observable beliefs rather than an unobservable abstract prior to be a primitive of the model. This makes the model more testable, as bids depend on actual beliefs of players rather than on the hypothetical beliefs of unrealised types.

Although it is not necessary that players with the same valuation have the same beliefs, we assume some structure on the beliefs of each player. The first assumption requires that the belief structure satisfies homogeneity of degree 0 ensuring that players i) believe valuations are highly correlated and ii) have no rank information about whether their valuation is higher or lower than their opponent's. Secondly weaker assumptions on the belief structure require agents to believe that the distribution is sufficiently well-behaved in order to ensure equilibrium existence. Finally we require the competitiveness of the auction to be common knowledge. This competitiveness of the auction is represented by γ^* which measures the probability of one player having a valuation very close to that of another player.

The concept of rank uncertainty appears in the literature on global games, including Carlsson & Van Damme (1993) and Morris & Shin (2003) among others. Although global games pursue a different objective these games have a strong link with the model suggested here, since they too exploit the players' rank uncertainty. In a recent working paper Compte & Postlewaite (2013) study simple auctions where players are boundedly rational and choose additively linear strategies.

The remainder of teh paper proceeds as follows. In the second section we introduce the general model as well as the key assumptions. Throughout the paper we consider the general case, where players are not required to have the same belief structure. Those readers not interested in the general case can ignore the player specific subscripts on beliefs. Clearly in this more restrictive setting all results remain valid. The third section presents results related to traditional auctions with two buyers, while the fourth section covers procurement auctions with two sellers. The final section concludes.

2 The general model

Consider a game with 2 players indexed by $i \in \{1, 2\}$, where each player privately observes his valuation $t_i \in \mathbb{R}_+$. Upon privately observing their valuation t_i, players simultaneously choose an action denoted by $a_i \in \mathbb{R}$. A pure strategy profile $\sigma_i(t_i) : \mathbb{R}_+ \mapsto \mathbb{R}$ maps a valuation to an action. The payoff function of player i is given by $\pi(a_i, a_j|t_i)$.

Players also have - perhaps player specific - interim beliefs which are defined as follows. Let $\Gamma_i(t_j|t_i)$ be the probability with which player i - after observing his own valuation to be t_i - believes the valuation of his opponent to be $\tilde{t}_j < t_j$. Moreover it is assumed that $\gamma_i(t_j|t_i) := \frac{\delta\Gamma_i(t_j|t_i)}{\delta t_j}$ exists whenever $t_i > 0$ and $t_j > 0$. The value function derived from players' payoff functions and possibly player-specific beliefs is defined as follows:

Definition 2.1.

$$V_i(a_i|\sigma_j, t_i) := \int_{\mathbb{R}_+} \pi(a_i, \sigma_j(t_j)|t_i) d\Gamma_i(t_j|t_i)$$

This gives the expected payoff of player i - after observing t_i - given that he plays a_i and his opponent is playing according to strategy profile σ_j. Note that probabilities are evaluated under the subjective prior of player i which is not necessarily the same as the subjective prior used by player j.

2.1 Modeling assumptions

The key assumptions are made directly on the players' interim beliefs rather than on a common prior. As discussed above, the decision to place assumptions on beliefs seems realistic in many situations. Note that these assumptions are placed precisely on the statistics required and allow players to otherwise differ in their beliefs: this implies the model can be solved without fully specifying the belief structure. This immediately shows that the belief structure need not be common knowledge among players. In the appendix we provide an example of a prior for which all belief assumptions hold, and therefore show that this model can be constructed starting from a common prior.

All of the following are assumed to be common knowledge among the players.

Assumption (Γ 1.1)**.**

$$\Gamma_1(1|1) = \Gamma_2(1|1) = \frac{1}{2}$$

This assumption means that upon observing a valuation $t_i = 1$, both players $i \in \{1, 2\}$ believe that the valuation of their opponent is less than 1 with probability $\frac{1}{2}$. Therefore each player - upon observing his valuation $t_i = 1$ - believes that his valuation will be higher than his opponents' with probability $\frac{1}{2}$.

While $\Gamma_i(1|1)$ denotes the probability that player i places on his opponent having valuation $t_j < 1$ given his own valuation is $t_i = 1$, $\gamma_i(1|1)$ denotes the density that player i puts on his opponent having valuation $t_j = 1$ given that he has valuation $t_i = 1$. This density determines the probability with which player i believes both valuations are extremely close, after observing $t_i = 1$. The next assumption is placed on this density:

Assumption (Γ 1.2)**.**

$$\gamma_1(1|1) = \gamma_2(1|1) = \gamma^*$$

This assumption requires the value $\gamma_i(1|1)$ to be the same for both players $i \in \{1, 2\}$. Informally - on observing a valuation $t_i = 1$ - both players hold similar beliefs about the chance of their opponent having a valuation $t_j \approx 1$. Both of the assumptions above are weak and satisfied by most (suitably normalised) standard auction models with symmetric bidders.

The final assumption is stronger and requires the belief structure of each player to be homogenous of degree 0:

Assumption (Γ 1.3)**.** *For all valuations $t_i > 0$, $t_j > 0$ and constants $k > 0$, :*

$$\Gamma_i(kt_j|kt_i) = \Gamma_i(t_j|t_i) \quad i \in \{1, 2\}$$

This assumption ensures that the structure of a player's beliefs is the same for any valuation he might have. After multiplying a player's valuation by a constant the player believes to be in a scaled up version of the game. Informally this means that his beliefs are preserved when multiplying his opponent's potential valuation by the same constant.

This assumption has two consequences. First it ensures that players have no rank information and do not know how their valuation compares to that of others. This is because when their valuation changes their beliefs also change in such a way that no rank inferences can be made. Secondly this assumption implies that players believe that their valuations are highly correlated. In models where valuations are highly correlated, agents find it difficult to make inferences about whether their valuation is high or low relative to an opponent. The appendix shows how this assumption approximately holds in settings where players' valuations are highly correlated, and how in such situations players cannot make rank inferences based on their valuation.

