Cost Sharing in a Condo Under Law's Umbrella

Bertrand Crettez *

Régis Deloche†‡

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Abstract

How to share the cost of an improvement in a condo? In France, as in most European civil law countries, the law generally does not provide any precise method for answering this question. This vagueness of the law calls for further study on this cost sharing problem. Because cooperative game theory focuses on how to distribute costs that are collectively incurred by a group of players, it is the most appropriate framework for dealing with this topic. In this theory, there is a classic and widely used method of deciding upon the distribution of the costs of any item: it is the Shapley value. We analyze the interest of using this concept to solve our problem. We show that taking into account law–in particular the requirement that any decision concerning improvements be taken by a two-third majority of the votes–affects the characteristic function of the game. Without loss of generality, we restrict ourselves to considering the case of a three-storey condo. We show that the Shapley value is almost never a relevant way to share the costs of an improvement. Indeed the Shapley value is not always in the core and, even if this is the case, it almost never receives an affirmative vote from the co-owner association.

Key words: Shapley value, Cost allocation, Condo.

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*Université Panthéon-Assas, Paris II, CRED, Paris Center for Law & Economics, bertrand.crettez@u-paris2.fr.

†Université Paris-Decartes, L.I.R.A.E.S., regis.deloche@parisdescartes.fr.

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1 Introduction

How to share the cost of an improvement in a condo? In France, as in most European civil law countries, the law generally does not provide any precise method for answering this question. It only specifies that these expenses are to be paid for by the co-owners of the apartments benefited thereby, and that the distribution of these expenses among these co-owners has to be made in such a way that each of them pays in proportion to the advantages he will receive from the planned work.

This vagueness of the law calls for further study on this cost sharing problem. Because cooperative game theory focuses on how to distribute costs that are collectively incurred by a group of players, it is the most appropriate framework for dealing with this topic. In this theory, there is a classic method of deciding upon the distribution of the costs of any item: it is the Shapley value. It has been used extensively and is the only sharing method which satisfies some reasonable properties or axioms (see, *e.g.*, Osborne and Rubinstein, 1994).[1] We analyze the interest of using this concept to share the cost of an improvement in a condo.

We pay a particular attention to the construction of the characteristic function of the cooperative game that we use in this paper. The characteristic function gives, for any coalition of co-owners, a security level to its members[2]. To build the characteristic function of our game, we take into account some key legal requirements in relation to this matter. To do that, we take the French apartment ownership legal system as a representative example of what exists in most member states of the European Union.

The idea that the characteristic function depends on law appeared originally in Aivazian and Callen (1981) (see also Benoît and Kornhauser, 2002). Aivazian and Callen provide an example where taking law into account when defining the characteristic function results in an empty core. These authors deduce from this fact that the Coase theorem breaks down since no agreement is stable.[3] Our perspective, however, is different. First, the core of our game is never empty. Second, we focus on the allocations which are in the core and receive an affirmative vote.

[1] For a survey of the numerous applications of the Shapley value, see, *e.g.*, Moretti and Patrone (2008).

[2] This is the result that can be achieved by the coalition regardless of what the outsiders do.

[3] There is a large discussion on the relevance of this conclusion (see, *e.g.*, Magnan de Borgnier, 1986).

Next, we show that the Shapley value is almost never a relevant way to share the costs of an improvement. Indeed it is not always in the core and, even if this is the case, it almost never receives an affirmative vote from the co-owner association.

Our finding that the Shapley value is of no usefulness to co-owners having to share the cost of an improvement contrasts with som recent contributions to the cooperative game theoretic approach to law-and-economics.[4] Indeed, Ben-Shahar *et al.* (2007, 2009) rely on the Shapley value to compute the relative values of storeys in high rises. They stress the fact that their approach could be of some use in law suits since it provides a normative benchmark. As the authors put it (*ibid*):

> The cost allocation mechanism may be further used, for example, by courts in order to compute required compensations in cases of tenant disputes regarding expansions and redevelopment

These authors, however, neither address the problem of decision making by majority vote, nor take into account the increase in the values of the apartments resulting from the elevator installation. More recently, cooperative game theory has been used to study the sharing of a damage that has been caused by several individuals (see Dehez and Ferey, 2012). By pointing that the Shapley value is of particular interest in this legal context because it is founded on axioms that are in line with fundamental principles of tort law, the authors show that the cooperative approach may bring useful insights into legal questions.

