Coordinating by Not Committing: Efficiency as the Unique Outcome*

Rohan Dutta† and Ryosuke Ishii‡

August, 2013

Abstract

An important form of commitment is the ability to restrict the set of future actions from which choices can be made. We study a simple dynamic game of complete information which incorporates this type of commitment. For a given initial game, the players engage in an endogenously determined number of commitment periods before choosing from the remaining actions. We show the existence of equilibria with pure strategies in the commitment periods. For important classes of games, including pure coordination games and the stag-hunt game the equilibrium outcome is unique and efficient. This is despite the synchronous move structure. Moreover, efficient coordination does not necessarily involve commitments on the equilibrium path: the option alone is sufficient.

JEL classification: C72, C73

Keywords: Dynamic Commitment, Endogenous Timing, Coordination Games, Uniqueness, Payoff Dominance, Stag Hunt.

*This paper is a significantly revised version of a chapter in the first author's PhD dissertation (2012) at Washington University in St. Louis. We thank Marcus Berliant, Martin Cripps, Haluk Ergin and Maher Said for their encouraging and helpful comments. Special thanks to Takuo Sugaya for making the collaboration of the two authors on this paper possible and to David Levine and John Nachbar for their detailed comments on earlier drafts. All errors are our own.

†CIREQ and Department of Economics, McGill University, Leacock 531, 855 Sherbrooke St. W, Montreal, QC, H3A 2T7, Canada; 514-398-3030 (ext. 00851); rohan.dutta@mcgill.ca.

‡Faculty of Business, Aichi Shukutoku University, 23 Sakuragaoka, Chikusa-ku, Nagoya, Aichi 464-8671, Japan; +81(0)52-781-1151(Switchboard); rishii@asu.aasa.ac.jp.

1 Introduction

The role of commitment in determining the outcome of strategic interactions has been studied in a variety of settings since the classic work of Schelling (1960). The particular form of commitment ability we focus on is the capability of agents to limit their own set of choices available for a given strategic interaction. Cited examples of such ability range from armies burning their bridges thereby eliminating the option of a retreat to making a sunk contribution towards a joint project making it impossible to contribute less than the latter amount. Our goal is to investigate the relationship between such commitment ability and coordination in environments that allow considerable freedom to the players regarding the timing and choice of their commitments.

Renou (2009), Bade et al. (2009) and Lazarev (2012) show that players' ability to unilaterally restrict their available set of actions before simultaneously playing the resulting game increases the set of outcomes of the original game which are supported by equilibrium arguments.[1] In particular, pure strategy Nash equilibria of the original game persist as equilibrium outcomes in the presence of such commitment ability.

Our analysis of such commitment ability, in contrast, delivers unique equilibrium outcomes for two classes of games which are particularly plagued with multiple Nash equilibria. Our study involves three significant points of departure from earlier studies. *Firstly*, committing to a smaller set of actions is assumed to be costly. This assumption is informed by the casual observation that whether such commitment involves the physical elimination of an option (burning bridges) or rendering an option infeasible by making it too costly (a President announcing publicly that he would veto a particular bill, making backing down prohibitively costly for re election prospects), the very act of constraining one's choices may involve a cost. Importantly, this cost may be small in comparison to the difference in payoffs across outcomes in the actual strategic game. While the cost of making a public announcement may be negligible in comparison to the difference in payoffs from a particular bill being passed or not, the cost is still strictly positive. *Secondly*, players have the ability to *commit to not commit*. This assumption is interesting for both normative and positive reasons. On the one hand it is common for people to take a *"no further comments"* stand when making a comment would potentially constrain their future choices. On the other

[1] Lazarev (2012) uses his model to analyze airline pricing.

hand, the normative analysis in this paper shows that having the ability to rule out future commitments may be hugely beneficial to both parties concerned, allowing them to avoid Pareto inefficient Nash equilibrium outcomes. *Finally*, the number of periods for which the game continues is determined endogenously. The players can continue to commit to progressively smaller subsets of their choice sets. The final strategic game is played following a period when all players who retain the ability to commit choose not to constrain their choices further.

More precisely, given a finite strategic game between two players, this paper embeds it in a larger multi stage game, referred to as a *dynamic commitment game*. In the first period the players decide whether they want to make a *strict commitment* by irreversibly eliminating some actions from their original choice sets. Alternatively, a player could *commit to not commit* thereby giving up the possibility of making any future commitments. A player could also play *passive*, involving no strict commitments, while leaving the option of future commitments open. The two players make their commitment decisions *simultaneously*. In the next period players who can still make commitments face the same set of commitment options as the previous period and make their decisions simultaneously. Only now their choice set does not contain the actions eliminated by them earlier. The game continues in this way until a period is reached when *all* players who have the ability to commit either play passive or commit to not commit. In the subsequent period a strategic game is played with each player choosing from actions not eliminated earlier. Given a strategy profile for the *dynamic commitment game*, a player gets the payoff from the outcome of the induced strategic game while paying the cost for all the *strict commitments* she made on the equilibrium path. The analysis focuses on the outcomes of the final period strategic game induced by subgame perfect strategy profiles for the *dynamic commitment game* with players using pure strategies in the commitment stages. Such outcomes are called *supportable*.

While we defer a detailed discussion about the motive and importance of our study of subgame perfect equilibrium (SPE) profiles that use pure strategies in the commitment stages to Section 5, such a restriction raises the thorny question of existence. The simultaneity of moves coupled with the use of pure strategies in the commitment stages makes this problem, to the best of our knowledge, unassailable by known existence results. A key contribution of this paper is to establish that such an SPE profile always exists for any initial *strict* strategic game for sufficiently small commitment costs(Theorem 1); thereby showing the notion of supportability to be permissive enough to always yield a prediction while being sufficiently restrictive to deliver the uniqueness results that follow.[2]

An immediate but important feature of the *dynamic commitment game* outlined above is that Nash equilibria of the original game need not be *supportable*(Section 3, Example 1). It turns out, however, that if there exists a Nash equilibrium that Pareto dominates all other outcomes then it will be supportable (Proposition 2). In fact such a Nash equilibrium can be supported without any player making any *strict commitments* in equilibrium. Interestingly, while they themselves may not be supportable, Nash equilibria of the original game systematically prevent some related outcomes from being supportable (Proposition 1).

This leads to the uniqueness results of the paper which show that for pure coordination games the Pareto dominant Nash equilibrium is the *unique* supportable outcome (Proposition 3). The uniqueness result also applies to the class of $n \times n$ games with n Pareto ranked Nash equilibria that satisfy a single crossing property (Proposition 4); a class that includes the generic stag hunt game. These propositions mark a significant departure from earlier results in two important ways. Firstly, it shows that such commitment ability can sharply reduce the set of equilibrium predictions. Secondly, it provides a natural setting in which players manage to coordinate on the efficient outcome without an asynchronous move structure.

To see the potential obstacle to efficient coordination that a simultaneous move structure creates, notice that coordination on an inefficient Nash equilibrium in a strategic game has its counterpart in a commitment game with the two players committing to a single action each, simultaneously, which jointly constitute an inefficient Nash equilibrium of the initial game. Indeed some earlier studies used precisely this feature to establish the existence of *commitment equilibrium*, by restricting the class of initial games to those admitting a pure strategy Nash equilibrium. Costly commitment, however, rules out such profiles from being an equilibrium since conditional on one player committing to a single action, the other player is always strictly better off not making a commitment. Of course, there still remains the possibility of the players simultaneously eliminating their action corresponding to the efficient profile while retaining enough flexibility to punish the other player (say by coordinating on an even worse outcome) were the latter to not make a commitment. A key contribu-

[2] A strict game requires no player to be indifferent across any two outcomes.

tion of this paper is to show how such profiles can be ruled out for the two classes of games considered in this study.

The finding that players coordinate on the efficient outcome without making any commitments; that merely the presence of such commitment ability suffices is similar in spirit to the *money burning* results in Ben-Porath and Dekel (1992). Similar to the signalling character of not burning money, not making a commitment signals to the other player that the former must believe that they will coordinate efficiently, since otherwise she would have committed to the action that would force such coordination. However, efficient coordination in the presence of money burning breaks down if both parties can simultaneously burn money. Not only does our uniqueness results allow for simultaneous commitments, by using subgame perfection as the equilibrium concept we avoid arguments involving iterated elimination of weakly dominated strategies.

The rest of the paper is as follows. Section 2 discusses the related literature. Section 3 formally introduces *dynamic commitment games*. Section 4 contains the results of the paper. Section 4.1 establishes the existence of the supportable outcomes. Section 4.2 states results relating to general strict games. Section 4.3 and 4.4 present the uniqueness results. Section 5 discusses the precise role of each of the key assumptions. Section 6 discusses how the commitment ability studied in this game does not necessarily lead to more efficient outcomes in general games. Section 7 concludes. All proofs are collected in the appendix.

2 Related Literature

Commitment Games. The studies closest to ours are those of Renou (2009), Bade et al. (2009) and Lazarev (2012). In these papers before playing a given strategic game players commit to a subset of their available actions and then proceed to play the game with these restricted action sets.[3] The possibility of committing to *any* subset of actions as opposed to a single action, allows for subtler tradeoffs between restricting available actions to make certain choices credible and retaining enough flexibility to deter one's opponent. While Renou (2009) and Lazarev (2012) allow for

[3] A number of studies such as Romano and Yildirim (2005) and Admati and Perry (1991) consider similar commitment games but with restrictions on the subsets of available actions a player can commit to. These restrictions arise naturally in certain strategic environments such as one of making sunk contributions towards a public project. Our study, by contrast, does not impose any such restrictions.

a single round of commitment, Bade et al. (2009) allows for multiple rounds.[4] In the particular class of strategic games studied in the latter paper, however, they show that the equilibrium prediction is independent of the number of rounds of commitment. This is not true in our analysis as is discussed in detail in Section 5.

While all these papers involve players voluntarily restricting their own choice sets, Nava (2008) considers games where players can restrict the set of action profiles (not merely their own actions) if they unanimously choose to do so. The requirement of unanimity retains the voluntary nature of commitment ability of the earlier studies.

A common finding in all these papers is that such commitment ability increases the set of outcomes supported by equilibrium arguments. While efficient outcomes that were not Nash Equilibria(NE) of the original game may now be achievable through commitment, the original set of NE continues to be outcomes supported by SPE strategies in the corresponding commitment games. Nava (2008), going further, shows that if the players have multiple rounds to make their joint commitments a folk theorem holds. Such a folk theorem also holds when players have the ability to make conditional commitments as in Kalai et al. (2010).

Endogenous Timing Games. The literature on endogenous timing games, starting with Hamilton and Slutsky (1990) and van Damme and Hurkens (1996) is an important precursor to that on commitment games. Their focus on commitment to single actions results in an equivalence between the choice of commitment and the choice of timing of actions with two possible periods where one could play. A significant contribution of these papers was to characterize the sequence of commitments and resulting outcome that would emerge endogenously when the players could decide the timing of their commitments as opposed to comparing outcomes across different exogenously fixed sequences of commitments. The present study is, in a way, a natural generalization of the endogenous timing framework, were one to allow for commitments to any subset of available actions and for multiple rounds of commitment.

Asynchronicity and Coordination. A number of studies have shown that players could coordinate on the efficient outcome in pure coordination games in the presence of an asynchronous move structure. Lagunoff and Matsui (1997), for instance, study infinitely repeated (cardinal) pure coordination games with the players moving asynchronously in the stage games.[5] Calcagno et al. (2010), in a setup more similar to

[4]The original action spaces in Bade et al. (2009) are closed intervals of the real line and they restrict commitments to only be subintervals of this space.