The following two lemmas show the implications of combining assumption (Γ1.3) with the previous two assumptions. In both cases the fact that beliefs are homogenous of degree zero means that an assumption on the case where a player has a valuation of 1 has implications for a player with any valuation.

Lemma 2.2. *Suppose assumptions* (Γ1.1) *and* (Γ1.3) *hold, then:*

$$\Gamma_i(t_i|t_i) = \frac{1}{2}$$

Hence player i believes that his opponent has a higher valuation than him with probability 1/2 and hence players believe they are equally likely to have the higher as they are to have the lower valuation.

Lemma 2.3. *Suppose assumptions* (Γ1.2) *and* (Γ1.3) *hold, then:*

$$t_i\gamma_i(t_i|t_i) = \gamma^*$$

We say that a belief structure satisfies (Γ1) if and only if it satisfies (Γ1.1), (Γ1.2) and (Γ1.3). Having made these assumptions we now define an equilibrium strategy profile. The equilibrium concept refers to the interim stage of the game:

Definition 2.4. *The strategy profile* (σ_1, σ_2) *is an equilibrium iff for all* i *for all* $t_i > 0$*:*

$$V_i(\sigma_i(t_i)|\sigma_j, t_i) \geq V_i(\hat{a}_i|\sigma_j, t_i) \quad \textit{for all} \quad \hat{a}_i \in \mathbb{R}$$

We use the term local equilibrium to refer to a strategy profile where players cannot gain by deviating to a nearby strategy. Formally this is defined as follows:

Definition 2.5. *A strategy profile* (σ_1, σ_2) *is a local equilibrium if and only if there exists an* $\epsilon > 0$ *such that for all* $t_i > 0$*:*

$$V_i(\sigma_i(t_i)|\sigma_j, t_i) \geq V_i(\hat{a}_i|\sigma_j, t_i) \quad \textit{for all} \quad \hat{a}_i \in [\sigma_i(t_i) - \epsilon, \sigma_i(t_i) + \epsilon]$$

In applications it is often easier to look for local equilibria first and the check whether the local equilibrium is an equilibrium. This is the tactic used below.

3 First price auction

In this section we focus on the first price auction where players have the following payoff function:

$$\pi(a_i, a_j|t_i) = \begin{cases} t_i - a_i & \text{if } a_i > a_j \\ 0 & \text{otherwise} \end{cases}$$

Note that this payoff function is homogenous of degree 1 and hence satisfies assumption (π). In addition we assume that the belief structure of each player satisfies assumption (Γ1).

3.1 Finding a local equilibrium:

In this section we search for a symmetric and linear strategy that is locally optimal for both players, this means that players cannot gain by unilaterally deviating to a nearby strategy. We refer to this as local equilibrium.

The symmetry restriction requires that $\sigma_1(t) = \sigma_2(t) = \sigma(t)$, while the linearity restriction requires that the strategy $\sigma(t) = \rho^* t$ for some ρ^*. [1] Having found a local equilibrium of this form, the next section will consider additional assumptions on the belief structure that ensure this local equilibrium is indeed an equilibrium.

When an opponent j plays according to the linear strategy $\sigma_j(t) = \rho^* t$, we abuse notation by writing $V(a|\sigma_j, t_i) = V(a|\rho^*, t_i)$. Now we look for a local equilibrium of the form $\sigma_i(t) = \sigma_j(t) = \rho^* t$ and without loss of generality consider the value function of player i:

$$V_i(a_i|\rho^*, t_i) = \Gamma_i\Big(\frac{a_i}{\rho^*}\Big|t_i\Big)(t_i - a_i)$$

Note that when player i chooses a bid a_i and his opponent is bidding according to strategy profile $\sigma_j(t_j) = \rho^* a_j$, then $a_i > a_j$ if and only if $\rho^* t_j < a_i$ which is the case if and only

[1] Note that while we are only looking equilibria of the linear form, we do not restrict the strategy set of players to be linear. Players may choose to deviate to arbitrary strategies. This is different to the approach taken by Compte & Postlewaite (2013)

if $t_j < \frac{a_i}{\rho^*}$. Hence player i evaluates his probability of winning the object to be $\Gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right)$ given that i) he values the object t_i, ii) he submits a bid a_i and iii) his opponent is bidding according to the strategy profile $\sigma_j(t) = \rho^* t$. Meanwhile the second term denotes the payoff $t_i - a_i$ that player i receives if he wins the auction.

Differentiating with respect to a leads to the first order condition. The first term represents the benefit of bidding higher since there is a probability of overtaking the opponent's bid and winning the object. Meanwhile the second term represents the disutility of paying more in the case when a player would have won the object in any case:

$$\frac{\delta V_i(a_i|\rho^*, t_i)}{\delta a_i} = \frac{1}{\rho^*}\gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right)(t_i - a_i) - \Gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right)$$

We look for a local equilibrium where both players play according to strategy profiles $\sigma_i(t) = \sigma_j(t) = \rho^* t$. Hence $a_i = \sigma_i(t_i) = \rho^* t_i$ and evaluating at this point leads to:

$$\begin{aligned}\frac{\delta V_i(a_i|\rho^*, t_i)}{\delta a_i}\Big|_{a_i=\rho^* t_i} &= \frac{1}{\rho^*}\gamma_i(t_i|t_i)(t_i - a_i) - \Gamma_i(t_i|t_i) \\ &= \frac{1}{\rho^*}t_i\gamma_i(t_i|t_i)(1 - \rho^*) - \Gamma_i(t_i|t_i)\end{aligned}$$

The local equilibrium condition requires that deviating to a nearby strategy is not profitable for any player. Since the value function is differentiable, for this to hold it is necessary for the derivative of the value function to be equal to 0. Hence:

$$\begin{aligned}\frac{\delta V_i(a_i|\rho^*, t_i)}{\delta a_i}\Big|_{a_i=\rho^* t_i} &= 0 \\ \frac{1}{\rho^*}t_i\gamma_i(t_i|t_i)(1 - \rho^*) - \Gamma_i(t_i|t_i) &= 0\end{aligned}$$

