PREEMPTION GAMES UNDER LÉVY UNCERTAINTY

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ABSTRACT. We study a stochastic version of Fudenberg–Tirole's preemption game. Two firms contemplate entering a new market with stochastic demand. Firms differ in sunk costs of entry. If the demand process has no upward jumps, the low cost firm enters first, and the high cost firm follows. If leader's optimization problem has an interior solution, the leader enters at the optimal threshold of a monopolist; otherwise, the leader enters earlier than the monopolist. If the demand admits positive jumps, then the optimal entry threshold of the leader can be lower than the monopolist's threshold even if the solution is interior; simultaneous entry can happen either as an equilibrium or a coordination failure; the high cost firm can become the leader. We characterize subgame perfect equilibrium strategies in terms of stopping times and value functions. Analytical expressions for the value functions and thresholds that define stopping times are derived.

Keywords: stopping time games, preemption, Lévy uncertainty

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1. INTRODUCTION

Stopping time games have important applications in economics and finance (see, for instance, review [48] and collection of papers [35]), such as, for example, investment timing (e.g., [18, 29, 37, 52]), capital accumulation (e.g., [4, 6, 29, 49]), product innovations (see, e.g., [21, 22]), asset sales (e.g., [21, 34]), pricing of convertible bonds (e.g., [46]), patenting (e.g., [32, 54]). Stopping time games arise also in the case when the firm's manager has time-inconsistent preferences and therefore, there is a game between her long-run self and multiple short-run selves (see, for example, [30] and

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references therein). Finally, games of strategic experimentation, which are used to represent social learning, are also stopping games (see, for example, [8, 33, 40, 41], and references therein).

In stopping time games, which are a special case of stochastic games, at each instant each player has only two available strategies: "wait" and "stop". The latter strategy is irreversible. In stopping games with two players, if one of the players plays "stop", then there are no strategic interactions from that moment on, and the game is either terminated or the players' payoffs become predetermined. If both players wait, the game environment evolves according to an exogenous stochastic process. If player i stops earlier than player j, then player i is the leader, and player j is the follower. The stopping time game for two players was initially formulated by Dynkin [23], as a generalization of optimal stopping problems for the case of a zero sum game in discrete time, and later generalized by various authors: see, for example, [24, 25, 34, 38, 48, 53] and the references therein. When players are restricted to stopping times, the value of the game does not necessarily exist, and the main part of the research on stopping games is focused on existence of the value of the game and optimal stopping strategies. While existence is extremely important as a theoretical question, deriving optimal stopping strategies and pricing game options is not less important, especially for economic and finance applications. Few of the existing papers dealing with game options provide analytical solutions for zero sum games (see, e.g., [2, 3, 36] and references therein). Optimal stopping problems are essential in stopping time games because an equilibrium can be computed by solving optimization problems of each agent separately as opposed to finding a fixed point of a certain mapping as, for example, in [4, 19, 20, 43], which is a less tractable approach. When entry thresholds of the firms are derived analytically, it becomes possible to calculate the probabilities of potentially observable outcomes of the preemption game as well as the expected waiting time until such outcomes may be observed, given the current state of the demand. Such calculations are important in identification models of oligopoly entry and, more generally, for estimation of dynamic games (see, for example, [1, 7, 44] and references therein).

Most of the existing literature on stopping games has assumed Gaussian uncertainty, for the sake of tractability. In this paper, we derive equilibrium strategies (in closed form) in preemption games with asymmetric players under jump-diffusion uncertainty. As an example, we consider two firms choosing optimal entry into a new market under demand uncertainty. Firms differ in the sunk costs of entry (as in [42]). A dynamic perspective on demand must allow for fluctuations for a number of reasons, including business cycle effects (e.g., Mankiw [39]) and industry-specific variability (e.g., Ghemawat and Nalebuff [27]). These demand fluctuations are often of a discontinuous nature when triggered by the advent of busts and booms, technology shocks, or new product introduction. In addition, for network goods, such as electronic devices, demand may jump up dramatically as soon as the "critical mass" of consumers using the same device or application reaches a certain level (see, e.g., Amir and Lazzati [5]). Cost asymmetry can be motivated by various factors such as regulations, liquidity constraints, credit history, etc (see [42] for other factors). We demonstrate that asymmetry between the players and jump-diffusion uncertainty can produce new non-trivial effects on equilibria in the preemption game. In order to focus on the effect of positive jumps on equilibrium strategies, we restrict our attention to the case of two players, but the framework of the paper can be extended to the case of multiple players. Such an extension leads to an interesting "accordion effect"studied in Bouis et al. [10] in the Gaussian setting: an exogenous demand shock results in a change of the wedge between investment thresholds (in case of sequential investment) of the odd and even numbered investors so that odd numbered firms delay and even numbered firms accelerate their investments.

Fudenberg and Tirole [26] developed equilibrium concepts for a continuous time preemption game for the case of symmetric players in a deterministic environment. Preemption games are a subset of stopping time games. In preemption games, both players have the first movers advantage in some region of the state space, called the preemption zone. In the preemption zone, it may happen that it is optimal to move for one player, but not for both, hence there is a coordination problem: who should move first. If players are symmetric, then there is a unique symmetric equilibrium in the preemption zone: each of the players may become the leader and the follower with positive probabilities, and the probability of simultaneous entry (coordination failure) is also positive for a game that starts inside the preemption zone. If the game starts when immediate action is not optimal for either player, then it ends as soon as the left boundary of the preemption zone is reached. In this case, each of the players becomes the leader with probability one half and the follower with probability one half; and the probability of simultaneous action is zero. Moreover, at the boundaries of the preemption zone, the value functions of the leader and the follower are the same, therefore, the players are indifferent between becoming the first or the second mover. This is called rent equalization.

The Fudenberg and Tirole [26] model was generalized for the case of non-standard payoff functions of the first mover by Hoppe and Lehmann-Grube [31] and also extended to a stochastic environment (see, for example, [18, 21, 28, 29, 50, 51, 52, 54] and references therein). The source of uncertainty in preemption games is typically a demand shock, and in the initial state the realization of the shock is low so that immediate action is not optimal. If the underlying uncertainty is Gaussian or there are no upward jumps in the stochastic process, then eventually, the left boundary of the preemption zone is reached, and the outcome is qualitatively the same as in the deterministic case. If the stochastic process admits upward jumps, then any point in the preemption zone can be reached. Furthermore, the preemption zone can be overshot, and then the unique equilibrium is when the players move simultaneously. Thijssen et al. [51] characterize symmetric subgame perfect equilibria in the preemption game (as in [26]) under Lévy uncertainty. However, their result is incomplete, because they characterize equilibrium strategies in terms of value functions, but do not provide expressions for the value functions in the general case. Moreover, Thijssen et al. [51] describe possible equilibria only in case when the stochastic process enters the premption zone earlier than any other action regions. Thijssen [50] studies preemption game with player specific Gaussian uncertainty.

Pawlina and Kort [42] consider the preemption game with asymmetric players under Gaussian uncertainty. They analyze the situation where two firms contemplating investment into a profit enhancing project differ in the sunk cost of investment. The cost asymmetry in [42] uniquely defines the role of the firms provided that the initial realization of the shock is low so that immediate investment is not optimal. The low cost firm is the leader and the high cost firm is the follower. In [42], strategies of players in the preemption zone are irrelevant, because the game ends no later than the left boundary of the preemption zone is reached. If the low cost firm had not invested earlier, then, at the boundary of the preemption zone, this firm strictly prefers to be the leader, and the high cost firm is indifferent between being the leader and the follower, therefore the low cost firm moves first, and coordination failure does not happen.

The present paper follows [42] in assuming that the initial realization of the demand shock is low enough so that the immediate investment is optimal to none of the firms (inaction region in the state space). As the stochastic demand increases, three scenarios may be possible: (i) only one of the players (the low cost firm) has the first mover's advantage; (ii) both players have the first mover's advantage, but simultaneous action is the worst outcome; (iii) simultaneous action is (weakly) optimal. Each of this scenarios corresponds to a certain action region of the state space. If the demand process is spectrally negative (i.e., there are no upward jumps), then qualitatively the results are the same as in [42] (see [17] for details). Namely, if the cost disadvantage is sufficiently high, then, if the shock ever enters an action region, it always enters the action region of the low cost firm earlier than any other action regions, and a sequential equilibrium (as in [42]) is played. If the sequential equilibrium happens, there are no strategic interactions in the sense that the low cost firm chooses its investment threshold as if it would be the monopolist in the market even though the payoff of the leader is affected by the future entry of the follower. If the cost disadvantage is not high, then, if the shock ever enters an action region, it always enters the preemption zone earlier than any other action regions, and a preemptive equilibrium (as in [42]) happens. In order to preempt the investment of the high cost firm and enjoy the first mover's advantage, the low cost firm has to invest earlier (that is as soon as the left boundary of the preemption zone is reached) than would have been optimal had the other firm precommitted to be the follower. Hence, as in [42], the roles of the firms are predetermined.

The spectrum of equilibria changes dramatically if the underlying process admits non-trivial positive jumps, and the roles of the players are no longer predetermined. If the cost disadvantage is sufficiently high, so that the high cost firm has no incentives to become the leader, preemption never occurs (the preemption zone is empty). In this case, the following scenarios are possible: (i) the demand shock may jump into a region where investment becomes optimal for the low cost firm but not for the high cost firm, then the sequential equilibrium will be played; (ii) alternatively, the shock may jump into a region where simultaneous entry is an equilibrium. In the first case, the low cost firm enjoys the first mover's advantage until the other firm follows.