[5]Takahashi (2005) makes an important generalization of this result to a large class of common

the present analysis where the strategic game is played just once at the end, study players with the ability to revise their actions according to a stochastic process till a deadline is reached when their most recent actions are played. The asynchronicity, in this case, is implicit in the assumption of independent stochastic arrival of revision possibilities. In Caruana and Einav (2008) the players can revise their actions in an asynchronous but deterministic way before a known deadline. Revising actions, however, is costly with the cost increasing as the deadline approaches.[6]

This relationship between asynchronicity and coordination is analyzed more generally in Dutta (2012); it turns out that sufficiently robust efficient coordination results require both asynchronicity in moves and a finite horizon. By contrast not only does the present analysis allow for simultaneous moves, the efficient outcome is achieved by the players *simultaneously* choosing not to make any commitments and then playing the Pareto dominant outcome in the resulting simultaneous game.

Delegation and Endogenous Games. Fershtman et al. (1991) in their work on delegation allow players to have agents play the game for them with the players committing to outcome contingent payments to the agents before the game. This effectively allows the players to change their own payoffs in the game. Allowing for simultaneous announcements of the contingent payment functions, however, forces the authors to use exogenous equilibrium selection arguments such as *mutual rational agents* to yield the efficient outcome as the unique prediction. While it is possible to interpret the commitment ability studied in this paper as a particular class of delegation games, the precise relationship becomes far more tentative with multiple rounds of commitment and the option of committing to not commit. In Jackson and Wilkie (2005) each player can commit to outcome contingent transfers to their opponent before playing a game. For pure coordination games, this analysis does yield the efficient outcome uniquely.[7] However the rationale behind such a result is very different from that involved in the present analysis.

interest games.

[6]Lipman and Wang (2000) in a related study involving revisions with switching costs, allow for simultaneous revisions but get a much weaker uniqueness result. For instance, for 2×2 symmetric coordination games, the efficient outcome is the unique prediction only if it is also the risk dominant outcome.

[7]We thank Matthew Jackson for his help in clarifying this issue.

3 Dynamic Commitment Games

Consider a set of two players denoted by $\mathcal{N} = \{1, 2\}$. The letter i is used to refer to a generic player in this set, while $-i$ is used for the *other* player. X_i denotes the *finite* set of actions available to Player i. With $X \equiv X_1 \times X_2$ capturing the set of all action profiles, ΔX denotes the set of all probability functions over elements of X. The payoff function for Player i is given by $u_i : X \to \mathbb{R}$ and $u = (u_1, u_2)$. $\Gamma(X, u)$ represents the simultaneous move game where the set of feasible action profiles and the payoff functions are X and u, respectively.

Given some *initial* simultaneous move game we are interested in the outcomes that arise from allowing players to credibly and unilaterally commit to not play some of their actions available in this initial game. We allow the players to not only make commitments over multiple periods but also credibly rule out their ability to make future commitments. To this end we shall use a state variable, c_i, to describe the commitment ability of Player i. $c_i = 1$ represents the state in which i cannot make any further commitments, while $c_i = 0$ represents the state in which Player i can indeed make a commitment.

For a given *initial* simultaneous move game $\Gamma(X, u)$ we now define its corresponding *dynamic commitment game* in a recursive way. Fix some $\epsilon > 0$. $g^\epsilon(X, u, c_1, c_2)$ is a dynamic commitment game that is defined as follows. If $c_i = 1$ for all $i \in \mathcal{N}$ then the simultaneous move game, $\Gamma(X, u)$ is played, following which the game ends. The set of actions available to Player i in this case would simply be X_i.

If $c_i = 0$ for at least one player, then all such players with $c_i = 0$ *simultaneously* choose from a set of feasible commitment stage actions(specified below), following which the game $g^\epsilon(X', u, c'_1, c'_2)$ is played in the next period. The new set of action profiles X' and the new state describing the players' commitment ability, c'_i, is determined by their choices in this period as follows.

If $c_i = 0$, then in the present period Player i has the following commitment stage actions available.

- Make a *strict commitment (SC)*, by choosing some non-empty $Y \subset X_i$.[8] In this case, Y, would be the set of actions available to Player i in the subsequent period; $X'_i = Y$. This also keeps the option of future commitments open if $|Y| > 1$; $c'_i = 0$. Committing to a single action, of course, makes future commitments

[8]The symbol $\subset$ is used to denote *"a strict subset of"*.

impossible; if $|Y| = 1$ then $c_i' = 1$.

- *Commit to not commit (NC)*, which would leave the set of actions available to Player i in the subsequent period unchanged; $X_i' = X_i$. Player i, however, cannot make any further commitments; $c_i' = 1$.

- Play *passive (PS)*, which would also leave the set of actions available to Player i in the subsequent period unchanged; $X_i' = X_i$. Whether or not i gets to make any further commitments depends on $-i$'s choice. In particular, if $-i$ either played PS or if $c_{-i} = 1$ then $c_i' = 1$. On the other hand, $c_i' = 0$ if $-i$ either makes a strict commitment or plays NC in this period.[9]

In other words, Player i playing *passive* leaves her commitment options open in the next period as long as the other player makes some commitment, whether it is a *strict commitment* or a commitment to not commit. However if every player that can make a strict commitment plays PS then the commitment stage of the game concludes and the resulting simultaneous game is played in the next period.

In the *dynamic commitment game*, $g^\epsilon(X, u, c_1, c_2)$ if $c_i = 1$ and $c_{-i} = 0$, then Player i has no commitment actions available. Moreover, $X_i' = X_i$ and $c_i' = c_i$. This concludes the description of the actions available to the two players in a *dynamic commitment game*.

Given the description of the game above it should be clear that the number of periods for which the game continues, though always finite, is determined endogenously. A typical non terminal history of the game, $g^\epsilon(X, u, c_1, c_2)$ consists of a sequence of action sets and commitment states, $((X^1, c_1^1, c_2^1), (X^2, c_1^2, c_2^2), \ldots, (X^T, c_1^T, c_2^T))$ with $X^1 = X$. Note that given the description of the game, it cannot be that $c_1^t = c_2^t = 1$ for any $t < T$. Indeed, any period that involves neither player having the ability to make commitments would lead to the resulting simultaneous game being played. A strategy, σ_i, for Player i specifies a mapping from the set of all non terminal histories to her available choices, outlined earlier.

A typical terminal history would involve a sequence of commitment actions followed by an outcome of the resulting simultaneous game,

$$h = ((X^1, c_1^1, c_2^1), (X^2, c_1^2, c_2^2), \ldots, (X^T, c_1^T, c_2^T), x)$$

[9] We often use subscripts with commitment actions, for example (PS_i, NC_{-i}), to clarify the identity of the player making the particular choice.

where $x \in X^T$. The payoff to Player i at such a terminal history is given by,

$$\tilde{\pi}_i(h) = u_i(x) - z_i(h)\epsilon \tag{1}$$

where $z_i(h)$ is the number of times Player i made a *strict commitment* in the history h.

A strategy profile $\sigma = (\sigma_i)_{i \in \mathcal{N}}$ generates a probability function over the set of all terminal histories. Player i's expected payoff from such a strategy profile is given by,

$$\pi_i(\sigma) = E(\tilde{\pi}_i(h)|\sigma) \tag{2}$$

A probability function over the set of all terminal histories implicitly describes a probability function over the elements of X as determined by the outcome of the final simultaneous game for each of these histories. For a given *dynamic commitment game*, $g^\epsilon(X, u, c_1, c_2)$ and a corresponding strategy profile σ, the resulting probability function over the elements of X is denoted by $\mu(\sigma)$. We refer to such a $\mu(\sigma)$ as an *outcome* of the dynamic commitment game. By a minor abuse of notation we call $x \in X$ an outcome of the dynamic commitment game if $\mu(\sigma)(x) = 1$. To distinguish generic strategy profiles in the dynamic commitment game from generic strategy profiles in a simultaneous move game we denote the latter by σ^{sim}.

The object of this study is to identify the outcomes of the *initial* game that are eventually played as part of some subgame perfect equilibrium strategy profile of the dynamic commitment game, which involves the use of pure strategies in the commitment stages. Further, since we are interested in settings where the cost of commitment is small in comparison to the stakes involved in the initial game we want to avoid scenarios where one outcome is preferred to another by some player, but that preference is reversed were she to take into account the commitments required to yield those outcomes.[10] To ensure that the cost of commitment does not reverse the preference rankings of any player we only consider sufficiently low values of ϵ. For a given pair of action sets $X = X_1 \times X_2$ and u, let $m(X) = \max\{|X_1|, |X_2|\}$ and $v(X, u) = \min_{x,y \in X,i} |(u_i(x) - u_i(y))|$. We only allow for values of ϵ that are smaller than $v(X, u)/m(X)$. We call a subgame perfect equilibrium of a dynamic commitment

[10]This is another key difference with papers on switching costs such as Caruana and Einav (2008). While our assumption is natural since we take the commitment ability as given, relaxing this assumption could very well be key in a study where the commitment ability itself is determined endogenously.

game that involves both players only using pure strategies in the commitment stages a *qualified subgame perfect equilibrium (QSPE)*. We can now define the notion of *supportable* outcomes.

Definition 1 *Given X and u, a probability function $\varphi \in \Delta(X)$ is said to be* ***supportable*** *if there exists a qualified subgame perfect equilibrium strategy profile, σ, for the game $g^{\epsilon}(X, u, 0, 0)$ for some $\epsilon < v(X, u)/m(X)$, such that $\varphi = \mu(\sigma)$.*

By a minor abuse of notation, given X and u, we refer to an outcome $x \in X$ as being **supportable** if φ is supportable and $\varphi(x) = 1$.

A simultaneous move game $\Gamma(X, u)$ is called a strict game if $\forall x, y \in X$, $x \neq y \Rightarrow u_i(x) \neq u_i(y)$, $\forall i \in \{1, 2\}$. This paper focuses entirely on the class of strict games. The set of Nash Equilibria of a simultaneous move game, $\Gamma(X, u)$ is denoted by $NE(\Gamma(X, u))$. The expected payoff profile from a given strategy profile, σ^{sim}, in a simultaneous move game, $\Gamma(X, u)$, is denoted by $\pi(\sigma^{sim}) = (\pi_1(\sigma^{sim}), \pi_2(\sigma^{sim}))$.

Example. Before delving into a thorough analysis of supportability an example would help give some flavor of the results that follow. The example explains why a Nash equilibrium of the initial game may not be supportable. Example 1 is a

Example 1: Nash Equilibrium not supportable.

	a_2	b_2
a_1	$3, 3$	$0, 7$
b_1	$2, 0$	$1, 1$

dominance solvable game with (b_1, b_2) as its unique Nash Equilibrium. The unique supportable profile, however, turns out to be (a_1, a_2). To see why (b_1, b_2) is not supportable note first that both players simultaneously committing to a single action each cannot be part of an SPE. For instance if players 1 and 2 commit to $\{b_1\}$ and $\{b_2\}$ respectively, the outcome would be (b_1, b_2) with a payoff of $1 - \epsilon$ where ϵ is the cost of making a *strict commitment*. If Player 2, on the other hand, deviates to *committing to not commit*, the strategic game $\Gamma(\{b_1\}, \{a_2, b_2\})$ must be played in the next period, resulting in the same outcome (b_1, b_2) but giving Player 2 a payoff of 1, thereby making it a profitable deviation. A similar argument applies more generally in ruling out both players simultaneously committing to a single action each from being part of an SPE. Consequently, any SPE must involve no more than one player making a strict commitment in the first period. If Player 2 is the one making the commitment she must get at least $3 - \epsilon$ since she can simply commit to $\{a_2\}$ guaranteeing herself her Stackelberg payoff, thereby rejecting possible support of the NE, (b_1, b_2). If Player 1 is the only one to make a commitment in period 1 and chooses $\{b_1\}$ the resultant payoff would be $1 - \epsilon$. Again deviating to playing either *passive* or *committing to not commit* would result in the original game being played in the next period, resulting in the same outcome. Player 1, however, would save on the ϵ cost from this deviation, thereby making the stated strategy profile fail subgame perfection. Finally consider the strategy profile involving the players choosing either PS or NC in the first period, followed by the profile (b_1, b_2) in the subsequent strategic game. Player 2 would again have the strictly profitable deviation to committing to $\{a_2\}$, forcing the outcome (a_1, a_2) in the subsequent strategic game, and getting a payoff of $3 - \epsilon$ as opposed to the original 1. (b_1, b_2), as a result, fails to be supportable.