By lemmas 2.2 and 2.3 above, we have that $\Gamma_i(t_i|t_i) = \frac{1}{2}$ and $t_i\gamma_i(t_i|t_i) = \gamma_i(1|1) = \gamma^*$. Hence:

$$\begin{aligned}
\frac{1}{\rho^*}t_i\gamma_i(t_i|t_i)(1-\rho^*) - \Gamma_i(t_i|t_i) &= 0 \\
\frac{\gamma^*}{\rho^*}(1-\rho^*) &= \frac{1}{2} \\
\rho^* &= \frac{2\gamma^*}{1+2\gamma^*}
\end{aligned}$$

This means that for all $t_i > 0$, player i has no profitable local deviations. Since i was chosen arbitrarily, the strategy profile $\sigma_1(t) = \sigma_2(t) = \rho^* t$ is a local equilibrium. Formally this result is summarised as follows:

Proposition 3.1. *Suppose* $(\Gamma 1)$ *is satisfied. Then the following the unique linear and symmetric local equilibrium is given by the following strategy profile:*

$$\sigma_i(t_i) = \frac{2\gamma^*}{2\gamma^*+1}t_i \quad \textit{for all } i$$

Note also that the above derivation has not only produced a linear symmetric local equilibrium it also shows that this local equilibrium is unique. Hence if a symmetric and linear equilibrium exists, then it is unique.

3.2 From local equilibrium to equilibrium

We now introduce an additional condition on the players' belief structures to ensure that the local equilibrium found above is indeed an equilibrium. First note that the linear strategy profile is strictly monotonic and continuous in a player's valuation. When a player determines his optimal bid, he may want to consider imitating a player having any of the valuations that may potentially be present given his true type. If he bids above the equilibrium bid of the player with the highest feasible valuation, he is certain to win the object. But this cannot be an equilibrium since he is strictly better of by imitating the player with the highest valuation, winning with certainty and paying strictly less. On the other hand bidding below the equilibrium bid of the lowest valuation is weakly dominated by imitating a player with the lowest valuation and receiving a payoff of zero with certainty.

When a player considers whether to imitate the strategy of a player with a different valuation, then for any valuation, t_i, such that $\Gamma_i(t_j|t_i) \in (0,1)$, the player considers how likely he is to go from winning the object to losing the object by lowering his bid slightly. This probability is reflected in the ratio of the likelihood that given the player has a valuation t_i, player j has a valuation just below t_j conditional on his valuation being at most t_j. This ratio is given by $\frac{\gamma(t_j|t_i)}{\Gamma(t_j|t_i)}$. The following condition ensures that no player wants to deviate by imitating another player's valuation:

Assumption (Γ2: DECREASING INVERSE HAZARD RATE)**.** *If* $t_j \leq t'_j$ *and* $\Gamma(t_j|t_i) > 0$ *then:*

$$\frac{\gamma(t_j|t_i)}{\Gamma(t_j|t_i)} \geq \frac{\gamma(t'_j|t_i)}{\Gamma(t'_j|t_i)}$$

This assumption means that the probability player i places on his opponent having a valuation of exactly t_j given that his oponnent has a valuation at most t_j is decreasing in t_j for all t_j that occur with positive probability. Hence players may not believe that on a given the support, the density of valuations increases too fast.

The assumption can also be interpreted in terms of competitiveness. Consider player i imitating the bidding strategy of any (feasible) valuation of player j. Suppose player i chooses a valuation that is above the true valuation of player j and hence makes player i bid higher than player j and win the auction. Then by lowering his bid slightly, player i is more likely to move from winning the auction to losing the auction the lower the type of player j.

Proposition 3.2. *Suppose assumptions* (Γ_1) *and* $(\Gamma 2)$ *are satisfied for all* $i \in \{1,2\}$*. Then* $\sigma_i(t) = \sigma_j(t) = \left(\frac{2\gamma^*}{1+2\gamma^*}\right)t$ *is an equilibrium*

Proof.

$$\frac{\delta V_i(a_i|\rho^*, t_i)}{\delta a_i} = \frac{1}{\rho^*}\gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right)(t_i - a_i) - \Gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right)$$

Whenever $\Gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right) \in (0, 1)$:

$$\frac{\delta V_i(a_i|\rho^*, t_i)}{\delta a_i} = \Gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right)\left[\frac{1}{\rho^*}\frac{\gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right)}{\Gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right)}(t_i - a_i) - 1\right]$$

Clearly any deviation $a_i > t_i$ is not profitable. When $a_i < t_i$ both $(t_i - a_i)$ and $\frac{\gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right)}{\Gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right)}$ are both positive and decreasing in a_i. Therefore:

$$\begin{array}{lccc}
\left[\frac{1}{\rho^*}\frac{\gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right)}{\Gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right)}(t_i - a_i) - 1\right] & \text{is decreasing in } a_i & & \\
\left[\frac{1}{\rho^*}\frac{\gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right)}{\Gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right)}(t_i - a_i) - 1\right] = 0 & \text{at most once} & \text{on interval } \Gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right) \in (0, 1) \\
\frac{\delta V_i(a_i|\rho^*, t_i)}{\delta a_i} & \text{obeys SCP} & \text{on interval } \Gamma_i\left(\frac{a_i}{\rho^*}\middle|t_i\right) \in (0, 1)
\end{array}$$

□

4 Full surplus extraction

In this section we introduce a simple mechanism an auctioneer can use to extract the full surplus from the players. Full surplus extraction is feasible because players cannot claim information rents. This is because - unlike in most auction models - players do not have information about their rank and hence their valuation relative to that of others.