If the cost disadvantage is not very high, then the preemption zone is not empty, and the stochastic demand may jump directly into this zone. In the preemption zone, there are three types of subgame perfect equilibria: (1) both firms enter with positive probabilities, and probability of coordination failure is positive; (2) the low cost firm enters as the leader with probability 1, and the high cost firm enters as the follower when the shock reaches its optimal entry threshold of the follower; (3) the high cost firm enters as the leader with probability 1, and the low cost firm enters as the follower when the shock reaches its optimal entry threshold of the follower.

If type (2) equilibrium is played in the preemption zone, then three equilibria are possible: (i) sequential equilibrium, which happens if the shock first reaches the region where only the low cost firm has first mover's advantage, (ii) preemptive equilibrium, with the low cost firm being the leader, and the high cost firm being the follower; this equilibrium happens if the shock first enters the preemption zone; (iii) simultaneous entry as described earlier. If type (1) or (3) equilibria are played in the preemption zone, then preemptive, sequential, and simultaneous entry equilibria are possible. If the shock first enters the action region of the low cost firm then this firm may enter with or without preemptive motives. If the shock first enters the preemption zone, then, in case of type (3) equilibrium, the high cost firm preempts with probability one; in case of type (1) equilibrium, any of the firms may preempt, or coordination failure occurs. For some realizations of uncertainty, the high cost firm enters earlier than or simultaneously with the low cost firm, and the low cost firm may not choose its optimal entry threshold as if the follower had not existed. The low cost firm has to take into consideration a positive probability of the event that the process will enter the preemption zone earlier than other action regions which will result in this firm becoming the follower with a positive probability and to adjust the entry threshold accordingly. Therefore strategic interactions do matter; and the sequential equilibrium where the low cost firm can make the entry decision without taking into consideration the future entry of the follower no longer exists. The low cost firm chooses the entry threshold that maximizes its value function subject to the constraint that the threshold is not higher than the left boundary of the preemption zone. When this problem has an interior solution, then the firm's optimal entry threshold is always lower than the optimal entry threshold of the monopolist. Therefore dynamic entry models that predict the number of firms in a market and use the monopolist's entry threshold to calculate the probability of the event that the low cost firm enters the market may come up with an incorrect probability if positive jumps in consumers' demand are possible.

Assuming that in each point of the preemption zone the same type of equilibria is played, we characterize the set of subgame perfect equilibria in terms of the value functions and optimal stopping times of the players. All optimal stopping times are of the threshold type; we prove their optimality in the class of all stopping times. We provide analytical solutions both for the exercise thresholds and value functions

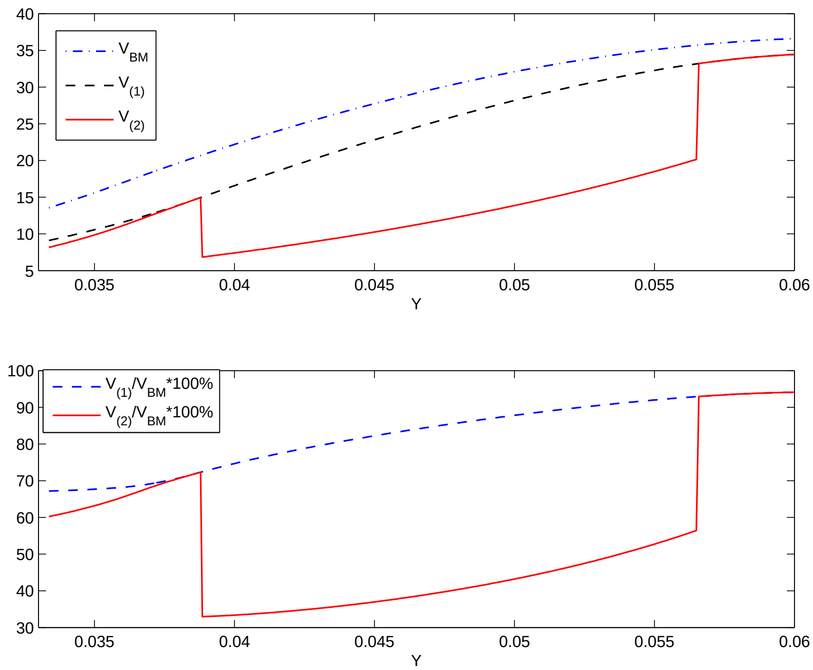

FIGURE 1. Upper panel: value functions of the low cost firm. Dash-dots – the demand shocks follow BM; for the BM with embedded positive jumps, dashes – type (2) equilibrium in the preemption zone; solid line – type (3) equilibrium in the preemption zone. Lower panel: value function of the low cost firm in case of the BM with embedded positive jumps as percentage of its value in the BM model. Dashes – type (2) equilibrium; solid line – type (3) equilibrium. $Y = e^x$ – demand shock. Annual discount rate is 5%.

using an efficient general methodology presented in [16]. This methodology allows one to solve optimal stopping problems under Lévy uncertainty in situations when payoff functions are non-monotone and/or discontinuous. In entry-exit problems with strategic interactions under non-Gaussian uncertainty and other stopping games, non-monotone payoffs are the rule rather than exception. In our model, the value function of the leader is non-monotone, and in addition, if type (2) or (3) equilibria are played in the preemption zone, then the value function of the low cost firm is discontinuous.

To illustrate importance of positive jumps, we compare the value functions of the low cost firm when (i) the log demand shocks follow the Brownian motion (BM) with zero drift and variance 4 per cent (common assumptions in the real options literature, see, e.g., [18]), (ii) the log demand shocks follow the BM with embedded positive jumps, and type (2) equilibrium is played in the preemption zone; (iii) the (log) demand shocks are as in (ii), and type (1) or (3) equilibrium is played in the preemption zone. To have a meaningful comparison, we keep the first (m_1) and second

(m_2) moments of the log of the stochastic factor fixed at $m_2 = 0.04, m_1 = -m_2/2$. We consider the situation where the cost disadvantage is moderately high (the sunk cost of investment of the high cost firm is 15 per cent higher than the investment cost of the low cost firm), so that the preemption zone exists, but the optimal entry threshold of the low cost firm is an interior solution to the value maximization problem. We assume that the probability of a positive jump is 15 per cent per year, and the average size of a positive jump is 20 per cent. Assume that the instantaneous revenue of each duopolist is 2/3 of the instantaneous revenue of the monopolist. In equilibrium of type (2), when the high cost firm does not attempt to enter in the preemption zone, the value function of the low cost firm drops by 30 per cent and more in the inaction region. Strategic interactions in equilibria of types (1) or (3) lead to a further drop by additional several per cent. See Fig. 1. Note that these sizable effects are observed even when the positive jump component is rather small, and the high cost firm is rather non-competitive. If the asymmetry decreases and/or the jump component increases, the value of the low cost firm drops further. For instance, our computations show that if the average size of jumps is 25 per cent, the value drops by 40-50 per cent in equilibrium of type (2), and by 45-58 per cent in equilibria of types (1) or (3).

The rest of the paper is organized as follows. Section 2 contains the model description and equilibrium concepts. In Section 3, main steps of solution are presented. Characterization of subgame perfect equilibria for the demand shocks following a jump-diffusion process with non-trivial positive jumps is given in Section 4. Section 5 concludes. In Appendix A, we recall basic facts about Lévy processes (jump-diffusion processes with independent identically distributed increments) and the Wiener-Hopf factorization method, which is a cornerstone of our approach, and present proofs of main optimal stopping theorems. Appendix B contains detailed study of the preemption zone. Proofs of the theorems describing subgame perfect equilibria are relegated to Appendix C.

2. Equilibrium concepts and strategies

2.1. **Model specification and main notation.** Suppose that two firms have a single investment opportunity in a new market. The firms differ in the sunk cost of investment. Assume that firm 1's sunk cost is $I_1 = I > 0$, and firm 2's sunk cost is $I_2 = kI$, where $k > 1$. Hence firm 1 is the low cost firm, and firm 2 is the high cost firm. Firms discount the future at the same constant rate $q > 0$. Observe that the sunk cost I_j can be viewed as the present value of the expenditure stream qI_j to which firm j ($j = 1, 2$) commits at the time of investment. When formulating optimization problems below, we will use expenditure streams instead of sunk costs.

After the investment is made, each firm can produce a single non-differentiated output good at rate one forever. For simplicity, assume that there are no variable costs of production. Since there are only two firms in the industry, the market supply is $Q \in \{0, 1, 2\}$. The investment is risky because the market demand is stochastic. We model the market demand as $p_t = e^{X_t} D(Q)$, where $D(Q)$ is a decreasing function so that $D(1) > D(2)$, and $\{X_t\}_{t\geq 0}$ is a jump-diffusion process with i.i.d. increments (Lévy process). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq\infty}, P)$ be the filtered probability space generated by the process X, satisfying the usual properties[1]. If firm i is a single producer on the market, its profit flow is $f_i^1(X_t) = e^{X_t}D(1) - qI_i$. If there are two producers, then producer i gets the profit flow $f_i^2(X_t) = e^{X_t}D(2) - qI_i$.