4 Supportable Outcomes

4.1 Existence

Earlier studies on commitment games have relied on the fact that pure strategy Nash equilibria of the original game continue to be supported by equilibrium arguments in these commitment games to establish existence. Example 1 above, however, makes it clear that in dynamic commitment games, Nash equilibria of the initial game may not persist as a supportable outcome. This necessitates an existence result. This task is made difficult by our definition of supportability which requires the use of pure strategies in the commitment stages that involve simultaneous moves; making it impossible (to the best of our knowledge) to use any known existence result. Thankfully, however, a supportable outcome always exists for any strict game. We prove this result by construction.

Theorem 1 *For any strict initial game* $\Gamma(\tilde{X}, u)$, $\epsilon < v(\tilde{X}, u)/m(\tilde{X})$ *and* $c_i \in \{0, 1\}$, *the dynamic commitment game* $g^\epsilon(\tilde{X}, u, c_1, c_2)$ *admits a qualified subgame perfect equilibrium.*

4.2 Supportability in General Strict Games

We begin with a handy observation that makes it easy to eliminate certain strategy profiles from constituting a QSPE for any general strict initial game. In particular, if one player chooses to commit to a single action, then the other player, in the same period, can never do worse by playing *NC*. Indeed, by ruling out further commitment ability not only does she retain greater flexibility in choosing her best response, she saves on the cost of making a strict commitment. An immediate implication of the observation is that QSPE strategy profiles for 2×2 strict initial games must involve *at most* one player making a strict commitment on the equilibrium path.

Observation 1 *Given a strict game* $\Gamma(X, u)$ *and its corresponding dynamic commitment game* $g^{\epsilon}(X, u, c = (0, 0))$*, a QSPE strategy profile for the latter cannot have both players simultaneously making a strict commitment with at least one player committing to a single action, following any history.*

The following proposition establishes a relationship between pure strategy Nash equilibria of an initial game with its corresponding supportable outcomes. As shown in Example 1, a Nash equilibrium outcome of an initial game is not necessarily supportable. We now show that any Nash equilibrium outcome of an initial game systematically eliminates the possibility of a set of *related outcomes* to be supportable. In particular, any outcome that involves *only one* of the players playing an action corresponding to some pure strategy Nash equilibrium of the strict game cannot be supportable.

Proposition 1 *Given a strict game,* $\Gamma(X, u)$*, if* $(x_1, x_2) \in NE(\Gamma(X, u))$ *then for all* $y \in X_{-i}$ *such that* $y \neq x_{-i}$*,* (x_i, y) *is not supportable.*

If a Nash Equilibrium of the strategic game Pareto dominates all other outcomes then it is in fact supportable. Subgame perfect strategy profiles required to support such an outcome does not necessarily involve any *strict commitments* to be made on the equilibrium path. Indeed the simplest strategy profile suffices.[11]

Proposition 2 *Given a strict game,* $\Gamma(X, u)$ *if there exists* $x \in X$ *such that* x *Pareto dominates all* $y \in X \setminus \{x\}$*, then* x *is supportable. Further there exists a qualified subgame perfect strategy profile that supports* x *involving no strict commitments on the equilibrium path.*

[11]Note that an outcome that Pareto dominates all other outcomes must be a Nash Equilibrium outcome.

To conclude this section we state a lemma that is useful for the uniqueness results that follow. We show that for any strict initial game with a Pareto dominant outcome, the payoff difference to a player between the Pareto dominant outcome and any other (possibly mixed) Nash equilibrium of any strict game that is a restriction of the initial game must be no smaller than the smallest payoff difference between any two pure outcomes of the initial strict game.

If a strict game, $\Gamma(X,u)$, has an outcome $x \in X$ that Pareto dominates all $y \in X \setminus \{x\}$ then we denote such an outcome as $\mathcal{P}(\Gamma(X,u))$. Let $\overline{NE}(\Gamma(X,u)) = (\cup_{\tilde{X}\subseteq X} NE(\Gamma(\tilde{X},u))) \setminus \mathcal{P}(\Gamma(X,u))$, denote the set of all Nash equilibria of all strict games that are restrictions of the initial game, $\Gamma(X,u)$.[12]

Lemma 1 *If a strict game $\Gamma(X,u)$ has a Pareto dominant outcome, $\mathcal{P}(\Gamma(X,u))$, then for each $i \in \mathcal{N}$,*

$$u_i(\mathcal{P}(\Gamma(X,u))) - \max_{\sigma^{sim}\in\overline{NE}(\Gamma(X,u))} E[u_i(\cdot)|\sigma^{sim}] \geq v(X,u) \tag{3}$$

4.3 Unique Supportable Outcome for Ordinal Pure Coordination Games

An *ordinal pure coordination game*(OPC game) is defined to be a simultaneous move strict game $\Gamma(X,u)$ such that

$$\forall x,y \in X, \qquad u_1(x) > u_1(y) \Leftrightarrow u_2(x) > u_2(y).$$

	a_2	b_2
a_1	$100, 102$	$10, 18$
b_1	$78, 20$	$80, 82$

	a_2	b_2	c_2	d_2
a_1	$1, -1$	$-2, -5$	$-4, -8$	$9, 11$
b_1	$2, 0$	$6, 8$	$8, 10$	$-1, -4$
c_1	$5, 6$	$0, -3$	$-3, -7$	$3, 2$

Figure 1: Ordinal Pure Coordination Games

A feature of OPC games is the existence of a Nash Equilibrium outcome that Pareto dominates all other outcomes. Given an OPC game, $\Gamma(X,u)$, its Pareto dominant outcome is denoted by $\mathcal{P}(\Gamma(X,u))$. Notice that following any sequence of strict commitments in an OPC game, the resulting game is also an OPC game.

[12] $\tilde{X} \subseteq X$ is used to denote $\tilde{X} = \tilde{X}_1 \times \tilde{X}_2$ with $\tilde{X}_i \subseteq X_i$.

The following lemma shows that for small enough values of ϵ, in a dynamic commitment game induced by an OPC game $\Gamma(X, u)$, if only one player has the ability to make commitments, the unique outcome is in fact the Pareto dominant one.

Lemma 2 *If $\Gamma(X, u)$ is an ordinal pure coordination game, then for any QSPE strategy profile of the game $g^\epsilon(X, u, c_i = 0, c_{-i} = 1)$ the unique outcome is $\mathcal{P}(\Gamma(X, u))$, for all $\epsilon < v(X, u)/m(X)$.*

Notice that the lemma above ruled out the scenario where Player i plays passive and then ends up coordinating with $-i$ on a Pareto inefficient Nash Equilibrium. Otherwise Player i would prefer to make a strict commitment, instead of playing passive, which would ensure that the Pareto dominant profile is the outcome. Since such a deviation is always available to Player i the only outcome that can result following Player i not making a strict commitment, in equilibrium, would have to be the Pareto dominant one.

The following proposition shows how this Pareto dominant outcome is in fact the unique supportable outcome. At this point it may help to consider the subtle role played by the commitment ability studied in this paper in avoiding Pareto inefficient Nash equilibria. Firstly, QSPE involving neither player making any commitments in the first period must result in the efficient outcome since otherwise both players would have an incentive to strictly commit to the single action that corresponds to the efficient profile, thereby forcing the efficient profile to be played eventually. A similar argument works for ruling out profiles yielding an inefficient outcome with only one player making a strict commitment in the first period.

Now consider the possibility of both players in the first period making simultaneous commitments. They could commit to a subset each and play an inefficient Nash equilibrium, allowing enough flexibility that a deviation to playing NC is punished by coordinating on an even worse Nash equilibrium. This logic of failed coordination, however, unravels in our setting. Crucially, such simultaneous commitments in the first period, ensure that both players retain their commitment ability for the next period. This in turn makes the choice of NC a profitable deviation for both players. From Lemma 2 we know that following such a deviation the efficient outcome of the game involving the restricted set of actions for the player who chose SC and all the actions of the one choosing NC would be played. Since this game contains all the action profiles from the restricted game following simultaneous commitments the outcome could be no worse for the deviating player than the one following the simultaneous commitments. Moreover, she saves on the cost of commitment. The fact that the preferences are perfectly aligned is key in making NC the best response to one's opponent making a strict commitment.

Proposition 3 *Given X and u such that $\Gamma(X, u)$ is an ordinal pure coordination game, $\mathcal{P}(\Gamma(X, u))$ is the unique supportable outcome.*

4.4 $n \times n$ Games with n Pareto Ranked Equilibria

An *$n \times n$ game with n Pareto ranked equilibria* (n-Eq game) is defined to be a simultaneous move strict game, $\Gamma(X, u)$ with $|X_i| = n$ such that

$$\forall x_i \in X_i, \exists x_{-i} \in X_{-i} \; s.t. \; (x_i, x_{-i}) \in NE(\Gamma(X, u))$$

$$and \qquad \forall x, y \in NE(\Gamma(X, u)) \text{ either } x \text{ Pareto dominates } y \text{ or } y \text{ Pareto dominates } x$$

	a_2	b_2
a_1	$100, 100$	$20, 90$
b_1	$90, 20$	$80, 80$

	a_2	b_2	c_2	d_2
a_1	$2, -8$	$9, 10$	$1, -1$	$7, 2$
b_1	$3, 7$	$6, 8$	$8, 9$	$-1, -4$
c_1	$5, 6$	$-2, 1$	$-3, 4$	$-6, -5$
d_1	$0, -7$	$4, 5$	$-4, -3$	$11, 12$

Figure 2: $n \times n$ Games with n Nash Equilibria

n-Eq games share a property with OPC games in that they too admit a Pareto dominant (thereby Nash equilibrium) profile. Other than this there is not much else that is common between the two classes of games. n-Eq games, without additional assumptions, can admit inefficient supportable outcomes. It turns out, however, that if an n-Eq game satisfies the single crossing property introduced by Milgrom and Shannon (1994) then it yields a unique efficient supportable outcome. Importantly, the generic stag hunt game falls within such a class of games.[13] We need a few additional definitions to describe the required condition.

There is a natural way in which the actions of each player in an n-Eq game can be ordered. For any two actions, x_i' and x_i, available to some Player i in an n-Eq game,

[13] See Skyrms (2004) for an extensive discussion about the importance and pervasiveness of stag hunt games.

x_i' is *greater* than x_i if the Nash equilibrium associated with x_i' yields a higher payoff to both players than that from x_i. Formally, we define $\succeq$ by,

$$x_i' \succeq x_i \textit{ if and only if } u_i(x_i', BR_{-i}(x_i')) \geq u_i(x_i, BR_{-i}(x_i))^{14}$$

Notice that $\succeq$ defines a total binary relation on X_i for each $i \in \mathcal{N}$.

Definition 2 *An n-Eq game is said to satisfy the* single crossing property *if for every $x_i', x_i \in X_i$ with $x_i' \succeq x_i$ and $y_{-i}', y_{-i} \in X_{-i}$ with $y_{-i}' \succeq y_{-i}$,*

$$u_i(x_i', y_{-i}) - u_i(x_i, y_{-i}) > 0 \Rightarrow u_i(x_i', y_{-i}') - u_i(x_i, y_{-i}') > 0$$

It can be verified that both the games in Figure 2 satisfy the single crossing property.