The full surplus extraction mechanism works by using a payment function between the first price and all-pay auction. If the agent submits the highest bid, he pays the full amount to the principal. Meanwhile if an agent does not submit the highest bid, he pays a proportion

$(1-c)$ of his bid to the principal. This payment rule leads to the following payoff function for the two agents:

$$\pi(a_i, a_j|t_i) = \begin{cases} t_i - a_i & \text{if} \quad a_i > a_j \\ -(1-c)a_i & \text{otherwise} \end{cases}$$

The limiting case $c = 1$ captures the first price auction studied above, while the limiting case $c = 0$ captures an all-pay auction.

4.1 Finding a local equilibrium

In order to find a local equilibrium we follow the same technique used above. In this case the value function is given as follows:

$$V_i^c(a_i|\rho^*, t_i) = \Gamma\Big(\frac{a_i}{\rho^*(c)}|t_i\Big)(t_i - ca_i) - (1-c)a_i$$

Differentiating leads to the following first order condition for player $t_i = 1$:

$$\frac{\delta V_i^c(a_i|\rho^*, 1)}{\delta a_i} = \frac{1}{\rho^*(c)}\gamma_i\Big(\frac{a_i}{\rho^*(c)}|t_i\Big)(1 - ca_i) - c\Gamma\Big(\frac{a_i}{\rho^*(c)}|t_i\Big) - (1-c)$$

Evaluating at $a_i = \tau^*$, setting this condition equal to 0 and re-arranging leads to the following expression for $\rho^*(c)$. Note that $\rho^*(1)$ coincides with the formula given before for the first price auction:

$$\rho^*(c) = \frac{2\gamma^*}{2\gamma^* c - c + 2}$$

As long as $\gamma^* \geq \frac{1}{2}$ and player's believe the conditional distribution around their type is sufficiently concentrated, then $\rho^*(c)$ is decreasing in c. This means that players bid higher as the payment rule moves away from being a first price auction and becomes closer to an all-pay auction. This - somewhat counterintuitive - result stems from the fact that when c is low a player receives a higher 'jump' in his payoff when he overtakes his opponent. This is because the extra payment the winning player must make is lower when c is low.

Hence an auction with an all pay element may lead to higher revenue for two reasons: first players bid higher because the extra gains from being the highest bidder are larger; secondly the auctioneer collects revenue from both players rather than just the highest bidder. This result contrasts with the revenue equivalence theorem for independent types, when many payment rules lead to the same revenue.

4.2 From local equilibrium to equilibrium

Due to the additional constraints arising from the fact that given a player is losing then lowering his bid results in a lower payment a stronger condition is needed to ensure that players have no global deviations. This condition says that players believe low valuations to occur more frequently than high valuations and ensures that the additional benefits from bidding higher are decreasing on the relevant interval:

Assumption (Γ5)**.**

$$\gamma(t_j|1) \geq \gamma(t'_j|1) \quad \textit{whenever} \quad t_j \leq t'_j$$

In addition we have to ensure that players do indeed want to submit a positive bid. This is the case when the following participation constraint holds.

Assumption (PC1)**.**

$$\frac{\Big(1 - c\rho^*(c)\Big)}{2} - (1-c)\rho^*(c) \geq 0$$

The numerator of the fraction represents the extra utility a player with valuation $t_i = 1$ receives if he is the highest bidder, which occurs with probability $\frac{1}{2}$. Meanwhile the second term represents the payment that a player will make regardless of whether or not he is the highest bidder. For a player to want to participate in the auction it must be the case that the expected gains from participating are greater than his expected payment, in particular including the payment incurred independent of winning or losing the auction.

These assumptions ensure that the local equilibrium found above is indeed an equilibrium.

Proposition 4.1. *Suppose assumptions* $(\Gamma 1)$, $(\Gamma 3)$ *snd* $(PC1)$ *are satisfied. Then the unique symmetric linear equilibrium of an auction with an all pay component is given by the strategy profile*

$$\sigma(t) = \Big(\frac{2\gamma^*}{2\gamma^* c - c + 2}\Big) t$$

4.3 A simple optimal mechanism

An optimal mechanism - given that full surplus extraction is possible - should leave 0 surplus to the agents. Hence a player with type $t_i = 1$ should be indifferent between participating and submitting a bid $a_i = 0$. Hence the participation constraint binds and:

$$\frac{\Big(1 - c^* \rho^*(c^*)\Big)}{2} - (1 - c^*)\rho^*(c^*) \geq 0$$

Re-arranging this expression leads to the following:

$$c^* = \frac{4\gamma^* - 2}{4\gamma^* - 1}$$

Note that if γ^* is high and the distribution is relatively concentrated, then c^* is close to 1 and the optimal mechanism closely resembles a first price auction. The reasoning for this is that when the distribution is concentrated the auction is already very competitive, and a first price auction does a good job of extracting all the surplus. However when γ^* is low the auction is much less competitive, and a mechanism more closely resembling an all-pay auction is required to fully extract the surplus.

It may at first seem puzzling that a relatively simple mechanism can lead to full surplus extraction. Formally this result stems from the fact that the participation constraint of every agent is the same (after multiplying by a suitable constant). This means that a mechanism designed to give players with type 1 zero surplus will also give zero surplus to every other player. Informally full-surplus extraction is not normally achievable without using complex lotteries because it is necessary to pay information rents to players. Here information rents do not arise because players do not have information about where their likely position in the distribution of types.

Proposition 4.2. *Suppose assumptions* (Γ_1) *and* $(\Gamma 3)$ *are satisfied. In the equilibrium where players play according to*

$$\sigma_i(t) = \frac{2\gamma^*}{2\gamma^* c - c + 2} t$$

the auctioneer can achieve full surplus extraction by choosing c^**, such that:*

$$c^* = \frac{4\gamma^* - 2}{4\gamma^* - 1}$$

5 Bertrand competition with costs

In this section we consider an application to Bertrand competition. Having gone through the first price auction example in detail, we immediately consider the general case, where firms have to pay a fraction c of their bid independent of winning or losing and pay the full bid in case they win. The standard case where players do not pay anything if they lose, is given by $c = 0$.

