Pairwise Comparison Dynamics for Games with Continuous Strategy Space*

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July 15, 2014

Abstract: This paper studies pairwise comparison dynamics for population games with continuous strategy space. We show that the pairwise comparison dynamic is well-defined if certain mild Lipschitz continuity conditions are satisfied. We establish Nash stationarity and positive correlation for pairwise comparison dynamics. Finally, we prove global convergence and local stability under general deterministic evolutionary dynamics in potential games, and global asymptotic stability under pairwise comparison dynamics in contractive games.

Keywords: Evolutionary dynamics, Population games, Continuous strategy space

JEL Classifications: C72, C73

*I thank my advisor Professor William H. Sandholm for very helpful discussions, comments and suggestions. The comments of an anonymous referee and an anonymous Associate Editor are also sincerely appreciated.

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1 Introduction

Over the past few decades, evolutionary game theory has been an active area of research in economics, biology, computer science, and sociology, and other fields that study interactions among large numbers of participants. However, most studies are restricted to settings in which the strategy space is finite. This restriction limits the use of population games in applications whose strategy spaces are naturally modeled as continuous, including games of timing, effort choice games, bargaining games, and oligopoly games, among others.

The games mentioned above have the same property that agents' actions are chosen from some intervals of real numbers. Another kind of games that involve continuous strategy sets is incomplete information games with a continuum of types, for example, auctions. This paper provides some new insights into an increased scope of the use of population games in applications with continuous strategy settings.[1]

When the strategy set is finite, a population state can be described by a real-valued vector with dimension equal to the cardinality of the strategy set. For the case of continuous strategy space, a population state is described by a probability measure over the strategy space. This introduces technical challenges.

For population games with continuous strategy sets, Bomze (1990, 1991) defines the replicator dynamic in the Banach space of finite signed measures with the variational norm. Bomze (1991) shows that the replicator dynamic is well-defined if certain Lipschitz continuity conditions are satisfied for the mean payoff function. Oechssler and Riedel (2001, 2002) follow this line of research on the replicator dynamic. They show that Bomze's conditions are always satisfied in pairwise encounters if the underlying pairwise payoff function is bounded, and give a more fruitful result on evolutionary stability.

Evolutionary dynamics describe the aggregate consequences of individual agents employing simple myopic rules to decide how to act. Different rules lead to different dynamics. The dynamic that attracts most people to study and is used most often in applications is the replicator dynamic, due to its origin from biology. The replicator dynamic is *imitative* in the sense that, under such dynamic, when an agent receives an opportunity to switch strategies, he chooses a candidate strategy at random according to the population state, i.e., according to how popular that candidate strategy is.

This paper introduces pairwise comparison dynamics for games with continuous strategy space. Unlike the replicator dynamic, pairwise comparison dynamics are *direct*[2] in the sense that a revising agent chooses a candidate strategy at random according to a fixed reference measure; in particular, a strategy's popularity does not influence the probability with which it is chosen as a candidate strategy.[3] Under pairwise comparison dynamics, the revising agent switches to the candidate strategy at a positive rate if and only if its payoff is higher than his current strategy's payoff. In the special case of the Smith (1984) dynamic, the rate is proportional to the difference between the candidate strategy's payoff and the payoff of the agent's current strategy.

[1]For some examples of the application of evolutionary dynamics in continuous strategy games, see Friedman and Ostrov (2013), Hofbauer, Oechssler and Riedel (2009), Hu (2011), Lahkar and Riedel (2013), Louge and Riedel (2012), and Oechssler and Riedel (2001, 2002).

[2]Cf. Sandholm, 2010a, Section 4.3.2.

[3]While the replicator dynamic is not direct but imitative (since the reference measure is not fixed but taken to be the current population state; see Remarks 1 and 2 in Section 2 below), the replicator dynamic is considered in the general framework and analysis of the present paper (e.g., the global convergence and local stability results for potential games apply to the replicator dynamic; see Remarks 4 and 5 in Section 5 below).

In the present paper, we provide a general framework to derive the *mean dynamic* for population games in continuous strategy settings. The framework and derivation are in the same spirit as those using *revision protocols* by Sandholm (2005, 2010a, 2010b) in the finite strategy case. Basically, revision protocols are the rules that individual agents follow to switch strategies. From the mean dynamic, not only pairwise comparison dynamics but also other deterministic evolutionary dynamics (such as the replicator dynamic, the BNN dynamic, and logit dynamics) can be derived.[4] Thus this paper provides some new insights into the modeling of dynamics when the strategy set is continuous.

Oechssler and Riedel (2001, 2002) only consider games in which agents are matched to play a two-player symmetric game. In the present paper, a population game with continuum strategy set $\mathcal{S}$ is defined as a weakly continuous map F from the space of probability measures over $\mathcal{S}$ to the space of bounded continuous functions on $\mathcal{S}$, i.e., F maps from the state space to the space of assignments of payoffs to each strategy. This broadens the study of population games with continuous strategy settings beyond matching to play a two-player symmetric game. We find that under mild Lipschitz continuity conditions, the pairwise comparison dynamic for F is well-defined. We then take a look at matching to play a two-player symmetric game, as considered by Oechssler and Riedel (2001, 2002), as an example (see Example 1), and obtain an existence and uniqueness result for solutions of pairwise comparison dynamics for this class of games.

Pairwise comparison dynamics for games with finite strategy space were first introduced by Sandholm (2010b). Sandholm (2010b) shows that pairwise comparison dynamics in finite strategy case have two nice properties: *Nash stationarity* (NS) and *positive correlation* (PC). These two properties relate the dynamics to the population game, and provide some tools for the analysis of convergence of the dynamics. Nash stationarity (NS) means that the rest points of the dynamic coincide with the set of Nash equilibria. Positive correlation (PC) means that the inner product of the payoff vector and the growth rate vector is positive whenever the dynamic is not at rest. In the present paper, we show that these two properties also hold in the continuous strategy case. Here, we define positive correlation (PC) for the continuous strategy case in a similar fashion by using the natural bilinear functional in the weak topology on the space of finite signed measures as the "inner product" function.

To study stability under evolutionary dynamics, we need to consider "closeness" and "neighborhoods" of population states, which depend on the choice of topology for the space of measures. One way to measure the distances between population states is to use the variational norm, which induces the strong topology. Another way is to use the Prohorov metric, which induces the weak topology. For the reasons outlined in Oechssler and Riedel (2002) and discussed further in Section 4 below, we use the weak topology to study dynamic stability. As a result, our definitions of Lyapunov stability and asymptotic stability, as well as ω-limit points, are defined in terms of the weak topology.

It is natural to ask what economic purpose is served by introducing dynamics for games with continuous strategy space, rather than constructing finite-strategy approximations for such games via discretization on the continuous strategy space, and then working with the corresponding finite-strategy dynamics. This question too comes down to topology. We argue in Section 4 that the appropriate notion of closeness to use for studying convergence and stability for continuous-strategy dynamics is the weak topology on the space of population states, which accounts for distances between the strategies themselves. On the other hand, evolutionary dynamics for finite strategy games are defined with respect to the usual topology on $\mathbb{R}^n$. However, the latter topology cannot

[4] See Remark 1 in Section 2 below.

capture the property that two homogeneous population states are close to each other if and only if their respective strategies are close to each other in the continuous strategy space. Once the discretization of the continuous strategy space is fixed, the distances between states for the approximating finite-strategy dynamic have no connection with the underlying metric on the strategy space. To work with the weak topology, it is most natural to work with continuous strategy spaces directly, rather than attempting to capture the notion of weak convergence by using increasingly fine discretizations.

We then focus on two classes of population games whose definitions we extend from the finite strategy setting to the continuous strategy setting. First, we consider potential games. Evolutionary dynamics for potential games in finite strategy case have been fully studied in Sandholm (2001, 2009). In the present paper, we define potential games for the continuous strategy case. We find that, as in the finite strategy case,[5] the potential function acts as an increasing strict Lyapunov function for dynamics satisfying positive correlation (PC). This allows us to obtain a global convergence result for general deterministic evolutionary dynamics in potential games. We also provide a local stability result for such dynamics in potential games.

Second, we consider contractive games. Contractive games (also known as stable games and negative semidefinite games)[6] and their dynamics, including pairwise comparison dynamics, in finite strategy case have been extensively studied in Hofbauer and Sandholm (2009). In the present paper, we define contractive games for the continuous strategy case. We find that, as in the finite strategy case,[7] contractive games are characterized by *self-defeating externalities* (SDE): when agents revise their strategies, the improvements in the payoffs of strategies to which revising agents are switching are always exceeded by the improvements in the payoffs of strategies which revising agents are abandoning. We show that the set of Nash equilibria is globally asymptotically stable for pairwise comparison dynamics in contractive games.

In the present paper, we focus on pairwise comparison dynamics. But most of our definitions (e.g., the definitions of population games, potential games, contractive games, Nash stationarity (NS), positive correlation (PC), Lyapunov functions, Lyapunov stability, asymptotic stability, and ω-limit points) and some of the results (e.g., the results in Lemmas 2–4 and Remarks 4–5) can certainly be applied to other deterministic evolutionary dynamics, like the replicator dynamic and the BNN dynamic.

In related work, Hofbauer, Oechssler and Riedel (2009) study the BNN dynamic for the continuous strategy case. They show that Nash stationarity (NS) is satisfied for the BNN dynamic, and provide stability results for doubly symmetric games (i.e., potential games generated by matching to play a two-player symmetric game with common interests; see Example 2) and negative semidefinite games. Lahkar and Riedel (2013) study logit dynamics with continuous strategy space, and use these dynamics to study price dispersion. Friedman and Ostrov (2010, 2013) consider population games in which each agent continuously adjusts his strategy at a velocity equals to the payoff gradient, which leads the resulting dynamic to follow a partial differential equation. Perkins and Leslie (2014) study stochastic fictitious play in games with continuous strategy spaces using stochastic approximation to relate the behavior of the stochastic process to a deterministic dynamic for a game with continuous strategy space.

This paper makes the following contributions to the literature of evolutionary dynamics. First,

[5] For the detailed results for evolutionary dynamics in potential games in finite strategy case, see Sandholm (2001).

[6] Cf. Sandholm, 2014, Section 7.2.

[7] For the detailed results for evolutionary dynamics in contractive games in finite strategy case, see Hofbauer and Sandholm (2009).

it extends the general properties[8] of pairwise comparison dynamics as well as global convergence and local stability results from finite strategy settings to continuous strategy settings. Second, it extends the study of evolutionary dynamics with continuous strategy spaces from games generated by pairwise matching to more general population games, which are defined as weakly continuous maps from the state space to the space of assignments of payoffs to each strategy. Third, it provides a microfoundation for deriving the mean dynamic (and thus microfoundations for the replicator dynamic, the BNN dynamic, and logic dynamics, etc.) for population games in continuous strategy settings, and so provides some new insights into the modeling of dynamics when the strategy set is continuous.

The rest of this paper is organized as follows. In Section 2, we define population games and pairwise comparison dynamics for the continuous strategy case. We show that, under mild Lipschitz continuity conditions, the pairwise comparison dynamic is well-defined, i.e., a unique solution exists for the dynamic from every initial strategy distribution. Section 3 establishes Nash stationarity (NS) and positive correlation (PC) for pairwise comparison dynamics. Section 4 discusses the weak and strong topologies on the space of finite signed measures in the context of evolutionary game dynamics. Section 5 defines potential games for the continuous strategy case, and provides a global convergence result as well as a local stability result for general deterministic evolutionary dynamics in potential games. Section 6 defines contractive games for the continuous strategy case, and provides a global asymptotic stability result for pairwise comparison dynamics in contractive games. Section 7 concludes.

2 Population Games and Pairwise Comparison Dynamics

2.1 Population Games and Mean Dynamics

Let $\mathbb{V}$ be a metrizable topological space with metric d, and let $\mathcal{S}$ be a compact convex subset of $\mathbb{V}$. Consider a unit mass of agents, each of whom chooses a pure strategy from $\mathcal{S}$. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathcal{S}$. For example, $\mathbb{V}$ may be $\mathbb{R}$, d the Euclidean metric in $\mathbb{R}$, $\mathcal{S}$ a compact interval in $\mathbb{R}$, and $\mathcal{B}$ the Borel σ-algebra on that interval.

If the strategy set $\mathcal{S}$ were finite, then the state of an evolutionary dynamic at a particular time could be described by a vector in $\mathbb{R}^{|\mathcal{S}|}$.[9] For the continuous strategy case, the state is instead described by a probability measure over $\mathcal{S}$. Denote by $\mathcal{M}_1^+(\mathcal{S})$ the space of probability measures on $(\mathcal{S}, \mathcal{B})$, and by $\mathcal{M}(\mathcal{S})$ the space of finite signed measures.[10] Then $\mathcal{M}(\mathcal{S})$ is a vector space and is the linear span of $\mathcal{M}_1^+(\mathcal{S})$. A *population state* is a distribution over strategies and is described by a probability measure $\mu \in \mathcal{M}_1^+(\mathcal{S})$.

We identify a *population game* with a map

$$F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$$

that is continuous with respect to the weak topology, where $\mathcal{C}_b(\mathcal{S})$ is the space of bounded continuous functions on $\mathcal{S}$ with the supremum norm.[11] We denote by $F_x(\mu)$ the payoff of pure strategy $x \in \mathcal{S}$

[8] I.e., existence and uniqueness of solutions, positive correlation (PC), and Nash stationarity (NS).

[9] $|A|$ denotes the number of elements in set A.

[10] Note that $\mathcal{M}_1^+(\mathcal{S})$ is a closed convex set in $\mathcal{M}(\mathcal{S})$.

[11] A sequence of measures $\mu_n \in \mathcal{M}(\mathcal{S})$ converges weakly to $\mu \in \mathcal{M}(\mathcal{S})$, written $\mu_n \xrightarrow{\text{w}} \mu$, if $\int_{\mathcal{S}} f \, d\mu_n \to \int_{\mathcal{S}} f \, d\mu$ for all $f \in \mathcal{C}_b(\mathcal{S})$. A map $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ is continuous with respect to the weak topology if $F(\mu_n) \to F(\mu)$ (in the supremum norm) for any sequence $\{\mu_n\} \subseteq \mathcal{M}_1^+(\mathcal{S})$ such that $\mu_n \xrightarrow{\text{w}} \mu$. We may call such map F *weakly continuous*.

at population state $\mu \in \mathcal{M}_1^+(\mathcal{S})$, and $F(\mu)$ specifies payoffs at all strategies in $\mathcal{S}$ at state μ. The (*population-weighted*) *average payoff* obtained by the unit mass of agents at state $\mu \in \mathcal{M}_1^+(\mathcal{S})$ is

$$\bar{F}(\mu) = \int_{\mathcal{S}} F_x(\mu)\, \mu(dx).$$

A *deterministic evolutionary dynamic* for population game F on the measurable space $(\mathcal{S}, \mathcal{B})$ is defined by a differential equation on $\mathcal{M}_1^+(\mathcal{S})$:

$$\dot{\mu}(A) = V^F(\mu)(A), \quad \forall A \in \mathcal{B},$$

where $V^F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{M}(\mathcal{S})$ satisfies the following condition:

$$\begin{cases} V^F(\mu)(\mathcal{S}) = 0, \quad \text{and} \\ \forall A \in \mathcal{B},\ V^F(\mu)(A) \geq 0 \text{ whenever } \mu(A) = 0. \end{cases} \tag{FI}$$

The first condition of (FI) says that any change in the population state does not change the population's total mass. The second condition says that any change in the population state does not reduce the mass on sets with measure zero.

The evolutionary process is described as follows. Suppose that the current state is $\mu \in \mathcal{M}_1^+(\mathcal{S})$, and that the current payoff profile over $\mathcal{S}$ is $\pi \in \mathcal{C}_b(\mathcal{S})$. Agents are selected uniformly at random, so that the strategy of the randomly selected agent follows distribution μ, i.e., the probability that the selected agent is using a strategy in $A \in \mathcal{B}$ is $\mu(A)$. The selected agents are given the opportunity to switch strategies. Let $\lambda \in \mathcal{M}_1^+(\mathcal{S})$ be the reference measure that describes the rates at which a revising agent chooses the various candidate strategies.[12] If the current strategy is $x \in \mathcal{S}$ and the candidate strategy is $y \in \mathcal{S}$, then the revising agent switches to strategy y with probability proportional to $\rho_{xy}(\pi)$. The map $\rho : \mathcal{S} \times \mathcal{S} \times \mathcal{C}_b(\mathcal{S}) \to \mathbb{R}_+$ describes the rates at which such switches occur, and thus $\rho_{xy}(\pi)$ is the conditional switch rate from strategy x to strategy y under payoff profile π. The system (λ, ρ) is called the *revision protocol*. Throughout this paper, we assume that $\rho_{xy}(\pi)$ is continuous in x, y and π, and that ρ_{xy} is bounded on bounded sets of payoff profiles π.

The *mean dynamic* with conditional switch rates ρ for population game F is defined as the following differential equation on $\mathcal{M}_1^+(\mathcal{S})$:

$$\dot{\mu}(A) = \int_{z\in\mathcal{S}} \int_{y\in A} \rho_{zy}(F(\mu))\, \lambda(dy)\, \mu(dz) - \int_{z\in\mathcal{S}} \int_{y\in A} \rho_{yz}(F(\mu))\, \mu(dy)\, \lambda(dz), \tag{M}$$

for all $A \in \mathcal{B}$. The first term on the RHS of (M) is the "inflow" of agents into strategies in A at state μ, and the second term is the "outflow" of agents from strategies in A at state μ. Clearly, condition (FI) is satisfied for the differential equation (M).

Remark 1 From the mean dynamic (M), we can derive different deterministic evolutionary dynamics by using different reference measure λ and conditional switch rates ρ. Examples are

- replicator dynamic (cf. Bomze (1990, 1991), Oechssler and Riedel (2001, 2002)) (by putting $\lambda = \mu$ and $\rho_{xy}(\pi) = [\pi(y) - \pi(x)]_+$ into (M)):[13]

$$\dot{\mu}(A) = \int_{y\in A} (F_y(\mu) - \bar{F}(\mu))\, \mu(dy), \quad \forall A \in \mathcal{B};$$

[12] In fact, we may allow λ to be any finite positive measure in $\mathcal{M}(\mathcal{S})$. This is innocuous since it only amounts to a change of speed in the dynamic. Also, here we do not assume λ to be a fixed measure, i.e., we allow λ to vary across time. For example, in deriving the replicator dynamic, we put $\lambda = \mu$ (see the first example under Remark 1). Later, when we define pairwise comparison dynamics, we will assume λ to be a fixed probability measure with full support.