It should be pointed out that general strict games satisfying the single crossing property (even with positive externalities) do not necessarily yield a unique supportable outcome.[15] In n-Eq games that satisfy the single crossing property the intuition for why only the Pareto dominant outcome is played following a first period with neither player making any commitment is identical to that described for OPC games. The difference in the logic of the uniqueness results lies in what deters player from simultaneously eliminating their action corresponding to the Pareto dominant profile.

Such simultaneous commitments must have a specific property to be part of a QSPE. Namely, among the actions that the players commit to the one that corresponds to the highest ranked Nash equilibria cannot have its best response eliminated by the other player in the first period. Since this Nash equilibrium Pareto dominates all other outcomes in the restricted game, a player who eliminates such an action could profitably deviate to committing to this same action alone. Moreover the outcome following such simultaneous commitments must be this highest ranked Nash equilibrium. In turn, given such an outcome both players have an incentive to deviate to NC in the first period instead, since it leaves the outcome the same. The reason for the outcome remaining the same relies entirely on the single crossing property.

Proposition 4 *If $\Gamma(X, u)$ is an n-Eq game that satisfies the single crossing property then $\mathcal{P}(\Gamma(X, u))$ is the unique supportable outcome.*

[14] BR_{-i} denotes Player $-i$'s best response function.

[15] There exist counterexamples for this case, the case of $n \times n$ games with n Pareto unranked equilibria with a Pareto dominant outcome as well as n-Eq games that do not satisfy the single crossing property.

5 Discussion regarding the assumptions

Cost of Commitment: The role played by the ϵ cost of making a *strict commitment* is obvious from Example 1. Without such a cost, all pure strategy Nash equilibria of the original game would be supportable. The presence of commitment costs allows the players to discriminate not only between outcomes of the initial game but between different sequences of commitments that yield the same initial game outcome. This richer strategic environment, however, comes at a price. The existence of an outcome supported by a QSPE profile in the commitment game is no longer trivial, even for initial games that admit a pure strategy Nash equilibrium.

The specific cost structure used in the model of dynamic commitment games was chosen to deliver the key message of the analysis while avoiding cumbersome notation. Neither the additive structure nor the fact that all strict commitments entail the same cost are fundamental for the results. The key assumptions we are interested in and that need to be embedded in any cost structure to deliver the same results are as follows.

(a) If there are two terminal histories, h and h', that result in the same outcome x of the initial game but involve Player i eliminating the set of actions A in h and B in h' with $B \subset A$ and $z(h') \leq z(h)$ then Player i weakly prefers achieving the outcome x following history h' over h. The preference is strict if $B = \emptyset$.

(b) Conditional on achieving outcome x by eliminating the set of actions A, Player i weakly prefers eliminating these actions in fewer periods as opposed to more.

(c) A player's preference over *pure* outcomes of the initial game does not get changed if she takes into account the commitments she must make to achieve those outcomes.

There is a variety of cost structures that have these features. For instance, each action x_i available to Player i could be assigned a particular cost $f(x_i) > 0$, which the player must incur should she choose to eliminate it. The cost incurred by committing to a set B when her set of available actions was A would then be $\Sigma_{x_i \in A \setminus B} f(x_i)$. For a given terminal history, the total cost could be the sum of these costs across all periods in which the player made strict commitments. To ensure that (c) is satisfied the cost would have to be scaled down by some factor, making the cost of commitment sufficiently smaller than the smallest difference between the payoffs of any two pure outcomes.

Committing to Not Commit: A surprising finding of the present analysis is the importance of the option of *committing to not commit*. In games involving a single round of commitment as studied in Renou (2009) and Bade et al. (2009) or (two period) endogenous timing games as in Hamilton and Slutsky (1990) there is no need for such an option since not making a commitment in the first period is strategically equivalent to NC. The possibility of multiple rounds of commitment breaks this equivalence. Choosing PS in a given period while ones opponent plays either SC or NC leaves the option of future commitments open. In particular if the opponent plays SC then the continuation commitment strategies by both players could be made to depend upon the particular subset of actions the opponent commits to in the present period. By contrast playing NC while ones opponent plays SC imposes greater structure on the set of possible continuation strategies that can arise in equilibrium. For instance, in equilibrium and following such a history, the opponent must get a payoff that is no less than what she can guarantee herself by strictly committing to a single action. Consequently for games in which the players' preferences are perfectly aligned, if one player makes a strict commitment in a given period, the other player could rule out any future coordination failure by giving up her commitment ability.

The following example shows how the belief that the presence of commitment costs alone deliver the uniqueness results, while tempting, is false. The cells which

Example 2: The importance of committing to not commit.

	a_2	b_2	c_2	d_2
a_1	$5, 7$	,	,	,
b_1	,	$4, 3$	,	,
c_1	,	,	$2, 2$	,
d_1	,	,	,	$10, 13$

have been left empty can be filled with a suitably chosen set of values, such that the resulting game is either an ordinal pure coordination game or an $n - Eq$ game which satisfies the single crossing property ($n = 4$) with the already filled in cells representing the Nash equilibria. Suppose the option of *committing to not commit* is not available to players. They may only choose between *strict commitments* and playing *passive*. In such a commitment game it can be shown that the Pareto inefficient Nash equilibrium profile (b_1, b_2) can be supportable.[16]

[16]It can be shown that QSPE strategy profiles for dynamic commitment games, induced by strict inital games, without the option NC with $\epsilon < v(\tilde{X}, u)/m(\tilde{X})$ always exist.

Consider the following strategy profile that supports (b_1, b_2). In period 1 players 1 and 2 commit to $\{b_1, c_1\}$ and $\{b_2, c_2\}$ respectively. In period 2 both players play *passive*. In the resulting third period strategic game, the profile (b_1, b_2) is played. If Player i deviates by playing *passive* in period 1 then in period 2 Player i commits to $\{b_i, c_i\}$, while Player $-i$ plays passive. In period 3 along this history, both players play *passive* and play the profile (b_1, b_2) in the period 4 strategic game. In the period 2 subgame after Player i's deviation in period 1 to playing *passive*, if Player i deviates by playing *passive* then in period 3 the strategic game with action sets $\{a_i, b_i, c_i, d_i\}$ and $\{b_{-i}, c_{-i}\}$ is played. In this strategic game the profile (c_1, c_2) is played. Qualified subgame perfect strategies are used for every other subgame. The resulting strategy profile is a QSPE with a payoff of $4 - \epsilon$ to Player 1 and $3 - \epsilon$ to Player 2. In equilibrium, Player i makes the commitment in period 1 because she knows that playing *passive* would simply mean she would have to make the same commitment in the next period, and she must make the latter commitment in the next period since failing to do so would result in coordinating on an even worse Nash equilibrium profile, namely (c_1, c_2). If Player i had the ability to commit to not commit, then she would simply deviate to NC in the first period, and not be forced to make the commitment in the subsequent period.

An implicit assumption that plays a crucial role in our results is that *committing to not commit* is costless. The assumption required for the results to hold, however, is that the cost of *committing to not commit* is less than that of making any *strict commitment*. It is very possible that in certain strategic environments this assumption would fail. The normative message of the paper, however, still holds in pointing out the benefit of making a *committing to not commit* option available and cheap for games in which preferences are well aligned and coordination is paramount.

Multiple Periods: As discussed earlier, the possibility of multiple rounds of commitment makes the option of committing to not commit relevant. One may conjecture that by considering a single round of commitment with costly commitment, we may still recover the uniqueness results in the absence of the NC option, which we know must be redundant in such a setting anyway. Such a conjecture, however, does not hold up.

Consider Example 2 again. With a single round of commitment (as in Renou (2009) or Lazarev (2012)) the players simultaneously make their commitments in the first period followed by the resulting restricted game being played in the second period. Consider the following strategies; Player i commits to the set $\{a_i, b_i\}$ in the first period. In the second period the action profile (a_1, a_2) is played. If following a deviation to playing passive by Player i the second period game is $\Gamma(\{a_i, b_i, c_i, d_i\} \times \{a_i, b_i\}, u)$, the action profile (b_1, b_2) is played. Some Nash equilibrium profile is played in the second stage for any other choice of first stage commitments. This strategy profile is easily seen to be a QSPE with payoffs of $(5 - \epsilon, 7 - \epsilon)$ resulting from the inefficient outcome, (a_1, a_2).

The key message of this example is that the coordination problem that arises from the lack of asynchronicity in simultaneous move games has an immediate counterpart when players have the ability to make a single round of commitment *simultaneously* even if these commitments are costly. Each player eliminates her action corresponding to the Pareto dominant profile, fearing that the failure to do so would result in an even worse outcome than the inefficient one she eventually achieves. It is precisely this logic of coordination failure that unravels in the presence of multiple rounds of commitment coupled with the option of *NC*.

The ability to make *strict commitments* in multiple periods has a critical bearing on the set of supportable outcomes beyond the issue of uniqueness. For certain initial games, an efficient profile may be supportable only in the presence of multiple periods of commitment. This feature of *dynamic commitment games* can be seen clearly in the following example.

Example 3: Efficiency achievable with multiple commitment periods but not with just one period.

	a_2	b_2	c_2
a_1	$3, 3$	$0, 7$	$-3, 2$
b_1	$2, 0$	$1, 1$	$-2, -1$
c_1	$7, -3$	$-1, -2$	$-4, -4$

The unique Nash Equilibrium of the strategic game in Example 3 is the profile (b_1, b_2). Not only is the game dominance solvable, the Stackelberg outcome with either of the players as leader is also (b_1, b_2). Allowing for players to make a single round of commitment does not make the efficient profile (a_1, a_2) supportable. On the other hand the following strategy profile in a *dynamic commitment game* does support the efficient profile. In period 1, Player 1 makes a strict commitment to $\{a_1, b_1\}$ while Player 2 plays *passive*. In period 2, Player 1 plays *passive* while Player 2 makes a strict commitment to $\{a_2, c_2\}$. In period 3 both players play passive resulting in the strategic game $\Gamma(\{a_1, b_1\} \times \{a_2, c_2\}, u)$ being played in period 4. In this strategic game the unique Nash Equilibrium profile, (a_1, a_2) is played. The payoff to both players from this profile is $3 - \epsilon$. It should be noted that (b_1, b_2) is also supportable.

The environment captured by Example 3 shares a property with the Prisoner's Dilemma in that if any player were to act cooperatively then her opponent would be better off not choosing the cooperative action and "defecting" instead. To achieve efficiency, the players would have to remove these defection actions, c_1 and b_2, from the game. However Player 2 cannot eliminate b_2 before Player 1 eliminates c_1, since otherwise Player 1 would commit to the single action c_1 getting herself a payoff of 7 minus her commitment costs. Player 1's unilateral elimination of c_1 does not suffer from such a fate and in turn makes it possible for Player 2 to *subsequently* make the necessary elimination of b_2. It is easy to generate examples where achieving a cooperative outcome involves (even larger) sequences of alternating commitments, with each round making it possible for the next round of commitment to be a "safe" one. The order of such commitments is uniquely pinned down by the particular strategic game in question. Such examples capture a definitive way in which certain strategic interactions cannot be hurried and require gradual "concessions" or "unilateral shows of goodwill" by the parties, one at a time, if a cooperative outcome is to be achieved. While such alternating acts of "concessions" may be given a behavioral interpretation such as reciprocity, they may be driven, as in Example 3, entirely by strategic motives.