The expected present value (EPV) of the flow $\mathbb{E}\left[\int_0^\infty e^{-qt}e^{X_t}dt\right]$ is finite iff $\mathbb{E}\left[e^{X_t}\right] < \infty$ and the no-bubble condition $q - \Psi(1) > 0$ holds. Here Ψ is the Lévy exponent of X definable from $\mathbb{E}[e^{\beta X_t}] = e^{t\Psi(\beta)}$. Indeed, if $\Psi(\beta) < q$, then, by Fubini's theorem,

$$\begin{aligned}\mathbb{E}^x\left[\int_0^{+\infty} e^{-qt}e^{\beta X_t}dt\right] &\equiv \mathbb{E}\left[\int_0^{+\infty} e^{-qt}e^{\beta X_t}dt|X_0 = x\right]\\ = \int_0^{+\infty} e^{-qt}\mathbb{E}^x\left[e^{\beta X_t}\right]dt &= \int_0^{+\infty} e^{-qt+\beta x+t\Psi(\beta)}dt = \frac{e^{\beta x}}{q-\Psi(\beta)}.\end{aligned}$$

Under the no-bubble condition, the EPV of the stream of profits of a monopolist is finite. Since the revenue flow of a monopolist is not less than the profit flow of a firm in the presence of another firm in the duopoly, the EPV of the profit flow of any of the two firms in the duopoly is finite as well.

Recall that a Lévy process $X = \{X_t\}_{t\geq 0}$ on $\mathbb{R}$ is defined in terms of the generating triplet $(\sigma^2, b, F(dx))$, where σ^2 is the (instantaneous) variance of the Brownian motion (BM) component, $F(dx)$ is the Lévy density (density of jumps), and $b \in \mathbb{R}$. For $(\alpha, \beta) \subset \mathbb{R} \setminus 0$, $F((\alpha, \beta))dt$ is the probability of a jump from 0 into (α, β) during an infinitesimally small time interval dt. If the density of jumps is zero, then X is the Brownian motion with the drift b and variance σ^2. The Lévy-Khintchine formula (see., e.g., [47, Thm. 8.1]) expresses Ψ in terms of the generating triplet. If X is a BM with embedded compound Poisson jumps or, more generally, if the jump part is a finite variation process, then the Lévy-Khintchine formula for Ψ can be written in the form

$$\Psi(\beta) = \frac{\sigma^2}{2}\beta^2 + b\beta + \int_{\mathbb{R}\setminus 0}(e^{\beta y} - 1)F(dy), \tag{2.1}$$

and b can be interpreted as the drift. Function Ψ will appear in the main formulas for the value functions. The Brownian motion (BM) is the only class of Lévy processes with continuous trajectories. In BM model, $F(dy) \equiv 0$, and $\Psi(\beta) = \sigma^2\beta^2/2 + b\beta$.

Most of the preliminary results of the paper (e.g., the follower problem and the study of the simultaneous entry zone and preemption zone) are valid for any Lévy process provided the no-bubble condition holds. However, the proofs of the main results of the paper require additional assumptions.

Assumption X. *X is a Lévy process such that*

(i) the supremum process is non-trivial;
(ii) the Lévy densities of positive and negative jumps are monotone;
(iii) conditions (2.6) *and* (2.7) *in Subsection 2.3 hold.*

[1]Namely, $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$, and the filtration $(\mathcal{F}_t)_{0\leq t<\infty}$ is right continuous.

Note that condition (i) is automatically satisfied for processes with the non-trivial BM component and other processes of infinite variation.

As the main example of a Lévy process with non-trivial densities of positive and negative jumps, which satisfy all sufficient conditions, the reader may have in mind the double-exponential jump-diffusion model (DEJD), with $\sigma > 0$ and the Lévy density

$$F(dy) = c_+\lambda_+ e^{-\lambda_+ y}\mathbb{1}_{(0,+\infty)}(y)dy + c_-(-\lambda_-)e^{-\lambda_- y}\mathbb{1}_{(-\infty,0)}(y)dy, \tag{2.2}$$

where $\lambda_- < 0 < \lambda_+$ and $c_\pm \geq 0$; if $c_+ = 0$ (respectively, $c_- = 0$), then there are no positive (respectively, negative) jumps. Substituting (2.2) into (2.1), we find

$$\Psi(\beta) = \frac{\sigma^2}{2}\beta^2 + b\beta + \frac{c_+\beta}{\lambda_+ - \beta} + \frac{c_-\beta}{\lambda_- - \beta}. \tag{2.3}$$

With the approach of this paper, DEJD model and more general models with the Lévy densities given by exponential polynomials are almost as tractable as the Brownian motion in optimal stopping problems. The most technically involved analytical parts of the proofs are calculations of expectations of functions of exponential random variables. See the monograph [15] for numerous examples of such calculations in various stopping problems. We will use these calculations to illustrate our results in the case of DEJD model.

2.2. **Strategy space and equilibrium concepts.** We consider a game of timing Γ, characterized by the following structure. Time $t \in \mathbb{R}_+$ is continuous; the game environment evolves according to an exogenous Lévy process X described in the previous subsection. There are two players (firms). At each point (t, X_t) player $i = 1, 2$ may make an irreversible stopping decision, conditional on the history of the game. Following the terminology in Laraki et al. [38], we define a pure plan of action of player i as a stopping set B_i (a Borel set) such that player i stops the first time the process X enters B_i. The stopping set B_i may depend on the history of the game. At each $t \geq 0$, the history of the game includes the path of the stochastic variable $\{X_s\}_{s\leq t}$ and actions of the players up to time t. As far as the actions are concerned, only two sorts of histories matter in the stopping game: (i) none of the firms has yet entered; (ii) at least one firm has entered. Let $t_i \in \mathbb{R}_+$ be the entry time of player i. Define function

$$\tilde{t}_i(t) = \begin{cases} t_i, & \text{if } t_i \leq t, \\ \infty, & \text{otherwise} \end{cases}.$$

Thus a typical history at time t is $h_t\left(\{X_s\}_{s\leq t}, \tilde{t}_1(t), \tilde{t}_2(t)\right)$.

Suppose that the history of the game is such that none of the players has acted. Let a random variable τ_i denote the first entrance time into B_i. Consider a sample path $\omega \in \Omega$. If $\tau_i(\omega) = \infty$, then player i never acts. If $\tau_i(\omega) < \tau_j(\omega) < \infty$, firm i is the leader and firm j is the follower. If $\tau_i(\omega) = \tau_j(\omega) < \infty$, then the firms enter simultaneously. Let $V^i_{\mathrm{lead}}(X_t)$, denote the value function of firm i at (t, X_t) after the history h_t such that $t = t_i$, $t_j > t$. Let $V^i_f(X_t)$, denote the value function of firm i at (t, X_t) after the history h_t such that $\tilde{t}_j(t) = t_j$, $t_j \leq t_i$. Let $P_i(X_t)$, denote the value function of firm i at (t, X_t) after the history h_t such that $t = t_i = t_j$. We present analytical expressions for all the above value functions in Section 3 and show that for each x, $V^i_{\text{lead}}(x) \geq P_i(x)$, and $V^i_f(x) \geq P_i(x)$ with strict inequalities in certain regions of the state space.

The most natural strategy concept for this stopping game would be to require each player to choose a stopping set B_i, provided the game starts when none of the players has yet acted. Unfortunately, this concept is not rich enough to describe equilibria in the state space region where both firms have the first mover's advantage $V^i_{\text{lead}}(x) > V^i_f(x)$ for $i = 1, 2$. This region is called the preemption zone PZ (it may be empty for some parameter values). Later, we show that in the PZ, simultaneous entry is the worst possible outcome: $V^i_f(x) > P_i(x)$ for $i = 1, 2$. Suppose that PZ is non-empty and denote by τ_{pz} the first entry time into PZ. Suppose $PZ \cap B_i \neq \emptyset$, $i = 1, 2$, then, if $\tau_1(\omega) = \tau_2(\omega) = \tau_{pz}(\omega)$, each player has an incentive to deviate from entry at $(\tau_{pz}(\omega), X_{\tau_{pz}(\omega)})$ because $V^i_f(X_{\tau_{pz}(\omega)}) > P_i(X_{\tau_{pz}(\omega)})$. Next, suppose that $PZ \cap B_i = \emptyset$, $i = 1, 2$, then if $\tau_{pz}(\omega) < \min_i\{\tau_i(\omega)\}$ each player has an incentive to deviate from prescribed strategy and enter at $(\tau_{pz}(\omega), X_{\tau_{pz}(\omega)})$ because $V^i_{\text{lead}}(X_{\tau_{pz}(\omega)}) > V^i_f(X_{\tau_{pz}(\omega)})$. Hence, it is necessary to extend the strategy space to account for situations when both players have the first mover's advantage.

We will use the concepts of simple and closed loop continuous time strategies introduced for the preemption game in the deterministic environment in [26] and later generalized to a stochastic environment in [51, 52]. In [26], a simple strategy of each player is a pair of real valued functions of time, one of which is a cumulative distribution function (the probability that the player moved by a given time t) and the second is the intensity function that measures the intensity of atoms in the interval $[t, t+dt]$. The intensity function replicates discrete time results that are lost in passing to the continuous time limit. In a stochastic environment, one has to use stochastic processes to define the simple strategies.

Below, we slightly modify the definitions from [51, 52].