[13] $[a]_+ := \max\{a, 0\}$.

- BNN dynamic (cf. Hofbauer, Oechssler and Riedel (2009)) (by taking λ to be a fixed probability measure with full support and putting $\rho_{xy}(F(\mu),\mu) = [F_y(\mu) - \bar{F}(\mu)]_+$ into (M)):[14]

$$\dot{\mu}(A) = \int_{y\in A} [F_y(\mu) - \bar{F}(\mu)]_+ \,\lambda(dy) - \mu(A) \int_{z\in\mathcal{S}} [F_z(\mu) - \bar{F}(\mu)]_+ \,\lambda(dz), \quad \forall A \in \mathcal{B};$$

- logit dynamic with noise level $\eta > 0$ (cf. Lahkar and Riedel (2013)) (by taking λ to be a fixed probability measure with full support and putting $\rho_{xy}(\pi) = \exp(\eta^{-1}\pi(y))/\int_{z\in\mathcal{S}} \exp(\eta^{-1}\pi(z))\,\lambda(dz)$ into (M)):

$$\dot{\mu}(A) = \frac{\int_{y\in A} \exp(\eta^{-1}F_y(\mu))\,\lambda(dy)}{\int_{z\in\mathcal{S}} \exp(\eta^{-1}F_z(\mu))\,\lambda(dz)} - \mu(A), \quad \forall A \in \mathcal{B}.$$

2.2 Pairwise Comparison Dynamics

To obtain pairwise comparison dynamics, we assume that the conditional switch rates ρ satisfy *sign-preservation* (SP):

$$\operatorname{sgn}(\rho_{xy}(\pi)) = \operatorname{sgn}([\pi(y) - \pi(x)]_+), \quad \forall x, y \in \mathcal{S}. \tag{SP}$$

Sign-preservation (SP) says that for any strategies $x, y \in \mathcal{S}$, the conditional switch rate from x to y is positive if and only if the payoff to y exceeds the payoff to x. Also, we assume that the reference measure λ is a *fixed* probability measure that has full support, so that a revising agent considers each strategy as possible candidate.

The *pairwise comparison dynamic* with conditional switch rates ρ for population game F is defined as the following differential equation on $\mathcal{M}_1^+(\mathcal{S})$:[15]

$$\dot{\mu}(A) = \int_{z\in\mathcal{S}} \int_{y\in A} \rho_{zy}(F(\mu))\,\lambda(dy)\,\mu(dz) - \int_{z\in\mathcal{S}} \int_{y\in A} \rho_{yz}(F(\mu))\,\mu(dy)\,\lambda(dz), \tag{PCD}$$

for all $A \in \mathcal{B}$. The formula in (M) and (PCD) are the same. The only difference between (M) and (PCD) is the assumptions on the reference measure λ and the conditional switch rates ρ. If we assume that $\rho_{xy}(\pi) = [\pi(y) - \pi(x)]_+$, then (PCD) becomes the Smith dynamic:

$$\dot{\mu}(A) = \int_{z\in\mathcal{S}} \int_{y\in A} [F_y(\mu) - F_z(\mu)]_+ \,\lambda(dy)\,\mu(dz) - \int_{z\in\mathcal{S}} \int_{y\in A} [F_z(\mu) - F_y(\mu)]_+ \,\mu(dy)\,\lambda(dz), \tag{SD}$$

for all $A \in \mathcal{B}$.

In continuous strategy settings, it could be more likely for revising agents to switch to strategies close to their current strategy (under the metric d) rather than to distant strategies. One example of conditional switch rates ρ that capture this property is $\rho_{xy}(\pi) = w(d(x,y))[\pi(y) - \pi(x)]_+$, where $w : \mathbb{R}_+ \to \mathbb{R}_+$ is a *weight function* that is continuous, decreasing, bounded, and bounded away from zero. All results in this paper apply to this special type of ρ.

[14] Note that, for the BNN dynamic, the conditional switch rates ρ is a map from $\mathcal{S} \times \mathcal{S} \times \mathcal{C}_b(\mathcal{S}) \times \mathcal{M}_1^+(\mathcal{S})$ to $\mathbb{R}_+$.

[15] Note that by setting $\mathcal{S} = \{1, \ldots, n\}$, $A = \{i\}$, $\mu(\{i\}) = x_i$, and letting λ be the uniform probability measure (i.e., $\lambda(\{j\}) = 1/n$ for all $j = 1, \ldots, n$) into (PCD), we get the usual formulation of pairwise comparison dynamics for the finite strategy case.

Remark 2 For imitative dynamics (e.g., the replicator dynamic), the reference measure λ is taken to be the population state μ. This means that when an agent receives an opportunity to switch strategies, he chooses a candidate strategy at random according to the population state, i.e., according to the popularity of the candidate strategy. The main difference between imitative dynamics and direct dynamics (e.g., pairwise comparison dynamics and the BNN dynamic) is that for direct dynamics the reference measure is fixed, but for imitative dynamics the reference measure is the population state and so evolves with the dynamics. Since population states in general are not of full support, it follows that imitative dynamics do not satisfy Nash stationarity (NS).[16]

Before discussing any properties of pairwise comparison dynamics, we first study under what conditions those dynamics are well-defined, i.e., solutions for the dynamics exist and are unique. Consider the variational norm on $\mathcal{M}(\mathcal{S})$ defined by

$$\|\varphi\| := \sup_{g} \Big| \int g \, d\varphi \Big|, \tag{1}$$

where the sup is taken over all measurable functions $g : \mathcal{S} \to \mathbb{R}$ bounded by 1 (i.e., $\sup_{s\in\mathcal{S}} |g(s)| \leq 1$). Endowed with the variational norm, $(\mathcal{M}(\mathcal{S}), \|\cdot\|)$ is a Banach space. We will use the following result from functional analysis to prove existence and uniqueness of solutions.

Fact 1 (cf. Zeidler, 1986, Corollary 3.9)[17] *Consider the ODE*

$$\dot{\psi}(t) = V(\psi(t)), \quad \psi(0) = \psi_0 \tag{2}$$

in a Banach space. Suppose that $V(\cdot)$ is bounded and is Lipschitz continuous. Then, a unique solution of the above ODE exists on $[0, \infty)$.

The derivative in (2) is defined with respect to the variational norm (1). That is, $\dot{\psi}(t) \in \mathcal{M}(\mathcal{S})$ is defined by the limit

$$\lim_{\tau\to 0} \left\| \frac{\psi(t+\tau) - \psi(t)}{\tau} - \dot{\psi}(t) \right\| = 0,$$

provided that this limit exists.

The following theorem specifies conditions that guarantee the existence of a unique forward solution from any initial condition for pairwise comparison dynamics. Moreover, under such conditions, solutions to the dynamic are continuous in their initial conditions.

Theorem 1 (Existence and Uniqueness of Solutions for Pairwise Comparison Dynamics for Population Games) *Let $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ be a population game, and let $\mu_0 = \mu(0) \in \mathcal{M}_1^+(\mathcal{S})$. Suppose that there is a map $\tilde{F} : \mathcal{M}(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ such that $\tilde{F}(\mu) = F(\mu)$ for all $\mu \in \mathcal{M}_1^+(\mathcal{S})$ (i.e., $\tilde{F}$ is an extension of F to $\mathcal{M}(\mathcal{S})$), and $\tilde{F}$ and ρ satisfy the following condition: there exist constants $0 \leq K, M < \infty$ such that for any $\psi, \xi \in \mathcal{M}(\mathcal{S})$,*[18]

$$\|\psi\|, \|\xi\| \leq 2 \;\Rightarrow\; \begin{cases} \sup_{y,z} |\rho_{yz}(\tilde{F}(\psi)) - \rho_{yz}(\tilde{F}(\xi))| \leq K\|\psi - \xi\| \\ \sup_{y,z} |\rho_{yz}(\tilde{F}(\psi))| \leq M. \end{cases} \tag{A}$$

[16] For the definition of Nash stationarity (NS), see Section 3.
[17] See also Oechssler and Riedel, 2001, Theorem 1.
[18] We emphasize that K, M are independent of ψ and ξ.

Then the pairwise comparison dynamic with ρ for F is well-defined, i.e., there exists a unique solution $\mu(t) \in \mathcal{M}_1^+(\mathcal{S})$ of the differential equation (PCD) *for $t \in [0, \infty)$. Furthermore, solutions to the dynamic are continuous in their initial conditions.*

Proof. First, we show existence and uniqueness of solutions. We follow the strategy for proving existence and uniqueness of solutions for the replicator dynamic in the continuous strategy case from Oechssler and Riedel (2001).[19] Define $V : \mathcal{M}(\mathcal{S}) \to \mathcal{M}(\mathcal{S})$ by

$$V(\mu)(A) := \int_{z\in\mathcal{S}} \int_{y\in A} \rho_{zy}(\tilde{F}(\mu))\,\lambda(dy)\,\mu(dz) - \int_{z\in\mathcal{S}} \int_{y\in A} \rho_{yz}(\tilde{F}(\mu))\,\mu(dy)\,\lambda(dz) \tag{3}$$

for $\mu \in \mathcal{M}(\mathcal{S})$ and $A \in \mathcal{B}$. Note that, if $\mu \in \mathcal{M}_1^+(\mathcal{S})$, then $V(\mu)(A)$ coincides with the RHS of (PCD) because $\tilde{F}(\mu) = F(\mu)$ for all $\mu \in \mathcal{M}_1^+(\mathcal{S})$. Since $V(\cdot)$ is not necessarily bounded and Lipschitz continuous on $(\mathcal{M}(\mathcal{S}), \|\cdot\|)$, we cannot directly apply Fact 1. Instead, we construct an auxiliary function $\tilde{V}(\cdot)$ (defined later) which has these properties and coincides with $V(\cdot)$ on $\mathcal{M}_1^+(\mathcal{S})$ (and hence coincides with the RHS of (PCD) on $\mathcal{M}_1^+(\mathcal{S})$). Then by Fact 1, the ODE

$$\dot{\mu}(t) = \tilde{V}(\mu(t)), \quad \mu(0) = \mu_0 \tag{4}$$

has a unique solution $(\mu(t))$. Since $\mu(0) = \mu_0 \in \mathcal{M}_1^+(\mathcal{S})$ and condition (FI) is satisfied, $\mu(t)$ never leaves $\mathcal{M}_1^+(\mathcal{S})$, which implies that $(\mu(t))$ also solves the differential equation (PCD) on $\mathcal{M}_1^+(\mathcal{S})$.

So, it remains to find $\tilde{V}(\cdot)$ such that $\tilde{V}(\cdot)$ is bounded and Lipschitz continuous on $(\mathcal{M}(\mathcal{S}), \|\cdot\|)$, and coincides with $V(\cdot)$ on $\mathcal{M}_1^+(\mathcal{S})$. Let $\tilde{V} : \mathcal{M}(\mathcal{S}) \to \mathcal{M}(\mathcal{S})$ be defined by

$$\tilde{V}(\mu) := [2 - \|\mu\|]_+ V(\mu). \tag{5}$$

Then $\tilde{V}(\mu)$ is zero for $\|\mu\| \geq 2$. Also, $\tilde{V}(\cdot)$ is bounded and coincides with $V(\cdot)$ on $\mathcal{M}_1^+(\mathcal{S})$ because probability measures have norm 1. It remains to show that $\tilde{V}(\cdot)$ is Lipschitz continuous, which is done in Lemma 5 in Appendix A.1.

Next, we show the continuity of solutions in their initial conditions. By Lemma 5, $\tilde{V}(\cdot)$ is Lipschitz continuous. Denote by $\tilde{K}$ the Lipschitz constant of $\tilde{V}(\cdot)$. Gronwall's lemma (see Zeidler, 1986, Propositions 3.10 and 3.11) implies that, if $\mu(t)$ and $\psi(t)$ are solutions to the ODE (4) with initial conditions μ_0 and ψ_0 respectively, then

$$\|\mu(t) - \psi(t)\| \leq e^{\tilde{K}t} \|\mu_0 - \psi_0\|.$$

Since the solution of ODE (4) is also the solution to the differential equation (PCD), the result follows.

□

[19] Note that the class of games studied in Oechssler and Riedel (2001) is restricted to matching to play a two-player symmetric game, i.e., the class of games that we will study in Example 1. See also Hofbauer, Oechssler and Riedel (2009) for the proof of existence and uniqueness of solutions for the BNN dynamic in the continuous strategy case. Same as Oechssler and Riedel (2001), their studies are restricted to matching to play a two-player symmetric game. The proofs of existence and uniqueness of solutions in Oechssler and Riedel (2001), Hofbauer, Oechssler and Riedel (2009), and the present paper are similar. The main difference is in the work to show $V(\psi)$ is Lipschitz for $\|\psi\| \leq 2$ (and hence $\tilde{V}(\cdot)$ is Lipschitz) because $V(\cdot)$ (and hence $\tilde{V}(\cdot)$) are different for different dynamics; see Lemma 5 in Appendix A.1.

Remark 3 A sufficient condition for (A) to hold is that $\tilde{F}(\psi)$ is Lipschitz continuous for $\|\psi\| \le 2$,[20] and ρ is Lipschitz continuous in the payoff profile argument. Note that, when $\rho_{yz}(\tilde{F}(\psi)) = [\tilde{F}_z(\psi) - \tilde{F}_y(\psi)]_+$, i.e., in the special case of the Smith dynamic, condition (A) becomes: there exist constants $0 \le K, M < \infty$ such that for any $\psi, \xi \in \mathcal{M}(\mathcal{S})$,

$$\|\psi\|, \|\xi\| \le 2 \;\Rightarrow\; \begin{cases} \sup\limits_{y,z} \left| [\tilde{F}_z(\psi) - \tilde{F}_y(\psi)]_+ - [\tilde{F}_z(\xi) - \tilde{F}_y(\xi)]_+ \right| \le K\|\psi - \xi\| \\ \sup\limits_{y,z} \; [\tilde{F}_z(\psi) - \tilde{F}_y(\psi)]_+ \le M. \end{cases} \tag{A$'$}$$

A sufficient condition for (A$'$) to hold is that $\tilde{F}(\psi)$ is Lipschitz continuous for $\|\psi\| \le 2$.

Example 1 *Matching to Play a Two-player Symmetric Game.* Consider a unit mass of agents who are matched to play a two-player symmetric game with continuous strategy set $\mathcal{S}$, which is convex and compact. Each pair of agents is matched exactly once. Let $h : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ be the single match payoff function, which is bounded and continuous. That is, $h(x, y)$ is the single match payoff of an agent playing pure strategy $x \in \mathcal{S}$ against an opponent playing pure strategy $y \in \mathcal{S}$. Then the *average payoff* of pure strategy $x \in \mathcal{S}$ at state $\mu \in \mathcal{M}_1^+(\mathcal{S})$ is[21]

$$F_x(\mu) = \int_{\mathcal{S}} h(x, y)\, \mu(dy). \tag{6}$$

Since h is bounded and continuous, we have

i) $\mu \mapsto F_x(\mu)$ is continuous with respect to the weak topology,[22] and

ii) $x \mapsto F_x(\mu)$ is bounded and continuous.

$F.(\mu)$ is the payoff profile over $\mathcal{S}$ at population state $\mu \in \mathcal{M}_1^+(\mathcal{S})$. We may just denote it as $F(\mu)$ and consider F to be a map from $\mathcal{M}_1^+(\mathcal{S})$ to $\mathcal{C}_b(\mathcal{S})$. Thus, the continuous (with respect to the weak topology) map $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ defines a population game.[23]

If we define $\tilde{F}$ by

$$\tilde{F}_x(\mu) := \int_{\mathcal{S}} h(x, y)\, \mu(dy) \tag{7}$$

for $\mu \in \mathcal{M}(\mathcal{S})$, then $\tilde{F}$ is an extension of F to $\mathcal{M}(\mathcal{S})$. So, by Theorem 1, solutions exist and are unique if condition (A) holds.

Note that $\tilde{F} : \mathcal{M}(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ defined by (7) is Lipschitz continuous because, for any $\psi, \xi \in \mathcal{M}(\mathcal{S})$,

$$|\tilde{F}_x(\psi) - \tilde{F}_x(\xi)| = \left| \int_{\mathcal{S}} h(x, y)\, (\psi - \xi)(dy) \right| \le \|h\|_\infty \|\psi - \xi\|, \quad \forall x \in \mathcal{S},$$

[20] Note that the Lipschitz continuity of $\tilde{F}(\psi)$ for $\|\psi\| \le 2$ implies that $\tilde{F}(\psi)$ is uniformly bounded for $\|\psi\| \le 2$.

[21] Instead of assuming a deterministic complete matching, one can alternatively assume that each agent is matched randomly with another agent in the population. Then $F_x(\mu)$ is the *expected payoff* of an agent playing pure strategy $x \in \mathcal{S}$ at state $\mu \in \mathcal{M}_1^+(\mathcal{S})$.

[22] Note that continuity with respect to the weak topology is a stronger condition than continuity with respect to the strong topology. For the meanings of the weak and strong topologies, see Section 4.

[23] Note that, by putting $\mathcal{S} = \{1, \ldots, n\}$ and $h(x, y) = A_{ij}$, we get the matching to play a two-player symmetric game for the finite strategy case. $A = (A_{ij})_{i,j=1}^n$ is the single match payoff matrix and the population game is the continuous map $F : \Delta \to \mathbb{R}^n$ defined by $F(x) = Ax$ for $x \in \Delta$, where Δ is the simplex in $\mathbb{R}^n$.

which implies

$$\|\tilde{F}(\psi) - \tilde{F}(\xi)\|_\infty = \sup_{x\in\mathcal{S}} |\tilde{F}_x(\psi) - \tilde{F}_x(\xi)| \leq \|h\|_\infty \|\psi - \xi\|.$$

Hence, a sufficient condition for (A) to hold is that ρ is Lipschitz continuous in the payoff profile argument. Since $\tilde{F}(\psi)$ is Lipschitz continuous, from the discussion in Remark 3, the Smith dynamic for F is always well-defined.