Mixed Strategies in the Commitment Stages: Allowing players to use mixed strategies in the commitment stages leads to a break down of the uniqueness results. However, there are a couple of reasons why we feel that the restriction to pure strategies in the commitment stages is compelling. Firstly, given that there always exists a QSPE for any strict game (Theorem 1) we believe it is plausible that the players would restrict their beliefs and choices to pure strategies in the commitment stages. Secondly SPE strategy profiles of dynamic commitment games that involve mixed strategies in the commitment stages yield outcomes that are sensitive to the particular structure of commitment costs. Unlike supportable outcomes these are not robust to different specifications of commitment costs that satisfy the general properties discussed earlier in this section. The following example may help to convey these issues.

Consider the following strategy profile. Player i commits to the single action set

Example 4: Mixed Strategy in the Commitment Stages and Inefficiency

	a_2	b_2
a_1	$2, 2$	$-4, -4$
b_1	$-3, -3$	$1, 1$

$\{b_i\}$, plays PS and plays NC with probabilities,

$$\frac{3 - 5\epsilon + \epsilon^2}{3 + \epsilon}, \frac{\epsilon(3 - \epsilon)}{3 + \epsilon} \ and \ \frac{3\epsilon}{3 + \epsilon}$$

respectively in period 1. Following a first period choice profile (PS_i, NC_{-i}), Player i commits to $\{a_i\}$ in the second period. In the simultaneous game $\Gamma(\{a_1, b_1\} \times \{a_2, b_2\}, u)$ the profile $(.4a_1 + .6b_1, .5a_2 + .5b_2)$ is played. SPE strategies are played in all other subgames. This strategy profile yields an inefficient outcome that is very "close" to (b_1, b_2).

Notice that the outcome above depends on the particular value of ϵ.[17] This is not true for supportable outcomes. Supportable outcomes persist as outcomes of dynamic commitment games for any cost structure satisfying the three basic properties outlined earlier.

6 Commitment and Efficiency

Does the commitment ability studied in this paper always have an efficiency enhancing effect on a general strategic game? The answer, unfortunately, is in the negative. The possibility of making commitments may lead players to reach an outcome that is Pareto inefficient even though the unique Nash Equilibrium in the original game involved a Pareto efficient outcome.

Example 5: Inefficiency due to commitment.

	a_2	b_2	c_2
a_1	$8, 8$	$10, 0$	$4, 2$
b_1	$0, 10$	$6, 6$	$3, 5$
c_1	$2, 4$	$5, 3$	$-1, -1$

[17]Examples exist where the outcome in the presence of mixed commitment is independent of the precise value of ϵ, but these rely heavily on all commitments incurring the same cost. Such examples are not robust to more general cost structures where the cost may vary across commitments.

Example 5 is yet another dominance solvable game. The unique Nash Equilibrium, (a_1, a_2), is an efficient outcome. While (a_1, a_2) continues to be supportable, *dynamic commitment games* also allow the inefficient profile (b_1, b_2) to be supportable. The following qualified subgame perfect strategy profile supports the outcome (b_1, b_2). Players 1 and 2 commit to the subsets (b_1, c_1) and (b_2, c_2), respectively, in the first period. In the second period both players choose to play *passive*. In the subsequent strategic game $\Gamma(\{b_1, c_1\}, \{b_2, c_2\}, u)$ the unique Nash Equilibrium (b_1, b_2) is played. Qualified subgame perfect strategies are played following every other subgame. The resulting payoff to each player is $6 - \epsilon$.

It has been shown that a Nash Equilibrium that Pareto dominates all other outcomes is supportable. However, it turns out that supportability does not necessarily extend to all Pareto efficient Nash Equilibria. Example 6 is a dominance solvable game with the unique Pareto efficient Nash Equilibrium, (b_1, a_2). The unique supportable profile, as can be easily verified, turns out to be (a_1, b_2).

Example 6: Pareto Efficient Nash Equilibrium not supportable.

	a_2	b_2
a_1	$1, 0$	$4, 3$
b_1	$2, 4$	$5, 1$

7 Conclusion

We have introduced a framework which can be used to analyze the import of players being able to unilaterally restrict their choices before some strategic interaction. While we have focused entirely on the relationship between such commitment ability and coordination our existence result makes it possible to study the notion of supportable outcomes in other settings. Our results on general strict games could be a useful starting point in this regard. Example 3 and the discussion following it, for instance, point out that such commitment ability may be used to explain sequences of unilateral commitments (concessions) that take place gradually, eventually making it possible to coordinate on an efficient outcome, which not only would have been impossible without the multiple rounds of commitment but also needed the precise (endogenously determined) sequence of commitments.

The fact that the framework allows for considerable flexibility in the timing and choice of commitments should make it useful in identifying the precise implications of additional structure on such commitment ability. Moreover, in the spirit of the literature on endogenous timing, we can potentially uncover which sequences of commitment arise endogenously instead of comparing the outcomes across such sequences.

Our uniqueness results show that efficient coordination is possible without any commitments being made and without any asynchronous moves in two important classes of games. However, the precise class of games for which it delivers a unique efficient prediction is yet to be characterized. As we have seen from some of our examples dynamic commitment games do not always have an efficiency enhancing effect on the underlying game. It also remains to be seen how the arguments presented here carry over to Bayesian games.

A Appendix

Proof of Theorem 1. Let $\Gamma(\tilde{X}, u)$ be a strict game. Fix some $\epsilon < v(\tilde{X}, u)/m(\tilde{X})$. Denote the set of all strict games that can arise from restricting the action sets of the original strict game, $\Gamma(\tilde{X}, u)$, as $\mathcal{X} = \{X | X_i \subseteq \tilde{X}_i \textit{ for } i \in \{1, 2\}\}$.[18] We begin by fixing a Nash equilibrium outcome for each strict game in $\mathcal{X}$. In particular, for any $X \in \mathcal{X}$, $\pi^*(X)$ denotes a payoff profile as defined below,

$$\pi^*(X) = (\pi_1^*(X), \pi_2^*(X)) = \pi\left(\arg \max_{\sigma^{sim} \in NE(\Gamma(X,u))} \pi_2\left(\sigma^{sim}\right)\right) \tag{4}$$

If for some $X \in \mathcal{X}$ there are multiple Nash equilibria which maximize Player 2's payoff, we simply select one of them to define $\pi^*(X)$. In the strategies we will construct, whenever the players end up playing the simultaneous move game, $\Gamma(X, u)$, with $X \in \mathcal{X}$ they will play the Nash equilibrium profile corresponding to the payoff profile $\pi^*(X)$. It is very important for the argument that the equilibrium selection mentioned earlier depends solely on the identity of the set X and entirely independent of how X may have been reached.

Next we pin down the payoff profile that emerges in equilibrium when only one of the two players has commitment ability. To arrive at this we define $W^i(X)$ to be the weak subset of X_i that gives Player i the highest payoff assuming that the payoff following any such choice is determined by the function π^*. Again, if there are multiple solutions to the maximization problem, $W^i(X)$ selects one of them in a way

[18]We suppress the dependence of the game on the payoff function u for notational convenience, but it should be understood that the relevant payoff function is simply the restriction of the original payoff function to the set of action profiles in the restricted game.

solely determined by the identity of X.

$$W^i(X) = \arg\max_{Y_i \subseteq X_i} \pi_i^*(Y_i \times X_{-i}) - I(Y_i|X)\epsilon \tag{5}$$

where $I(Y_i|X)$ is equal to 0 if $Y_i = X_i$ and 1 otherwise. The expression above allows for the possibility that Player i gets her highest payoff by not making any further strict commitment and simply playing the Nash equilibrium of the game $\Gamma(X, u)$ corresponding to the payoff profile, $\pi^*(X)$. The relevant payoff profiles are defined as follows,

$$\begin{aligned} V^1(X) &= (\pi_1^*(X_1 \times W^2(X)), \pi_2^*(X_1 \times W^2(X)) - I(W^2|X)\epsilon) \qquad (6) \\ V^2(X) &= (\pi_1^*(W^1(X) \times X_2) - I(W^1|X)\epsilon, \pi_2^*(W^1(X) \times X_2)) \qquad (7) \end{aligned}$$

For any $X \in \mathcal{X}$, $V^i(X)$ gives the payoff profile that emerges in our subsequent equilibrium construction if Player i commits to not commit while Player $-i$ retains her commitment ability, when the set of available action profiles is X. Notice that these profiles are entirely independent of how X may have been reached.

Next we define a *terminal payoff profile* $U^{T_k}(X)$ corresponding to every simultaneous game $\Gamma(X, u)$ with $X \in \mathcal{X}$ and $k \in \{1, 2\}$. If $|X_1| = 1$ or $|X_2| = 1$, then the terminal payoff profile $U^{T_k}(X)$ is the (necessarily unique) Nash equilibrium payoff profile in $\Gamma(X, u)$, for either value of k. Terminal payoff profiles for $\Gamma(X, u)$ with $|X_1| \geq 2$ and $|X_2| \geq 2$ are defined inductively. To avoid cluttering we first state the set of inequalities that are used to define these terminal payoff profiles.

$$\begin{aligned} \pi_1^*(X) &\geq U_1^{T_2}(Z_1(X) \times X_2) - \epsilon, V_1^1(X) \\ \textit{and } \pi_2^*(X) &\geq U_2^{T_1}(X_1 \times Z_2(X)) - \epsilon, V_2^2(X) \end{aligned} \tag{8}$$

$$\begin{aligned} U_1^{T_2}(Z_1(X) \times X_2) - \epsilon &> \pi_1^*(X) \\ &\geq V_1^1(X) \end{aligned} \tag{9}$$

$$\begin{aligned} U_2^{T_1}(X_1 \times Z_2(X)) - \epsilon &> \pi_2^*(X) \\ &\geq V_2^2(X) \end{aligned} \tag{10}$$

$$V_k^k(X) > \pi_k^*(X), U_k^{T_{-k}}(Z_k(X) \times X_{-k}) - \epsilon \tag{11}$$

For any $X \in \mathcal{X}$ with $|X_1| \geq 2$ and $|X_2| \geq 2$ the corresponding terminal payoff profile for some $k \in \{1,2\}$ is defined as follows,

$$U^{T_k}(X) = \begin{cases} \pi^*(X) & \text{if} \quad 8 \text{ holds} \\ \left(U_1^{T_2}(Z_1(X), X_2) - \epsilon, U_2^{T_2}(Z_1(X), X_2)\right) & \text{if} \quad 9 \text{ holds} \\ \left(U_1^{T_1}(X_1, Z_2(X)), U_2^{T_1}(X_1, Z_2(X)) - \epsilon\right) & \text{if} \quad 10 \text{ holds and } 9 \text{ fails} \\ V^k(X) & \text{if} \quad 11 \text{ holds} \\ V^{-k}(X) & \text{otherwise} \end{cases}$$

where

$$Z_1(X) = \arg\max_{Y_1 \subset X_1} U_1^{T_2}(Y_1, X_2)$$

$$Z_2(X) = \arg\max_{Y_2 \subset X_2} U_2^{T_1}(X_1, Y_2)$$

Notice that the definition above, by strong induction, ensures that $U^{T_k}(X)$ is well defined for every simultaneous game $\Gamma(X,u)$ with $X_i \subseteq \tilde{X}_i$ and $k \in \{1,2\}$. In particular, $U^{T_k}(X)$ is well defined for any X with $|X_i| = 1$ for some $i \in \{1,2\}$. Now suppose that terminal payoff profiles are well defined for *all* games $\Gamma(X,u)$ with $|X_i| \cdot |X_{-i}| \leq n$. The definition above, then, ensures that $U^{T^k}(X)$ is well defined for all $\Gamma(X,u)$ with $|X_i| \cdot |X_{-i}| = n+1$. It can be verified that for a fixed $k \in \{1,2\}$ and for any $X \in \mathcal{X}$, of the inequalities 8, 9, 10 and 11 only 9 and 10 can hold simultaneously. When none of these inequalities hold it must be that, $\pi_k^* \geq U_k^{T_{-k}}(Z_k(X) \times X_{-k}) - \epsilon, V_k^k(X)$ and $V_{-k}^{-k}(X) > \pi_{-k}^*, U_{-k}^{T_k}(Z_{-k}(X) \times X_k) - \epsilon$. For every $X \in \mathcal{X}$ let $\alpha(X) = 2$ if 9 holds, with $\alpha(X) = 1$ otherwise.