This paper also contributes to the comparative law-and-economics analysis of the provision of public goods in condos introduced by De Geest (1992). De Geest focuses on the majority requirements most condos choose to decide on maintenance and improvement projects. He argues that the choice between simple majority, qualified majority or unanimity hinges on whether the public goods supplied in a condo are or not "ex ante determinable": the more "ex ante determinable" the public good, the higher the majority requirement. This paper does not address this issue. Rather, we take

[4] For an early application of cooperative game theory to law-and-economics see, *e.g.*, the study of the link between the core and the law in Bergstrom (1975). See also Telser (1994) for an application of core theory to competition law and the analysis of the bankruptcy problem from the Talmud provided by Aumann and Maschler (1985).

the law as given, and we focus on the question of whether the Shapley value can be useful as a cost-sharing scheme.

This paper unfolds as follows. In section 2, we present the model of a condo which contemplates the realization of an improvement. We restrict ourselves without loss of generality to the case of a three-storey building. In section 3 we study the cost-sharing of an improvement when there are majority requirements. In section 4, we study the cost sharing relying on the Shapley surplus sharing. Section 5 concludes. All the proofs are gathered in an appendix.

2 The Setup

In the European Union, there are two main types of apartment ownership (European University Institute, 2005). First, there are legal systems, like the Dutch one, in which the buildings are jointly owned and the co-owners are granted exclusive rights to use a specific apartment. Second, there are legal systems, like the French one, in which apartment ownership is a combination of separate ownership of the apartments and joint ownership of the building's common structures (*e.g.*, stairwell, elevator, roof, exterior walls, and so on). In such a legal system, a splitting deed regulates not only the various property rights and their demarcation among the owners, but also other matters, like the allocation of costs for utilities. Together, the owners form an association whose statutes are generally governed by the law. This association decides on the maintenance and management of the building. Various voting schemes are used for different types of decisions.

After presenting the key points of the French apartment ownership Act of 1965, we will address the question of the cost sharing of an improvement in a three-storey condo building, and we shall consider a specific example in which the improvement project is to install an elevator.

2.1 Improvements in a condo and the French Law

In France, the system of apartment ownership (*copropriété*) is governed by the statute 65-557 of 10 July 1965.[5] As concerns the decisions on improvements in a condo, several articles of the French

[5] An English translation of this law is available at http://tenantslife.com/pdf/france/english/coproeng.pdf.

law play a key role. Article 5 of the law states that "the share of the common parts pertaining to each lot is proportionate to the value of each private part in relation to the value of all said parts, as the value resulting from the establishment of the property, the substance and the situation of the lots without regards to their use".[6] Article 22 of the law states that each "co-owner has a number of votes which corresponds to his share in the common parts."[7] Article 26 of the law states that decisions concerning the works involving improvements "are taken by a majority of the members of the management association representing at least two thirds of the votes". Article 30 of the law states that the general assembly of the co-owners establishes, with the same majority, the distribution of the expenses for the improvement works "in proportion to the advantages that will result from the planned works for each of the co-owners, except when taking into account the consent of some of them to support a higher part of the expenses". In addition, article 30 of the law states that, when the general assembly refuses the authorization defined in article 25[8], "any co-owner or group of co-owners can be authorized by the High Court to execute any improvement works", such as the transformation of one or several existing equipments.

To take all these legal points into account without actually building too complex a model, we shall consider a three-storey condo with one apartment unit on each floor, and we shall assume that each co-owner (one co-owner for each apartment unit) has one vote. Thus, we keep the cost-sharing problem interesting and we obtain results that are qualitatively the same as those which would have been obtained by building a heavier model.

[6] When a co-owner has a share in the common parts that is superior to half of the common parts, the number of votes that he has is reduced to the total amount of votes belonging to the other co-owners (art. 22 of the French law, modified by art. 7, Statute 2009-526 of May 12, 2009).

[7] De Geest (1992) explains why votes in a condo are generally allocated this way in Europe rather than on a "one person, one vote" basis by pointing out that unit values reflect the intensity of preferences better than per capita voting.