Definition 2.1. A simple strategy for player $i \in \{1, 2\}$ in the game starting at $(t_0, X_0) \in [0, \infty) \times \mathbb{R}$ is given by a pair of real valued functions $(G^{t_0,X_0}_i, \alpha^{t_0,X_0}_i)$: $[t_0, \infty) \times \Omega \to [0, 1] \times [0, 1]$ such that

(i) the processes $G^{t_0,X_0}_i(t, \cdot)$ and $\alpha^{t_0,X_0}_i(t, \cdot)$ are adapted to the filtration $(\mathcal{F}_t)_{t_0 \leq t < \infty}$;

(ii) $G^{t_0,X_0}_i(\cdot, \omega)$ is non-decreasing, and, at each point, is either left continuous with right limits or right continuous with left limits, a.s.;

(iii) $\alpha^{t_0,X_0}_i(\cdot, \omega)$ at each point is either left continuous with right limits or right continuous with left limits, a.s.

$G^{t_0,X_0}_i(t, \omega)$ is the probability that firm i has invested no later than at t (equivalently, $G^{t_0,X_0}_i(t, \omega)$ is the probability that $\tau_i(\omega) \leq t$), and $\alpha^{t_0,X_0}_i(t, \omega)$ is the "intensity" of atoms in the interval $(t, t + dt)$ (see [51] for detailed explanation). In [51, 52], the authors impose an additional condition on the intensity functions: if $\alpha^{t_0,X_0}_i(t, \omega) = 0$ and $t = \inf\{t' \geq t_0 |\, \alpha^{t_0,X_0}_i(t') > 0\}$, then the right derivative of $\alpha^{t_0,X_0}_i(t, \omega)$ exists and it is positive. This condition is essential to define the value functions of the symmetric firms at the boundary of the preemption zone. With asymmetric firms, this condition plays no role, therefore we omit it. Also, in [51, 52], simple strategies are required to be right continuous with left limits. We need to allow for points where $G_i^{t_0,X_0}(\cdot,\omega), \alpha_i^{t_0,X_0}(\cdot,\omega)$ are left continuous, but not right continuous because this situation naturally arises in specification of strategies off the equilibrium path.

The definition of simple strategies does not exclude the possibility that firms will choose the intensity functions α_1, α_2, which are inconsistent with the cumulative distribution functions G_1, G_2. To ensure consistency, we modify the notion of α-consistency as in [51, 52] to allow for points at which the simple strategies are left continuous, but not right continuous. Let $\alpha_i^{t_0,X_0}(t-,\omega) = \lim_{s\uparrow t}\alpha_i^{t_0,X_0}(s,\omega)$, $G_i^{t_0,X_0}(t-,\omega) = \lim_{s\uparrow t} G_i^{t_0,X_0}(s,\omega)$, $\alpha_i^{t_0,X_0}(t+,\omega) = \lim_{s\downarrow t}\alpha_i^{t_0,X_0}(s,\omega)$, $G_i^{t_0,X_0}(t+,\omega) = \lim_{s\downarrow t} G_i^{t_0,X_0}(s,\omega)$.

Definition 2.2. A pair of simple strategies $(G_i^{t_0,X_0}, \alpha_i^{t_0,X_0})$ $(i = 1,2)$ for the game starting at (t_0, X_0) is α-consistent, if for all $\omega \in \Omega$ and $i, j = 1, 2, i \neq j$,

$$(i) \qquad \alpha_i^{t_0,X_0}(t,\omega) - \alpha_i^{t_0,X_0}(t-,\omega) \neq 0 \Rightarrow G_i^{t_0,X_0}(t,\omega) - G_i^{t_0,X_0}(t-,\omega) = \\ = \frac{\left(1 - G_i^{t_0,X_0}(t-,\omega)\right)\alpha_i^{t_0,X_0}(t,\omega)}{\alpha_i^{t_0,X_0}(t,\omega) + \alpha_j^{t_0,X_0}(t,\omega) - \alpha_i^{t_0,X_0}(t,\omega)\alpha_j^{t_0,X_0}(t,\omega)};$$

(ii) same as (i), if one replaces $t-$ (respectively, t) with t (respectively, $t+$) in (i).

We denote the set of pairs of simple α-consistent strategies by $\mathcal{M}$.

Consider the game that starts at (t_0, X_0) and assume that none of the players has yet acted at the start of the game. This means that the initial realization of the demand shock X_0 is sufficiently low, so that it is unprofitable to enter. If the realization of the shock becomes sufficiently high, it may become optimal for one or both players to act. We will prove that the entry rules are of the threshold type, that is, one of the firms or both enter when the underlying process X reaches or crosses a certain threshold $h \in \mathbb{R}$ from below. Given a strategy profile

$$s^{t_0,X_0} = \left(s_1^{t_0,X_0}, s_2^{t_0,X_0}\right) = \left((G_1^{t_0,X_0}, \alpha_1^{t_0,X_0}), (G_2^{t_0,X_0}, \alpha_2^{t_0,X_0})\right),$$

one may observe the following outcomes: (i) none of the firms enters; (ii) only one of the firms enters; (iii) firms enter sequentially; (iv) firms enter simultaneously. The first two outcomes are possible only if the first instantaneous moment of X is negative, so that with a positive probability the realization of the shock does not become sufficiently high (see Lemma A.1 for the exact statement and an explicit formula for this probability) for entry to become profitable[2]. If the first instantaneous moment of X is non-negative, X enters the interval $[h, +\infty)$ with probability 1. In this case only outcomes (iii) or (iv) can be observed. As it will be demonstrated later, sequential

[2]We are grateful to an anonymous referee for pointing out to us that this subtle issue should be addressed in the paper.

entry (iii) may happen with or without preemptive motives, and simultaneous entry (iv) may occur as an equilibrium, or as a coordination failure.

If firm $i \in \{1,2\}$ has invested no later than at time $t \geq t_0$ but firm $j \neq i$ has not, firm i's instantaneous profit flow is $D(1)e^{X_t} - qI_i$. If, by time t, both firms have invested, firm i's instantaneous profit flow is $D(2)e^{X_t} - qI_i$. If firm i has not invested by time t, its instantaneous profit flow is zero. Introduce the joint distribution $\mathcal{G}^{t_0,X_0}(t,\omega)$ – the probability that both firms enter no later than at time t. Given the strategy profile s^{t_0,X_0}, firm i's value at (t_0, X_0) is

$$W^i_{t_0}(s^{t_0,X_0}) = \mathbb{E}^x \left[\int_{t_0}^{+\infty} e^{-q(t-t_0)} \left(\left(G_i^{t_0,X_0}(t) D(1) + \mathcal{G}^{t_0,X_0}(t) \left(D(2) - D(1) \right) \right) e^{X_t} - qI_i \right) dt \right].$$

The integrand is an adapted process that is uniformly bounded from below, and bounded from above by $D(1)e^{X_t - qt}$. Under the no-bubble condition, $\int_0^{+\infty} \mathbb{E}^x \left[e^{X_t - qt} \right] dt < \infty$, therefore, applying Fubini's theorem, we conclude that $W^i_{t_0}(s^{t_0,X_0})$ is well-defined and finite.

Definition 2.3. A pair of strategies $(s_i^{t_0,X_0}(t,\omega), s_j^{t_0,X_0}(t,\omega)) \in \mathcal{M}$ is a Nash equilibrium for the game starting at (t_0, X_0) if for every $(i,j) \in \{(1,2),(2,1)\}$,

$$W^i_{t_0}(s_i^{t_0,X_0}, s_j^{t_0,X_0}) \geq W^i_{t_0}(\tilde{s}_i^{t_0,X_0}, \hat{s}_j^{t_0,X_0}), \ \forall\, \tilde{s}_i^{t_0,X_0} \text{ s.t. } (\tilde{s}_i^{t_0,X_0}, \hat{s}_j^{t_0,X_0}) \in \mathcal{M},$$

where $\hat{s}_j^{t_0,X_0} = (\hat{G}_j^{t_0,X_0}, \alpha_j^{t_0,X_0})$, and $\hat{G}_j^{t_0,X_0}$ replaces $G_j^{t_0,X_0}$ according to the α-consistency condition if $\tilde{\alpha}_i^{t_0,X_0} \neq \alpha_i^{t_0,X_0}$ and $\alpha_j^{t_0,X_0}(t,\omega) - \alpha_j^{t_0,X_0}(t-,\omega) \neq 0$.

Following Laraki et al. [38], and Dutta and Rustichini [21], we define, for any time $t \geq t_0$, a proper subgame as the timing game that starts at the end of the history h_t, that is at the decision node originating at (t, X_t), and the payoffs are evaluated at time t. In order to define a subgame perfect equilibrium, one needs to use the notion of closed loop strategies, which are simple strategies that satisfy some intertemporal consistency conditions. The intertemporal consistency conditions below are different from those in [26] due to the reasons explained in [51].

Definition 2.4. A closed loop strategy profile for player $i \in \{1,2\}$ is a collection of simple strategies $\left\{ (G_i^{t,X_t}(\cdot,\omega), \alpha_i^{t,X_t}(\cdot,\omega)) \right\}_{t \geq 0, \omega \in \Omega}$ that satisfies the following two intertemporal consistency conditions both of which hold $\forall\, \omega \in \Omega$:

(i) $\forall\, 0 \leq t \leq u \leq v < \infty : v = \inf\{\tau > t |\, X_\tau(\omega) = X_v(\omega)\} \ \Rightarrow\ G_i^{t,X_t}(v,\omega) = G_i^{u,X_u}(v,\omega)$;

(ii) $\forall\, 0 \leq t \leq u \leq v < \infty : v = \inf\{\tau > t |\, X_\tau(\omega) = X_v(\omega)\} \ \Rightarrow\ \alpha_i^{t,X_t}(v,\omega) = \alpha_i^{u,X_u}(v,\omega)$.