We can now construct a qualified subgame perfect strategy profile, σ^*, for $g^\epsilon(\tilde{X}, u, c)$. At any period t of the game $g^\epsilon(\tilde{X}, u, c)$ with $X^t \in \mathcal{X}$, if $c^t = (1,1)$ then each player plays the strategy in the simultaneous game $\Gamma(X^t, u)$ that corresponds to the payoff profile $\pi^*(X)$. In other words, whenever an action stage is reached in the dynamic commitment game the players play the particular Nash equilibrium profile we had pinned down earlier in our definition of π^*. Importantly, the equilibrium profile is independent of how X^t was reached.

Now consider a period t with only one of the players having the ability to commit. In particular let $X^t \in \mathcal{X}$ with $(c_i^t = 0, c_{-i}^t = 1)$. By the rules of the game, Player $-i$ has no choice to make. Player i plays PS if $W^i(X^t) = X_i^t$. Otherwise she strictly commits to $W^i(X^t)$. By the definition of W^i, the choice made by Player i depends solely on X^t and the fact that she alone can make commitments, $(c_i^t = 0, c_{-i}^t = 1)$.

Finally consider a period t in which both players have the ability to commit, $c^t = (0,0)$, with $X^t \in \mathcal{X}$. Suppose the history of play up to period t is captured by h^t. Let $k = 2$ if $t \geq 2$ and $\alpha(X^{t-1}) = 2$, where X^{t-1} captures the action profiles available to the players in the previous period according to h^t. Let $k = 1$ otherwise. As for period t, if $U^{T_k}(X^t) = \pi^*(X^t)$ then both players play PS. If $U^{T_k}(X^t) = \left(U_1^{T_2}\left(Z_1(X^t), X_2^t\right) - \epsilon, U_2^{T_2}\left(Z_1(X^t), X_2^t\right)\right)$ then Player 1 strictly commits to $Z_1(X^t)$ while Player 2 plays PS. If instead $U^{T_k}(X^t) = \left(U_1^{T_1}\left(X_1^t, Z_2(X^t)\right), U_2^{T}\left(X_1^t, Z_2(X^t)\right) - \epsilon\right)$ then Player 1 plays PS while Player 2 strictly commits to $Z_2(X^t)$. Notice that in the cases above the value taken by k makes no difference. For instance if $U^{T_1}(X^t) = \pi^*(X^t)$ then $U^{T_2}(X^t) = \pi^*(X^t)$. In other words, if σ^* requires either both players to play PS or for just one player to strictly commit while the other plays PS at a given X^t then it will require the players to do precisely the same irrespective of the history (how X^t was reached). The part of the strategy profile which is history dependent follows. If $U^{T_k}(X^t) = V^k(X^t)$ then Player k plays NC, giving up her commitment ability, while Player $-k$ plays PS. The only other scenario involves $U^{T_k}(X^t) = V^{-k}(X^t)$, in which case Player k plays PS while $-k$ plays NC. This concludes the description of the proposed QSPE strategy profile, σ^*.

We can now proceed to verify if σ^* constitutes a QSPE of the game $g^\epsilon(\tilde{X}, u, c)$. We will do so by checking if any player has an incentive to deviate from σ^* at any period t with the set of available action profiles $X^t \in \mathcal{X}$ and following any history. There are three scenarios to consider depending on the values taken by the commitment state variable, c^t. Suppose first that for the period t under consideration with $X^t \in \mathcal{X}$, $c^t = (1,1)$, implying that neither player has the ability to make commitments. At this period therefore, the players are simply playing the simultaneous game $\Gamma(X^t, u)$. σ^* requires the players to play their respective actions corresponding to a Nash equilibrium which in turn corresponds to the payoff profile $\pi^*(X^t)$. Evidently, neither player has a profitable deviation in such a period t.

Consider next a period t with $X^t \in \mathcal{X}$ and $(c_i = 0, c_{-i} = 1)$; only Player i has the ability to make commitments. We need only verify that Player i has no incentive to deviate from σ^* in this period, since $-i$ has no actions available to her in period t. If σ^* requires that Player i play PS then she receives a payoff of $\pi_i^*(X^t)$, which by construction of σ^* must be no less than what she would get by deviating to her most profitable strict commitment. Notice that her most profitable strict commitment cannot result in another round of commitment from her next period. Such a commitment, say W_i^t must necessarily be followed by i playing PS in $t+1$ and playing the Nash equilibrium of the game $\Gamma(W_i^t \times X_{-i}^t)$ corresponding to the payoff profile, $\pi^*(W_i^t \times X_{-i}^t)$ in $t+2$. Consequently when considering Player i's optimal choice in period t it is sufficient to consider the strict commitment in t which maximizes her continuation payoff assuming that such a commitment would be followed by the resulting simultaneous game being played without any further commitments. Since $\arg\max_{Y_i \subseteq X_i^t} \pi_i^*(Y_i \times X_{-i}^t) - I(Y_i|X)\epsilon = X_i^t$ if σ^* requires i to play PS in period t, she cannot do any better by deviating. Remember that playing NC in such a case wouldn't change her payoff. If σ^* instead requires Player i to strictly commit to some set, say $A \subset X_i^t$ then it must be that $A = W^i(X^t)$. Again by the definition of W^i she has no profitable deviations.

Before verifying the final case of $c^t = (0,0)$ a few remarks about the proposed strategy profile, σ^*, are in order. Firstly, along some history at any period t if both players have commitment ability with $X^t \in \mathcal{X}$, $U^{t_k}(X^t)$ gives the continuation payoff from following σ^*. k takes a value of 2 if $t \geq 2$ and $\alpha(X^{t-1}) = 2$ and a value of 1 otherwise, where X^{t-1} denotes the action profiles available in the previous period according to the history under consideration. Secondly, by construction, σ^* involves at most one player making a strict commitment at any period along the equilibrium path. Thirdly, a more subtle yet critical feature of the construction consists in no player making a strict commitment in two consecutive periods according to σ^*. For instance if at a given period t with available action profiles X^t, say Player 1 commits to Z_1^t followed by another commitment Z_1^{t+1} in the next period according to σ^* (while Player 2 plays PS in both) then it must mean that

$$U_1^{T_k}(X^t) = U_1^{T_2}(Z_1(X^t) \times X_2^t) - \epsilon = U_1^{T_2}(Z_1^t \times X_2^t) - \epsilon = U_1^{T_2}(Z_1^{t+1} \times X_2^t) - 2\epsilon \quad (12)$$

with $Z_1(X^t) = \arg\max_{Y_1 \subset X_1^t} U_1^{T_2}(Y_1, X_2^t)$.
This would imply that $U_1^{T_2}(Z_1^{t+1} \times X_2^t) > \max_{Y_1 \subset X_1^t} U_1^{T_2}(Y_1, X_2^t)$, which is clearly a contradiction since $Z_1^{t+1} \subset X_1^t$. An identical argument works for Player 2 making successive strict commitments while 1 plays PS. This verifies the claim that play according to σ^* never involves a player making strict commitments in consecutive periods.

We can now consider a period t with $X^t \in \mathcal{X}$ and $c^t = (0,0)$. If in period t, σ^* involves the action profile (PS_1, PS_2) then the inequalities in 8 must hold. Indeed this is true *irrespective* of the history (if $U^{T_k}(X^t) = \pi^*(X^t)$ then it must also be true that $U^{T_{-k}}(X^t) = \pi^*(X^t)$). Since $V_i^i(X)$ denotes the continuation payoff to Player i from deviating to NC and $U_i^{T_{-i}}(Z_i(X) \times X_{-i}) - \epsilon$, her highest continuation payoff from making a strict commitment, (8) ensures that she has no incentive to deviate.

Suppose instead that σ^* involves the play of (NC_i, PS_{-i}) in period t with $X^t \in \mathcal{X}$ and $c^t = (0,0)$. This must mean that $V_i^i(X^t) > \pi_i^*(X^t), U_i^{T_{-i}}(Z_i(X^t) \times X_{-i}^t) - \epsilon$. In other words, Player i's continuation payoff from deviating to either PS or making the most profitable strict commitment is strictly worse than that from playing NC_i, given that $-i$ is playing PS. As for Player $-i$, her continuation payoff from playing PS_{-i} in period t is $V_{-i}^i(X^t)$, given that i plays NC_i and the continuation strategy induced by σ^*. Notice that given the continuation strategies induced by σ^*, $-i$'s payoff is the highest payoff she can get by either making a strict commitment or playing PS in the next period, $V_{-i}^i(X) = \max_{Y_i \subseteq X_i} \pi_i^*(Y_i \times X_{-i}) - I(Y_i|X)\epsilon$. Indeed, playing PS_{-i} is always a best response to NC_i at any period, since it allows Player $-i$ not only to keep her commitment options open in the next period but also to not eliminate any further actions allowing her to play the corresponding simultaneous game as is. Since $-i$ optimally chooses among these options in the next period in any case according to σ^*, there is nothing to be gained by deviating from PS_{-i} in the present period.

Finally consider the case in which σ^* involves (SC_i, PS_{-i}) being played in period t with $X^t \in \mathcal{X}$ and $c^t = (0,0)$. If it is Player 1 making the strict commitment ($i = 1$) then by construction it must be that Player 1 commits to $Z_1(X^t)$ and her payoff from doing so (given the QSPE continuation strategies induced by σ^*) is $U_1^{T_2}(Z_1(X^t) \times X_2^t) - \epsilon$. Moreover, such a payoff must be strictly greater than $\pi_1^*(X^t)$, which she would achieve were she to deviate to playing PS and no less than $V_1^1(X)$, which would be her payoff from deviating to playing NC. As a result Player 1 would have no incentive to deviate. An identical argument works for Player 2 if $i = 2$. The argument that establishes $-i$ not having any profitable deviations is more subtle and in particular relies on the history dependent nature of σ^*. Recall that σ^* does not involve the same player making a strict commitment in two consecutive periods while the other player chooses PS for both. In other words, according to σ^*, the play of (SC_i, PS_{-i}) in period t must be followed by either (PS_i, y) with $y \in \{NC_{-i}, PS_{-i}, SC_{-i}\}$ or (NC_i, PS_{-i}) in period $t+1$. Suppose σ^* involves (PS_i, y) for some $y \in \{NC_{-i}, PS_{-i}, SC_{-i}\}$ being played in period $t+1$. By the arguments above it should be clear that in period $t+1$ Player $-i$ cannot do any better than to play the y prescribed by σ^*. But this in turn means that $-i$ can do no better than play PS_{-i} in period t. To see this suppose that $-i$ could profitably deviate to making some strict commitment, A_{-i}, in period t then it must be that holding period t play as is, in period $t+1$ she could profitably deviate to committing to A_{-i}. Similarly if deviating to NC_{-i} in period t gives her a higher payoff then it must be profitable to deviate to NC_{-i} in period $t+1$. Since no such profitable deviation exists in $t+1$, $-i$ can do no better than to play PS_{-i} in period t.