[8] Article 25 of the law states that are only adopted by a majority of the votes of all co-owners decisions concerning the authorization given to certain co-owners to execute at their own expense the works affecting the common parts in accordance to the occupancy of the building.

2.2 A General three-person cooperative game

The above-mentioned three co-owners contemplate an improvement project. If this project is realized and if a co-owner i participates in this project, he gets a benefit, *i.e.*, an individual advantage, equal to b_i. We assume that this project is a club good, *i.e.*, a non-rival but excludable good: one person's consumption of this good does not diminish another person's, but those who do not pay for this good can be excluded from the consumption of it.

The project cost depends on the number of participants. We let $c(S)$ be a cost function which gives the total cost of the project if the set S of co-owners participate in the project. Here $S \subseteq N \equiv \{1,2,3\}$.

To address the cost sharing of the improvement we follow a cooperative game approach. A cooperative game is defined by a set of players, N, and a characteristic function, v. The values taken by v may depend on the law (see, *e.g.*, Benoît and Kornhauser, 2002).

Specifically, we will rely on a cooperative surplus sharing game whose players are the co-owners. We first define the maximal available surplus $v(S)$ of a coalition S of co-owners as follows:

$$v(S) = \max_{T \subseteq S} \left\{ 0, \sum_{i \in T} b_i - c(T) \right\}. \tag{1}$$

The maximal available surplus to a given coalition is the largest surplus from serving any of its subcoalitions, including zero because we take into account the possibility of inaction.

Second, we define the characterisic function of this game. To do this we consider the majority requirements. Let then a coalition containing at least two agents be given. In that case the improvement project of this coalition cannot be defeated in a majority vote. Then, the value of the characteristic function of the cooperative surplus sharing game is equal to $v(S)$.

Now consider a one-player coalition. According to article 25 of the French Law each improvement project of this coalition can be defeated in a majority vote. Moreover according to article 30 of the French Law, such an improvement project can be authorized or rejected by the high court.

As there is no certainty with regard to the high court decision we assume that a judge authorizes a single co-owner to carry out his project with probability p, $p \in [0,1]$. Therefore, we set the value of the characteristic function of a co-owner i to be equal to $pv(i)$.[9]

To summarize the preceding presentation, if v^l denotes the characteristic function of our suplus sharing game, we have:

$$v^l(S) = \begin{cases} v(S) \text{ if } \#S \geq 2, \\ pv(S) \text{ otherwise.} \end{cases} \quad (2)$$

where $\#S$ is the cardinal of coalition S.

2.3 An Example

We consider a condo which contemplates the installation of an elevator. The cost of an elevator comprises the following elements:

- a fixed cost: c_f,
- a unit cost of building a storey: c_s.

For instance, the cost of building a three-storey elevator (with access to all the storeys) is equal to: $c_f + 3c_s$.

We further assume that building an elevator increases the storey i's apartment value by b_i (where b_i is a positive real number). As it is often observed, walk-up apartment buildings worth less than elevator apartment buildings.[10]. Therefore we assume that $b_1 < b_2 < b_3$.

We define the function v for the elevator surplus game as follows:

$$v(S) = \max_{T \subseteq S} \left\{0, \sum_{i \in T} b_i - \max\{c_z, z \in T\}\right\}, \quad (3)$$

[9] According to Maschler, Solan et Zamir (2013) (page 660): "The real number v(S) is called the worth of the coalition S. The significance of this is that if the members of S agree to form the coalition S, then as a result they can produce (or *expect to receive*) the sum of v(S) units of money, independently of the actions of the players who are not members of S". Emphasis is ours. For an example of an application of stochastic cooperative game, see Dinar *et. al.* (2006).

[10] In France, according to Quignard (2005, p.124), units in buildings with an elevator worth up to fifty percent more compared to walk-up units (*ceteris paribus*).

where: c_z is the cost of an an elevator which reaches story z. More precisely, we set $c_0 = 0$ and $c_z = c_f + zc_s$ where S is a coalition of co-owners.

3 Cost-Sharing Under Law's Umbrella

In this section, we first present a general cost-sharing scheme of an improvement, which relies on the cooperative surplus sharing game introduced in the preceding section. Next, we study the cost-sharings which will not be dominated, that is, the cost-sharings which not only meet the coalitional rationality requirement but also receive an affirmative vote.