◇

3 Nash Stationarity and Positive Correlation

In the finite strategy case, pairwise comparison dynamics (including the Smith dynamic) have two nice properties: *Nash stationarity* (NS) and *positive correlation* (PC). In this section, we will show that these two properties also hold in the continuous strategy case.

Nash stationarity (NS) means that the rest points of the dynamic coincide with the set of Nash equilibria. Formally, Nash stationarity (NS) for the continuous strategy case is defined as follows.

Denote by $S(\mu)$ the support of measure $\mu \in \mathcal{M}(\mathcal{S})$.[24] A population state $\mu^* \in \mathcal{M}_1^+(\mathcal{S})$ is a *Nash equilibrium* (NE) of population game F if the following condition holds:

$$F_y(\mu^*) \leq F_z(\mu^*), \quad \forall z \in S(\mu^*),\ \forall y \in \mathcal{S}. \tag{NE}$$

In words, a population state $\mu^* \in \mathcal{M}_1^+(\mathcal{S})$ is a Nash equilibrium if at state μ^*, the payoffs to the strategies in the support of μ^* are no less than the payoffs to any other strategies in $\mathcal{S}$. Equivalently, μ^* is a Nash equilibrium of F if and only if

$$[F_y(\mu^*) - F_z(\mu^*)]_+ = 0, \quad \forall z \in S(\mu^*),\ \forall y \in \mathcal{S}. \tag{8}$$

We say the dynamic $\dot{\mu} = V^F(\mu)$ for population game F satisfies *Nash stationarity* (NS) if

$$\mu \text{ is a NE of } F \quad \Longleftrightarrow \quad V^F(\mu) = 0 \ \text{ (i.e., } V^F(\mu)(A) = 0 \text{ for all } A \in \mathcal{B}\text{)}. \tag{NS}$$

That is, the dynamic $\dot{\mu} = V^F(\mu)$ for F satisfies Nash stationarity (NS) if every Nash equilibrium of F is a rest point for the dynamic and vice versa.

In the finite strategy case, positive correlation (PC) means that the inner product of the payoff vector and the growth rate vector (the direction of motion) is positive whenever the dynamic is not at rest. We define positive correlation (PC) for the continuous strategy case in a similar fashion.

Since we are dealing with the weak topology on the space of finite signed measures, there is a natural bilinear functional $\langle\cdot,\cdot\rangle : \mathcal{C}_b(\mathcal{S}) \times \mathcal{M}(\mathcal{S}) \to \mathbb{R}$, which is defined by $\langle g, \nu\rangle = \int_{\mathcal{S}} g\, d\nu$, that we can use as the "inner product" of the payoff profile $F(\mu)$ and the direction of motion $V^F(\mu)$. We say the dynamic $\dot{\mu} = V^F(\mu)$ for F satisfies *positive correlation* (PC) if

$$V^F(\mu) \neq 0 \quad \Rightarrow \quad \langle F(\mu), V^F(\mu)\rangle \equiv \int_{\mathcal{S}} F_x(\mu)\, V^F(\mu)(dx) > 0. \tag{PC}$$

[24] For positive measure μ in $\mathcal{M}(\mathcal{S})$, its support $S(\mu)$ is defined as the largest subset of $\mathcal{S}$ for which every open neighborhood of every point on this subset has positive μ-measure, i.e., $S(\mu) := \{x \in \mathcal{S} : \mu(\mathfrak{U}) > 0 \text{ for each neighborhood } \mathfrak{U} \text{ of } x\}$. Note that $S(\mu)$ must be closed. For signed measure μ in $\mathcal{M}(\mathcal{S})$, write $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are positive measures that form the Jordan decomposition of μ, then the support of μ is defined as the union of the supports of μ^+ and μ^-, i.e., $S(\mu) := S(\mu^+) \cup S(\mu^-)$.

Intuitively, positive correlation (PC) requires that the covariance between strategies' payoffs and growth rates is positive under the uniform probability distribution on strategies whenever the dynamic is not at rest. To see this, suppose $\mathcal{S}$ is a compact convex subset of $\mathbb{R}^n$, fix $\mu \in \mathcal{M}_1^+(\mathcal{S})$. Let $\mathcal{U}$ be the uniform probability measure over $\mathcal{S}$. Suppose that the Radon-Nikodym derivative $\frac{dV^F(\mu)}{d\mathcal{U}}$ exists. This requires the signed measure $V^F(\mu)$ to have a density $\frac{dV^F(\mu)}{d\mathcal{U}}$ over $\mathcal{S}$. The density function $\frac{dV^F(\mu)}{d\mathcal{U}}$ is the (relative) growth rate profile over $\mathcal{S}$ at state μ. Then the expected payoff at state μ with respect to the uniform probability distribution is

$$\mathbb{E}_{\mathcal{U}}[F(\mu)] = \int_{\mathcal{S}} F_x(\mu)\mathcal{U}(dx),$$

and the covariance between strategies' payoffs and growth rates under the uniform probability distribution is

$$\begin{aligned}
\mathrm{Cov}_{\mathcal{U}}(F(\mu), \frac{dV^F(\mu)}{d\mathcal{U}}) &= \int_{\mathcal{S}} F_x(\mu)\frac{dV^F(\mu)}{d\mathcal{U}}(x)\mathcal{U}(dx) - \Big(\int_{\mathcal{S}} F_x(\mu)\mathcal{U}(dx)\Big)\Big(\int_{\mathcal{S}} \frac{dV^F(\mu)}{d\mathcal{U}}(x)\mathcal{U}(dx)\Big) \\
&= \int_{\mathcal{S}} F_x(\mu)\, V^F(\mu)(dx) - \Big(\int_{\mathcal{S}} F_x(\mu)\mathcal{U}(dx)\Big)\Big(\int_{\mathcal{S}} V^F(\mu)(dx)\Big) \\
&= \langle F(\mu), V^F(\mu)\rangle - \mathbb{E}_{\mathcal{U}}[F(\mu)]\cdot V^F(\mu)(\mathcal{S}) \\
&= \langle F(\mu), V^F(\mu)\rangle, \qquad (9)
\end{aligned}$$

which is positive if $V^F(\mu) \neq 0$ by (PC).[25]

The following proposition shows that Nash stationarity (NS) and positive correlation (PC) are satisfied by any pairwise comparison dynamics.

Proposition 1 (Nash Stationarity and Positive Correlation for Pairwise Comparison Dynamics) *Every pairwise comparison dynamic satisfies Nash stationarity* (NS) *and positive correlation* (PC).

The proof of Proposition 1 relies on the following lemma:

Lemma 1 *Let $\dot{\mu} = V^F(\mu)$ be the pairwise comparison dynamic for population game F. Then the following are equivalent:*

a) $V^F(\mu) = 0$, *i.e.*, $V^F(\mu)(A) = 0$ *for all* $A \in \mathcal{B}$;

b) $\langle F(\mu), V^F(\mu)\rangle \equiv \int_{\mathcal{S}} F_x(\mu)\, V^F(\mu)(dx) = 0$;

c) μ *is a NE of* F.

Proof. First, we claim the following:
Claim: For any $\mu \in \mathcal{M}_1^+(S)$,

$$\int_{\mathcal{S}} F_x(\mu)\, V^F(\mu)(dx) = \int_{x\in\mathcal{S}} \int_{z\in\mathcal{S}} [F_x(\mu) - F_z(\mu)]\rho_{zx}(F(\mu))\, \mu(dz)\, \lambda(dx).$$

[25] Note that the last equality of (9) is due to $V^F(\mu)(\mathcal{S}) = 0$ by condition (FI).

Let $V^F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{M}(\mathcal{S})$ be defined by the RHS of (PCD). We define $V^I : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{M}(\mathcal{S})$ and $V^O : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{M}(\mathcal{S})$ by

$$V^I(\mu)(A) := \int_{y\in A}\int_{z\in\mathcal{S}} \rho_{zy}(F(\mu))\,\mu(dz)\,\lambda(dy), \tag{10}$$

$$V^O(\mu)(A) := \int_{y\in A}\int_{z\in\mathcal{S}} \rho_{yz}(F(\mu))\,\lambda(dz)\,\mu(dy), \tag{11}$$

for $\mu \in \mathcal{M}_1^+(\mathcal{S})$ and $A \in \mathcal{B}$. Then (PCD) can be written as

$$\dot\mu = V^F(\mu) = V^I(\mu) - V^O(\mu).$$

$V^I(\mu)(A)$ is the "inflow" of agents into strategies in A at state μ, and $V^O(\mu)(A)$ is the "outflow" of agents from strategies in A at state μ.

Let $\mu \in \mathcal{M}_1^+(S)$. By (10), we have $V^I(\mu) \ll \lambda$.[26] So the Radon-Nikodym derivative $\frac{dV^I(\mu)}{d\lambda}$ exists and

$$\frac{dV^I(\mu)}{d\lambda} = \int_{z\in\mathcal{S}} \rho_{z\cdot}(F(\mu))\,\mu(dz). \tag{12}$$

Similarly, by (11), we have $V^O(\mu) \ll \mu$. So the Radon-Nikodym derivative $\frac{dV^O(\mu)}{d\mu}$ exists and

$$\frac{dV^O(\mu)}{d\mu} = \int_{z\in\mathcal{S}} \rho_{\cdot z}(F(\mu))\,\lambda(dz). \tag{13}$$

So, we have

$$\begin{aligned}
&\int_{\mathcal{S}} F_x(\mu)\,V^F(\mu)(dx)\\
=\ &\int_{\mathcal{S}} F_x(\mu)\,V^I(\mu)(dx) - \int_{\mathcal{S}} F_x(\mu)\,V^O(\mu)(dx)\\
=\ &\int_{\mathcal{S}} F_x(\mu)\frac{dV^I(\mu)}{d\lambda}(x)\,\lambda(dx) - \int_{\mathcal{S}} F_x(\mu)\frac{dV^O(\mu)}{d\mu}(x)\,\mu(dx)\\
=\ &\int_{x\in\mathcal{S}}\int_{z\in\mathcal{S}} F_x(\mu)\rho_{zx}(F(\mu))\,\mu(dz)\,\lambda(dx) - \int_{x\in\mathcal{S}}\int_{z\in\mathcal{S}} F_x(\mu)\rho_{xz}(F(\mu))\,\lambda(dz)\,\mu(dx)\\
=\ &\int_{x\in\mathcal{S}}\int_{z\in\mathcal{S}} F_x(\mu)\rho_{zx}(F(\mu))\,\mu(dz)\,\lambda(dx) - \int_{z\in\mathcal{S}}\int_{x\in\mathcal{S}} F_z(\mu)\rho_{zx}(F(\mu))\,\lambda(dx)\,\mu(dz)\\
=\ &\int_{x\in\mathcal{S}}\int_{z\in\mathcal{S}} [F_x(\mu) - F_z(\mu)]\rho_{zx}(F(\mu))\,\mu(dz)\,\lambda(dx).
\end{aligned}$$

Hence, the Claim is proved.

Now, we prove the lemma.

(a)⇒(b): Trivial.

[26]For any signed measure φ and any positive measure ν, we say φ is absolutely continuous with respect to ν, written $\varphi \ll \nu$, if for any $A \in \mathcal{B}$, $\nu(A) = 0$ implies $\varphi(A) = 0$. Note that the definition of absolute continuity requires ν to be a positive measure.

(b)⇒(c): $\langle F(\mu), V^F(\mu)\rangle \equiv \int_{\mathcal{S}} F_x(\mu)\, V^F(\mu)(dx) = 0$. By the Claim, this implies

$$\int_{x\in\mathcal{S}}\int_{z\in\mathcal{S}} [F_x(\mu) - F_z(\mu)]\rho_{zx}(F(\mu))\,\mu(dz)\,\lambda(dx) = 0.$$

Since λ has full support and $[F_x(\mu) - F_z(\mu)]\rho_{zx}(F(\mu)) \geq 0$ for any $x, z \in \mathcal{S}$, by continuity, we have

$$\int_{z\in\mathcal{S}} [F_x(\mu) - F_z(\mu)]\rho_{zx}(F(\mu))\,\mu(dz) = 0, \quad \forall x \in \mathcal{S}.$$

By continuity again, we have

$$[F_x(\mu) - F_z(\mu)]\rho_{zx}(F(\mu)) = 0, \quad \forall z \in S(\mu),\ \forall x \in \mathcal{S},$$

which implies

$$[F_x(\mu) - F_z(\mu)]_+ = 0, \quad \forall z \in S(\mu),\ \forall x \in \mathcal{S}.$$

This is precisely the condition for Nash equilibrium (see (8)). So, μ is a NE of F.
(c)⇒(a): Let $A \in \mathcal{B}$. $V^F(\mu)(A)$ can be written as

$$V^F(\mu)(A) = \int_{y\in A}\int_{z\in\mathcal{S}} \rho_{zy}(F(\mu))\,\mu(dz)\,\lambda(dy) - \int_{y\in\mathcal{S}}\int_{z\in A} \rho_{zy}(F(\mu))\,\mu(dz)\,\lambda(dy). \tag{14}$$

Note that by sign-preservation (SP) and (8), we have for any ρ of the pairwise comparison dynamic for F, μ^* is a NE of F if and only if

$$\rho_{zy}(F(\mu^*)) = 0, \quad \forall z \in S(\mu^*),\ \forall y \in \mathcal{S}. \tag{15}$$

So, when μ is a NE, both terms on the RHS of (14) are zero. Hence, $V^F(\mu)(A) = 0$ for all $A \in \mathcal{B}$.

□

Proof of Proposition 1. Nash stationarity (NS) follows from Lemma 1 since (a) and (c) are equivalent.

To show positive correlation (PC), it is equivalent to show

$$\int_{\mathcal{S}} F_x(\mu)\, V^F(\mu)(dx) \geq 0, \quad \forall \mu \in \mathcal{M}_1^+(\mathcal{S}), \tag{16}$$

and

$$\int_{\mathcal{S}} F_x(\mu)\, V^F(\mu)(dx) = 0 \quad \Rightarrow \quad V^F(\mu) = 0. \tag{17}$$

Let $\mu \in \mathcal{M}_1^+(\mathcal{S})$. Using the Claim and the fact that $[F_x(\mu) - F_z(\mu)]\rho_{zx}(F(\mu)) \geq 0$ for any $x, z \in \mathcal{S}$, we have

$$\int_{\mathcal{S}} F_x(\mu)\, V^F(\mu)(dx) = \int_{x\in\mathcal{S}}\int_{z\in\mathcal{S}} [F_x(\mu) - F_z(\mu)]\rho_{zx}(F(\mu))\,\mu(dz)\,\lambda(dx) \geq 0.$$

Hence, (16) holds. Also, (17) holds by Lemma 1 since (a) and (b) are equivalent.

□

4 Weak Topology and Strong Topology

In order to study stability under evolutionary dynamics, we need to consider "closeness" and "neighborhoods" of population states (e.g., what it means for a population state $\mu \in \mathcal{M}_1^+(\mathcal{S})$ to be close to another population state $\nu \in \mathcal{M}_1^+(\mathcal{S})$), which depend on the choice of topology for the space of measures. See Oechssler and Riedel (2002) for an extensive discussion on these issues. We review some of the points therein below for our use.

Recall the variational norm on $\mathcal{M}(\mathcal{S})$ defined in Section 2. The topology induced by the variational norm is called the strong topology. The strong topology has the advantage that it turns the vector space $\mathcal{M}(\mathcal{S})$ into a Banach space, as we have seen in Section 2. For any two probability measures $\mu, \nu \in \mathcal{M}_1^+(\mathcal{S})$, the distance between them under the variational norm is[27]

$$\|\mu - \nu\| = 2 \sup_{A \in \mathcal{B}} |\mu(A) - \nu(A)|.$$

The variational norm is a very strong measure of distance. For example, under the variational norm, a population state $\mu \in \mathcal{M}_1^+(\mathcal{S})$ is ε-close to a homogeneous population state δ_x ($x \in \mathcal{S}$) only if μ places at least mass $1 - \varepsilon$ on x. Also, the distance between two different homogeneous population states δ_x, δ_u ($x, u \in \mathcal{S}$ and $x \neq u$) is always maximal in the strong topology since $\|\delta_x - \delta_u\| = 2$.

So, if we use the strong topology to study dynamic stability, we only consider perturbations to population states by a (possibly large) change of strategic play from a small fraction of players (e.g., change from δ_x to $(1-\varepsilon)\delta_x + \varepsilon\delta_u$ with $x \neq u$ and ε small). However, in applications, one would also like to consider perturbations to population states by a small change of strategic play from a large fraction of players (e.g., change from δ_x to δ_u with $d(x, u) < \varepsilon$). To capture this second kind of perturbation, we need to use the weak topology.

The weak topology is related to weak convergence of measures. The weak topology on $\mathcal{M}(\mathcal{S})$ is the coarsest topology (i.e., the topology with the fewest open sets) on $\mathcal{M}(\mathcal{S})$ such that $\mu \mapsto \int_{\mathcal{S}} f \, d\mu$ is continuous for all $f \in \mathcal{C}_b(\mathcal{S})$.[28] Suppose that $\mathcal{S}$ is separable. Then the weak topology on $\mathcal{M}_1^+(\mathcal{S})$ is metrized by the Prohorov metric κ, which is defined by

$$\kappa(\mu, \nu) := \inf\{\varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon, \ \forall A \in \mathcal{B}\},$$

where $A^\varepsilon := \{x \in \mathcal{S} : d(x, y) < \varepsilon \text{ for some } y \in A\}$.[29] Specifically, $\mu_n \in \mathcal{M}_1^+(\mathcal{S})$ converges weakly to $\mu \in \mathcal{M}_1^+(\mathcal{S})$ if and only if $\kappa(\mu_n, \mu) \to 0$. In other words, weak convergence and convergence in the Prohorov metric are equivalent for separable $\mathcal{S}$.