For the case of (SC_i, PS_{-i}) in period t being followed by (NC_i, PS_{-i}) in $t+1$ it would be clearer to specifically consider the case of $i = 1$. Also let X^{t+1} denote the set of action profiles that survive following the strict commitment by Player i in period t according to σ^*. If $i = 1$ then $\alpha(X^t) = 2$ since σ^* would prescribe (SC_1, PS_2) only if 9 holds for $X = X^t$. As a result the continuation payoffs in period $t+1$ are determined by $U^{T_2}(X^{t+1})$ $(k = 2)$. Since σ^* requires (NC_1, PS_2) to be played in period $t+1$ it must therefore mean that $\pi_2^*(X^{t+1}) \geq U_2^{T_1}(X_1^{t+1} \times Z_2(X^{t+1})) - \epsilon, V_2^2(X^{t+1})$ and $V_1^1(X^{t+1}) > \pi_1^*(X^{t+1}), U_1^{T_2}(Z_1(X^{t+1}) \times X_2^{t+1})$. Notice that Player 2's payoff from deviating to some strict commitment in t could be no larger than $U_2^{T_1}(X_1^{t+1} \times Z_2(X^{t+1})) - \epsilon$. Her continuation payoff in period t from following σ^* and playing PS is $V_2^1(X^{t+1})$ which of course can be no less than $\pi_2^*(X^{t+1})$. Remember that with Player 2 alone having commitment ability in X^{t+1} she could always play PS and achieve $\pi_2^*(X^{t+1})$ and therefore her payoff from optimally using her commitment ability, as is captured by $V_2^1(X^{t+1})$, can be no less than $\pi_2^*(X^{t+1})$. But we also know that $\pi_2^*(X^{t+1}) \geq U_2^{T_1}(X_1^{t+1} \times Z_2(X^{t+1})) - \epsilon$. This shows that Player 2 cannot profitably deviate to making some strict commitment in period t. If Player 2 were to deviate to NC in period t then her continuation payoff would be $V_2^2(X^{t+1})$. Again, since σ^* prescribes (NC_1, PS_2) to be played in $t+1$ with $\alpha(X^t) = 2$ it must be that $\pi_2^*(X^{t+1}) \geq V_2^2(X^{t+1})$. As argued earlier, we also know that her payoff from following σ^* and playing PS in t, namely $V_2^1(X^{t+1})$, is no less than $\pi_2^*(X^{t+1})$. Consequently Player 2 does not gain by deviating to NC in period t. A symmetric argument establishes the case with $i = 2$. It has therefore been shown that for play according to σ^*, if in period t the profile (SC_i, PS_{-i}) is meant to be played then neither player has any incentive to deviate. ■

Proof of Observation 1. Let σ denote a QSPE strategy profile for $g^\epsilon(X, u, c = (0,0))$. Consider a subgame where both players still have the ability to make commitments, $g^\epsilon(X^t, u, c^t = (0,0))$.[19] By contradiction, suppose that under σ both players make strict commitments in period t, with Player $-i$ committing to a single action. In particular, $X_{-i}^{t+1} = \{x_{-i}\}$ while $X_i^{t+1} \subset X_i^t$. The equilibrium payoff to Player i could then be no larger than $\max_{y \in X_i^{t+1}} u_i(y, x_{-i}) - \epsilon - z_i\epsilon$ where z_i is the number of strict commitments made by Player i to reach the subgame $g^\epsilon(X^t, u, c^t = (0,0))$. If Player i, instead deviates to playing NC, the game must move to the next period where $g^\epsilon(\tilde{X}^{t+1}, u, c^{t+1} = (1,1))$ is played with $\tilde{X}_i^{t+1} = X_i^t$ and $\tilde{X}_{-i}^{t+1} = \{x_{-i}\}$. Notice that $g^\epsilon(\tilde{X}^{t+1}, u, c = (1,1))$ is simply the simultaneous game $\Gamma(\tilde{X}^{t+1}, u)$. Since σ is a QSPE strategy profile, the unique outcome of such a simultaneous move game would then be (w, x_{-i}) where w is Player i's unique best response to x_{-i} in the set $\tilde{X}_i^{t+1} = X_i^t$, resulting in a payoff of $\max_{y \in X_i^t} u_i(y, x_{-i}) - z_i\epsilon$ to Player i. Such a deviation must be strictly profitable since $\max_{y \in X_i^t} u_i(y, x_{-i}) - z_i\epsilon > \max_{y \in X_i^{t+1}} u_i(y, x_{-i}) - \epsilon - z_i\epsilon$. This contradicts the initial premise of σ being a QSPE strategy profile. ∎

Proof of Proposition 1. Let $(x_1, x_2) \in NE(\Gamma(X, u))$. By contradiction, suppose for some $g^\epsilon(X, u, c = 0)$ with $\epsilon < v(X, u)/m(X)$ a QSPE strategy profile σ induces a simultaneous move game with outcome (x_i, y), where $y \neq x_{-i}$. Further let the history that precedes the simultaneous move game on the equilibrium path be $h^T = ((X^1, c^1), (X^2, c^2), \ldots, (X^T, c^T))$. It must be that $x_i \in X_i^t$, $\forall 1 \leq t \leq T$. Consider the deviation by Player $-i$ in the first period involving the strict commitment to the single action set $\{x_{-i}\}$. The resulting subgame beginning in period 2 would be $g^\epsilon(X_i^2 \times \{x_{-i}\}, u, c_i^2 = 0, c_{-i}^2 = 1)$. If Player i does not make a strict commitment in period 2, her payoff would be determined by the (necessarily unique) Nash Equilibrium outcome of the induced simultaneous move game, $\Gamma(X_i^2 \times \{x_{-i}\}, u)$ with $X_i^2 = X_i^1$, namely (x_1, x_2). Any strict commitment by Player i in period 2 would give her no more than $\max_{y \in X_i^1} u_i(y, x_{-i}) - \epsilon$. Since $\max_{y \in X_i^1} u_i(y, x_{-i}) - \epsilon < u_i(x_1, x_2)$, such strategies would not constitute a QSPE. To be a QSPE, σ must therefore require Player i to not make a strict commitment in period 2 following the deviation by Player $-i$ in period 1. Consequently, Player $-i$'s initial deviation guarantees her a payoff of $u_{-i}(x_1, x_2)$ that is strictly higher than her payoff of $u_{-i}(x_i, y)$, since x_{-i} is her unique best response to x_i in X_{-i}^1. This contradicts the initial premise of σ being a QSPE strategy profile.
∎

Proof of Proposition 2. Consider the following strategies for the game $g^\epsilon(X, u, c =$

[19] Remember that $c_i^t = 0 \Rightarrow |X_i^t| \geq 2$.

0) with $\epsilon < v(X,u)/m(X)$. Both players play passive in period 1. In the original strategic game, $\Gamma(X,u)$, played in period 2, the Pareto dominant Nash Equilibrium profile x is played. Qualified subgame perfect strategies are used for every other subgame. It is clear that no deviation by any player, given these strategies, can give the said player a higher payoff. In fact, deviation to any strict commitment would give the deviating player a strictly lower payoff. ■

Proof of Lemma 1. If the Nash Equilibrium σ^{sim} from the set $\overline{NE}(\Gamma(X,u))$ that maximizes Player i's utility is in pure strategies then the inequality in Equation 3 follows immediately from the definition of $v(X,u)$. Since $\Gamma(X,u)$ is a strict game so is $\Gamma(\tilde{X},u)$ for any $\tilde{X} \subseteq X$. In any strict game Player i has a unique pure strategy best response to any pure strategy of Player $-i$. Consequently any Nash Equilibrium $\sigma^{sim} \in \overline{NE}(\Gamma(X,u))$ that is not in pure strategies must involve neither player using a pure strategy. Let $\sigma^{sim} \in \overline{NE}(\Gamma(X,u))$ be one such equilibrium that is not in pure strategies. Consider Player i's expected payoff from such an equilibrium. Since Player i does not use a pure strategy her strategy must put a positive probability on some action y_i that is not her action in the Pareto dominant profile; $y_i \neq \mathcal{P}_i(\Gamma(X,u))$. Since σ^{sim} is a Nash Equilibrium it must be that $E[u_i(\cdot)|\sigma^{sim}] = E[u_i(y_i,\cdot)|\sigma^{sim}_{-i}]$. Moreover, $E[u_i(y_i,\cdot)|\sigma^{sim}_{-i}] < \max_{y_{-i}\in X_{-i}} u_i(y_i,y_{-i})$. It follows then that

$$u_i(\mathcal{P}(\Gamma(X,u))) - E[u_i(\cdot)|\sigma^{sim}] > u_i(\mathcal{P}(\Gamma(X,u))) - \max_{y_{-i}\in X_{-i}} u_i(y_i,y_{-i}) \geq v(X,u)$$

This readily yields Inequality 3. ■

Proof of Lemma 2. In the first period of the game $g^{\epsilon}(X,u,c_i=0,c_{-i}=1)$, only Player i has the ability to make strict commitments. It is clear that $u_i(\mathcal{P}(\Gamma(X,u))) > u_i(y)$, for all $y \in X \setminus \{\mathcal{P}(\Gamma(X,u))\}$. To show that no other outcome can be supportable in this game, consider a strategy profile, σ, that results in an outcome that is not $\mathcal{P}(\Gamma(X,u))$. For σ to be a QSPE, the outcome must correspond to a Nash Equilibrium of some simultaneous game, $\Gamma(\tilde{X}_i \times X_{-i},u)$ with $\tilde{X}_i \subseteq X_i$; $\sigma^{sim} \in \overline{NE}(\Gamma(X,u))$. Consider the outcome if Player i deviates by committing to the single action $\mathcal{P}_i(\Gamma(X,u))$. Following such a commitment the game $\Gamma(\{\mathcal{P}_i(\Gamma(X,u))\} \times X_{-i},u)$ is played in the next period. The outcome, $\mathcal{P}_i(\Gamma(X,u))$, from such a deviation brings Player i a payoff of $u_i(\mathcal{P}(\Gamma(X,u))) - \epsilon$ as opposed to the original payoff that could be no larger than $Eu_i(\sigma^{sim})$. By Lemma 1 such a deviation must be a profitable one. This rules out the possibility of the game $g^{\epsilon}(X,u,c_i=0,c_{-i}=1)$ yielding an outcome different from

$\mathcal{P}(\Gamma(X,u))$.

To show that there exist QSPE strategies that result in the Pareto dominant outcome consider the following strategy profile. Player i plays passive in the first period. In the subsequent subgame $g^\epsilon(X,u,c=(1,1))$, the profile $\mathcal{P}(\Gamma(X,u))$ is played. Qualified subgame perfect strategies are played following every other history. It is easy to see that these strategies constitute a QSPE of $g^\epsilon(X,u,c_i=0,c_{-i}=1)$, since any deviation must result either in a Pareto dominated outcome or in an outcome involving a commitment cost of ϵ. ■

Proof of Proposition 3. It was established in Proposition 2 that the Pareto dominant outcome, $\mathcal{P}(\Gamma(X,u))$, is indeed supportable. It remains to be shown that there are no other supportable outcomes.

The set of all strategy profiles for the game $g^\epsilon(X,u,c=(0,0))$ can be classified on the basis of the actions prescribed for the first period in the following way. Given any strategy profile, (σ_1,σ_2), in the first period either *(i)* both players play SC and eliminate their action corresponding to the efficient outcome, namely $\mathcal{P}_i(X,u)$, or *(ii)* otherwise.