3.1 From surplus sharing to cost sharings

Following Littlechild (1975), we model the cost-sharing by following a two-step approach. We first concentrate on the net surplus generated by an improvement. More to the point, we study all the surplus allocations which are in the core of the surplus sharing game. All these allocations satisfy a natural coalitional rationality requirement. Namely, no coalition would gain by breaking away from the grand coalition (the coalition which comprises all the co-owners). We *define* the cost-share of the improvement for a co-owner i as the difference between his benefit (b_i) and his share of the surplus. In so doing, we assume that $v^l(N) = v(N) = \sum_i b_i - c(N)$ (that is, the improvement will benefit all the co-owners).

Let us recall the definition of the core for our surplus-sharing game.

Definition 1. *A surplus allocation $(x_i)_i$ is in the core of the surplus-sharing game if:*

$$\sum_{i \in N} x_i = v^l(N), \tag{4}$$

$$\sum_{i \in S} x_i \geq v^l(S), \forall S \subseteq N. \tag{5}$$

In our setting these inequations reduce to:

$$x_1 + x_2 + x_3 = v(123), \tag{6}$$

$$x_1 + x_2 \geq v(12), \tag{7}$$

$$x_1 + x_3 \geq v(13), \tag{8}$$

$$x_2 + x_3 \geq v(23), \tag{9}$$

$$x_1 \geq pv(1), \tag{10}$$

$$x_2 \geq pv(2), \tag{11}$$

$$x_3 \geq pv(3). \tag{12}$$

To guarantees that the core is always nonempty, we will assume that our game is convex when $p = 1$. A game is convex if for two coalitions A and B, we have $v(A \cup B) \geq v(A) + v(B) - v(A \cap B)$. This property captures the idea that larger coalitions generate economies of scales. With this property we have the following Proposition.

Proposition 1. *Assume that the game is convex when $p = 1$ (i.e., when $v^l(S) = v(S)$ for all S). Then the core of the surplus-sharing game is nonempty for all $p \in [0, 1]$.*

The Proposition results from the fact that if the core is nonempty with $p = 1$, then by definition the inequations (6)-(12) are satisfied with $p = 1$. But this implies that these inequations are still satisfied when $p < 1$ since then the values of their right-hand sides are either independent or decreasing with respect to p.[11]

Now, let (x_i) be an allocation in the core of the surplus game. We *define* the cost y_i borne by a co-owner i as the difference between his benefit (b_i) and his share of the surplus (x_i):

$$y_i = b_i - x_i, \ \forall i \in N. \tag{13}$$

[11] We notice that even if the game is convex when $p = 1$, this game is not necessarily convex when $p < 1$ since if $v^l(A) = v(A) > 0$, $v^l(B) = v(B) > 0$ and $A \cap B \neq \emptyset$, the inequation $v(A \cup B) \geq v(A) + v(B) - v(A \cap B)$ does not imply that $v^l(A \cup B) = v(A \cup B) \geq v^l(A) + v^l(B) - v^l(A \cap B) = v(A) + v(B) - pv(A \cap B)$.

3.2 Cost sharing and majority voting

To study the relationship between cost sharing and majority voting we introduce the following definition:

Definition 2. *A cost-sharing y is dominated by another cost-sharing y' if there is a majority who will vote for y' rather than for y in the pair-wise vote between these two alternatives. That is, there is a cost-sharing y' which satisfies $y'_i < y_i$ and $y'_j < y_j$ for some i and j in $\{1,2,3\}$, $i \neq j$.*

In the above definition, to the cost-sharing y and y' correspond two allocations x and x' in the core of the surplus game which are such that for all i we have respectively: $y_i = b_i - x_i$ and $y'_i = b_i - x'_i$. Therefore, a cost-sharing y is dominated by a cost-sharing y' if the associated allocations in the core of the surplus game satisfy: $x'_i > x_i$ and $x'_j > x_j$ for some i and j in $\{1,2,3\}$, $i \neq j$. We will rely on this remark to study the cost-sharings which are not dominated.

Notice that we require that the inequations involving x_i or x_j be strictly satisfied (indifference is not taken into account). Moreover, we require that the domination be made using an allocation in the core of the surplus game.