Players' payoff functions are given as follows:

$$\pi(a_i, a_j|t_i) = \begin{cases} a_i - t_i & \text{if} \quad a_i < a_j \\ -ca_i & \text{otherwise} \end{cases}$$

5.1 Finding a local equilibrium

Again we initially look for linear and symmetric equilibrium of the form $\sigma_i(t) = \sigma_j(t) = \rho^* t$. The probability firm i places on being the lowest bidder given i) he submits a bid a_i, ii) his total costs in the case where he wins the contract are t_i and iii) his opponent is bidding according to the strategy profile $\sigma_j(t) = \rho^* t$ is given by $\left[1 - \Gamma_i\left(\frac{a_i}{\rho^*}|t_i\right)\right]$. Meanwhile firm i always incurs costs of ϵa_i regardless of whether he has the highest bid or the lowest bid, and if he has the lowest bid he incurs additional costs of t_i and receives payment of $a_i(1+c)$. This leads to the following value function for player i:

$$V_i(a_i|\rho^*, t_i) = \Big[1 - \Gamma_i\Big(\frac{a_i}{\rho^*}|t_i\Big)\Big]\Big[a_i(1+\epsilon) - t_i\Big] - ca_i$$

Differentiating this expression with respect to a_i leads to the following first order condition:

$$\frac{\delta V_i(a_i|\rho^*, t_i)}{\delta a_i} = \Big[1 - \Gamma_i\Big(\frac{a_i}{\rho^*}|t_i\Big)\Big](1+c) - \frac{1}{\rho^*}\gamma_i\Big(\frac{a_i}{\rho^*}|t_i\Big)\Big[a_i(1+c) - t_i\Big] - c$$

To find a symmetric, linear equilibrium we set $a_i = \rho^* t_i$. Setting the first order condition equal to 0 yields:

$$\Big[1 - \Gamma_i(t_i|t_i)\Big](1+c) = \frac{1}{\rho^*} t_i \gamma_i(t_i|t_i)\Big[\rho^*(1+c) - 1\Big] - c$$

Note that $\Gamma_i(t_i|t_i) = \frac{1}{2}$ and $t_i\gamma_i(t_i|t_i) = \gamma^*$. Hence:

$$\frac{1+c}{2} = \frac{\gamma^*}{\rho^*}\Big[\rho^*(1+c) - 1\Big] - c$$

Re-arranging:

$$\rho^*(c) = \frac{2\gamma^*}{2\gamma^*(1+c) - (1+3c)}$$

This leads to the following result:

Proposition 5.1. *Suppose assumptions* ($\Gamma 1$) *is satisfied. Then in the unique pure strategy symmetric linear equilibrium of the Betrand competition game, if it exists, firms bid according to the following bid function:*

$$\sigma(t_i) = \Big[\frac{2\gamma^*}{2\gamma^*(1+c) - (1+3c)}\Big] t_i$$

5.2 From local equilibrium to equilibrium

Recall that in the upward auction we needed an additional condition on the the likelihood of just losing the object by lowering one's bid slightly given that one is currently winning the object. In case of a procurement auction the condition required is reversed. The condition required is not a decreasing inverse hazard rate but an increasing hazard rate. This hazard rate is related to the probability of losing the object by raising one's bid slightly given that one is currently winning the object.

Assumption ($\Gamma 4$: Increasing hazard rate)**.** *If* $t_j \leq t'_j$ *and* $\Gamma(t_j|t_i) < 1$ *then:*

$$\frac{\gamma(t_j|t_i)}{1-\Gamma(t_j|t_i)} \leq \frac{\gamma(t'_j|t_i)}{1-\Gamma(t'_j|t_i)}$$

Note that unlike in the case of an upward auction, this condition is sufficient both when the losing firm does not pay anything and when the losing firm pays a fraction of its bid. This follows from the fact that raising one's bid has two negative effects. First the firm's probability of winning the object decreases. Secondly the price the firm has to pay independent of winning or losing increases. This means that overall, firms have less incentive to deviate by imitating a higher player with a higher valuation.

Recall that for the upward auction these effects move in different directions. By lowering his bid slightly a player is less likely to win the object, but at the same time the amount he has to pay independent of winning or losing decreases. Therefore stronger conditions are needed for upward auctions with an all pay component.

Due to the all pay component, we require the following participation constraint to hold:

Assumption (PC2)**.**

$$\frac{(1+c)\rho^*(c)-1}{2} - c\rho^*(c) \geq 0$$

In case of procurement auctions with an all pay component, assumption ($\Gamma 4$) along with ($PC2$) are sufficient to ensure the following:

Proposition 5.2. *Suppose* $(\Gamma 1)$, $(\Gamma 4)$ *and* $(PC2)$ *are satisfied for all* $i \in \{1,2\}$. *Then the following is an equilibrium strategy profile:*

$$\sigma_i(t) = \sigma_j(t) = \Big[\frac{2\gamma(1|1)}{2\gamma(1|1)(1+c) - (1+3c)}\Big]t$$

is an equilibrium of the Bertrand competition game with costs.

The proof is similar to that of the first price auction (proposition 3.2) and is relegated to the appendix.

5.3 Entry paradox

In this section we investigate a version of the Bertrand paradox referred to as the entry paradox. Consider a situation where two firms compete for a procurement contract. Each firm submits a bid a_i and the lower bidder wins the contract incurring total costs of t_i and receiving payment a_i. Meanwhile the higher bidder incurs bidding costs ca_i proportional to his bid. Additionally, if one player chooses to bid 0 then the auction is deemed uncompetitive and is postponed: neither firm wins the contract, and bidder i incurs a cost of ca_i due to auction participation costs.