The weak topology allows us to consider the second kind of perturbation described above. If $\mu = (1-\varepsilon)\delta_x + \varepsilon\delta_u$ with $0 \leq \varepsilon \leq 1$, then $\kappa(\mu, \delta_x) = \min\{\varepsilon, d(x, u)\}$. In particular, the distance between two homogeneous population states in the weak topology agrees with the underlying metric d in the continuous strategy space $\mathcal{S}$, i.e., $\kappa(\delta_x, \delta_u) = d(x, u)$, when x and u are close to each other in $\mathcal{S}$. Also, the weak topology has the advantage that $\mathcal{M}_1^+(\mathcal{S})$ is compact in the weak topology (but is not so in the strong topology).[30] The compactness of $\mathcal{M}_1^+(\mathcal{S})$ is important for two reasons. First,

[27] Cf. Shiryaev, 1995, p. 360.

[28] Cf. Ekeland and Témam, 1999, p. 6.

[29] Cf. Billingsley, 1999, p. 72–73.

[30] The compactness of $\mathcal{M}_1^+(\mathcal{S})$ in the weak topology follows from Alaoglu's theorem (which states that the closed unit ball of the dual space of a normed vector space is compact in the weak* topology; cf. Conway, 1990, Chapter 5, Section 3), that $\mathcal{M}(\mathcal{S})$ and $\mathcal{C}_b(\mathcal{S})$ are dual spaces, and that any closed subset of a compact set is compact (clearly, $\mathcal{M}_1^+(\mathcal{S})$ is closed). To see that $\mathcal{M}_1^+(\mathcal{S})$ is not compact in the strong topology, consider a sequence of probability measures $\mu_n \in \mathcal{M}_1^+(\mathcal{S})$ which have densities on $\mathcal{S}$ converging weakly to a probability measure $\mu \in \mathcal{M}_1^+(\mathcal{S})$ which has a positive mass at some point in $\mathcal{S}$. Then no subsequence of $\{\mu_n\}$ converges in the variational norm.

it ensures the existence of ω-limits (see Section 4.2). Second, it ensures that any closed subset of $\mathcal{M}_1^+(\mathcal{S})$ is compact (see footnote 31), which is needed in the proof of the Lyapunov's Theorem (see Theorem 6 and its proof in Appendix A.2).

For the above reasons, we use the weak topology in studying stability under evolutionary dynamics. From now on, we assume $\mathcal{S}$ is separable and use the Prohorov metric κ to measure the distances between population states.

4.1 Definitions for the Study of Local Stability

Let $\mu \in \mathcal{M}_1^+(\mathcal{S})$ and $Y \subseteq \mathcal{M}_1^+(\mathcal{S})$. The distance between μ and the set Y in the weak topology is

$$\kappa(\mu, Y) := \inf\{\kappa(\mu, \nu) : \nu \in Y\}.$$

The ε-neighborhood of Y (in the weak topology) is

$$Y^\varepsilon := \{\mu \in \mathcal{M}_1^+(\mathcal{S}) : \kappa(\mu, Y) < \varepsilon\}.$$

Let

$$\dot{\mu} = V(\mu) \tag{D}$$

be a differential equation on $\mathcal{M}_1^+(\mathcal{S})$ that admits a unique forward solution from each initial condition, and suppose that solutions to (D) are continuous in their initial conditions. Let $Z \subseteq \mathcal{M}_1^+(\mathcal{S})$ be a closed set.[31] We say Z is *Lyapunov stable* under (D) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\kappa(\mu(0), Z) < \delta \quad \Rightarrow \quad \kappa(\mu(t), Z) < \varepsilon \ \text{ for all } t \geq 0,$$

i.e., every solution $(\mu(t))$ of (D) that starts in Z^δ is contained in Z^ε. We say Z is *attracting* if there exists $\varepsilon > 0$ such that

$$\kappa(\mu(0), Z) < \varepsilon \quad \Rightarrow \quad \kappa(\mu(t), Z) \to 0,$$

i.e., every solution $(\mu(t))$ of (D) that starts in Z^ε converges weakly to Z. We say Z is *globally attracting* if

$$\kappa(\mu(t), Z) \to 0 \ \text{ for any } \mu(0) \in \mathcal{M}_1^+(\mathcal{S}),$$

i.e., every solution $(\mu(t))$ of (D) converges weakly to Z for any initial condition in $\mathcal{M}_1^+(\mathcal{S})$. Finally, we say Z is *asymptotically stable* if it is Lyapunov stable and attracting, and Z is *globally asymptotically stable* if it is Lyapunov stable and globally attracting.

4.2 Definitions for the Study of Global Convergence

Let $\xi \in \mathcal{M}_1^+(\mathcal{S})$, and let $\{\mu_t\}_{t\in[0,\infty)}$ be the solution trajectory to (D) with $\mu_0 = \xi$. The *ω-limit* $\omega(\xi)$ is the set of all points that the solution trajectory from ξ approaches arbitrarily closely infinitely often in the weak topology:

$$\omega(\xi) := \{\psi \in \mathcal{M}_1^+(\mathcal{S}) : \exists\{t_k\}_{k=1}^\infty \text{ with } \lim_{k\to\infty} t_k = \infty \text{ such that } \mu_{t_k} \xrightarrow{\text{w}} \psi \text{ as } k \to \infty\}. \tag{18}$$

[31] Note that $\mathcal{M}_1^+(\mathcal{S})$ is compact in the weak topology and that any closed subset of a compact set is compact. So, Z is compact in the weak topology.

Since $\mathcal{M}_1^+(\mathcal{S})$ is compact in the weak topology, $\omega(\xi)$ is nonempty.[32] The set

$$\Omega := \bigcup_{\xi \in \mathcal{M}_1^+(\mathcal{S})} \omega(\xi)$$

denotes the set of all ω-limit points of all solution trajectories, which provides a basic notion of recurrence for deterministic dynamics.

4.3 Continuous-Strategy Dynamic vs. its Approximating Finite-Strategy Dynamic

To study the behavior of an evolutionary dynamic for a continuous strategy game, one might ask whether we could construct a finite-strategy approximation for the game via discretization on the continuous strategy space, and then work with the corresponding finite-strategy dynamic. This raises the question of whether the behavior, especially the convergence and stability properties, of the original continuous-strategy dynamic and the approximating finite-strategy dynamic are the same.

This question comes down to topology. As we have argued, the appropriate notion of closeness to use for studying convergence and stability for continuous-strategy dynamics is the weak topology on $\mathcal{M}_1^+(\mathcal{S})$, which accounts for distances between the strategies themselves.[33] On the other hand, evolutionary dynamics for finite strategy games are defined with respect to the usual topology on $\mathbb{R}^n$. However, the latter topology cannot capture the property that two homogeneous population states are close to each other if and only if their respective strategies are close to each other in the continuous strategy space. Once the discretization of the continuous strategy space is fixed, the distances between states for the approximating finite-strategy dynamic have no connection with the underlying metric on the strategy space.

Denote by N the fineness of the discretization of the continuous strategy space,[34] and let $\{\mu_t^N\}_{t\geq 0}$ denote the solution trajectory of the approximating finite-strategy dynamic under discretization N. If one wanted to capture the notion of weak convergence, one would need to consider a sequence of discretizations. One possibility is to take $t \to \infty$ and then $N \to \infty$ to look at weak convergence of the limit points of the corresponding sequence of finite-strategy dynamics. With any fixed discretization, the limit (as time goes to infinity) in the approximating finite-strategy dynamic may not agree with the limit (as time goes to infinity) in the original continuous-strategy dynamic. So at some point one would have to use weak convergence anyway to make the connection. In part this is a matter of preference or convenience, whether one does this via discretization or directly works with the continuous-strategy dynamic. However, it may become cumbersome to analyze via discretization, say, when the continuous strategy space is multi-dimensional (e.g., $\mathbb{V} = \mathbb{R}^m$ for some $m \geq 2$) or even infinite-dimensional (e.g., $\mathbb{V}$ is the space of bounded measurable functions), but come out cleanly with a continuum.

Alternatively, one can take both limits simultaneously. For taking both limits simultaneously, one may take a sequence of times $\{\tau^N\}_{N=1}^{\infty}$ where $\tau^N \to \infty$ as $N \to \infty$, and look at $\lim_{N\to\infty} \mu_{\tau^N}^N$,

[32] Cf. Robinson, 1995, Theorem 5.4.1. Note that we define $\omega(\xi)$ in terms of the weak topology instead of the strong topology. If we define $\omega(\xi)$ in terms of the strong topology (i.e., change $\mu_{t_k} \xrightarrow{w} \psi$ to $\mu_{t_k} \to \psi$ in (18)), then $\omega(\xi)$ could be an empty set since $\mathcal{M}_1^+(\mathcal{S})$ is not compact in the strong topology.

[33] Recall that the distance between two homogeneous population states in the weak topology on $\mathcal{M}_1^+(\mathcal{S})$ agrees with the underlying metric d in the continuous strategy space $\mathcal{S}$, i.e., $\kappa(\delta_x, \delta_u) = d(x, u)$, when x and u are close to each other in $\mathcal{S}$.

[34] What we mean is, e.g., discretizing $[0,1]$ into $\{0, \frac{1}{N}, \frac{2}{N}, \ldots, 1\}$, or $\{0, \frac{1}{2^N}, \frac{2}{2^N}, \ldots, 1\}$.

which would be the limit of μ_t^N when N and t are taken to infinity simultaneously. However, this is not obviously preferable to working directly with the continuous-strategy dynamic.

5 Potential Games

5.1 Definition and Examples

Consider a function

$$f : \mathcal{M}(\mathcal{S}) \to \mathbb{R}.$$

Suppose that f is Fréchet-differentiable when $\mathcal{M}(\mathcal{S})$ is endowed with the strong topology.[35] The Fréchet-derivative of f at $\mu \in \mathcal{M}(\mathcal{S})$ is a continuous linear map

$$Df(\mu) : \mathcal{M}(\mathcal{S}) \to \mathbb{R}$$

that maps tangent vectors $\zeta \in \mathcal{M}(\mathcal{S})$ to rates of change in the value of f when one moves from μ in the direction ζ. Since $Df(\mu)$ is a linear map from $\mathcal{M}(\mathcal{S})$ to $\mathbb{R}$, by the Riesz representation theorem, there is an element $\nabla f(\mu)$ of $\mathcal{C}_b(\mathcal{S})$ (the dual space of $\mathcal{M}(\mathcal{S})$) that represents $Df(\mu)$ in the sense that

$$Df(\mu)\zeta = \int_{\mathcal{S}} \nabla f(\mu)\, d\zeta \equiv \langle \nabla f(\mu), \zeta \rangle. \tag{19}$$

This $\nabla f(\mu)$ is called the gradient of f at μ.

We define potential games for the continuous strategy case as follows:

Definition 1 *Population game $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ is a potential game if there exists a Fréchet-differentiable and continuous (with respect to the weak topology) function $f : \mathcal{M}(\mathcal{S}) \to \mathbb{R}$ with its gradient ∇f satisfying*

$$\nabla f(\mu) = F(\mu), \quad \forall \mu \in \mathcal{M}_1^+(\mathcal{S}). \tag{PG}$$

The function f is called the potential function.[36]

Condition (PG) says that the gradient of the potential function at $\mu \in \mathcal{M}_1^+(\mathcal{S})$ equals the payoff profile at state μ, which is analogous to the condition for potential games in finite strategy case: the gradient of the potential function equals the payoff vector.

By Definition 1, if the state moves from μ in direction ζ, the change in potential is

$$Df(\mu)\zeta = \langle \nabla f(\mu), \zeta \rangle = \langle F(\mu), \zeta \rangle.$$

In particular, if agents switch from pure strategy $y \in \mathcal{S}$ to pure strategy $z \in \mathcal{S}$, the change in potential is

$$Df(\mu)(\delta_z - \delta_y) = \langle F(\mu), \delta_z - \delta_y \rangle = F_z(\mu) - F_y(\mu).$$

Thus profitable changes in strategy increase potential.

[35] If X and Y are Banach spaces, we say $g : X \to Y$ is Fréchet-differentiable at x if there exists a continuous linear map $T : X \to Y$ such that $g(x + \vartheta) = g(x) + T\vartheta + o(\|\vartheta\|)$ for all ϑ in some neighborhood of zero in X. If it exists, this T is called the Fréchet-derivative of g at x, and is written as $Dg(x)$. Cf. Zeidler, 1986, Chapter 4.

[36] Note that Fréchet-differentiability implies continuity with respect to the strong topology, but continuity with respect to the weak topology is a stronger requirement than continuity with respect to the strong topology.

Example 2 *Matching to Play a Two-player Symmetric Game with Common Interests.* Consider the same setting as in Example 1. In addition, we assume that the single match payoff function $h : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ is doubly symmetric, i.e., $h(x, y) = h(y, x)$ for all $x, y \in \mathcal{S}$. In this case, we say the game exhibits *common interests* since two matched players always receive the same payoffs. We have the following claim:

Claim: The population game $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ defined by (6) is a potential game with potential function

$$f(\mu) := \frac{1}{2} \int_{\mathcal{S}} \int_{\mathcal{S}} h(x, y)\, \mu(dy)\, \mu(dx).$$

Note that $f : \mathcal{M}(\mathcal{S}) \to \mathbb{R}$ is continuous with respect to the weak topology. In order to prove the Claim, we need to show

$$\nabla f(\mu) = F(\mu), \quad \forall \mu \in \mathcal{M}_1^+(\mathcal{S}).$$

Let $\mu \in \mathcal{M}_1^+(\mathcal{S})$ and $\zeta \in \mathcal{M}(\mathcal{S})$. Then

$$\begin{aligned}
f(\mu + \zeta) &= \frac{1}{2} \int_{x \in \mathcal{S}} \int_{y \in \mathcal{S}} h(x, y)\, (\mu + \zeta)(dy)\, (\mu + \zeta)(dx) \\
&= \frac{1}{2} \int_{x \in \mathcal{S}} \int_{y \in \mathcal{S}} h(x, y)\, \mu(dy)\, \mu(dx) + \frac{1}{2} \int_{x \in \mathcal{S}} \int_{y \in \mathcal{S}} h(x, y)\, \mu(dy)\, \zeta(dx) \\
&\quad + \frac{1}{2} \int_{x \in \mathcal{S}} \int_{y \in \mathcal{S}} h(x, y)\, \zeta(dy)\, \mu(dx) + \frac{1}{2} \int_{x \in \mathcal{S}} \int_{y \in \mathcal{S}} h(x, y)\, \zeta(dy)\, \zeta(dx) \\
&= f(\mu) + \frac{1}{2} \int_{x \in \mathcal{S}} \int_{y \in \mathcal{S}} h(x, y)\, \mu(dy)\, \zeta(dx) + \frac{1}{2} \int_{y \in \mathcal{S}} \int_{x \in \mathcal{S}} h(y, x)\, \zeta(dx)\, \mu(dy) + o(\|\zeta\|) \\
&= f(\mu) + \frac{1}{2} \int_{x \in \mathcal{S}} \int_{y \in \mathcal{S}} h(x, y)\, \mu(dy)\, \zeta(dx) + \frac{1}{2} \int_{y \in \mathcal{S}} \int_{x \in \mathcal{S}} h(x, y)\, \zeta(dx)\, \mu(dy) + o(\|\zeta\|) \\
&= f(\mu) + \int_{x \in \mathcal{S}} \int_{y \in \mathcal{S}} h(x, y)\, \mu(dy)\, \zeta(dx) + o(\|\zeta\|).
\end{aligned}$$

So, we have

$$Df(\mu)\zeta = \int_{x \in \mathcal{S}} \int_{y \in \mathcal{S}} h(x, y)\, \mu(dy)\, \zeta(dx) = \int_{x \in \mathcal{S}} F_x(\mu)\, \zeta(dx) = \langle F(\mu), \zeta \rangle.$$

Hence, by (19),

$$\langle \nabla f(\mu), \zeta \rangle = \langle F(\mu), \zeta \rangle. \tag{20}$$

Since (20) is true for any $\zeta \in \mathcal{M}(\mathcal{S})$, we have $\nabla f(\mu) = F(\mu)$ for all $\mu \in \mathcal{M}_1^+(\mathcal{S})$. Hence, the Claim is proved.

Note that, if $\mu \in \mathcal{M}_1^+(\mathcal{S})$, then

$$f(\mu) = \frac{1}{2} \int_{\mathcal{S}} \int_{\mathcal{S}} h(x, y)\, \mu(dy)\, \mu(dx) = \frac{1}{2} \int_{\mathcal{S}} F_x(\mu)\, \mu(dx) = \frac{1}{2} \bar{F}(\mu).$$

So, the potential function coincides with one-half of the average payoff function when μ is restricted on $\mathcal{M}_1^+(\mathcal{S})$. Therefore, by Lemma 2 (see below), any evolutionary dynamic satisfying positive correlation (PC) improves social outcomes.

◇

Example 3 *Games Generated by Variable Pricing Schemes.*[37] When an agent interacts with other agents in a population game, his choice of strategy affects not only his own payoff, but also the payoffs of other agents. One way to internalize this kind of externalities among agents is to introduce pricing schemes. Suppose that population game $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ has an extension to $\mathcal{M}(\mathcal{S})$. With an abuse of notation, we denote by $F : \mathcal{M}(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ the extension of F to $\mathcal{M}(\mathcal{S})$. We assume that $F : \mathcal{M}(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ is continuous with respect to the weak topology. Also, we assume that F is Fréchet-differentiable so that the Fréchet-derivative of F, DF, is well-defined. Recall that the average payoff function $\bar{F}$ is defined by

$$\bar{F}(\mu) := \int_{\mathcal{S}} F_x(\mu)\, \mu(dx) = \int_{\mathcal{S}} F(\mu)\, d\mu.$$

In this example, we allow μ to be any finite signed measure, i.e., $\bar{F} : \mathcal{M}(\mathcal{S}) \to \mathbb{R}$.

Consider the augmented game $\hat{F} : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ defined by[38]

$$\hat{F}_x(\mu) := F_x(\mu) + \int_{\mathcal{S}} (DF(\mu)\delta_x)(y)\, \mu(dy). \tag{21}$$

The second term on the RHS of (21) represents the (aggregate) marginal effect that an agent choosing strategy $x \in \mathcal{S}$ has on the payoffs of other agents. We interpret it as a price (either subsidy or tax) imposed by a social planner. We have the following claim:

Claim: The augmented game $\hat{F}$ is a potential game with potential function being the average payoff function of the original game F.