The next Proposition gives some conditions under which a cost-sharing obtained from a surplus-sharing will or will not receive a majority approval.

Proposition 2. *Let an allocation x be in the core. Assume that at least two inequations among inequations (7), (8), (9) are strictly satisfied and that the common variable x_i in these inequations satisfies $x_i > pv(i)$. Then there is an allocation x' which dominates x.*

The intuition of this Proposition is as follows. Under the assumption of the Proposition one can always decrease x_i and increase the surplus of the other co-owners, while insuring that no coalition of co-owners would be better of by leaving the grand coalition. In that case, the two remaining co-owners would vote againt allocation x.

The next Proposition gives some condition underwhich an allocation is not dominated.

Proposition 3. *Let an allocation x be in the core. Assume that there is at most one inequation among inequations (7), (8), (9) which is strictly satisfied, or two but with the common value x_i satisfying $x_i = pv(i)$. Then the allocation x is not dominated.*

Under the conditions given in the above Propositionn, it is no more possible to devise an alternative allocation that would be preferred to x by two co-owners. This is because one cannot decrease the share of the co-owner for whom $x_i = pv(i)$, or because when one decreases the share of a co-owner, a coalition of two co-owners is better of by leaving the grand coalition (the alternative allocation is not in the core of the surplus-sharing game).

It is easy to show that when at most one inequation among (7), (8), (9) is satisfied with strict inequality, the surplus-sharing schemes are given by the following expressions:

$$x_i = v(ik) + v(ij) - v(123), \tag{14}$$

$$x_j = v(123) - v(ik), \tag{15}$$

$$x_k = v(123) - v(ij), \tag{16}$$

where $i, j, k \in \{1,2,3\}$, $i \neq j \neq k$. From then, we can compute the induced cost-sharing schemes for the different possible schemes. If, furthermore, we assume that all coalitions of at least two co-owners find worthwile to realize the improvement for itself (*i.e.*, $\sum_i b_i \geq c(S)$ for all S, where S comprises two co-owners) then the following cost-sharings are not dominated:

$$\begin{array}{llll}
y_1 = c(12) + c(13) - c(123) & y_1 = c(123) - c(23) & y_1 = c(123) - c(23) & (17) \\
y_2 = c(123) - c(13) & y_2 = c(12) + c(23) - c(123) & y_2 = c(123) - c(13) & (18) \\
y_3 = c(123) - c(12) & y_3 = c(123) - c(12) & y_3 = c(13) + c(23) - c(123) & (19)
\end{array}$$

For our elevator example, we can show that these cost-sharings reduce to the following expressions:

$$\begin{array}{llll}
y_1 = c_f + 2c_e & y_1 = 0 & y_1 = 0 & (20) \\
y_2 = 0 & y_2 = 2c_e + c_f & y_2 = 0 & (21) \\
y_3 = c_e & y_3 = c_e & y_3 = c_f + 3c_e & (22)
\end{array}$$

We notice that in each of the three above cost-sharings, a co-owner is exploited by the two others. Moreover, these cost-sharings are not proportional to the individual advantages. As a consequence, the exploited co-owner could sue the two others. Here, the proportionnality requirement conflicts with the majority requirement.

We conclude this section by noting that sharing cost in a way that complies with law and insures the efficiency of the surplus-sharing seems to be a particularly hard task[12]. We now take up the analysis of the Shapley value.

4 Cost Sharing according to the Shapley value

The above-defined core of the surplus sharing game has an important desirable property of efficiency, but it also has one important drawback: it generally consists of whole range of outcomes. So, in most cases, this concept does not determine an outcome uniquely. Other concepts do better in this respect by perfectly meeting the uniqueness requirement. The best known is the Shapley value. It is among the most popular concepts in cooperative game theory.[13]

To introduce the Shapley value, let us consider all the different ways to form the grand coalition of all co-owners by step-by-step increases in membership. For each of these ways, we can compute the marginal contribution of a co-owner i to the grand coalition. This marginal contribution is the difference between the surplus of the coalition containing the first co-owners and the surplus of the coalition composed of these co-owners and co-owner i. The Shapley solution gives to each co-owner the average of his marginal contributions to the grand coalition.[14]

[12] This conclusion particularly applies to the second case considered in Proposition 3. It can be shown in this case, that if $x_i + x_j > v(ij)$, $x_j + x_k > v(jk)$, $x_j = pv(j)$, $i, j, k \in \{1,2,3\}$, $i \neq j \neq k$, then the equation $v(123) = v(ik) + pv(j)$ must hold. This is a stringent condition.