This game models several real-life scenarios, when uncompetitive procurement auctions are canceled. The payoff function is given as follows:

$$\pi(a_i, a_j|t_i) = \begin{cases} a_i - t_i & \text{if} \quad a_i < a_j \text{ and } \min\{a_i, a_j\} > 0 \\ -ca_i & \text{otherwise} \end{cases}$$

First note that the pure strategy equilibria found in the previous section continue to be equilibria. This is because - as long as c is low enough - all firms make positive profits in expectation and do not want to deviate to bidding 0. Hence positive bids will be made in equilibrium. Secondly note that this is not the case in standard models with a bounded common prior with support $[\underline{t}, \overline{t}]$ and monotonic bidding. In this case a firm with high marginal cost close to $\bar{t}$ knows it has very little chance of winning, and his participation constraint will be violated. This means that such a firm will not participate in the auction. This is the base case.

For the inductive step suppose all firms with marginal cost above t^* will not participate in the auction and consider firms in the interval $[\epsilon - t^*, t^*]$. Such firms cannot receive a positive payoff unless their rival is also in the same interval: a firm with marginal cost less that $t^* - \epsilon$ would bid less and win the auction, while a firm with marginal cost above t^* would bid 0 ensuring the auction is uncompetitive. If ϵ is sufficiently small the benefits from submitting a positive bid outweigh the costs, and so a bid of 0 will be made. Using this inductive argument it can be seen that in equilibrium no positive bids will be made by any firm.

This effect is driven by a similar reasoning to that of Abreu and Brunnermeier (2003).

6 Discussion

Returning to the first price auction example and the way it is solved, we make the following observation. Standard models are usually solved by noting that a bidder with the lowest type bids his valuation and then calculating the equilibrium bids of other types using this information. These solutions work their way from the lowest valuations to the highest valuations. Furthermore they typically involve the use of a types' virtual valuation which is given by his valuation minus the inverse hazard rate at his type and is therefore given by $t - \frac{1-F(t)}{f(t)}$. This concept lacks intuition. Note that when there are at least two players competing for a single object, a player with the lowest valuation has no chance of winning the object. Nevertheless all players with higher valuations take into account his bid. In reality it is difficult to imagine a player knowing his valuation chooses his equilibrium bid, by first going through all the lower types.

In contrast the model suggested in this paper is solved by starting from the top leading to very intuitive conditions. A player who knows he has the highest valuation decides his bid by considering how likely he is to go from winning the object to losing the object by lowering his bid slightly. This ratio can be described by $\frac{f(t)}{F(t)}$ and represents the likelihood of an opponent having a valuation of exactly t reflected by $f(t)$ given that he has a valuation of t or below as given by $F(t)$. In order to solve the model we therefore need to place an assumption on this ratio.

This natural solution approach becomes particularly clear when comparing the assumption needed in an upward first prize auction to a downward auction such as represented by Bertrand competition. As one might expect, in this case the relevant ratio is given by the hazard rate, $\frac{f(t)}{1-F(t)}$. When deciding its mark-up above marginal cost, a firm considers how likely it is to go from winning the market and having the lowest price to losing the market by increasing its mark-up slightly.

The intuition behind this result is that players have more to gain from increasing their bid slightly, because the gain in payoff when moving from losing the auction to winning is greater. In a first price auction this payoff is given by the difference between a players total bid and his valuation. In an all pay auction however players have to pay their bid independent of winning or losing. If a slightly higher bid results in winning the auction the player's payoff increases by his valuation minus the slight increase in his bid and hence almost his entire valuation. In other words, when deciding whether to increase his bid, his current bid can be considered a sunk cost.

Further research We believe that conducting an experiment may provide additional insights and justification for the model proposed in this paper. In the experiment we suggest, participants are shown a number of art objects that were auctioned off in a true auction at an established auction house. For each of the objects the players are assigned a valuation according to some distribution satisfying our assumptions. Players then choose a bid below their valuation. If this bid is above the true price of the object the player is awarded a (fraction of) the difference between his valuation and the price. This approach allows us to test whether - in a setting where this is appropriate - players do indeed bid a constant fraction of their valuation. While this approach does not immediately verify the suggested theory, it adds credibility in case of not falsifying it. Due to its simplicity the theory is particularly easy to falsify in case it is incorrect.

Furthermore note that the approach of using assumptions on beliefs rather than on underlying primitives allowed us to place the assumptions and restrictions precisely on the points where they are required. This approach allows for the highest possible flexibility of the model. It means that as long as these assumptions hold, any other aspect of the model can be changed without altering the outcome. This means that the model proposed is particularly robust and one can immediately see what drives the result. Once more we would like to stress that players in this model are fully rational and that the result do not arise due to irrational behaviour. The argument only says that the results still go through when players' beliefs are not fully rational as long as they are rational on some critical points.

We believe that in many other settings a similar approach may lead to interesting insights and hence provides opportunities for future research.

7 Appendix

7.1 Proof of proposition 4.1

Proof. Recall the first order condition:

$$\frac{\delta V_i^c(a_i|\rho^*, 1)}{\delta a_i} = \frac{1}{\rho^*(c)}\gamma_i\Big(\frac{a_i}{\rho^*(c)}|t_i\Big)(1 - ca_i) - c\Gamma\Big(\frac{a_i}{\rho^*(c)}|t_i\Big) - (1 - c)$$

Note that assumption ($\Gamma 3$) ensures that this expression is decreasing in a_i. Finally the participation constraint ensures that players do indeed want to submit a positive bid. This completes the proof.