Note that $\bar{F} : \mathcal{M}(\mathcal{S}) \to \mathbb{R}$ is continuous with respect to the weak topology. To prove the Claim, we will show for any $\mu \in \mathcal{M}_1^+(\mathcal{S})$,

$$\nabla \bar{F}(\mu)(x) = F_x(\mu) + \int_{\mathcal{S}} (DF(\mu)\delta_x)(y)\, \mu(dy), \quad \forall x \in \mathcal{S}.$$

Let $\mu \in \mathcal{M}_1^+(\mathcal{S})$ and $\zeta \in \mathcal{M}(\mathcal{S})$. Then

$$\begin{aligned}
\bar{F}(\mu + \zeta) &= \int_{\mathcal{S}} F(\mu + \zeta)\, d(\mu + \zeta) \\
&= \int_{\mathcal{S}} \big(F(\mu) + DF(\mu)\zeta + o(\|\zeta\|)\big)\, d(\mu + \zeta) \\
&= \int_{\mathcal{S}} F(\mu)\, d\mu + \int_{\mathcal{S}} F(\mu)\, d\zeta + \int_{\mathcal{S}} DF(\mu)\zeta\, d\mu + \int_{\mathcal{S}} DF(\mu)\zeta\, d\zeta + o(\|\zeta\|) \\
&= \bar{F}(\mu) + \int_{\mathcal{S}} F(\mu)\, d\zeta + \int_{\mathcal{S}} DF(\mu)\zeta\, d\mu + o(\|\zeta\|).
\end{aligned}$$

Hence,

$$D\bar{F}(\mu)\zeta = \int_{\mathcal{S}} F(\mu)\, d\zeta + \int_{\mathcal{S}} DF(\mu)\zeta\, d\mu.$$

Since $D\bar{F}(\mu)\zeta = \int_{\mathcal{S}} \nabla \bar{F}(\mu)\, d\zeta$ by (19), we have

$$\int_{\mathcal{S}} \nabla \bar{F}(\mu)\, d\zeta = \int_{\mathcal{S}} F(\mu)\, d\zeta + \int_{\mathcal{S}} DF(\mu)\zeta\, d\mu,$$

[37] Cf. Sandholm, 2010a, Example 3.4.1.

[38] Note that $DF(\mu)$ is a continuous linear map from $\mathcal{M}(\mathcal{S})$ to $\mathcal{C}_b(\mathcal{S})$. For any $\zeta \in \mathcal{M}(\mathcal{S})$, $DF(\mu)\zeta$ means that the continuous linear map $DF(\mu)$ acts on ζ.

i.e.,

$$\int_{\mathcal{S}} \nabla \bar{F}(\mu)(y)\, \zeta(dy) = \int_{\mathcal{S}} F_y(\mu)\, \zeta(dy) + \int_{\mathcal{S}} (DF(\mu)\zeta)(y)\, \mu(dy). \tag{22}$$

Since (22) holds for any $\zeta \in \mathcal{M}(\mathcal{S})$, by putting $\zeta = \delta_x$, we have

$$\nabla \bar{F}(\mu)(x) = F_x(\mu) + \int_{\mathcal{S}} (DF(\mu)\delta_x)(y)\, \mu(dy).$$

Hence, the Claim is proved.

Since in potential games profitable changes in strategy increase potential, by the Claim, when agents switch strategies in response to the combination of original payoffs and prices, average payoff of the original game (and hence efficiency) increases.

$\diamond$

The following lemma shows that in potential games, any evolutionary dynamic satisfying positive correlation (PC) ascends potential.

Lemma 2 *Let F be a potential game with potential function f. Suppose that the dynamic $\dot{\mu} = V^F(\mu)$ for F satisfies positive correlation* (PC)*. Then along any solution trajectory $(\mu(t))$, we have $\frac{d}{dt} f(\mu(t)) > 0$ whenever $\dot{\mu}(t) \neq 0$.*

Proof. Since F is a potential game with potential function f, (PC) becomes

$$V^F(\mu) \neq 0 \quad \Rightarrow \quad \langle \nabla f(\mu), V^F(\mu) \rangle = \int_{\mathcal{S}} F_x(\mu)\, V^F(\mu)(dx) > 0.$$

Then, by the chain rule,

$$\frac{d}{dt} f(\mu(t)) = Df(\mu(t))\dot{\mu}(t) = Df(\mu(t))V^F(\mu(t)) = \langle \nabla f(\mu(t)), V^F(\mu(t)) \rangle \geq 0,$$

and the inequality is strict if $V^F(\mu(t)) \neq 0$.

$\square$

We say a weakly continuous and Fréchet-differentiable function $L : \mathcal{M}_1^+(\mathcal{S}) \to \mathbb{R}$ is an (*increasing*) *Lyapunov function* for the differential equation $\dot{\mu} = V(\mu)$ if $\dot{L}(\mu) \equiv \langle \nabla L(\mu), V(\mu) \rangle \geq 0$ for all $\mu \in \mathcal{M}_1^+(\mathcal{S})$. If, in addition, equality holds only at rest points (i.e., only when $V(\mu) = 0$), then we call L an (*increasing*) *strict Lyapunov function.* Thus Lemma 2 shows that for any potential game F, if the dynamic $\dot{\mu} = V^F(\mu)$ for F satisfies positive correlation (PC), then the potential function acts as an increasing strict Lyapunov function for the dynamic.

5.2 Global Convergence

Let $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ be a population game, and let $\dot{\mu} = V^F(\mu)$ be a deterministic evolutionary dynamic for F. We denote by $RP(V^F)$ the set of rest points under the dynamic $\dot{\mu} = V^F(\mu)$, i.e.,

$$RP(V^F) := \{\psi \in \mathcal{M}_1^+(\mathcal{S}) : V^F(\psi) = 0\}.$$

Also, we denote by $NE(F)$ the set of Nash equilibria for F, i.e.,

$$NE(F) := \{\psi \in \mathcal{M}_1^+(\mathcal{S}) : [F_y(\psi) - F_z(\psi)]_+ = 0,\ \forall z \in S(\psi),\ \forall y \in \mathcal{S}\}.$$

The following theorem provides a global convergence result to the set of Nash equilibria for pairwise comparison dynamics in potential games. It shows that every solution trajectory of such dynamics converges to the set of Nash equilibria.

Theorem 2 (Global Convergence for Pairwise Comparison Dynamics for Potential Games) *Let F be a potential game with potential function f, and let $\dot{\mu} = V^F(\mu)$ be a pairwise comparison dynamic for F. Suppose that the conditions in Theorem 1 are satisfied so that a unique forward solution exists from each initial condition and solutions to the dynamic are continuous in their initial conditions. Then $\Omega = RP(V^F) = NE(F)$.*

Proof. By Proposition 1, positive correlation (PC) is satisfied. So by Lemma 2, the potential function f acts as an increasing strict Lyapunov function for the dynamic. Hence, by Theorem 5 in Appendix A.2, $\Omega = RP(V^F)$. Also, Nash stationarity (NS) is satisfied by Proposition 1. Thus, we have $\Omega = RP(V^F) = NE(F)$.

□

Remark 4 For general deterministic dynamics, if only positive correlation (PC) is satisfied but not Nash stationarity (NS) (e.g., the replicator dynamic and other imitative dynamics do not satisfy Nash stationarity (NS)), then we still have $\Omega = RP(V^F)$. If both positive correlation (PC) and Nash stationarity (NS) are satisfied, then we have the same conclusion as in Theorem 2.

5.3 Local Stability

Let $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ be a potential game with potential function $f : \mathcal{M}(\mathcal{S}) \to \mathbb{R}$. A set $Z \subseteq \mathcal{M}_1^+(\mathcal{S})$ is a *local maximizer set* of the potential function f if (i) Z is connected, (ii) f is constant on Z, and (iii) there exists a neighborhood $Y \subseteq \mathcal{M}_1^+(\mathcal{S})$ of Z in the weak topology such that $f(\mu) > f(\psi)$ for all $\mu \in Z$ and all $\psi \in Y \backslash Z$.[39]

The following lemma shows that in potential games, all local maximizer sets of the potential function consist entirely of Nash equilibria of the game.

Lemma 3 *Let F be a potential game with potential function f. If μ is a local maximizer of f, then $\mu \in NE(F)$.*

Proof. Let $\dot{\mu} = V^F(\mu)$ be a pairwise comparison dynamic for F. By Proposition 1, the dynamic satisfies positive correlation (PC). So by Lemma 2, if μ is not a rest point of the dynamic, i.e., $V^F(\mu) \neq 0$, then $\dot{f}(\mu) > 0$, which implies that μ is not a local maximizer of f. Therefore, if μ is a local maximizer of f, then μ is a rest point of the dynamic, which implies that μ is a Nash equilibrium of F since the dynamic also satisfies Nash stationarity (NS) by Proposition 1.

□

By Lemma 3, local maximizer sets of f are subsets of $NE(F)$. Also, since f is continuous with respect to the weak topology, all local maximizer sets of f are closed. We call a closed set $Z \subseteq NE(F)$ *isolated* in $NE(F)$ if there exists a neighborhood $Y \subseteq \mathcal{M}_1^+(\mathcal{S})$ of Z in the weak topology such that Y does not contain any Nash equilibria other than those in Z.

The following theorem provides a local stability result for pairwise comparison dynamics in potential games. It shows that local maximizer sets of the potential function are Lyapunov stable; and if, in addition, those sets are isolated in $NE(F)$, then they are asymptotically stable.

Theorem 3 (Local Stability for Pairwise Comparison Dynamics for Potential Games) *Let F be a potential game with potential function f, and let $\dot{\mu} = V^F(\mu)$ be a pairwise comparison dynamic for F. Suppose that the conditions in Theorem 1 are satisfied so that a unique forward*

[39]We call $Y \subseteq \mathcal{M}_1^+(\mathcal{S})$ a neighborhood of $Z \subseteq \mathcal{M}_1^+(\mathcal{S})$ in the weak topology if Y is open relative to $\mathcal{M}_1^+(\mathcal{S})$ in the weak topology and contains Z.

solution exists from each initial condition and solutions to the dynamic are continuous in their initial conditions. Let $Z \subseteq \mathcal{M}_1^+(\mathcal{S})$ be a local maximizer set of f. Then

i) *Z is Lyapunov stable;*

ii) *if Z is also isolated in $NE(F)$, then Z is asymptotically stable.*

Proof. By Proposition 1, the dynamic satisfies positive correlation (PC) and Nash stationarity (NS). Then part (i) follows immediately from positive correlation (PC), Lemma 2, and Theorem 6 in Appendix A.2. Since Z is isolated in $NE(F)$, there exists a neighborhood $Y \subseteq \mathcal{M}_1^+(\mathcal{S})$ of Z in the weak topology such that $Y \backslash Z$ does not contain any Nash equilibrium. By Nash stationarity (NS), $V^F(\mu) \neq 0$ for all $\mu \in Y \backslash Z$. Then by positive correlation (PC), $\dot{f}(\mu) = \langle \nabla f(\mu), V^F(\mu) \rangle > 0$ for all $\mu \in Y \backslash Z$. Therefore, by Corollary 2 in Appendix A.2, Z is asymptotically stable.

□

Remark 5 For general deterministic dynamics, if only positive correlation (PC) is satisfied but not Nash stationarity (NS), then the conclusion in part (i) still holds. If both positive correlation (PC) and Nash stationarity (NS) are satisfied, then the conclusion in part (ii) also holds.

The following corollary is immediate from Theorem 3.

Corollary 1 *Suppose that the conditions in Theorem 2 are satisfied. If the potential game F has a unique Nash equilibrium μ^*, then $\{\mu^*\}$ is globally asymptotically stable.*

Example 2 (Continued) Recall that in Example 2, we have shown that the population game $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ defined by $F_x(\mu) = \int_{\mathcal{S}} h(x,y)\,\mu(dy)$, where $h : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ satisfies $h(x,y) = h(y,x)$ for all $x, y \in \mathcal{S}$, is a potential game with potential function

$$f(\mu) := \frac{1}{2} \int_{\mathcal{S}} \int_{\mathcal{S}} h(x,y)\,\mu(dy)\,\mu(dx).$$

State $\mu^* \in \mathcal{M}_1^+(\mathcal{S})$ is *evolutionarily robust*[40] if there exists $\varepsilon > 0$ such that for all $\psi \neq \mu^*$ ($\psi \in \mathcal{M}_1^+(\mathcal{S})$) with $\kappa(\psi, \mu^*) < \varepsilon$,

$$\int_{\mathcal{S}} \int_{\mathcal{S}} h(x,y)\,\psi(dy)\,\mu^*(dx) > \int_{\mathcal{S}} \int_{\mathcal{S}} h(x,y)\,\psi(dy)\,\psi(dx). \tag{ER}$$

We have the following claim:

Claim: If $\mu^* \in \mathcal{M}_1^+(\mathcal{S})$ is evolutionarily robust, then $\{\mu^*\}$ is asymptotically stable under pairwise comparison dynamics.

It suffices to show that μ^* is a local maximizer of the potential function f and that $\{\mu^*\}$ is isolated in $NE(F)$. Then by Theorem 3, the result follows. For all $\psi \neq \mu^*$ ($\psi \in \mathcal{M}_1^+(\mathcal{S})$) with $\kappa(\psi, \mu^*) < \varepsilon$,

$$\begin{aligned} f(\mu^*) &= \frac{1}{2} \int_{\mathcal{S}} \int_{\mathcal{S}} h(x,y)\,\mu^*(dy)\,\mu^*(dx) - \frac{1}{2} \int_{\mathcal{S}} \int_{\mathcal{S}} h(x,y)\,\psi(dy)\,\mu^*(dx) \\ &\quad + \frac{1}{2} \int_{\mathcal{S}} \int_{\mathcal{S}} h(x,y)\,\psi(dy)\,\mu^*(dx) \\ &\geq \frac{1}{2} \int_{\mathcal{S}} \int_{\mathcal{S}} h(x,y)\,\psi(dy)\,\mu^*(dx) \end{aligned}$$

[40] Cf. Oechssler and Riedel, 2002, Definition 5.

$$
\begin{aligned}
&> \frac{1}{2}\int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\psi(dy)\,\psi(dx) \\
&= f(\psi).
\end{aligned}
$$

The first inequality is due to the fact that evolutionary robustness implies ESS, which in turn implies Nash equilibrium,[41] and the assumption that $h(x,y) = h(y,x)$ for all $x, y \in \mathcal{S}$. The second inequality is by (ER). So μ^* is a strict local maximizer of f. Also, (ER) implies that all those $\psi \neq \mu^*$ ($\psi \in \mathcal{M}_1^+(\mathcal{S})$) with $\kappa(\psi, \mu^*) < \varepsilon$ are not Nash equilibria (see (27) below). Hence, $\{\mu^*\}$ is isolated in $NE(F)$.

◇

6 Contractive Games

6.1 Definition and Examples

We define contractive games for the continuous strategy case as follows:

Definition 2 *Population game $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ is a contractive game if*

$$
\langle F(\mu) - F(\psi), \mu - \psi \rangle \equiv \int_{\mathcal{S}} (F(\mu) - F(\psi))\, d(\mu - \psi) \leq 0, \quad \forall \mu, \psi \in \mathcal{M}_1^+(\mathcal{S}). \tag{CG}
$$

If the inequality in (CG) *holds strictly whenever $\mu \neq \psi$, we call F a strictly contractive game; whereas if this inequality always binds, we call F a null contractive game.*

Given the state space $\mathcal{M}_1^+(\mathcal{S})$, we can define the tangent space $T\mathcal{M}_1^+(\mathcal{S})$ as in the finite strategy case. The tangent space of $\mathcal{M}_1^+(\mathcal{S})$ is

$$
T\mathcal{M}_1^+(\mathcal{S}) = \mathcal{M}_0(\mathcal{S}) \equiv \{\mu \in \mathcal{M}(\mathcal{S}) : \mu(\mathcal{S}) = 0\}.
$$

Recall that $\mathcal{M}(\mathcal{S})$ is a vector space and is the linear span of $\mathcal{M}_1^+(\mathcal{S})$. $T\mathcal{M}_1^+(\mathcal{S})$ is the smallest subspace of $\mathcal{M}(\mathcal{S})$ that contains all vectors describing motions between population states in $\mathcal{M}_1^+(\mathcal{S})$. In other words, if $\mu, \psi \in \mathcal{M}_1^+(\mathcal{S})$, then $\mu - \psi \in T\mathcal{M}_1^+(\mathcal{S})$, and $T\mathcal{M}_1^+(\mathcal{S})$ is the linear span of all vectors of this form. The restriction $\mu(\mathcal{S}) = 0$ embodies the fact that changes in the population state leave the population's mass constant.

Assume that population game $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ is C^1 in the sense of Fréchet-differentiability.[42] We say F satisfies *self-defeating externalities* (SDE) if

$$
\langle DF(\psi)\zeta, \zeta \rangle \leq 0, \quad \text{for any } \psi \in \mathcal{M}_1^+(\mathcal{S}) \text{ and } \zeta \in \mathcal{M}_0(\mathcal{S}). \tag{SDE}
$$

For intuition, suppose that the measure $\zeta \in \mathcal{M}_0(\mathcal{S})$ takes the form $\zeta = \delta_z - \delta_y$, representing switches by agents from pure strategy $y \in \mathcal{S}$ to pure strategy $z \in \mathcal{S}$. Then (SDE) requires that $\int_{\mathcal{S}} DF(\psi)\zeta\, d\delta_z \leq \int_{\mathcal{S}} DF(\psi)\zeta\, d\delta_y$, i.e., $(DF(\psi)\zeta)(z) \leq (DF(\psi)\zeta)(y)$. Thus, the improvements in the payoffs of strategies to which revising agents are switching are always exceeded by the improvements in the payoffs of strategies which revising agents are abandoning.

The following lemma shows that when the population game $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ is C^1, conditions (CG) and (SDE) are equivalent.

[41] See Oechssler and Riedel, 2002, Proposition 1 and Figure 2.

[42] To be more precise, we assume that F is C^1 in the interior of $\mathcal{M}_1^+(\mathcal{S})$, and the values of F on the boundary of $\mathcal{M}_1^+(\mathcal{S})$ are determined such that F and DF are continuous on $\mathcal{M}_1^+(\mathcal{S})$.