Strategy profiles of type *(ii)* that result in an outcome different from $\mathcal{P}(X,u)$ cannot constitute a QSPE. By contradiction, suppose such a profile, σ, does indeed constitute a QSPE. The outcome of such a strategy profile must correspond to a Nash Equilibrium of some simultaneous move game, $\Gamma(X^T,u)$ where $X_i^T \subseteq X_i$. Let this Nash Equilibrium strategy profile be denoted by σ^{sim}. By definition, $\sigma^{sim} \in \overline{NE}(\Gamma(X,u))$. Let Player $-i$ be the one, who by assumption, did not eliminate her action corresponding to the efficient outcome in the first period; $\mathcal{P}_{-i}(X,u) \in X^2_{-i}$. If Player i according to σ, in period 1, made a strict commitment to $\{\mathcal{P}_i(X,u)\}$ then the subgame beginning in the subsequent period must be $g^\epsilon(\{\mathcal{P}_i(X,u)\} \times X^2_{-i}, u, c_i = 1, c_{-i} = 0)$. By Lemma 2 the unique outcome of such a subgame must be $\mathcal{P}(X,u)$. So for σ to not yield $\mathcal{P}(X,u)$ as the outcome, Player i cannot be strictly committing to $\{\mathcal{P}_i(X,u)\}$ in the first period. However, if that were true then Player i would have a profitable deviation in the first period to making a strict commitment to the single action set $\{\mathcal{P}_i(X,u)\}$. Doing so would lead to the game $g^\epsilon(\{\mathcal{P}_i(X,u)\} \times X^2_{-i}, u, c_i = 1, c_{-i} = 0)$ in the subsequent period. Again by Lemma 2 we know that for σ to be a QSPE the outcome following such a subgame must be $\mathcal{P}(X,u)$. Player i's payoff from such a deviation would be $u_i(\mathcal{P}(X,u)) - \epsilon$, which strictly exceeds $Eu_i(\sigma^{sim})$ by Lemma 1 and the fact that $\epsilon < v(X,u)/m(X)$. As a result strategy profiles of type *(ii)* that constitute a QSPE cannot result in an outcome different from $\mathcal{P}(X, u)$.

Strategy profiles of type (i) cannot constitute a QSPE since either player can profitably deviate in period 1 by choosing NC. In particular, let σ be a strategy profile of type (i) with $X_1^2 \subset X_1$ and $X_2^2 \subset X_2$ denoting the first period strict commitments of the two players. Player i's payoff from such a strategy profile cannot exceed $u_i(\mathcal{P}(X^2, u)) - \epsilon$.[20] Instead, by choosing NC in the first period, Player i guarantees that the game $g^\epsilon(X_i \times X_{-i}^2, u, c_i^2 = 1, c_{-i}^2 = 0)$ is played in the next period. By Lemma 2 the unique outcome of this game must be $\mathcal{P}(X_i \times X_{-i}^2, u)$ for σ to be a QSPE. Player i's payoff from such a deviation would therefore be $u_i(\mathcal{P}(X_i \times X_{-i}^2, u))$, which strictly exceeds $u_i(\mathcal{P}(X^2, u)) - \epsilon$. This concludes the proof of why no Pareto dominated outcome is supportable in an OPC game. ■

Proof of Proposition 4. It was established in Proposition 2 that the Pareto dominant outcome, $\mathcal{P}(\Gamma(X, u))$, is indeed supportable. It remains to be shown that there are no other supportable outcomes.

We again classify the set of all strategy profiles for the game $g^\epsilon(X, u, c = (0, 0))$ on the basis of the actions prescribed for the first period into two classes. Given any strategy profile, (σ_1, σ_2), in the first period either *(i)* both players play SC and eliminate their action corresponding to the efficient outcome, namely $\mathcal{P}_i(X, u)$, or *(ii)* otherwise.

The argument for why strategy profiles of type *(ii)* that result in an outcome other than $\mathcal{P}(\Gamma(X, u))$ does not constitute a QSPE is identical to the one in Proposition 3. In particular, if Player i did not eliminate her action $\mathcal{P}_i(\Gamma(X, u))$, then Player $-i$ can make a profitable deviation by committing to the single action $\mathcal{P}_{-i}(\Gamma(X, u))$ in the first period.

Consider now a strategy profile of type *(i)*, say σ. By contradiction, assume that σ is a QSPE that yields an outcome different from $\mathcal{P}(\Gamma(X, u))$. Let X_1^2 and X_2^2 be the strict commitments made by the two players in the first period according to σ. By assumption $\mathcal{P}_i(\Gamma(X, u)) \notin X_i^2$. Let $\hat{x}_i$ be the maximal element in X_i^2 according to the order $\succeq$.

We first show that for σ to be a QSPE it must be that $(\hat{x}_1, \hat{x}_2) \in NE(\Gamma(X, u))$. In other words the first period commitments must be such that the highest action according to $\succeq$ still available to one player must have its Nash counterpart from

[20] Remember that $\Gamma(X^2, u)$ is also an OPC game and therefore admits a Pareto dominant profile, $\mathcal{P}(X^2, u)$.

the original game still available to the other player. Indeed, if this were not true then there would be some $i \in \mathcal{N}$ whose $\hat{x}_i$ corresponds to a Nash equilibrium of the original game that is Pareto dominated by the Nash equilibrium corresponding to $\hat{x}_{-i}$. Formally,

$$u_i(BR_i(\hat{x}_{-i}; X_i^1), \hat{x}_{-i}) > u_i(\hat{x}_i, BR_{-i}(\hat{x}_i; X_{-i}^1)) \text{ and } BR_i(\hat{x}_{-i}; X_i^1) \notin X_i^2$$

This is simply the result of the equilibria being Pareto ranked. Importantly for such a Player i it must also be true that

$$u_i(BR_i(\hat{x}_{-i}; X_i^1), \hat{x}_{-i}) > u_i(x) \qquad \forall x \in X^2$$

The outcome of σ can bring Player i a payoff no higher than $\max_{x \in X^2} u_i(x)$. Instead if Player i were to deviate to committing to the single action $BR_i(\hat{x}_{-i})$ in the first period she could guarantee herself a payoff of $u_i(BR_i(\hat{x}_{-i}; X_i^1), \hat{x}_{-i}) - \epsilon$. Since $\epsilon < v(X, u)/m(X)$ this would indeed be a profitable deviation. As a result, for σ to be a QSPE it must be that $(\hat{x}_1, \hat{x}_2) \in NE(\Gamma(X, u))$. This, in turn, ensures that for σ to be a QSPE, its outcome must be $(\hat{x}_1, \hat{x}_2)$. To see this recall that $(\hat{x}_1, \hat{x}_2)$ Pareto dominates every other outcome in X^2, by virtue of this being an $n - Eq$ game. Moreover $\hat{x}_i \in X_i^2$. So if σ were to give rise to some other outcome, Player i could deviate to committing to the single action $\hat{x}_i$ in the first period. This would induce the outcome $(\hat{x}_1, \hat{x}_2)$, which would be a profitable deviation by Lemma 1.

It has been established that for σ to be a QSPE it must result in the outcome $(\hat{x}_1, \hat{x}_2)$, yielding to Player i a payoff no higher than $u_i(\hat{x}) - \epsilon$. The next step is to show that for such a strategy profile, σ, each player could profitably deviate in the first period by playing NC instead since such a deviation leaves the outcome unchanged while saving the deviator the cost of commitment. To see this consider the game $g^\epsilon(X_i \times X_{-i}^2, (c_i = 1, c_{-i} = 0))$, which would result from such a deviation by Player i. Since Player i has given up her ability to commit, the outcome of such a subgame must be a Nash equilibrium of a game in which Player i has access to all her actions in X_i while Player $-i$'s available actions are some subset of X_{-i}^2. Player $-i$ could guarantee herself a payoff of $u_{-i}(\hat{x}) - 2\epsilon$ by strictly committing to the single action $\hat{x}_{-i}$. By virtue of $\Gamma(X, u)$ being an n-Eq game, it is true that

$$u_{-i}(\hat{x}) > u_{-i}(x) \qquad \forall x \in X_i \times X_{-i}^2 \text{ with } \hat{x}_i \succeq x_i$$

In other words any pure outcome in the game $\Gamma(X_i \times X^2_{-i}, u)$ different from $(\hat{x}_1, \hat{x}_2)$ would give Player i a strictly lower payoff than that from $(\hat{x}_1, \hat{x}_2)$ so long as Player i does not play an action *higher* than $\hat{x}_i$. The same is obviously true if we looked at the game $\Gamma(X_i \times Y_{-i}, u)$ where Y_{-i} is a subset of X^2_{-i}. Due to the assumption of $\Gamma(X, u)$ satisfying the single crossing property, it turns out that $\hat{x}_i$ strictly dominates any action y_i with $y_i \succeq \hat{x}_i$ and $y_i \neq \hat{x}_i$ in any game $\Gamma(X_i \times Y_{-i}, u)$ with $Y_{-i} \subseteq X^2_{-i}$. This of course means that whatever actions Player $-i$ chooses to eliminate in $g^\epsilon(X_i \times X^2_{-i}, (c_i = 1, c_{-i} = 0))$, in the induced simultaneous game, Player i would not play any action *higher* than $\hat{x}_i$. As a result Player $-i$ could do not better than with the outcome $\hat{x}$. The game $g^\epsilon(X_i \times X^2_{-i}, (c_i = 1, c_{-i} = 0))$ must therefore have a unique outcome, $\hat{x}$ since for any other outcome Player $-i$ would receive a lower payoff while having the ability to profitably deviate to committing to the single action $\hat{x}_{-i}$ and forcing the outcome $\hat{x}$. Therefore Player i can profitably deviate from σ in the first period by choosing NC instead. This delivers the required contradiction. ■

References

[1] Bade, S., Haeringer, G., Renou, L., 2009. Bilateral commitment. Journal of Economic Theory 144, 1817-1831.

[2] Ben-Porath, E., Dekel, E., 1992. Signaling future actions and the potential for sacrifice. Journal of Economic Theory 57, 36-51.

[3] Calcagno, R., Kamada, Y., Lovo, S., Sugaya, T., 2013. Asynchronicity and coordination in common and opposing interest games. Theoretical Economics, Forthcoming.

[4] Caruana, G., Einav, L., 2008. A theory of endogenous commitment. Review of Economic Studies 75, 99-116.

[5] van Damme, E., Hurkens, S., 1996. Commitment robust equilibria and endogenous timing. Games and Economic Behavior 15, 290-311.

[6] Dutta, P.K., 2012. Coordination need not be a problem. Games and Economic Behavior 76, 519-534.

[7] Fershtman, C., Judd, K., Kalai, E., 1991. Observable contracts, strategic delegation and cooperation. International Economic Review 32, 551-559.

[8] Hamilton, J.H., Slutsky, S.M., 1990. Endogenous timing in duopoly games: Stackelberg or Cournot equilibria. Games and Economic Behavior 2, 29-46.

[9] Jackson, M., Wilkie, S., 2005. Endogenous games and mechanisms: Side payments among players. Review of Economic Studies 72, 543-566.

[10] Kalai, A.T., Kalai, E., Lehrer, E., Samet, D., 2010. A commitment folk theorem. Games and Economic Behavior 69, 127-137.

[11] Lagunoff, R., Matsui, A., 1997. Asynchronous choice in repeated coordination games. Econometrica 65(6), 1467-1477.

[12] Lazarev, J., 2012. Getting more from less: understanding airline pricing. *mimeo*.

[13] Lipman, B.L., Wang, R., 2000. Switching costs in frequently repeated games. Journal of Economic Theory 93, 149-190.

[14] Milgrom, P., Shannon, C., 1994. Monotone comparative statics. Econometrica 62(1), 157-180.

[15] Nava, F., 2008. Joint commitment: Implementation without public randomization. *mimeo*

[16] Renou, L., 2009. Commitment games. Games and Economic Behavior 66, 488-505.

[17] Romano, R., Yildirim, H., 2005. On the endogeneity of Cournot-Nash and Stackelberg equilibria: games of accumulation. Journal of Economic Theory 120, 73-107.

[18] Schelling, T., 1960. The strategy of conflict. Harvard University Press, Cambridge, MA.

[19] Skyrms, B., 2004. The stag hunt and the evolution of social structure. Cambridge University Press, Cambridge, UK.

[20] Takahashi, S., 2005. Infinite horizon common interest games with perfect information. Games and Economic Behavior 53, 231-247.