□

7.2 Proof of proposition 5.2

Proof.

$$\frac{\delta V_i(a_i|\rho^*,t_i)}{\delta a_i} = \Big[1-\Gamma_i\Big(\frac{a_i}{\rho^*}|t_i\Big)\Big](1+c) - \frac{1}{\rho^*}\gamma_i\Big(\frac{a_i}{\rho^*}|t_i\Big)\Big[a_i(1+\epsilon)-t_i\Big] - c$$

Whenever $\Gamma_i\Big(\frac{a_i}{\rho^*}|t_i\Big) \in (0,1)$:

$$\frac{\delta V_i(a_i|\rho^*,t_i)}{\delta a_i} = \Big[1-\Gamma_i\Big(\frac{a_i}{\rho^*}|t_i\Big)\Big]\Big[1+c-\frac{1}{\rho^*}\frac{\gamma_i\Big(\frac{a_i}{\rho^*}|t_i\Big)\Big[a_i(1+c)-t_i\Big]}{1-\Gamma_i\Big(\frac{a_i}{\rho^*}|t_i\Big)} - \frac{\epsilon}{1-\Gamma_i\Big(\frac{a_i}{\rho^*}|t_i\Big)}\Big]$$

Note that:

$$\begin{array}{lll}
\frac{\epsilon}{1-\Gamma_i\left(\frac{a_i}{\rho^*}|t_i\right)} & \text{is increasing in } a_i & \\
\frac{\gamma_i\left(\frac{a_i}{\rho^*}|t_i\right)\left[a_i(1+c)-t_i\right]}{1-\Gamma_i\left(\frac{a_i}{\rho^*}|t_i\right)} & \text{is increasing in } a_i & \\
\Big[1+c-\frac{1}{\rho^*}\frac{\gamma_i\left(\frac{a_i}{\rho^*}|t_i\right)\left[a_i(1+c)-t_i\right]}{1-\Gamma_i\left(\frac{a_i}{\rho^*}|t_i\right)} - \frac{\epsilon}{1-\Gamma_i\left(\frac{a_i}{\rho^*}|t_i\right)}\Big] & \text{is decreasing in } a_i & \\
\Big[1+c-\frac{1}{\rho^*}\frac{\gamma_i\left(\frac{a_i}{\rho^*}|t_i\right)\left[a_i(1+c)-t_i\right]}{1-\Gamma_i\left(\frac{a_i}{\rho^*}|t_i\right)} - \frac{\epsilon}{1-\Gamma_i\left(\frac{a_i}{\rho^*}|t_i\right)}\Big] = 0 & \text{at most once} & \text{on interval } \Gamma_i\Big(\frac{a_i}{\rho^*}|t_i\Big) \in (0,1) \\
\frac{\delta V_i(a_i|\rho^*,t_i)}{\delta a_i} & \text{obeys SCP} & \text{on interval } \Gamma_i\Big(\frac{a_i}{\rho^*}|t_i\Big) \in (0,1)
\end{array}$$

This - combined with the participation constraint - is enough to show that ρ^* is indeed a global maximum. The inverse is trivial. Clearly if the participation constraint is not satisfied, then it cannot be a global equilibrium since deviating to $a_i = 0$ is profitable.

□

7.3 Foundation using a Harsanyi prior

We now calculate the distribution $\gamma(t|1)$ when using the base distribution $t \sim U[k\lambda, \lambda]$ and $\lambda \sim exp$. Since t_j is uniformly distributed given λ, $f(t_j|\lambda)$ is given as follows:

$$f(t_j|\lambda) = \begin{cases} \frac{1}{\lambda(1-k)} & \text{if} \quad t_j \in [k\lambda, \lambda] \\ 0 & \text{otherwise} \end{cases}$$

Using this expression we can calculate $g(\lambda|t_i)$. Note $g(\lambda|t_i) = 0$ if $\lambda \notin \left[\frac{t_i}{k}, t_i\right]$. Meanwhile if $\lambda \in \left[\frac{t_i}{k}, t_i\right]$:

$$\begin{aligned} g(\lambda|t_i) &= \frac{f(t_i|\lambda)g(\lambda)}{\int_{\Re_+} f(t_i|\tilde{\lambda})g(\tilde{\lambda})d\tilde{\lambda}} \\ &= \frac{1}{\lambda^2(1-k)} \frac{1}{\int_{t_i}^{\frac{t_i}{k}} \frac{1}{\lambda^2(1-k)} d\tilde{\lambda}} \\ &= \frac{1}{\lambda^2} \frac{1}{\left[-\frac{1}{\lambda}\right]_{t_i}^{\frac{t_i}{k}} d\tilde{\lambda}} \\ &= \frac{t_i}{\lambda^2(1-k)} \end{aligned}$$

We now calculate the interim beliefs:

$$\gamma(t_j|t_i) = \int_0^\infty g(\lambda|t_i)f(t_j|\lambda)d\lambda$$

First when $t_j \in \left[kt_i, t_i\right]$ it must be the case that $\lambda \in \left[t_i, \frac{t_j}{k}\right]$. Then:

$$
\begin{aligned}
\gamma(t_j|t_i) &= \int_{t_i}^{\frac{t_j}{k}} \frac{t_i}{\lambda^2(1-k)} \frac{1}{\lambda(1-k)} d\lambda \\
&= \Big[-\frac{t_i}{2\lambda^2(1-k)^2} \Big]_{t_i}^{\frac{t_j}{k}} \\
&= \frac{1}{2t_i(1-k)^2} - \frac{k^2 t_i}{2t_j^2(1-k)^2} \\
&= \frac{t_j^2 - k^2 t_i^2}{2t_i t_j^2 (1-k)^2} \\
&= \frac{(t_j - kt_i)(t_j + kt_i)}{2t_i t_j^2 (1-k)^2}
\end{aligned}
$$

Integrating over the interval $\tilde{t}_j \in \Big[kt_i, t_j\Big]$

$$
\begin{aligned}
\Gamma(t_j|t_i) &= \int_{kt_i}^{t_j} \gamma(t_j|t_i) d\tilde{t}_j \\
&= \int_{kt_i}^{t_j} \frac{1}{2t_i(1-k)^2} - \frac{k^2 t_i}{2\tilde{t}_j^2(1-k)^2} d\tilde{t}_j \\
&= \frac{1}{2(1-k)^2} \Big[\frac{\tilde{t}_j}{t_i} + \frac{k^2 t_i}{\tilde{t}_j} \Big]_{kt_i}^{t_j} \\
&= \frac{1}{2(1-k)^2} \Big[\frac{t_j}{t_i} + \frac{k^2 t_i}{t_j} - k - k \Big] \\
&= \frac{t_j^2 + k^2 t_i^2 - 2kt_i t_j}{2(1-k)^2 t_i t_j} \\
&= \frac{(t_j - kt_i)^2}{2(1-k)^2 t_i t_j}
\end{aligned}
$$

We now show that $\frac{\gamma(t_j|t_i)}{\Gamma(t_j|t_i)}$ is decreasing in t_j for all k:

$$\begin{aligned}\frac{\gamma(t_j|t_i)}{\Gamma(t_j|t_i)} &= \frac{t_j + kt_i}{t_j(t_j - kt_i)} \\ &= \frac{1 + k\frac{t_i}{t_j}}{(t_j - kt_i)}\end{aligned}$$

The numerator is decreasing in t_j and the denominator is increasing in t_j, and hence the whole expression is decreasing in t_j.