Lemma 4 *Suppose that the population game* $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ *is* C^1*. Then* F *is a contractive game if and only if it satisfies self-defeating externalities* (SDE)*.*

Proof.
($\Rightarrow$) Suppose that F is a contractive game. Fix $\psi \in \mathcal{M}_1^+(\mathcal{S})$ and $\zeta \in \mathcal{M}_0(\mathcal{S})$. Since F is C^1, it is enough to consider ψ in the interior of $\mathcal{M}_1^+(\mathcal{S})$. Let $\mu_\varepsilon = \psi + \varepsilon\zeta$, where $\varepsilon > 0$ is small enough that $\mu_\varepsilon \in \mathcal{M}_1^+(\mathcal{S})$. Then

$$\begin{aligned} F(\mu_\varepsilon) &= F(\psi + \varepsilon\zeta) \\ &= F(\psi) + DF(\psi)\varepsilon\zeta + o(\varepsilon\|\zeta\|), \end{aligned}$$

which implies

$$F(\mu_\varepsilon) - F(\psi) = \varepsilon DF(\psi)\zeta + o(\varepsilon\|\zeta\|). \tag{23}$$

Integrating both sides of (23) with respect to the measure $\mu_\varepsilon - \psi = \varepsilon\zeta$, we have

$$\int_{\mathcal{S}} (F(\mu_\varepsilon) - F(\psi))\, d(\mu_\varepsilon - \psi) = \varepsilon \int_{\mathcal{S}} DF(\psi)\zeta\, d(\mu_\varepsilon - \psi) + o(\varepsilon^2\|\zeta\|^2),$$

i.e.,[43]

$$\langle F(\mu_\varepsilon) - F(\psi), \mu_\varepsilon - \psi\rangle = \varepsilon^2 \langle DF(\psi)\zeta, \zeta\rangle + o(\varepsilon^2). \tag{24}$$

Since $\mu_\varepsilon, \psi \in \mathcal{M}_1^+(\mathcal{S})$, by (CG), LHS of (24) is nonpositive. So we have

$$\langle DF(\psi)\zeta, \zeta\rangle + \frac{o(\varepsilon^2)}{\varepsilon^2} \leq 0.$$

Taking $\varepsilon \to 0$ yields $\langle DF(\psi)\zeta, \zeta\rangle \leq 0$. Since $\psi \in \mathcal{M}_1^+(\mathcal{S})$ and $\zeta \in \mathcal{M}_0(\mathcal{S})$ are arbitrary, (SDE) holds.

($\Leftarrow$) Suppose that (SDE) holds. Let $\mu, \psi \in \mathcal{M}_1^+(\mathcal{S})$ and $\alpha(t) = t\mu + (1-t)\psi$. Then the fundamental theorem of calculus implies that[44]

$$F(\mu) - F(\psi) = \int_{[0,1]} DF(\alpha(t))(\mu - \psi)\, dt. \tag{25}$$

Integrating both sides of (25) with respect to the measure $\mu - \psi$, we have

$$\begin{aligned} \int_{\mathcal{S}} (F(\mu) - F(\psi))\, d(\mu - \psi) &= \int_{\mathcal{S}} \int_{[0,1]} DF(\alpha(t))(\mu - \psi)\, dt\, d(\mu - \psi) \\ &= \int_{[0,1]} \int_{\mathcal{S}} DF(\alpha(t))(\mu - \psi)\, d(\mu - \psi)\, dt, \end{aligned}$$

i.e.,

$$\langle F(\mu) - F(\psi), \mu - \psi\rangle = \int_{[0,1]} \langle DF(\alpha(t))(\mu - \psi), \mu - \psi\rangle\, dt.$$

[43] Note that $\zeta \in \mathcal{M}_0(\mathcal{S})$ is fixed.

[44] For the fundamental theorem of calculus in Banach spaces, see, e.g., Hamilton (1982).

Since $\alpha(t) \in \mathcal{M}_1^+(\mathcal{S})$ for $t \in [0,1]$ and $\mu - \psi \in \mathcal{M}_0(\mathcal{S})$, by (SDE), we have

$$\langle DF(\alpha(t))(\mu - \psi), \mu - \psi \rangle \leq 0,$$

and hence

$$\langle F(\mu) - F(\psi), \mu - \psi \rangle \leq 0. \tag{26}$$

Since (26) holds for any $\mu, \psi \in \mathcal{M}_1^+(\mathcal{S})$, F is a contractive game.

□

Example 4 *Matching to Play a Two-player Symmetric Zero-sum Game.* Consider the same setting as in Example 1. In addition, we assume that the single match payoff function $h : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ satisfies $h(x,y) + h(y,x) = 0$ for all $x, y \in \mathcal{S}$. For any $\mu, \psi \in \mathcal{M}_1^+(\mathcal{S})$,

$$\begin{aligned}
\langle F(\mu) - F(\psi), \mu - \psi \rangle &= \int_{\mathcal{S}} (F(\mu) - F(\psi))\, d(\mu - \psi) \\
&= \int_{\mathcal{S}} F(\mu)\, d\mu - \int_{\mathcal{S}} F(\mu)\, d\psi - \int_{\mathcal{S}} F(\psi)\, d\mu + \int_{\mathcal{S}} F(\psi)\, d\psi.
\end{aligned}$$

Note that, since $F_x(\mu) = \int_{\mathcal{S}} h(x,y)\, \mu(dy)$, we have

$$\begin{aligned}
\int_{\mathcal{S}} F(\mu)\, d\psi + \int_{\mathcal{S}} F(\psi)\, d\mu &= \int_{x\in\mathcal{S}} \int_{y\in\mathcal{S}} h(x,y)\, \mu(dy)\, \psi(dx) + \int_{x\in\mathcal{S}} \int_{y\in\mathcal{S}} h(x,y)\, \psi(dy)\, \mu(dx) \\
&= \int_{x\in\mathcal{S}} \int_{y\in\mathcal{S}} h(x,y)\, \mu(dy)\, \psi(dx) + \int_{y\in\mathcal{S}} \int_{x\in\mathcal{S}} h(y,x)\, \psi(dx)\, \mu(dy) \\
&= \int_{x\in\mathcal{S}} \int_{y\in\mathcal{S}} \big(h(x,y) + h(y,x)\big)\, \mu(dy)\, \psi(dx) \\
&= 0.
\end{aligned}$$

In particular, we have $\int_{\mathcal{S}} F(\mu)\, d\mu = 0 = \int_{\mathcal{S}} F(\psi)\, d\psi$. Therefore, $\langle F(\mu) - F(\psi), \mu - \psi \rangle = 0$ and hence F is a null contractive game.

◇

Example 5 *Matching to Play a Two-player Symmetric Game with an Interior ESS or NSS.*[45] Consider the same setting as in Example 1. Note that

$$\begin{aligned}
\mu \in NE(F) \quad &\Leftrightarrow \quad F_z(\mu) \geq F_x(\mu),\ \forall z \in S(\mu),\ \forall x \in \mathcal{S} \\
&\Leftrightarrow \quad \int_{\mathcal{S}} F_z(\mu)\, \mu(dz) \geq F_x(\mu),\ \forall x \in \mathcal{S} \\
&\Leftrightarrow \quad \int_{\mathcal{S}} F_z(\mu)\, \mu(dz) \geq \int_{\mathcal{S}} F_x(\mu)\, \psi(dx),\ \forall \psi \in \mathcal{M}_1^+(\mathcal{S}).
\end{aligned}$$

Since $F_x(\mu) = \int_{\mathcal{S}} h(x,y)\, \mu(dy)$, we have

$$\mu \in NE(F) \quad \Leftrightarrow \quad \int_{\mathcal{S}} \int_{\mathcal{S}} h(x,y)\, \mu(dy)\, \mu(dx) \geq \int_{\mathcal{S}} \int_{\mathcal{S}} h(x,y)\, \mu(dy)\, \psi(dx),\ \forall \psi \in \mathcal{M}_1^+(\mathcal{S}). \tag{27}$$

State $\mu \in \mathcal{M}_1^+(\mathcal{S})$ is an *evolutionarily stable state* (ESS) if the following two conditions are satisfied:

[45] Cf. Maynard Smith and Price (1973) and Maynard Smith (1982).

i) $\int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\mu(dy)\,\mu(dx) \geq \int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\mu(dy)\,\psi(dx),\ \forall \psi \in \mathcal{M}_1^+(\mathcal{S})$;

ii) $\left[\int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\mu(dy)\,\mu(dx) = \int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\mu(dy)\,\psi(dx) \text{ and } \mu \neq \psi\right]$ imply that $\int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\psi(dy)\,\mu(dx) > \int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\psi(dy)\,\psi(dx)$.

If condition (ii) is weakened to

ii′) $\left[\int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\mu(dy)\,\mu(dx) = \int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\mu(dy)\,\psi(dx) \text{ and } \mu \neq \psi\right]$ imply that $\int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\psi(dy)\,\mu(dx) \geq \int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\psi(dy)\,\psi(dx)$,

then μ is called a *neutrally stable state* (NSS). We have the following claim:

Claim: If there exists $\mu \in \operatorname{int}(\mathcal{M}_1^+(\mathcal{S}))$ which is an ESS, then F is a strictly contractive game.

The proof of the Claim is presented in Appendix A.3. Similar reasoning shows that, if F admits an interior NSS, then F is a contractive game.

◇

For more examples of contractive games, see Hofbauer, Oechssler and Riedel, 2009, Section 7.

6.2 Global Asymptotic Stability

For pairwise comparison dynamics, we have assumed that the conditional switch rates ρ satisfy sign-preservation (SP):

$$\operatorname{sgn}(\rho_{xy}(\pi)) = \operatorname{sgn}([\pi(y) - \pi(x)]_+), \quad \forall x, y \in \mathcal{S}. \tag{SP}$$

To obtain results for global asymptotic stability (and hence global convergence) for contractive games, an additional condition is needed, namely, *impartiality* (IP):

$$\rho_{xy}(\pi) = \tau_y(\pi(y) - \pi(x)) \text{ for some continuous functions } \tau_y : \mathbb{R} \to \mathbb{R}_+. \tag{IP}$$

Under impartiality (IP), the conditional switch rate from x to y only depends on the payoff difference $\pi(y) - \pi(x)$ and the strategy to which the revising agent is switching. In particular, it does not depend on the agent's current strategy. The differential equation for pairwise comparison dynamic (PCD) becomes

$$\dot{\mu}(A) = \int_{z\in\mathcal{S}}\int_{y\in A} \tau_y(F_y(\mu) - F_z(\mu))\,\lambda(dy)\,\mu(dz) - \int_{z\in\mathcal{S}}\int_{y\in A} \tau_z(F_z(\mu) - F_y(\mu))\,\mu(dy)\,\lambda(dz), \tag{PCD′}$$

for all $A \in \mathcal{B}$. For the special case of the Smith dynamic, $\rho_{xy}(\pi) = [\pi(y) - \pi(x)]_+$ depends only on the payoff difference $\pi(y) - \pi(x)$, and thus impartiality (IP) is satisfied.

The following theorem shows that sign-preservation (SP) and impartiality (IP) together ensure that the set of Nash equilibria is globally asymptotically stable for pairwise comparison dynamics in contractive games.

Theorem 4 (Global Asymptotic Stability for Pairwise Comparison Dynamics for Contractive Games) *Let $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ be a contractive game that is C^1, and let $\dot{\mu} = V^F(\mu)$ be an impartial pairwise comparison dynamic for F. Suppose that the conditions in Theorem 1 are satisfied so that a unique forward solution exists from each initial condition and solutions to the dynamic are continuous in their initial conditions. Define the function $H : \mathcal{M}_1^+(\mathcal{S}) \to \mathbb{R}_+$ by*

$$H(\mu) := \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \eta_y(F_y(\mu) - F_z(\mu))\,\mu(dz)\,\lambda(dy),$$

where

$$\eta_y(d) := \int_0^d \tau_y(r)\, dr$$

is the definite integral of τ_y*. Then* $H^{-1}(0) = NE(F)$*. Moreover,* $\dot{H}(\mu) \leq 0$ *for all* $\mu \in \mathcal{M}_1^+(\mathcal{S})$*, with equality if and only if* $\mu \in NE(F)$*, and so* $NE(F)$ *is globally asymptotically stable.*

The proof of Theorem 4 is presented in Appendix A.3.

7 Conclusion

To summarize, we defined a population game with continuum strategy set $\mathcal{S}$ as a weakly continuous map from the space of probability measures over $\mathcal{S}$ to the space of bounded continuous functions on $\mathcal{S}$. We provided a general framework to derive the mean dynamic for population games in continuous strategy settings. We showed that, under mild Lipschitz continuity conditions, a unique solution exists for the pairwise comparison dynamic from every initial strategy distribution, where solutions are defined with respect to the variational norm. We established two nice properties—Nash stationarity (NS) and positive correlation (PC)—for pairwise comparison dynamics. We studied the global convergence and local stability of these dynamics, defining neighborhoods of population states as well as ω-limit points in terms of the weak topology. We provided global convergence and local stability results for general deterministic dynamics in potential games, and a global asymptotic stability result for pairwise comparison dynamics in contractive games.

We defined a population game as a weakly continuous map $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$. One might ask whether we could generalize this definition, for example, by

i) defining a population game as a continuous map $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ (instead of a weakly continuous map);[46]

ii) defining a population game as a weakly continuous map $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathfrak{M}_b(\mathcal{S})$ (instead of $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$), where $\mathfrak{M}_b(\mathcal{S})$ denotes the space of bounded measurable functions on $\mathcal{S}$;

iii) defining a population game as a continuous map $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathfrak{M}_b(\mathcal{S})$ (instead of a weakly continuous map $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$).

The definition in (iii) should be the broadest definition of a population game. In all cases, we assume that F is continuous in the specified sense, because we would like that small changes in the population state only lead to small changes in the payoffs of strategies in a population game. In fact, our existence and uniqueness result (Theorem 1) still holds when any one of the above definitions is being used.

Since $F(\mu)$ is an assignment of payoffs to each strategy, one may consider $F(\mu)$ to be a bounded measurable function on $\mathcal{S}$, i.e., $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathfrak{M}_b(\mathcal{S})$, as in (ii) and (iii) above. However, for Nash stationarity (NS) and positive correlation (PC) (Proposition 1), we need $F(\mu) \in \mathcal{C}_b(\mathcal{S})$, which means that there are no jumps in the payoff profile and hence near-by strategies give similar payoffs. Which one of the assumptions, $F(\mu) \in \mathcal{C}_b(\mathcal{S})$ or $F(\mu) \in \mathfrak{M}_b(\mathcal{S})$, makes more sense depends on the application in question.

[46] Recall that *weakly continuous* is a stronger condition than *continuous*.

Finally, when we consider the global convergence and local stability of the dynamics, we need to assume that F is weakly continuous, because the weak topology is used in our definitions of stability and ω-limit points.

A Appendix

A.1 Lipschitz continuity of $\tilde{V}(\cdot)$

Lemma 5 *Let $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ be a population game, and let $\mu_0 = \mu(0) \in \mathcal{M}_1^+(\mathcal{S})$. Suppose that the conditions in Theorem 1 are satisfied. Let $V(\cdot)$ be defined by* (3)*, and $\tilde{V}(\cdot)$ be defined by* (5)*. We have the following:*

a) *$V(\psi)$ is Lipschitz continuous for $\|\psi\| \leq 2$, and*

b) *$\tilde{V}(\cdot)$ is Lipschitz continuous.*

We need to use some results from Appendix of Oechssler and Riedel (2001). Let ψ, ξ be finite positive measures. Let $\varphi = a\psi - b\xi$, for some $0 \leq a, b < \infty$, be a finite signed measure. Assume that there is a finite positive measure ν such that $\psi \ll \nu$ and $\xi \ll \nu$ (i.e., ψ and ξ are absolutely continuous with respect to ν), and hence $\varphi \ll \nu$. Then the Radon-Nikodym derivatives $\frac{d\psi}{d\nu}$, $\frac{d\xi}{d\nu}$ and $\frac{d\varphi}{d\nu} = a\frac{d\psi}{d\nu} - b\frac{d\xi}{d\nu}$ exist. We have

Fact 2 (cf. Oechssler and Riedel, 2001, Theorem 5) *The variational norm of φ is given by*

$$\|\varphi\| = \int_{\mathcal{S}} \left|\frac{d\varphi}{d\nu}\right| d\nu.$$

In particular,

$$\|\psi - \xi\| = \int_{\mathcal{S}} \left|\frac{d\psi}{d\nu} - \frac{d\xi}{d\nu}\right| d\nu.$$

Proof.[47] Let $g : \mathcal{S} \to \mathbb{R}$ be a measurable function bounded by 1 (i.e., $\sup_{s\in\mathcal{S}} |g(s)| \leq 1$). Then

$$\left|\int_{\mathcal{S}} g\, d\varphi\right| = \left|\int_{\mathcal{S}} g\frac{d\varphi}{d\nu}\, d\nu\right| \leq \int_{\mathcal{S}} \left|g\frac{d\varphi}{d\nu}\right| d\nu \leq \int_{\mathcal{S}} \left|\frac{d\varphi}{d\nu}\right| d\nu.$$

Taking sup over g, we have $\|\varphi\| \leq \int_{\mathcal{S}} \left|\frac{d\varphi}{d\nu}\right| d\nu$. To show equality, set $A = \{\frac{d\varphi}{d\nu} > 0\}$ and $g = \mathbb{1}_A - \mathbb{1}_{A^c}$. Then g is a measurable function bounded by 1, and hence

$$\|\varphi\| \geq \left|\int_{\mathcal{S}} g\, d\varphi\right| = \left|\int_A \frac{d\varphi}{d\nu}\, d\nu - \int_{A^c} \frac{d\varphi}{d\nu}\, d\nu\right| = \int_{\mathcal{S}} \left|\frac{d\varphi}{d\nu}\right| d\nu$$

because $\frac{d\varphi}{d\nu} = \left|\frac{d\varphi}{d\nu}\right|$ on A and $-\frac{d\varphi}{d\nu} = \left|\frac{d\varphi}{d\nu}\right|$ on A^c.

□

Proof of Lemma 5.

[47] The proof is simple and can be found in Oechssler and Riedel, 2001, Appendix A.1. We state it here for readers' convenience.