Finally, we define a subgame perfect equilibrium in continuous time as in [26, 52].

Definition 2.5. A pair of closed loop strategies is a subgame perfect equilibrium if for every $t \in [0,\infty)$ and $\omega \in \Omega$, the corresponding strategy profile $s^{t,X_t}(\cdot,\omega) \in \mathcal{M}$ is a Nash equilibrium.

2.3. **Auxiliary notation.** The solutions of the follower and leader problems below are based on the general results on optimal stopping summarized in Appendix A. The main objects used in the formulation of the results and proofs are:
(i) τ_h^+, the first entrance time of X into the semi-infinite interval $[h, +\infty)$;
(ii) the normalized EPV operator

$$(\mathcal{E}_q f)(x) = \mathbb{E}^x \left[\int_0^{+\infty} q e^{-qt} f(X_t) dt \right] \tag{2.4}$$

that calculates the EPV of the stream of payoffs f; notice that $(\mathcal{E}_q e^{\cdot})(x) = qe^x/(q - \Psi(1))$ under the no-bubble condition;
(iii) the supremum and infimum processes $\overline{X}_t = \sup_{0\le s\le t} X_s$ and $\underline{X}_t = \inf_{0\le s\le t} X_s$, respectively; the (normalized) expected present value operators $\mathcal{E}_q^+$ and $\mathcal{E}_q^-$ under these processes

$$(\mathcal{E}_q^+ f)(x) = \mathbb{E}^x \left[\int_0^{+\infty} q e^{-qt} f(\overline{X}_t) dt \right], \quad (\mathcal{E}_q^- f)(x) = \mathbb{E}^x \left[\int_0^{+\infty} q e^{-qt} f(\underline{X}_t) dt \right];$$

(iv) the notation $\kappa_q^\pm(\beta) = \left(\mathcal{E}_q^\pm e^{\beta x}\right)|_{x=0}$, and
(v) the operator version of the Wiener-Hopf factorization formula which states that

$$\mathcal{E}_q = \mathcal{E}_q^+ \mathcal{E}_q^- = \mathcal{E}_q^- \mathcal{E}_q^+ \tag{2.5}$$

as operators in spaces of semi-bounded measurable functions (see, e.g.,[15, (11.16)]). Evidently, $\mathcal{E}_q$ and $\mathcal{E}_q^\pm$ are positive operators. For the list of the other properties used in the proofs below, see Lemma A.2.

For any Lévy process, $\kappa_q^+(\beta) < \infty, \forall\ \beta \le 0$, and $\kappa_q^-(\beta) < \infty, \forall\ \beta \ge 0$. If the no-bubble condition $q - \Psi(1) > 0$ holds, then $\kappa_q^+(\beta) < \infty, \forall\ \beta \le 1$.

Clearly, $\mathcal{E}_q^+ f(x) = \mathbb{E}[f(x + \overline{X}_{T_q})]$, $\mathcal{E}_q^- f(x) = \mathbb{E}[f(x + \underline{X}_{T_q})]$, where T_q is an exponential random variable of mean $1/q$, independent of the process X. The last technical conditions are: (i) the pdf of $\overline{X}_{T_q}$ is of the following form

$$k_q^+(x) = \int \mu_q(d\beta) \beta e^{-\beta x}, \tag{2.6}$$

where $d\mu \ge 0$ is a measure of total mass 1, supported at a subset of $(1, +\infty)$; and (ii) the inverse $(\mathcal{E}_q^+)^{-1}$ admits the following representation

$$(\mathcal{E}_q^+)^{-1} g(x) = c_{q0}^+ g(x) - c_{q1}^+ g'(x) - \int_0^{+\infty} g(x+y) k_q^{+-}(y) dy, \tag{2.7}$$

where c_{q0}^+ and c_{q1}^+ are constants, and k_q^{+-} is a non-negative, non-increasing on $(0, +\infty)$ function (for details see [12, 15, 14]).

The verification of our conditions on the process and calculations of the entry thresholds and value functions are straightforward if X is the Brownian motion (BM) with drift b and volatility $\sigma > 0$, or DEJD model with drift b, volatility $\sigma > 0$, and the Lévy density (2.2). In the BM case, it is well-known (see, for example, [9]) that $\overline{X}_{T_q}$ is an exponentially distributed random variable on $\mathbb{R}_+$ of mean $1/\beta^+$, and $\underline{X}_{T_q}$ is an exponentially distributed random variable on $\mathbb{R}_-$ of mean $1/\beta^-$, where $\beta^- < 0 < \beta^+$ are the roots of the characteristic equation $q - \Psi(\beta) = 0$, and $\Psi(\beta) = b\beta + \sigma^2\beta^2/2$. Hence, for the Brownian motion, we can write the EPV-operators $\mathcal{E}^\pm_q$ as convolution operators with exponential kernels $\mathcal{E}^+_q = I^+_{\beta^+}$, $\mathcal{E}^-_q = I^-_{\beta^-}$, where

$$I^+_{\beta^+} = \int_0^{+\infty} \beta^+ e^{-\beta^+ y} f(x+y)dy, \quad I^-_{\beta^-} = \int_{-\infty}^0 (-\beta^-)e^{-\beta^- y} f(x+y)dy, \tag{2.8}$$

and $\kappa^\pm_q(\beta) = \beta^\pm_q/(\beta^\pm_q - \beta)$.

In DEJD model, $\mathcal{E}^\pm_q = \sum_{j=1,2} a^\pm_j(q) I^\pm_{\beta^\pm_j(q)}$, where $\beta^\pm_j(q)$ are the solutions of the equation $q - \Psi(\beta) = 0$, Ψ is given by (2.3), $a^\pm_j(q) > 0$ can be expressed in terms of $\beta^\pm_1(q)$ and $\beta^\pm_2(q)$ ([15, eqn. (11.29),(11.30)])

$$a^\pm_1 = \frac{\beta^\pm_2}{\beta^\pm_2 - \beta^\pm_1} \cdot \frac{\lambda^\pm - \beta^\pm_1}{\lambda_\pm}, \quad a^\pm_2 = \frac{\beta^\pm_1}{\beta^\pm_1 - \beta^\pm_2} \cdot \frac{\lambda^\pm - \beta^\pm_2}{\lambda_\pm}, \tag{2.9}$$

and ([15, eqn. (11.24)])

$$\kappa^\pm_q(\beta) = \frac{\beta^\pm_1}{\beta^\pm_1 - \beta} \cdot \frac{\beta^\pm_2}{\beta^\pm_2 - \beta} \cdot \frac{\lambda^\pm - \beta}{\lambda^\pm}. \tag{2.10}$$

Under the no-bubble condition $q - \Psi(1) > 0$, we have $\beta^-_2 < \lambda_- < \beta^-_1 < 0 < 1 < \beta^+_1 < \lambda_+ < \beta^+_2$ (see, e.g., [14, pp. 10 and 45] and [15, pp. 199–201] for details).

3. Main steps of solution

Once one of the firms has entered, the other firm faces the standard optimal stopping problem, which can be easily solved. Thus, when considering subgame perfect equilibria, we will first examine subgames when one of the firms has entered, and then move to subgames where neither firm has entered as yet. To simplify the notation, we suppress the dependence of value functions on the other player's strategy.

3.1. **Follower's problem.** Consider a subgame that starts after the history such that only one of the firms has entered. Let the subgame start at the decision node originating at (t, X_t); set $t = 0$, $x = X_t$. Assuming that firm j has entered, firm i solves the optimization problem

$$V^i_f(x) = \sup_{\tau\in\mathcal{T}_0} \mathbb{E}^x \left[\int_\tau^{+\infty} e^{-qs} f^2_i(X_s)ds\right], \tag{3.1}$$

where $f^2_i(x) = D(2)e^x - qI_i$, and $\mathcal{T}_0$ is a family of stopping times w.r.t. to filtration $\{\mathcal{F}_t\}$. The solution is well-known (see, e.g., [14, Thm. 3.7] or [15, Thm. 11.4.5] or Theorem A.3). We recall the scheme of the proof since the main idea of the proof appears in many situations in the paper. Fix $h \in \mathbb{R}$. Let $\tau = \tau^+_h$, then the stochastic expression on the RHS can be represented as $q^{-1}(\mathcal{E}^+_q \mathbb{1}_{[h,+\infty)} w)(x)$, where $w(x) = \mathcal{E}^-_q f^2_i(x) = \mathcal{E}^-_q(D(2)e^{\cdot} - qI_i)(x) = \kappa^-_q(1)D(2)e^x - qI_i$. Clearly, function w is monotone and changes sign. Since the EPV operator $\mathcal{E}^+_q$ is positive, the value function is maximized if $h = h^i_f$ is the unique solution of the equation $\mathcal{E}^-_q(D(2)e^x - qI_i) = 0$, which can be written as $\kappa^-_q(1)D(2)e^x - qI_i = 0$. Thus,

$$e^{h^i_f} = \frac{qI_i}{\kappa^-_q(1)D(2)}. \tag{3.2}$$

By Theorem A.6, $\tau^i_f := \tau^+_{h^i_f}$ is optimal in the class $\mathcal{T}_0$. In the case of DEJD, equation (3.2) assumes the form (see (2.10))

$$e^{h^i_f} = \frac{qI_i\beta^-_1\beta^-_2(\lambda_+ - 1)}{\lambda_+(\beta^-_1 - 1)(\beta^-_2 - 1)D(2)}. \tag{3.3}$$