[13] A modern general treatment of cooperative game theory is Maschler *et al.* (2013), chapters 16-20. It can be shown that the Shapley is the only sharing method wich satisfies the following the four following axioms. Axiom 1: the sum of the sums received by each agent must be equal to the surplus to be distributed. Axiom 2: a player for whom the marginal contribution to the surplus of the co-ownership is the same, no matter which group is currently considered, should receive this contribution. Axiom 3: two players for whom the marginal contribution is the same, no matter which group is considered, should receive the same. Axiom 4: the benefit received by any player for two different co-ownerships should be the sum of the payments for the two co-ownerships separately.

[14] In a cooperative game with n players, the Shapley value of a player i in a cooperative game is:

$$\varphi_i(v) = \sum_{S \subset N, i \notin S} \frac{(n - \#S - 1)!(\#S)!}{n!}(v(S \cup \{i\}) - v(S)).$$

There is, however, no guarantee that the Shapley value results in a relevant allocation for our problem. More precisely, there are two potential difficulties associated to the use of the Shapley value as a cost-sharing means. First, when $p < 1$, the surplus-sharing game is not necessarily convex and therefore, the Shapley value may not belong to the core. Second, even if the Shapley value belongs to the core, it does not always satisfy the majority requirement. We analyze these two potential difficulties in this section. Before addressing these issues we first recall how to compute the Shapley value for our game.

4.1 Computation of the Shapley Value

The marginal contributions of the three players are given in the following table. In this table the first column gives the order in which the co-owner meet, and the last three give the marginal contributions of players 1, 2, 3 respectively.

	1	2	3
123	$v^l(1)$	$v^l(12) - v^l(1)$	$v^l(123) - v^l(12)$
132	$v^l(1)$	$v^l(123) - v^l(13)$	$v^l(13) - v^l(1)$
213	$v^l(12) - v^l(2)$	$v^l(2)$	$v^l(123) - v^l(12)$
231	$v^l(123) - v^l(23)$	$v^l(2)$	$v^l(23) - v^l(2)$
312	$v^l(13) - v^l(3)$	$v^l(123) - v^l(13)$	$v^l(3)$
321	$v^l(123) - v^l(23)$	$v^l(23) - v^l(3)$	$v^l(3)$

From there, we can find the Shapley value (in terms of the function v and p) by computing the average marginal contributions for each co-owner:

$$Sh_1 = \frac{pv(1)}{3} + \frac{v(12) - pv(2) + v(13) - pv(3)}{6} + \frac{v(123) - v(23)}{3}, \tag{23}$$

$$Sh_2 = \frac{pv(2)}{3} + \frac{v(12) - pv(1) + v(23) - pv(3)}{6} + \frac{v(123) - v(13)}{3}, \tag{24}$$

$$Sh_3 = \frac{pv(3)}{3} + \frac{v(13) - pv(1) + v(23) - pv(2)}{6} + \frac{v(123) - v(12)}{3}. \tag{25}$$

We obtain the cost-sharing accordingly:

$$Sh_1^c = b_1 - \Big(\frac{pv(1)}{3} + \frac{v(12) - pv(2) + v(13) - pv(3)}{6} + \frac{v(123) - v(23)}{3}\Big), \tag{26}$$

$$Sh_2^c = b_2 - \Big(\frac{pv(2)}{3} + \frac{v(12) - pv(1) + v(23) - pv(3)}{6} + \frac{v(123) - v(13)}{3}\Big), \tag{27}$$

$$Sh_3^c = b_3 - \Big(\frac{pv(3)}{3} + \frac{v(13) - pv(1) + v(23) - pv(2)}{6} + \frac{v(123) - v(12)}{3}\Big). \tag{28}$$