Part (a): Define $V^I : \mathcal{M}(\mathcal{S}) \to \mathcal{M}(\mathcal{S})$ and $V^O : \mathcal{M}(\mathcal{S}) \to \mathcal{M}(\mathcal{S})$ by

$$V^I(\psi)(A) := \int_{y\in A}\int_{z\in\mathcal{S}} \rho_{zy}(\tilde{F}(\psi))\,\psi(dz)\,\lambda(dy), \tag{28}$$

$$V^O(\psi)(A) := \int_{y\in A}\int_{z\in\mathcal{S}} \rho_{yz}(\tilde{F}(\psi))\,\lambda(dz)\,\psi(dy), \tag{29}$$

for $\psi \in \mathcal{M}(\mathcal{S})$ and $A \in \mathcal{B}$. Then $V(\psi) = V^I(\psi) - V^O(\psi)$ for any $\psi \in \mathcal{M}(\mathcal{S})$. Note that the definitions of $V^I(\cdot)$ and $V^O(\cdot)$ are different from those in (10) and (11), because now the domains are $\mathcal{M}(\mathcal{S})$ instead of $\mathcal{M}_1^+(\mathcal{S})$.

Let $\psi, \xi \in \mathcal{M}(\mathcal{S})$. In below, for any $\phi \in \mathcal{M}(S)$, we denote $|\phi| := \phi^+ + \phi^-$, where the positive measures ϕ^+ and ϕ^- form the Jordan decomposition of ϕ, i.e., $\phi = \phi^+ - \phi^-$. We show $V(\psi)$ is Lipschitz continuous for $\|\psi\| \le 2$ in two steps.

Step 1: Show $V^I(\psi)$ is Lipschitz for $\|\psi\| \le 2$.
Note that $V^I(\psi) \ll \lambda$ by (28). So the Radon-Nikodym derivative $\frac{dV^I(\psi)}{d\lambda}$ exists and

$$\frac{dV^I(\psi)}{d\lambda} = \int_{z\in\mathcal{S}} \rho_{z\cdot}(\tilde{F}(\psi))\,\psi(dz).$$

Using Fact 2 and (A), we have for any $\psi, \xi \in \mathcal{M}(\mathcal{S})$ with $\|\psi\|, \|\xi\| \le 2$,

$$\begin{aligned}
\|V^I(\psi) - V^I(\xi)\| &= E^\lambda\Big|\frac{dV^I(\psi)}{d\lambda} - \frac{dV^I(\xi)}{d\lambda}\Big| \\
&= \int_{y\in\mathcal{S}}\Big|\int_{z\in\mathcal{S}} \rho_{zy}(\tilde{F}(\psi))\,\psi(dz) - \int_{z\in\mathcal{S}} \rho_{zy}(\tilde{F}(\xi))\,\xi(dz)\Big|\,\lambda(dy) \\
&= \int_{y\in\mathcal{S}}\Big|\int_{z\in\mathcal{S}} \rho_{zy}(\tilde{F}(\psi))\,\psi(dz) - \int_{z\in\mathcal{S}} \rho_{zy}(\tilde{F}(\psi))\,\xi(dz) \\
&\qquad + \int_{z\in\mathcal{S}} \rho_{zy}(\tilde{F}(\psi))\,\xi(dz) - \int_{z\in\mathcal{S}} \rho_{zy}(\tilde{F}(\xi))\,\xi(dz)\Big|\,\lambda(dy) \\
&\le \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \rho_{zy}(\tilde{F}(\psi))\,|\psi-\xi|(dz)\,\lambda(dy) \\
&\quad + \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \big|\rho_{zy}(\tilde{F}(\psi)) - \rho_{zy}(\tilde{F}(\xi))\big|\,|\xi|(dz)\,\lambda(dy) \\
&\le M\int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} |\psi-\xi|(dz)\,\lambda(dy) \\
&\quad + K\|\psi-\xi\|\int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} |\xi|(dz)\,\lambda(dy) \\
&= M\cdot\lambda(\mathcal{S})\int_{\mathcal{S}} d|\psi-\xi| + K\|\psi-\xi\|\cdot\lambda(\mathcal{S})\int_{\mathcal{S}} d|\xi| \\
&\le M\lambda(\mathcal{S})\|\psi-\xi\| + K\lambda(\mathcal{S})\|\psi-\xi\|\|\xi\| \\
&\le (M+2K)\lambda(\mathcal{S})\|\psi-\xi\|.
\end{aligned}$$

Hence, $V^I(\psi)$ is Lipschitz for $\|\psi\| \le 2$.

Step 2: Show $V^O(\psi)$ is Lipschitz for $\|\psi\| \le 2$.

Note that $V^O(\psi) \ll |\psi|$ by (29).[48] So the Radon-Nikodym derivative $\frac{dV^O(\psi)}{d|\psi|}$ exists and

$$\frac{dV^O(\psi)}{d|\psi|} = \int_{z\in\mathcal{S}} \rho_{\cdot z}(\tilde{F}(\psi))\,\lambda(dz)\,\frac{d\psi}{d|\psi|}.$$

Let $\nu = \frac{|\psi|+|\xi|}{2}$. Then $|\psi|, |\xi|$ and ν are finite positive measures. Also, $\psi, |\psi|, \xi, |\xi| \ll \nu$. Hence, the Radon-Nikodym derivatives $\frac{d\psi}{d\nu}, \frac{d|\psi|}{d\nu}, \frac{d\xi}{d\nu}$ and $\frac{d|\xi|}{d\nu}$ exist. Since $V^O(\psi) \ll |\psi|$ and $|\psi| \ll \nu$, we have $V^O(\psi) \ll \nu$. Similarly, $V^O(\xi) \ll \nu$. So the Radon-Nikodym derivatives $\frac{dV^O(\psi)}{d\nu}$ and $\frac{dV^O(\xi)}{d\nu}$ exist. Using Fact 2 and (A), we have for any $\psi, \xi \in \mathcal{M}(\mathcal{S})$ with $\|\psi\|, \|\xi\| \le 2$,

$$\begin{aligned}
\|V^O(\psi) - V^O(\xi)\| &= E^\nu\Big|\frac{dV^O(\psi)}{d\nu} - \frac{dV^O(\xi)}{d\nu}\Big| \\
&= E^\nu\Big|\frac{dV^O(\psi)}{d|\psi|}\frac{d|\psi|}{d\nu} - \frac{dV^O(\xi)}{d|\xi|}\frac{d|\xi|}{d\nu}\Big| \\
&= \int_{y\in\mathcal{S}}\Big|\int_{z\in\mathcal{S}} \rho_{yz}(\tilde{F}(\psi))\,\lambda(dz)\frac{d\psi}{d|\psi|}(y)\frac{d|\psi|}{d\nu}(y) \\
&\qquad - \int_{z\in\mathcal{S}} \rho_{yz}(\tilde{F}(\xi))\,\lambda(dz)\frac{d\xi}{d|\xi|}(y)\frac{d|\xi|}{d\nu}(y)\Big|\,\nu(dy) \\
&= \int_{y\in\mathcal{S}}\Big|\int_{z\in\mathcal{S}} \rho_{yz}(\tilde{F}(\psi))\,\lambda(dz)\frac{d\psi}{d\nu}(y) - \int_{z\in\mathcal{S}} \rho_{yz}(\tilde{F}(\xi))\,\lambda(dz)\frac{d\xi}{d\nu}(y)\Big|\,\nu(dy) \\
&= \int_{y\in\mathcal{S}}\Big|\int_{z\in\mathcal{S}} \rho_{yz}(\tilde{F}(\psi))\,\lambda(dz)\frac{d\psi}{d\nu}(y) - \int_{z\in\mathcal{S}} \rho_{yz}(\tilde{F}(\psi))\,\lambda(dz)\frac{d\xi}{d\nu}(y) \\
&\qquad + \int_{z\in\mathcal{S}} \rho_{yz}(\tilde{F}(\psi))\,\lambda(dz)\frac{d\xi}{d\nu}(y) - \int_{z\in\mathcal{S}} \rho_{yz}(\tilde{F}(\xi))\,\lambda(dz)\frac{d\xi}{d\nu}(y)\Big|\,\nu(dy) \\
&\le \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \rho_{yz}(\tilde{F}(\psi))\,\lambda(dz)\Big|\frac{d\psi}{d\nu}(y) - \frac{d\xi}{d\nu}(y)\Big|\,\nu(dy) \\
&\qquad + \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \big|\rho_{yz}(\tilde{F}(\psi)) - \rho_{yz}(\tilde{F}(\xi))\big|\,\lambda(dz)\Big|\frac{d\xi}{d\nu}(y)\Big|\,\nu(dy) \\
&\le M\int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \lambda(dz)\Big|\frac{d\psi}{d\nu}(y) - \frac{d\xi}{d\nu}(y)\Big|\,\nu(dy) \\
&\qquad + K\|\psi-\xi\|\int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \lambda(dz)\Big|\frac{d\xi}{d\nu}(y)\Big|\,\nu(dy) \\
&= M\cdot\lambda(\mathcal{S})\int_{\mathcal{S}}\Big|\frac{d\psi}{d\nu} - \frac{d\xi}{d\nu}\Big|\,d\nu \;+\; K\|\psi-\xi\|\cdot\lambda(\mathcal{S})\int_{\mathcal{S}}\Big|\frac{d\xi}{d\nu}\Big|\,d\nu \\
&= M\lambda(\mathcal{S})\|\psi-\xi\| + K\lambda(\mathcal{S})\|\psi-\xi\|\|\xi\| \\
&\le (M+2K)\lambda(\mathcal{S})\|\psi-\xi\|.
\end{aligned}$$

Hence, $V^O(\psi)$ is Lipschitz for $\|\psi\| \le 2$.

From Steps 1 and 2, we conclude that $V(\psi)$ is Lipschitz for $\|\psi\| \le 2$. More precisely, for any $\psi, \xi \in \mathcal{M}(\mathcal{S})$ with $\|\psi\|, \|\xi\| \le 2$,

$$\|V(\psi) - V(\xi)\| = \|V^I(\psi) - V^O(\psi) - V^I(\xi) + V^O(\xi)\|$$

[48] Note that, for any $A \in \mathcal{B}$, $|\psi|(A) = 0 \Rightarrow \psi^+(A) = \psi^-(A) = 0 \Rightarrow V^O(\psi)(A) = 0$.

$$\begin{aligned}
&\leq \|V^I(\psi)-V^I(\xi)\|+\|V^O(\psi)-V^O(\xi)\| \\
&\leq (M+2K)\lambda(\mathcal{S})\|\psi-\xi\|+(M+2K)\lambda(\mathcal{S})\|\psi-\xi\| \\
&= 2(M+2K)\lambda(\mathcal{S})\|\psi-\xi\|. \qquad (30)
\end{aligned}$$

Part (b): To show $\tilde{V}(\cdot)$ is Lipschitz, we need to distinguish between three cases. Let $\psi, \xi \in \mathcal{M}(\mathcal{S})$.

Case 1: $\|\psi\|, \|\xi\| \geq 2$.
Then $\tilde{V}(\psi) = \tilde{V}(\xi) = 0$ and there is nothing to show.

Case 2: $\|\psi\| \geq 2 \geq \|\xi\|$.
Then $\tilde{V}(\psi) = 0$ and hence

$$\begin{aligned}
\|\tilde{V}(\psi)-\tilde{V}(\xi)\| &= (2-\|\xi\|)\|V(\xi)\| \\
&= (2-\|\xi\|)\|V^I(\xi)-V^O(\xi)\| \\
&\leq (2-\|\xi\|)(\|V^I(\xi)\|+\|V^O(\xi)\|).
\end{aligned}$$

First, we consider $\|V^I(\xi)\|$. We know from Step 1 of Part (a) that $V^I(\xi) \ll \lambda$ and $\frac{dV^I(\xi)}{d\lambda}$ exists, and

$$\frac{dV^I(\xi)}{d\lambda} = \int_{z\in\mathcal{S}} \rho_{z\cdot}(\tilde{F}(\xi))\,\xi(dz).$$

Using Fact 2 and (A), we have

$$\begin{aligned}
\|V^I(\xi)\| &= E^\lambda\Big|\frac{dV^I(\xi)}{d\lambda}\Big| \\
&= \int_{y\in\mathcal{S}}\Big|\int_{z\in\mathcal{S}} \rho_{zy}(\tilde{F}(\xi))\,\xi(dz)\Big|\,\lambda(dy) \\
&\leq \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \rho_{zy}(\tilde{F}(\xi))\,|\xi|(dz)\,\lambda(dy) \\
&\leq M\cdot\lambda(\mathcal{S})\int_{\mathcal{S}} d|\xi| \\
&\leq M\lambda(\mathcal{S})\|\xi\|.
\end{aligned}$$

Next, we consider $\|V^O(\xi)\|$. We know from Step 2 of Part (a) that $V^O(\xi) \ll |\xi|$ and $\frac{dV^O(\xi)}{d|\xi|}$ exists, and

$$\frac{dV^O(\xi)}{d|\xi|} = \int_{z\in\mathcal{S}} \rho_{\cdot z}(\tilde{F}(\xi))\,\lambda(dz)\,\frac{d\xi}{d|\xi|}.$$

Using Fact 2 and (A), we have

$$\begin{aligned}
\|V^O(\xi)\| &= E^{|\xi|}\Big|\frac{dV^O(\xi)}{d|\xi|}\Big| \\
&= \int_{y\in\mathcal{S}}\Big|\int_{z\in\mathcal{S}} \rho_{yz}(\tilde{F}(\xi))\,\lambda(dz)\frac{d\xi}{d|\xi|}(y)\Big|\,|\xi|(dy) \\
&\leq \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \rho_{yz}(\tilde{F}(\xi))\,\lambda(dz)\Big|\frac{d\xi}{d|\xi|}(y)\Big|\,|\xi|(dy)
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad M \cdot \lambda(\mathcal{S}) \int_{\mathcal{S}} \left| \frac{d\xi}{d|\xi|} \right| d|\xi| \\
&= \quad M\lambda(\mathcal{S})\|\xi\|.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\|V(\xi)\| &\leq \|V^I(\xi)\| + \|V^O(\xi)\| \\
&\leq M\lambda(\mathcal{S})\|\xi\| + M\lambda(\mathcal{S})\|\xi\| \\
&= 2M\lambda(\mathcal{S})\|\xi\| \qquad (31)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\tilde{V}(\psi) - \tilde{V}(\xi)\| &\leq (2 - \|\xi\|)(\|V^I(\xi)\| + \|V^O(\xi)\|) \\
&\leq (\|\psi\| - \|\xi\|)2M\lambda(\mathcal{S})\|\xi\| \\
&\leq 4M\lambda(\mathcal{S})\|\psi - \xi\|.
\end{aligned}
$$

So, in this case, $\tilde{V}(\cdot)$ is Lipschitz.

Case 3: $\|\psi\|, \|\xi\| \leq 2$.
Then

$$
\begin{aligned}
\|\tilde{V}(\psi) - \tilde{V}(\xi)\| &= \|\ (2 - \|\psi\|)V(\psi) - (2 - \|\xi\|)V(\xi)\ \| \\
&= \|\ (2 - \|\psi\|)V(\psi) - (2 - \|\psi\|)V(\xi) + (2 - \|\psi\|)V(\xi) - (2 - \|\xi\|)V(\xi)\ \| \\
&\leq (2 - \|\psi\|)\ \|V(\psi) - V(\xi)\| + \|V(\xi)\|\ \big|\|\psi\| - \|\xi\|\big|.
\end{aligned}
$$

Using (30) and (31), we have

$$
\begin{aligned}
\|\tilde{V}(\psi) - \tilde{V}(\xi)\| &\leq 2 \cdot 2(M + 2K)\lambda(\mathcal{S})\|\psi - \xi\| + 2M\lambda(\mathcal{S})\|\xi\| \cdot \|\psi - \xi\| \\
&\leq 4(M + 2K)\lambda(\mathcal{S})\|\psi - \xi\| + 4M\lambda(\mathcal{S})\|\psi - \xi\| \\
&= 8(M + K)\lambda(\mathcal{S})\|\psi - \xi\|.
\end{aligned}
$$

So, in this case, $\tilde{V}(\cdot)$ is Lipschitz.

Therefore, we conclude that $\tilde{V}(\cdot)$ is Lipschitz with Lipschitz constant $\tilde{K} = 8(M + K)\lambda(\mathcal{S})$.

□

A.2 Some theorems from dynamical systems theory

Let

$$
\dot{\mu} = V(\mu) \qquad \text{(D)}
$$

be a differential equation on $\mathcal{M}_1^+(\mathcal{S})$ that admits a unique forward solution from each initial condition, and suppose that solutions to (D) are continuous in their initial conditions. We have the following two theorems:

Theorem 5 (cf. Sandholm, 2010a, Theorem 7.B.3) *Let $Y \subseteq \mathcal{M}_1^+(\mathcal{S})$, and let $L : Y \to \mathbb{R}$ be a decreasing Lyapunov function for* (D)*, i.e., L is weakly continuous and Fréchet-differentiable with $\dot{L}(\mu) \equiv \langle \nabla L(\mu), V(\mu) \rangle \leq 0$ for all $\mu \in Y$. Then $\omega(\mu_0) \subseteq \{\mu \in Y : \dot{L}(\mu) = 0\}$ for all $\mu_0 \in Y$. Thus, if $\dot{L}(\mu) = 0$ implies $V(\mu) = 0$ (i.e., L is a strict Lyapunov function), then $\omega(\mu_0) \subseteq RP(V) \cap Y$.*

The proof of Theorem 5 follows similarly as that of Theorem 7.B.3 in Sandholm (2010a), and thus is omitted. The conclusions of Theorem 5 are the same if L is an increasing Lyapunov function for (D).

Theorem 6 (Lyapunov's Theorem)(cf. Oechssler and Riedel, 2002, Proposition 6)[49] *Let $Z \subseteq \mathcal{M}_1^+(\mathcal{S})$ be a closed set, and let $Y \subseteq \mathcal{M}_1^+(\mathcal{S})$ be a neighborhood of Z in the weak topology. Let $L : Y \to \mathbb{R}_+$ be a decreasing Lyapunov function for* (D)*, i.e., L is weakly continuous and Fréchet-differentiable with $\dot{L}(\mu) \equiv \langle \nabla L(\mu), V(\mu) \rangle \leq 0$ for all $\mu \in Y$. Suppose that $L^{-1}(0) = Z$. Then Z is Lyapunov stable under* (D).