Since $I_2 = kI_1$, we have $h^2_f > h^1_f$. In the action region $x \ge h^i_f$,

$$V^i_f(x) = q^{-1}(\mathcal{E}_q f^2_i)(x) = \frac{D(2)e^x}{q - \Psi(1)} - I_i. \tag{3.4}$$

In the inaction region $x < h^i_f$, we need a more general formula (see Theorem A.3)

$$\begin{aligned} V^i_f(x) &= q^{-1}(\mathcal{E}^+_q \mathbb{1}_{[h^i_f,+\infty)}\mathcal{E}^-_q f^2_i)(x) \\ &= q^{-1}(\mathcal{E}^+_q \mathbb{1}_{[h^i_f,+\infty)}(\kappa^-_q(1)D(2)e^{\cdot} - qI_i))(x) \end{aligned} \tag{3.5}$$

If X is DEJD, the RHS in (3.5) can be calculated explicitly, and we obtain (see, e.g., [14, p. 17] and [15, p. 210] for details)

$$V^i_f(x) = I_i \sum_{j=1,2} \frac{a^+_j e^{\beta^+_j(x-h^i_f)}}{\beta^+_j - 1}, \quad x < h^i_f \tag{3.6}$$

3.2. **Simultaneous entry.** Recall that $P_i(X_t)$ denotes the value function of firm i after the history h_t such that $t = t_i = t_j$. Consider a subgame that starts after such a history at the decision node originating at (t, X_t), and set $t = 0$, $x = X_t$. Clearly, $P_i(x) = q^{-1}(\mathcal{E}_q f^2_i)(x)$, hence $P_i(x) = V^i_f(x)$ if $x \ge h^i_f$. If $x < h^i_f$, then, using (2.5), we obtain

$$V^i_f(x) - P_i(x) = q^{-1}\left(\mathcal{E}^+_q \mathbb{1}_{[h^i_f,+\infty)}\mathcal{E}^-_q f^2_i(x) - \mathcal{E}_q f^2_i(x)\right) = -q^{-1}\mathcal{E}^+_q \mathbb{1}_{(-\infty,h^i_f)}\mathcal{E}^-_q f^2_i(x).$$

The RHS is positive, because by definition of h^i_f, $\mathcal{E}^-_q f^2_i(x) = \kappa^-_q(1)D(2)e^x - qI_i < 0$ for $x < h^i_f$, and $\mathcal{E}^+_q$ is a positive operator. Hence

$$V^i_f(x) > P_i(x), \quad \text{for } x < h^i_f, \tag{3.7}$$

so being the follower is better than simultaneous investment.

3.3. **Leader's value at entry.** Recall that $V^i_{\rm lead}(X_t)$ denotes the value of firm i after the history h_t such that $t = t_i$, $t_j > t$. Consider a subgame that starts after such a history at the decision node originating at (t, X_t), and set $t = 0$, $x = X_t$. Then $x < h^j_f$, and

$$\begin{aligned} V^i_{\rm lead}(x) &= \mathbb{E}^x\left[\int_0^{\tau^j_f} e^{-qs} f^1_i(X_s)ds + \int_{\tau^j_f}^{\infty} e^{-qs} f^2_i(X_s)ds\right] \\ &= \mathbb{E}^x\left[\int_0^{\infty} e^{-qs} f^1_i(X_s)ds\right] - \mathbb{E}^x\left[\int_{\tau^j_f}^{\infty} e^{-qs}(f^1_i(X_s) - f^2_i(X_s))ds\right] \quad (3.8) \\ &= \mathbb{E}^x\left[\int_0^{\infty} e^{-qs} f^2_i(X_s)ds\right] + \mathbb{E}^x\left[\int_0^{\tau^j_f} e^{-qs}(f^1_i(X_s) - f^2_i(X_s))ds\right] \quad (3.9) \end{aligned}$$

Had firm i been the monopolist in the market forever ($\tau^j_f = +\infty$), its value at the moment of entry would have been

$$V^i_m(x) = q^{-1}\mathcal{E}_q f^1_i(x) = \frac{D(1)e^x}{q - \Psi(1)} - I_i, \tag{3.10}$$

which is the first term on the RHS of (3.8). We write (3.8) as

$$V^i_{\rm lead}(x) = V^i_m(x) - V^i_{\rm loss}(x), \tag{3.11}$$

where $V^i_{\rm loss}(x)$ is the second term, which represents the loss in the monopolist's value due to the entrance of the follower. Applying Theorem 11.4.5 in [15] and taking into account that $f^1_i(x) - f^2_i(x) = (D(1) - D(2))\, e^x$, we obtain

$$V^i_{\rm loss}(x) = (D(1) - D(2))\, q^{-1}\mathcal{E}^+_q \mathbb{1}_{[h^j_f, +\infty)}\mathcal{E}^-_q e^x \tag{3.12}$$

In the case of DEJD model, the loss in the monopolist's value can be calculated explicitly (see [15, eqn. (11.54)])

$$V^i_{\rm loss}(x) = \left(\frac{D(1)}{D(2)} - 1\right) I_j \sum_{k=1,2} \frac{a^+_k \beta^+_k e^{\beta^+_k(x - h^j_f)}}{\beta^+_k - 1}. \tag{3.13}$$

The first term on the RHS of (3.9) is $P_i(x)$, the value of entering the market at the same time as the other firm, and the second term is the gain from the leadership. The second term is positive because $f^1_i(x) - f^2_i(x) = (D(1) - D(2))\, e^x > 0$. Therefore, for $x < h^j_f$, $V^i_{\rm lead}(x) > P_i(x)$, i.e., being the leader is better than simultaneous entry.

Remark 3.1. If $x \ge h^j_f$, then firm j enters as well, and we set $V^i_{\rm lead}(x) = V^i_f(x) = P_i(x)$. Thus, we have for $x < \min\{h^i_f, h^j_f\} = h^1_f$, $P_i(x) < \min\left\{V^i_{\rm lead}(x), V^i_f(x)\right\}$, $i = 1, 2$, and for $x \ge \max\{h^i_f, h^j_f\} = h^2_f$, $P_i(x) = V^i_{\rm lead}(x) = V^i_f(x)$ $(i = 1, 2)$. Hence simultaneous entry can be optimal only if $x \ge h^2_f$.

We call the semi-infinite interval SEZ$= [h^2_f, +\infty)$ the *simultaneous entry zone*. Let $\tau_{\rm sez}$ denote the first entrance time into the interval SEZ.

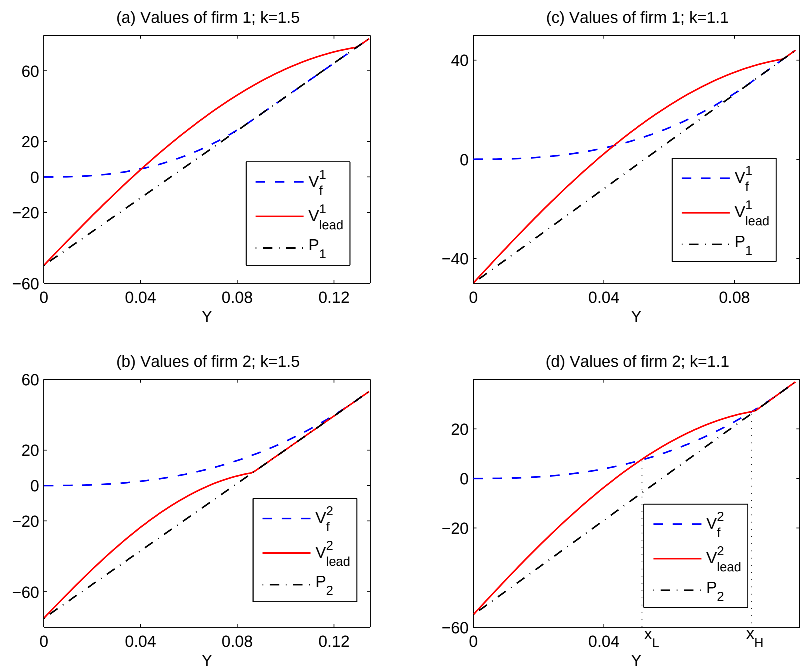

FIGURE 2. Value functions of the low cost and high cost firms. In panels (a) and (b), cost disadvantage is high, and the preemption zone is empty. In panels (c) and (d), cost disadvantage is small, and the preemption zone is non-empty. $Y = e^x$ – demand shock.

3.4. **Preemption zone.** Introduce the difference $DV_f^i(k,x) = V_f^i(x) - V_{\text{lead}}^i(x)$. Preemption zone is a set (it may be empty for some parameter values), where $DV_f^i(k,x) < 0$ for both players. In Appendix B, we show that $\{x \in \mathbb{R} |\, DV_f^2(k,x) < 0\} \subset \{x \in \mathbb{R} |\, DV_f^1(k,x) < 0\}$, therefore the preemption zone is $\text{PZ} = \{x \in \mathbb{R} |\, DV_f^2(k,x) < 0\}$. In Lemma B.3, we show that there exists $k^* > 1$ such that for all $k \geq k^*$, the preemption zone is empty: if the cost disadvantage is too big, the high cost firm never finds it optimal to be the first entrant. If $k < k^*$, then the preemption zone is an interval: PZ=$(x_L(k), x_H(k))$, and $DV_f^2(k,x) > 0$ if $x < x_L(k)$ or $x \in (x_H(k), h_f^2)$. See Fig. 2 for illustration. The boundaries of the preemption zone are defined by $DV_f^2(k,x) = 0$. Let τ_{pz} denote the first entrance time into the preemption zone.