4.2 Shapley value and the core

By definition an allocation of the surplus is in the core of the surplus game if the inequations (7)-(12) are satisfied. It can be shown that the inequations (10)-(12) are always satisfied by the Shapley value. Moreover, the equality (6) is satisfied since the sum of the individual shares of the surplus is equal to $v(123)$ (by construction of the Shapley value). Let us study the remaining inequations (namely inequations (7)-(9)). Using the expressions of the Shapley value, these inequations reduce to:

$$p\big(v(1) + v(2) - 2v(3)\big) \geq 4\big(v(12) - v(123)\big) + v(13) + v(23), \tag{29}$$

$$p\big(v(1) + v(3) - 2v(2)\big) \geq 4\big(v(13) - v(123)\big) + v(12) + v(23), \tag{30}$$

$$p\big(v(2) + v(3) - 2v(1)\big) \geq 4\big(v(23) - v(123)\big) + v(12) + v(13). \tag{31}$$

Proposition 4. *Assume that the following inequations hold:*

$$v(1) \leq v(2) \leq v(3), \tag{A1}$$

$$v(12) \leq v(13) \leq v(23). \tag{A2}$$

Then a sufficient condition for the Shapley value to belong to the core for all p is: $4\big(v(23) - v(123)\big) + v(12) + v(13) \leq 0$.

Assumptions A1 and A2 are certainly satisfied if the improvement consists in installing an elevator. Indeed, in this case, the incremental cost of servicing an additional higher floor is constant, and it is generally much less than the incremental value of this service, which in addition is increasing with the level of the floor served.

The sufficient condition provided in the above Proposition is in fact the necessary and sufficient condition for Shapley value to belong to the core, for all p, when the individual marginal contributions are nil (that is $v(i) = 0$, for all i).

We next study this condition in our example in which a three-storey condo contemplates the installation of an elevator.

A three-storey example. The condition $4\big(v(23) - v(123)\big) + v(12) + v(13) \leq 0$ reduces to:

$$b_2 + b_3 - 2b_1 - 5c_s - 2c_f \leq 0. \tag{32}$$

This inequation, in turn, is satisfied if: $b_1 \in [c_s + c_f, 2c_s + c_f]$, $b_2 \in [2c_s + c_f, 3c_s + c_f]$, and $b_3 \in [3c_s + c_f, 4c_s + 2c_f]$.

Indeed, in that case we have:

$$\begin{aligned} 2b_1 + 5c_s + 2c_f - (b_2 + b_3) &\geq 2c_s + 2c_f + 5c_s + 2c_f - (3c_s + c_f + 4c_s + 2c_f), && (33) \\ &\geq c_f, && (34) \\ &\geq 0. && (35) \end{aligned}$$

Il remains to be shown that the inequations stated in assumption A.2 above are true. This is so if : $b_2 + c_s \leq b_3$. But this inequation can be satisfied since the difference in the upper-bounds of the intervals containing b_2 and b_3 respectively is equal to: $c_s + c_f$.

We now provide an example which shows that the Shapley value might not be in the core.[15] Assume that: $c_s = 1$, $c_f = 5$, $b_1 = 1$, $b_2 = 8$, $b_3 = 11$ and $p = 0$. Then the sufficient condition of Proposition 4 is not satisfied and the Shapley value is not in the core because $x_2 + x_3 = 64/6 < v(23) = 11$. In this particular case where $p = 0$, the condition of Proposition 4 is in fact both necessary and sufficient for the Shapley value to be in the core.

[15] The fact that the Shapley value is not always in the core is also noticed in the paper by Dinar *et al.* (2006). But the model and the results of the present paper are different from Dinar *ibid*'s.

4.3 Shapley value and majority voting

We now rely on the previous section to see if the Shapley sharing can receive an affirmative vote by satisfying a two-third majority requirement.

It turns out that there will almost never an affirmative vote for the Shapley value. To see this, we rely on Proposition 3 and the comments that follow. First of all we can neglect the case where at most two inequations (7)-(9) are strictly satisfied (non-domination is realized only if a particular condition with regard the characteristic function $v(.)$ is satisfied). Let us concentrate on the case where at most one inequation (among (7)-(9)) is strictly satisfied. Consider the allocation given by the equations (14)-(16). Now comparing these equations and the Shapley value for co-owner 1, *i.e.*, (23-25), we see that the allocations coincide only if p takes a particular value, a condition that is almost never satisfied.