Secondly when $t_j \in \left[t_i, \frac{t_i}{k}\right]$ it must be the case that $\lambda \in \left[t_j, \frac{t_i}{k}\right]$. Then:

$$\begin{aligned}\gamma(t_j|t_i) &= \int_{t_j}^{\frac{t_i}{k}} \frac{t_i}{\lambda^2(1-k)} \frac{1}{\lambda(1-k)} d\lambda \\ &= \left[-\frac{t_i}{2\lambda^2(1-k)^2}\right]_{t_j}^{\frac{t_i}{k}} \\ &= \frac{t_i}{2t_j^2(1-k)^2} - \frac{k^2}{2t_i(1-k)^2} \\ &= \frac{t_i^2 - k^2 t_j^2}{2t_i t_j^2(1-k)^2} \\ &= \frac{(t_i - kt_j)(t_i + kt_j)}{2t_i t_j^2(1-k)^2}\end{aligned}$$

Integrating over the interval $\tilde{t}_j \in \left[t_j, \frac{t_i}{k}\right]$:

$$
\begin{aligned}
1-\Gamma(t_j|t_i) &= \int_{t_j}^{\frac{t_i}{k}} \gamma(t_j|t_i) d\tilde{t}_j \\
&= \int_{t_j}^{\frac{t_i}{k}} \frac{t_i}{2\tilde{t}_j^2(1-k)^2} - \frac{k^2}{2t_i(1-k)^2} d\tilde{t}_j \\
&= \Big[-\frac{t_i}{2\tilde{t}_j(1-k)^2} - \frac{k^2\tilde{t}_j}{2t_i(1-k)^2} \Big]_{t_j}^{\frac{t_i}{k}} \\
&= \frac{t_i}{2t_j(1-k)^2} + \frac{k^2 t_j}{2t_i(1-k)^2} - \frac{k}{2(1-k)^2} + \frac{kt_i}{2t_i(1-k)^2} \\
&= \frac{t_i^2 + k^2 t_j^2 - 2kt_i t_j}{2t_j t_i (1-k)^2} \\
&= \frac{(t_i - kt_j)^2}{2t_j t_i (1-k)^2}
\end{aligned}
$$

Calculating $\frac{\gamma(t_j|t_i)}{1-\Gamma(t_j|t_i)}$:

$$
\frac{\gamma(t_j|t_i)}{1-\Gamma(t_j|t_i)} = \frac{t_i + kt_j}{t_j(t_i - kt_j)}
$$

Integrating over the interval $\tilde{t}_j \in \Big[t_i, t_j\Big]$:

$$
\begin{aligned}
\Gamma(t_j|t_i) &= \Gamma(t_i|t_i) + \int_{t_i}^{t_j} \gamma(t_j|t_i) d\tilde{t}_j \\
&= \Gamma(t_i|t_i) + \int_{t_i}^{t_j} \frac{t_i}{2\tilde{t}_j^2(1-k)^2} - \frac{k^2}{2t_i(1-k)^2} d\tilde{t}_j \\
&= \frac{1}{2} + \Big[-\frac{t_i}{2\tilde{t}_j(1-k)^2} - \frac{k^2\tilde{t}_j}{2t_i(1-k)^2} \Big]_{t_i}^{t_j} \\
&= \frac{1}{2} + \frac{1}{2(1-k)^2} - \frac{t_i}{2t_j(1-k)^2} + \frac{k^2}{2(1-k)^2} - \frac{k^2 t_j}{2t_i(1-k)^2}
\end{aligned}
$$

Now note that for the uniform distribution $f(t|\lambda) = \frac{1}{\lambda}$ for all $t \in [\lambda k, \lambda]$ and is equal to zero otherwise.

Moreover we know that $g(\lambda|t) = \frac{1}{\lambda^2}$ for all $\lambda \geq 1$ and is equal to zero otherwise. Obviously λ cannot be lower than the type observed, since otherwise this type is not feasible. Furthermore the density of the distribution decreases as λ increases, ie observing a particular type $t_i = 1$ when $\lambda = 1$ is twice as likely as observing this type when $\lambda = 2$ creating a $\frac{1}{\lambda}$ term in the density. The second term comes from the fact, that the exponential distribution means that the unconditional likelihood of any λ decreases at a rate of $\frac{1}{\lambda}$.

$$\begin{aligned}
\gamma(t|1) &= \int_1^\infty g(\lambda|1) f(t|\lambda) d\lambda \\
&= \int_{max\{1,t\}}^\infty \frac{1}{\lambda^3} d\lambda
\end{aligned}$$

First consider the case $t \leq 1$:

$$\begin{aligned}
\gamma(t|1) &= \int_1^\infty \frac{1}{\lambda^3} d\lambda \\
&= -\frac{1}{2\lambda^2}\Big|_1^\infty \\
&= \int_0^1 \frac{1}{2} dt \\
&= = \frac{1}{2}
\end{aligned}$$

This means that all types in the interval $[0, 1]$ are equally likely, since it is known that they are feasible.

Now consider the case $t > 1$:

$$\begin{aligned}\gamma(t|1) &= \int_t^\infty \frac{1}{\lambda^3} d\lambda \\ &= -\frac{1}{2\lambda^2}\Big|_t^\infty \\ &= \frac{1}{2t^2}\end{aligned}$$

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