3.5. **Subgame perfect equilibria in the preemption zone.** If $k < k^*$, the preemption zone is non-empty. From now on, we let $\omega \in \Omega$ be fixed. For notational convenience, we drop ω as an argument in players' strategies, and write τ instead of $\tau(\omega)$.

Theorem 3.2. *Consider a subgame starting at $(t, X_t) = (\tau_{\mathrm{pz}}, X_{\tau_{\mathrm{pz}}})$ after such a history that none of the players has yet acted. Then there are three subgame perfect equilibria given by the following pairs of closed loop strategies.*

(1) For $i, j \in \{1,2\}$, $i \neq j$,

$$G_i^{t,X_t}(s) = \begin{cases} \frac{\alpha_i^{t,X_t}(t)}{\alpha_i^{t,X_t}(t)+\alpha_j^{t,X_t}(t)-\alpha_i^{t,X_t}(t)\alpha_j^{t,X_t}(t)} \text{ if } & t \leq s < \tau_f^i \\ 1 \text{ if } & s \geq \tau_f^i; \end{cases} \tag{3.14}$$

$$\alpha_i^{t,X_t}(s) = \begin{cases} \frac{V_{\mathrm{lead}}^j(X_t)-V_f^j(X_t)}{V_{\mathrm{lead}}^j(X_t)-P_j(X_t)} \text{ if } & t \leq s < \tau_f^i \\ 1 \text{ if } & s \geq \tau_f^i. \end{cases} \tag{3.15}$$

(2) $G_1^{t,X_t}(s) = \alpha_1^{t,X_t}(s) = 1$ for all $s \geq t$. $G_2^{t,X_t}(t) = \alpha_2^{t,X_t}(t) = 0$. If $G_1^{t,X_t}(t) = 1$, then

$$G_2^{t,X_t}(s) = \alpha_2^{t,X_t}(s) = \begin{cases} 0 & \text{if} \quad t < s < \tau_f^2, \\ 1 & \text{if} \quad s \geq \tau_f^2. \end{cases}$$

If $G_1^{t,X_t}(t) = 0$, then for all $s > t$, $\alpha_i^{t,X_t}(s)$ are given by the RHS of (3.15), and $G_i^{t,X_t}(s)$ are given by the RHS of (3.14) with $\alpha_i^{t,X_t}(t)$ and $\alpha_j^{t,X_t}(t)$ being replaced by $\alpha_i^{t,X_t}(t+)$ and $\alpha_j^{t,X_t}(t+)$, respectively ($i, j \in \{(1,2),(2,1)\}$).

(3) Same as (2), but subscripts 1 and 2 interchange throughout.

The firms' equilibrium payoffs are: $W_t^1(s^{t,X_t}) = V_f^1(X_t)$, $W_t^2(s^{t,X_t}) = V_f^2(X_t)$ in equilibrium (1); $W_t^1(s^{t,X_t}) = V_{\mathrm{lead}}^1(X_t)$, $W_t^2(s^{t,X_t}) = V_f^2(X_t)$ in equilibrium (2); $W_t^1(s^{t,X_t}) = V_f^1(X_t)$, $W_t^2(s^{t,X_t}) = V_{\mathrm{lead}}^2(X_t)$ equilibrium (3).

Proof in Subsection C.1[3]. Theorem 3.2 establishes the fact that in any subgame that starts in the preemption zone, three types of equilibria are possible. In type (1) equilibrium, both firms enter with positive probabilities. In type (2) equilibrium, firm 1 enters with probability one immediately, and firm 2 follows when the follower's optimal threshold is crossed. In type (3) equilibrium, the roles of the firms are reversed. In each of these equilibria, the first entrant enjoys the Stackelberg leader's advantage until the other firm follows.

If type (1) equilibrium is played, then there are several things to note. First, the intensity functions are such that ex ante, a player who enters with positive probability has the same expected value as a player who stays away from entry and becomes the follower. Therefore a weak form of the rent-equalization property holds in the preemption region. Ex post, if only one of the firms enters, the leader gets higher value than the follower. Next, the probability of coordination failure in the preemption zone

[3]We are thankful for an anonymous referee for pointing out an important omission in the earlier version of the statement of Theorem 3.2.

is positive. Indeed, the probability of simultaneous entry is (see [51] for details)

$$\mathcal{G}^{t,X_t}(t) = \frac{\alpha_1^{t,X_t}(t)\alpha_2^{t,X_t}(t)}{\alpha_1^{t,X_t}(t) + \alpha_2^{t,X_t}(t) - \alpha_1^{t,X_t}(t)\alpha_2^{t,X_t}(t)} > 0,$$

because both intensities are positive in the preemption zone on the strength of (3.15). Finally, as $X_t \downarrow x_L$ (or $X_t \uparrow x_H$), $\alpha_1^{t,X_t}(t) \downarrow 0$, $\alpha_2^{t,X_t}(t) \to \text{const} > 0$, $G_1^{t,X_t}(t) \downarrow 0$, and $G_2^{t,X_t}(t) \uparrow 1$. Hence if the game starts in the preemption zone close to one of the boundaries, the high cost firm is more likely to be the first mover, and the probability of simultaneous investment tends to zero.

3.6. **Problem of low cost firm.** Consider a subgame starting at (t, X_t) after the history such that none of the players has yet acted. Assume also that $V_f^i(X_t) > V_{\text{lead}}^i(X_t)$ $(i = 1, 2)$. Set $t = 0$ and $x = X_t$. Let $\mathcal{V}^1$ denote the value function of firm 1 when this firm contemplates to become the leader. Consider first the case when $k \geq k^*$ so that firm 2 finds it non-optimal to enter at any level $x < h_f^2$. The results below are also applicable to the case when $k < k^*$ but firm 2 can precommit not to enter before the threshold h_f^2 is reached or crossed. Firm 1 solves the following optimization problem

$$\mathcal{V}^1(x) = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}^x \left[e^{-q\tau} V_{\text{lead}}^1(X_\tau) \right]. \tag{3.16}$$

Theorem 3.3. *Let $F(dy)$ be non-decreasing on $(-\infty, 0)$; define*

$$e^{h_l} = \frac{qI}{\kappa_q^-(1)D(1)}. \tag{3.17}$$

Then $\tau_{h_l}^+$ is the optimal stopping time of firm 1 in the class $\mathcal{T}_0$, and

$$\mathcal{V}^1(x) = \mathcal{E}_q^+ \mathbb{1}_{[h_l,+\infty)} \left(\mathcal{E}_q^+\right)^{-1} V_{\text{lead}}^1(x). \tag{3.18}$$

Proof in Subsection A.4.

Now consider the case when the preemption zone is not empty and precommitment of firm 2 is impossible. In Lemmata B.4, B.5, we show that if the preemption zone is not empty, then $h_l < h_f^1 < x_H$, therefore waiting in the area $[x_H, \infty)$ is not optimal for firm 1. Thus, if none of the players have entered before $\tau_{x_H}^+$, firm 1 enters immediately, and the value at entry is $V(X_t) = V_{\text{lead}}^1(X_t)$, where $t = \tau_{x_H}^+$. If none of the firms have entered before τ_{pz}, then the solution of the subgame that starts in the preemption zone gives us the value of firm 1 at $t = \tau_{\text{pz}}$:

$$V(X_t) = \begin{cases} V_{\text{lead}}^1(X_t) & \text{in type (2) equilibrium} \\ V_f^1(X_t) & \text{in type (1) or (3) equilibrium.} \end{cases}$$

If firm 1 decides to enter at some $t < \tau_{x_L}^+$, then the value at entry is $V(X_t) = V_{\text{lead}}^1(X_t)$. Hence firm 1 solves the following optimization problem

$$\mathcal{V}^1(x) = \sup_{\tau \leq \tau_{x_L}^+} \mathbb{E}^x \left[e^{-q\tau} V(X_\tau) \right], \tag{3.19}$$

where

$$V(x) = V^1_{\text{lead}}(x), \tag{3.20}$$

if type (2) equilibrium is played in the preemption zone. If type (1) or type (3) equilibrium is played in the preemption zone, then

$$V(x) = V^1_{\text{lead}}(x) - \mathbb{1}_{(x_L,x_H)}(x)(V^1_{\text{lead}} - V^1_f)(x). \tag{3.21}$$

Theorem 3.4. *Let $k < k^*$, and, if the density of positive jumps is non-trivial, let the equilibrium at points $x \in (x_L, x_H)$ be of type (2). Then the optimal stopping time of firm 1 is $\tau^+_{\min\{h_l,x_L\}}$.*

See Subsection A.5 for the proof. If $h_l \le x_L = x_L(k)$, the low cost firm chooses the optimal entry threshold given by (3.17) as if it were a monopolist and gets the value $\mathcal{V}^1(x)$ given by (3.18). If $h_l > x_L = x_L(k)$, then the optimal entry threshold for firm 1 is x_L, and the value function of this firm is

$$\mathcal{V}^1(x) = \mathcal{E}^+_q \mathbb{1}_{[x_L,+\infty)} \left(\mathcal{E}^+_q\right)^{-1} V^1_{\text{lead}}(x). \tag{3.22}$$

The firms' equilibrium payoffs are $W^1_t(s^{t,X_t}) = \mathcal{V}^1(X_t)$, $W^2_t(s^{t,X_t}) = V^2_f(X_t)$.