5 Conclusion

We have considered a three-storey condo which contemplates the realization of an improvement and we have used cooperative game theory to shed new light on this topic. Taking the Shapley value as a means for cost sharing has the benefit of using the marginal contribution of the co-owners. That leads, however, to two main difficulties. First, the Shapley value may not belong to the core. Second, there is almost never an affirmative vote for the Shapley value.

Although our results are proved by way of a simple example with three co-owners, the ideas we have developped have broad-ranging applications. The results of this paper extend to the n-storey case. In that case, there would be more coalitions who cannot insure an affirmative vote for their project (if the remaining storeys oppose this project). These coalitions would also seek to obtain a judge's approval. But as this approval is uncertain, the characteristic function would more often take expected rather than certain values. It would be more difficult for the Shapley value to belong to the core.

A natural extension of this paper is to study the case where the values of the individual advantages of an improvement are uncertain. This would lead us to take into account both legal and economic uncertainties in the analysis.

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Appendix

Proof of Proposition 1 . Assume first that $p = 1$. Then, since the game is convex, it has a nonempty core (see, *e.g.*, Osborne and Rubinstein, 1994). Now assume that $0 \leq p < 1$. Then, it is easy to see that the core of the game when $p = 1$ is included in the core of the game when $p < 1$. The result follows. $\square$

Proof of Proposition 2. Let us first consider without loss of generality the case where $x_1 + x_2 > v(12)$ and $x_1 + x_3 > v(13)$. Then, since by assumption $x_1 > pv(1)$, there exist $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$, such that $x_1 - \varepsilon_2 - \varepsilon_3 \geq pv(1)$ and such that:

$$x_1 - \varepsilon_3 + x_2 \geq v(12), \tag{36}$$

$$x_1 - \varepsilon_2 + x_3 \geq v(13), \tag{37}$$

and :

$$x_1' = x_1 - \varepsilon_2 - \varepsilon_3, \tag{38}$$

$$x_2' = x_2 + \varepsilon_2, \tag{39}$$

$$x_3' = x_3 + \varepsilon_3. \tag{40}$$

It is easy to see that x' is in the core and dominates x. We observe that If $x_1 = pv(1)$ we cannot build a dominating allocation. $\square$

Proof of Proposition 3. Now consider the case where only one inequation is strictly satisfied. Assume without loss of generality that we have: $x_1 + x_2 > v(12)$. We have also by assumption: $x_1 + x_3 = v(13)$ and $x_2 + x_3 = v(23)$. Assume that we can decrease x_1 by an amount $\triangle$ such that: $x_1 - \triangle + x_2 \geq v(12)$ (this requires that $x_1 > pv(1)$). Necessarily, we must increase x_3 by $\triangle$ to satisfy $x_1 + x_3 = v(13)$. Therefore, there is no room for increasing x_2 and there is thus no majority which vote against x. $\square$

Proof of Proposition 4. We must show that the following inequations hold for all $p \in [0,1]$:

$$p\big(v(1)+v(2)-2v(3)\big) \geq 4\big(v(12)-v(123)\big)+v(13)+v(23), \tag{41}$$

$$p\big(v(1)+v(3)-2v(2)\big) \geq 4\big(v(13)-v(123)\big)+v(12)+v(23), \tag{42}$$

$$p\big(v(2)+v(3)-2v(1)\big) \geq 4\big(v(23)-v(123)\big)+v(12)+v(13). \tag{43}$$

Since we have assume that $v(.)$ is convex, all inequations (41)-(43) are satisfied when $p=1$.[16]

Under assumption A1, we see that the left-hand part of (41) is non-positive, and the left-hand part of (43) is non-negative.

This implies that the right-hand part of (41) is non-positive and then that this inequation is satisfied for all $p \in [0,1]$, since it is true for $p=1$.

As for the remaining inequations, if their right-hand part is non-positive, they are always satisfied. This is because these inequations are true when $p=1$ and this remains so if p decreases.

Under assumption A2, both the right-hand sides of (42) and (43) are non-positive. Indeed, if the above assumption holds and if $4\big(v(23)-v(123)\big)+v(12)+v(13) \leq 0$, we remark that $4\big(v(13)-v(123)\big)+v(12)+v(23) \leq 0$. $\square$

[16] See, *e.g.*, Mas-Colell *et al.* (1995), Proposition 18.AA.1, page 683.