Proof.[50] Let $\varepsilon > 0$ be such that $\mathrm{cl}(Z^\varepsilon) \equiv \{\mu \in \mathcal{M}_1^+(\mathcal{S}) : \kappa(\mu, Z) \leq \varepsilon\} \subseteq Y$. The boundary of $\mathrm{cl}(Z^\varepsilon)$, denoted by $\mathrm{bd}(Z^\varepsilon)$, is closed and hence compact in the weak topology. Since L is continuous with respect to the weak topology, $m := \min_{\mu \in \mathrm{bd}(Z^\varepsilon)} L(\mu)$ exists, and $m > 0$. Now choose $\delta > 0$ such that $\kappa(\mu, Z) < \delta$ implies $L(\mu) < m$. If $(\mu(t))$ is a solution trajectory of (D) with $\kappa(\mu(0), Z) < \delta$, then $L(\mu(t))$ decreases in t because $\dot{L}(\mu) \leq 0$ for all $\mu \in \mathcal{M}_1^+(\mathcal{S})$. Hence, we have $L(\mu(t)) < m$ for all $t \geq 0$, and thus $\kappa(\mu(t), Z) < \varepsilon$ for all $t \geq 0$.

□

The above two theorems imply the following corollary:

Corollary 2 (cf. Sandholm, 2010a, Corollary 7.B.6) *Let $Z \subseteq \mathcal{M}_1^+(\mathcal{S})$ be a closed set, and let $Y \subseteq \mathcal{M}_1^+(\mathcal{S})$ be a neighborhood of Z in the weak topology. Let $L : Y \to \mathbb{R}_+$ be a decreasing Lyapunov function for* (D)*, i.e., L is weakly continuous and Fréchet-differentiable with $\dot{L}(\mu) \equiv \langle \nabla L(\mu), V(\mu) \rangle \leq 0$ for all $\mu \in Y$. Suppose that $L^{-1}(0) = Z$. If $\dot{L}(\mu) < 0$ for all $\mu \in Y \backslash Z$, then Z is asymptotically stable under* (D)*. If in addition $Y = \mathcal{M}_1^+(\mathcal{S})$, then Z is globally asymptotically stable under* (D).

A.3 Some proofs from Section 6

Proof of the Claim in Example 5. To prove the Claim, it suffices to show that

$$\langle DF(\psi)\zeta, \zeta \rangle < 0, \ \forall \psi \in \mathcal{M}_1^+(\mathcal{S}) \text{ and } \zeta \in \mathcal{M}_0(\mathcal{S}) \text{ with } \zeta \neq 0. \tag{32}$$

First, we find $DF(\psi)\zeta$. Let $\psi \in \mathrm{int}(\mathcal{M}_1^+(\mathcal{S}))$, $\zeta \in \mathcal{M}_0(\mathcal{S})$, and let $\varepsilon > 0$ be small enough such that $\psi + \varepsilon\zeta \in \mathcal{M}_1^+(\mathcal{S})$. Then

$$\begin{aligned} F_x(\psi + \varepsilon\zeta) &= \int_{\mathcal{S}} h(x,y)\, (\psi + \varepsilon\zeta)(dy) \\ &= \int_{\mathcal{S}} h(x,y)\, \psi(dy) + \varepsilon \int_{\mathcal{S}} h(x,y)\, \zeta(dy) \\ &= F_x(\psi) + \varepsilon \int_{\mathcal{S}} h(x,y)\, \zeta(dy). \end{aligned}$$

So, we have

$$(DF(\psi)\zeta)(x) = \int_{\mathcal{S}} h(x,y)\, \zeta(dy). \tag{33}$$

[49] See also Sandholm, 2010a, Theorem 7.B.2.

[50] The proof can be found in Oechssler and Riedel, 2002, Appendix B. We state it here for the sake of clarity because the definitions of Lyapunov function in Oechssler and Riedel (2002) and in the present paper are a bit different. Also, we generalize the theorem a bit from "a measure is Lyapunov stable" to "a closed set of measures is Lyapunov stable".

Next, we show (32). Since μ is an interior NE, μ has full support. So, we have $F_z(\mu) = F_x(\mu)$ for all $x, z \in \mathcal{S}$, which implies $\int_{\mathcal{S}} F_z(\mu)\,\mu(dz) = F_x(\mu)$ for all $x \in \mathcal{S}$, and hence $\int_{\mathcal{S}} F_z(\mu)\,\mu(dz) = \int_{\mathcal{S}} F_x(\mu)\,\psi(dx)$ for all $\psi \in \mathcal{M}_1^+(\mathcal{S})$, i.e.,

$$\int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\mu(dy)\,\mu(dx) = \int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\mu(dy)\,\psi(dx),\ \forall \psi \in \mathcal{M}_1^+(\mathcal{S}). \tag{34}$$

So, by condition (ii), for any $\psi \neq \mu$,

$$\int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\psi(dy)\,\mu(dx) > \int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\psi(dy)\,\psi(dx),$$

which implies

$$\int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\psi(dy)\,(\psi-\mu)(dx) < 0,\ \forall \psi \neq \mu. \tag{35}$$

From (34), we have

$$\int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\mu(dy)\,(\psi-\mu)(dx) = 0,\ \forall \psi \in \mathcal{M}_1^+(\mathcal{S}). \tag{36}$$

Subtracting (36) from (35) yields

$$\int_{\mathcal{S}}\Big(\int_{\mathcal{S}} h(x,y)\,\psi(dy) - \int_{\mathcal{S}} h(x,y)\,\mu(dy)\Big)\,(\psi-\mu)(dx) < 0,\ \forall \psi \neq \mu,$$

i.e.,

$$\int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,(\psi-\mu)(dy)\,(\psi-\mu)(dx) < 0,\ \forall \psi \neq \mu. \tag{37}$$

Since $\mu \in \mathrm{int}(\mathcal{M}_1^+(\mathcal{S}))$, any $\zeta \in \mathcal{M}_0(\mathcal{S})$ with $\zeta \neq 0$ is proportional to $\psi - \mu$ for some $\psi \in \mathcal{M}_1^+(\mathcal{S})$ with $\psi \neq \mu$. So, (37) implies that

$$\int_{\mathcal{S}}\int_{\mathcal{S}} h(x,y)\,\zeta(dy)\,\zeta(dx) < 0,\ \forall \zeta \in \mathcal{M}_0(\mathcal{S}) \text{ with } \zeta \neq 0.$$

Then, by (33), we have

$$\int_{\mathcal{S}} (DF(\psi)\zeta)(x)\,\zeta(dx) < 0,\ \forall \psi \in \mathcal{M}_1^+(\mathcal{S}) \text{ and } \zeta \in \mathcal{M}_0(\mathcal{S}) \text{ with } \zeta \neq 0,$$

i.e.,

$$\langle DF(\psi)\zeta, \zeta\rangle < 0,\ \forall \psi \in \mathcal{M}_1^+(\mathcal{S}) \text{ and } \zeta \in \mathcal{M}_0(\mathcal{S}) \text{ with } \zeta \neq 0.$$

□

Proof of Theorem 4. Sign-preservation (SP) and impartiality (IP) together imply that

$$\mathrm{sgn}(\tau_y(d)) = \mathrm{sgn}([d]_+),\quad \forall d \in \mathbb{R}.$$

The first claim is proved as follows:

$$\begin{aligned}\mu \in NE(F) &\Leftrightarrow F_z(\mu) \geq F_y(\mu),\ \forall z \in S(\mu),\ \forall y \in \mathcal{S} \\ &\Leftrightarrow \eta_y(F_y(\mu) - F_z(\mu)) = 0,\ \forall z \in S(\mu),\ \forall y \in \mathcal{S} \\ &\Leftrightarrow H(\mu) = 0.\end{aligned}$$

The last "⇔" follows from the continuity of $\eta_y(d)$ in y and d.

To prove the second claim, we calculate $\dot{H}(\mu)$.

$$\begin{aligned}\dot{H}(\mu) &= \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \tau_y(F_y(\mu) - F_z(\mu))[(DF(\mu)\dot{\mu})(y) - (DF(\mu)\dot{\mu})(z)]\,\mu(dz)\,\lambda(dy) \\ &\quad + \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \eta_y(F_y(\mu) - F_z(\mu))\,\dot{\mu}(dz)\,\lambda(dy) \\ &= \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \rho_{zy}(F(\mu))(DF(\mu)\dot{\mu})(y)\,\mu(dz)\,\lambda(dy) \\ &\quad - \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \rho_{zy}(F(\mu))(DF(\mu)\dot{\mu})(z)\,\mu(dz)\,\lambda(dy) \\ &\quad + \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \eta_y(F_y(\mu) - F_z(\mu))\,\dot{\mu}(dz)\,\lambda(dy). \qquad (38)\end{aligned}$$

We will consider the three terms on the RHS of (38) separately.

Recall the functions $V^I : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{M}(\mathcal{S})$ and $V^O : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{M}(\mathcal{S})$ defined by (10) and (11) respectively in the proof of Lemma 1. Using (12), we have

$$\begin{aligned}&\int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \rho_{zy}(F(\mu))(DF(\mu)\dot{\mu})(y)\,\mu(dz)\,\lambda(dy) \\ =\ &\int_{y\in\mathcal{S}} (DF(\mu)\dot{\mu})(y) \int_{z\in\mathcal{S}} \rho_{zy}(F(\mu))\,\mu(dz)\,\lambda(dy) \\ =\ &\int_{y\in\mathcal{S}} (DF(\mu)\dot{\mu})(y) \frac{dV^I(\mu)}{d\lambda}(y)\,\lambda(dy) \\ =\ &\int_{y\in\mathcal{S}} (DF(\mu)\dot{\mu})(y)\,V^I(\mu)(dy) \\ =\ &\int_{\mathcal{S}} DF(\mu)\dot{\mu}\,dV^I(\mu).\end{aligned}$$

Using (13), we have

$$\begin{aligned}&\int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \rho_{zy}(F(\mu))(DF(\mu)\dot{\mu})(z)\,\mu(dz)\,\lambda(dy) \\ =\ &\int_{z\in\mathcal{S}} (DF(\mu)\dot{\mu})(z) \int_{y\in\mathcal{S}} \rho_{zy}(F(\mu))\,\lambda(dy)\,\mu(dz) \\ =\ &\int_{z\in\mathcal{S}} (DF(\mu)\dot{\mu})(z) \frac{dV^O(\mu)}{d\mu}(z)\,\mu(dz) \\ =\ &\int_{z\in\mathcal{S}} (DF(\mu)\dot{\mu})(z)\,V^O(\mu)(dz) \\ =\ &\int_{\mathcal{S}} DF(\mu)\dot{\mu}\,dV^O(\mu).\end{aligned}$$

Hence, by Lemma 4 and the fact that

$$\dot{\mu} = V^F(\mu) = V^I(\mu) - V^O(\mu) \in \mathcal{M}_0(\mathcal{S}), \tag{39}$$

we have

$$\begin{aligned}
& \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \rho_{zy}(F(\mu))(DF(\mu)\dot{\mu})(y)\,\mu(dz)\,\lambda(dy) - \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \rho_{zy}(F(\mu))(DF(\mu)\dot{\mu})(z)\,\mu(dz)\,\lambda(dy) \\
= & \int_{\mathcal{S}} DF(\mu)\dot{\mu}\,dV^I(\mu) - \int_{\mathcal{S}} DF(\mu)\dot{\mu}\,dV^O(\mu) \\
= & \int_{\mathcal{S}} DF(\mu)V^F(\mu)\,dV^F(\mu) \\
= & \langle DF(\mu)V^F(\mu), V^F(\mu)\rangle \\
\leq & \ 0.
\end{aligned}$$

Denote by $K(\mu)$ the last term on the RHS of (38). Using (12), (13) and (39), we have

$$\begin{aligned}
K(\mu) &= \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \eta_y(F_y(\mu) - F_z(\mu))\,\dot{\mu}(dz)\,\lambda(dy) \\
&= \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \eta_y(F_y(\mu) - F_z(\mu))\,V^I(\mu)(dz)\,\lambda(dy) - \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \eta_y(F_y(\mu) - F_z(\mu))\,V^O(\mu)(dz)\,\lambda(dy) \\
&= \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \eta_y(F_y(\mu) - F_z(\mu))\frac{dV^I(\mu)}{d\lambda}(z)\,\lambda(dz)\,\lambda(dy) \\
&\quad - \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \eta_y(F_y(\mu) - F_z(\mu))\frac{dV^O(\mu)}{d\mu}(z)\,\mu(dz)\,\lambda(dy) \\
&= \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \eta_y(F_y(\mu) - F_z(\mu))\int_{x\in\mathcal{S}} \rho_{xz}(F(\mu))\,\mu(dx)\,\lambda(dz)\,\lambda(dy) \\
&\quad - \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \eta_y(F_y(\mu) - F_z(\mu))\int_{x\in\mathcal{S}} \rho_{zx}(F(\mu))\,\lambda(dx)\,\mu(dz)\,\lambda(dy) \\
&= \int_{y\in\mathcal{S}}\int_{x\in\mathcal{S}} \eta_y(F_y(\mu) - F_x(\mu))\int_{z\in\mathcal{S}} \rho_{zx}(F(\mu))\,\mu(dz)\,\lambda(dx)\,\lambda(dy) \\
&\quad - \int_{y\in\mathcal{S}}\int_{z\in\mathcal{S}} \eta_y(F_y(\mu) - F_z(\mu))\int_{x\in\mathcal{S}} \rho_{zx}(F(\mu))\,\lambda(dx)\,\mu(dz)\,\lambda(dy) \\
&= \int_{z\in\mathcal{S}}\int_{x\in\mathcal{S}}\int_{y\in\mathcal{S}} [\eta_y(F_y(\mu) - F_x(\mu)) - \eta_y(F_y(\mu) - F_z(\mu))]\rho_{zx}(F(\mu))\,\lambda(dy)\,\lambda(dx)\,\mu(dz) \\
&= \int_{z\in\mathcal{S}}\int_{x\in\mathcal{S}} \rho_{zx}(F(\mu))\int_{y\in\mathcal{S}} [\eta_y(F_y(\mu) - F_x(\mu)) - \eta_y(F_y(\mu) - F_z(\mu))]\,\lambda(dy)\,\lambda(dx)\,\mu(dz).
\end{aligned}$$

To evaluate the last expression, first observed that, if $F_x(\mu) > F_z(\mu)$, then $\rho_{zx}(F(\mu)) > 0$ and $F_y(\mu) - F_x(\mu) < F_y(\mu) - F_z(\mu)$; since each η_y is nondecreasing, it follows that $\eta_y(F_y(\mu) - F_x(\mu)) - \eta_y(F_y(\mu) - F_z(\mu)) \leq 0$. In fact, when $y = x$, the comparison between payoff differences becomes $0 < F_y(\mu) - F_z(\mu)$; since each η_y is strictly increasing on $[0, \infty)$, it follows that $\eta_y(0) - \eta_y(F_y(\mu) - F_z(\mu)) < 0$. Since λ has full support, by continuity, we have $\int_{y\in\mathcal{S}}[\eta_y(F_y(\mu) - F_x(\mu)) - \eta_y(F_y(\mu) - F_z(\mu))]\,\lambda(dy) < 0$. One can therefore conclude that, if $F_x(\mu) > F_z(\mu)$, then $\rho_{zx}(F(\mu)) > 0$ and $\int_{y\in\mathcal{S}}[\eta_y(F_y(\mu) - F_x(\mu)) - \eta_y(F_y(\mu) - F_z(\mu))]\,\lambda(dy) < 0$. On the other hand, if $F_z(\mu) \geq F_x(\mu)$, we immediately have $\rho_{zx}(F(\mu)) = 0$. Altogether, we have

$$K(\mu) \equiv \int_{z\in\mathcal{S}} \int_{x\in\mathcal{S}} \rho_{zx}(F(\mu)) \int_{y\in\mathcal{S}} [\eta_y(F_y(\mu) - F_x(\mu)) - \eta_y(F_y(\mu) - F_z(\mu))]\, \lambda(dy)\, \lambda(dx)\, \mu(dz) \leq 0,$$

with equality if and only if

$$\rho_{zx}(F(\mu)) = 0, \quad \forall z \in S(\mu),\ \forall x \in \mathcal{S},$$

i.e., $\mu \in NE(F)$ (see (15)).

By the above calculation,

$$\dot{H}(\mu) = \langle DF(\mu)V^F(\mu), V^F(\mu)\rangle + K(\mu), \tag{40}$$

and both terms on the RHS of (40) are nonpositive. Since this is true for any $\mu \in \mathcal{M}_1^+(\mathcal{S})$, we have $\dot{H}(\mu) \leq 0$ for all $\mu \in \mathcal{M}_1^+(\mathcal{S})$. By Proposition 1, Nash stationarity (NS) is satisfied and thus $NE(F) = RP(V^F)$. So, if $\dot{H}(\mu) = 0$, then $K(\mu) = 0$, which implies $\mu \in NE(F)$. For the reverse direction, if $\mu \in NE(F)$, then $K(\mu) = 0$; also, since $\mu \in NE(F) = RP(V^F)$, we have $V^F(\mu) = 0$, which implies $\langle DF(\mu)V^F(\mu), V^F(\mu)\rangle = 0$ and hence $\dot{H}(\mu) = 0$. Therefore, $\dot{H}(\mu) = 0$ if and only if $\mu \in NE(F)$, and so the second claim is proved. Finally, since $F : \mathcal{M}_1^+(\mathcal{S}) \to \mathcal{C}_b(\mathcal{S})$ is continuous with respect to the weak topology, $H : \mathcal{M}_1^+(\mathcal{S}) \to \mathbb{R}_+$ is also continuous with respect to the weak topology. Hence, the function $H : \mathcal{M}_1^+(\mathcal{S}) \to \mathbb{R}_+$ acts as a decreasing strict Lyapunov function for the dynamic. Therefore, by Corollary 2 in Appendix A.2, $NE(F)$ is globally asymptotically stable. □

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