Observe that if the underlying stochastic process is spectrally negative, that is, there are no upward jumps in the market demand, then, for the outcome of the game, it is irrelevant which equilibrium is played in the preemption zone, because eventually, the process X that starts at $x < x_e(k) := \min\{x_L(k), h_l\}$ either never enters $[x_e(k), \infty)$, and then neither firm enters; or the process X enters $[x_e(k), \infty)$ at $x_e(k)$. Indeed, the trajectories of the supremum process are continuous if the process is spectrally negative. If $h_l \le x_L(k)$, and the process X reaches h_l, then the low cost firm enters at h_l, the threshold which is optimal for the monopolist, i.e., firm 1 becomes the leader without preemptive motives. If $x_L(k) < h_l$, and the process reaches $x_L(k)$, then the low cost enters as the leader at $x_L(k)$ in order to preempt the entry of the high cost firm. In either case, the high cost firm becomes the follower (if the first instantaneous moment is negative, it is possible that the high cost firm does not enter for a given sample path).

Theorem 3.5. *Let the density of positive jumps be non-trivial, let $k < k^*$, and let the equilibria at points $x \in (x_L, x_H)$ be of types (1) or (3). Then*

(a) if equation $(\mathcal{E}^+_q)^{-1}V(x) = 0$ has a solution on $(-\infty, x_L(k)]$, this solution is unique; denote it $h_{lp} = h_{lp}(k)$;

(b) if $h_{lp}(k) \le x_L(k)$ exists, then the optimal stopping time of firm 1 is $\tau^+_{h_{lp}}$ and the value function of this firm is

$$\mathcal{V}^1(x) = q^{-1}\mathcal{E}^+\mathbb{1}_{[h_{lp},+\infty)}(\mathcal{E}^+)^{-1}V(x);$$

otherwise, the optimal stopping time of firm 1 is $\tau^+_{x_L}$ and the value function of this firm is

$$\mathcal{V}^1(x) = q^{-1}\mathcal{E}^+\mathbb{1}_{[x_L,+\infty)}(\mathcal{E}^+)^{-1}V(x);$$

(c) if $h_l \le x_L(k)$, then $h_{lp}(k)$ exists, and $h_{lp}(k) < h_l$.

See Subsection A.6 for the proof. Thus, if equilibria of types (1) or (3) are played in the preemption zone, $V(x) = V^1_{\text{lead}}(x)$ for $x \le x_L$ and for $x \ge x_H$, but, on (x_L, x_H), the value of firm 1 at entry is smaller: $V(x) = V^1_f(x)$. Since positive jumps may bring the leader into the preemption region (x_L, x_H), where the value function drops to the value of the follower, the low cost firm enters earlier than when type (2) equilibrium is played in the preemption zone whenever $h_{lp}(k) < h_l \le x_L(k)$, or $h_{lp}(k) < x_L(k) < h_l$. If $h_{lp}(k) < h_l \le x_L(k)$, the sequential equilibrium, where firm 1 chooses the optimal entry threshold of the monopolist, no longer exists.

Firm 1's equilibrium payoff is $W^1_t(s^{t,X_t}) = \mathcal{V}^1(X_t)$. If type (1) equilibrium is played in the preemption zone, then the equilibrium payoff of firm 2 is $W^2_t(s^{t,X_t}) = V^2_f(X_t)$. If type (3) equilibrium is played in the preemption zone, then the equilibrium payoff of firm 2 is

$$W^2_t(s^{t,X_t}) = V^2_f(X_t) + \mathbb{1}_{(x_L,x_H)}(X_t)(V^2_{\text{lead}} - V^2_f)(X_t).$$

4. Subgame perfect equilibria for demand process with non-trivial positive jumps.

Assume that in the initial state, entry is not optimal for any of the firms, and that a sample path ω is fixed. Let LCZ denote the low cost entry zone, that is the state space region, when entry is optimal for the low cost firm but not for the high cost firm. We characterize this region in each of the theorems below. Let $\tau_{\text{lcz}} = \tau_{\text{lcz}}(\omega)$ denote the first entrance time into the set LCZ. If there are non-trivial upward jumps in the demand process, then any of the three action zones – LCZ, PZ, or SEZ – can be the first one entered by X_t.

Suppose first that $k \ge k^*$, so that the preemption zone is empty. If the shock enters the low cost entry zone earlier than the simultaneous entry zone then sequential equilibrium happens. If the shock enters the simultaneous entry zone earlier than the other action region, then the simultaneous investment occurs as an equilibrium. These results are summarized in

Theorem 4.1. *Let the density of positive jumps be non-trivial, and let $k \ge k^*$. Then*
(a) LCZ=$[h_l, h^2_f)$.
(b) A perfect equilibrium in a subgame starting at (t_0, X_0) such that $t_0 < \min\{\tau_{\text{lcz}}, \tau_{\text{sez}}\}$ is given by the following pair of closed loop strategies.

- *For $t < \min\{\tau_{\text{lcz}}, \tau_{\text{sez}}\}$, $G^{t_0,X_0}_i(t) = \alpha^{t_0,X_0}_i(t) = 0$ $(i = 1, 2)$.*
- *For $t \ge \min\{\tau_{\text{lcz}}, \tau_{\text{sez}}\}$,*

(i) if $\tau_{\text{lcz}} < \tau_{\text{sez}}$, then

$$G^{t_0,X_0}_1(t) = \alpha^{t_0,X_0}_1(t) = 1 \ \forall\, t \ge \tau_{\text{lcz}}; \quad G^{t_0,X_0}_2(t) = \alpha^{t_0,X_0}_2(t) = \begin{cases} 0 \text{ if} & t < \tau_{\text{sez}} \\ 1 \text{ if} & t \ge \tau_{\text{sez}} \end{cases} \tag{4.1}$$

(ii) if $\tau_{\text{sez}} < \tau_{\text{lcz}}$, then

$$G^{t_0,X_0}_1(t) = G^{t_0,X_0}_2(t) = \alpha^{t_0,X_0}_1(t) = \alpha^{t_0,X_0}_2(t) = \begin{cases} 0 & \text{if} \quad t < \tau_{\text{sez}} \\ 1 & \text{if} \quad t \ge \tau_{\text{sez}} \end{cases} \tag{4.2}$$

Hence, under the conditions of Theorem 4.1, firm 1 chooses the entry threshold given by (3.17) as if it had it been a monopolist in the market. In other words, the low cost firm simply maximizes the value of its investment opportunity as if the high cost firm does not exist. Proofs of Theorem 4.1 and all results in the rest of this Section can be found in Appendix C.

If the preemption zone is non-empty, then the results depend on the type of equilibrium played in the preemption zone.

Theorem 4.2. *Let the density of positive jumps be non-trivial, let $k < k^*$, and let the equilibrium at points $x \in (x_L, x_H)$ be of type (1). Then*

(a) LCZ=$[h_{lp}, h_f^2) \setminus (x_L, x_H)$, if $h_{lp}(k) \leq x_L(k)$ exists. Otherwise LCZ=$[x_H, h_f^2)$.

(b) A perfect equilibrium in a subgame starting at (t_0, X_0) such that $t_0 < \min\{\tau_{\text{lcz}}, \tau_{\text{sez}}, \tau_{\text{pz}}\}$ is given by the following pairs of closed loop strategies.

- *For $t < \min\{\tau_{\text{lcz}}, \tau_{\text{sez}}, \tau_{\text{pz}}\}$, $G_i^{t_0,X_0}(t) = \alpha_i^{t_0,X_0}(t) = 0$ ($i = 1, 2$).*
- *For $t \geq \min\{\tau_{\text{lcz}}, \tau_{\text{sez}}, \tau_{\text{pz}}\}$,*

(i) if $\tau_{\text{pz}} < \min\{\tau_{\text{lcz}}, \tau_{\text{sez}}\}$, then

$$G_i^{t_0,X_0}(t) = \begin{cases} \frac{\alpha_i^{t_0,X_0}(\tau_{\text{pz}})}{\alpha_i^{t_0,X_0}(\tau_{\text{pz}})+\alpha_j^{t_0,X_0}(\tau_{\text{pz}})-\alpha_i^{t_0,X_0}(\tau_{\text{pz}})\alpha_j^{t_0,X_0}(\tau_{\text{pz}})} & \text{if} \quad \tau_{\text{pz}} \leq t < \tau_f^i \\ 1 \quad \text{if} & \qquad t \geq \tau_f^i; \end{cases} \tag{4.3}$$

$$\alpha_i^{t_0,X_0}(t) = \begin{cases} \frac{V_{\text{lead}}^j(X_{\tau_{\text{pz}}})-V_f^j(X_{\tau_{\text{pz}}})}{V_{\text{lead}}^j(X_{\tau_{\text{pz}}})-P_j(X_{\tau_{\text{pz}}})} & \text{if} \quad \tau_{\text{pz}} \leq t < \tau_f^i \\ 1 \quad \text{if} & \qquad t \geq \tau_f^i \end{cases} \tag{4.4}$$

(ii) if $\tau_{\text{lcz}} < \min\{\tau_{\text{pz}}, \tau_{\text{sez}}\}$, then the equilibrium strategies are given by (4.1)*; if $\tau_{\text{sez}} < \min\{\tau_{\text{pz}}, \tau_{\text{lcz}}\}$, then the equilibrium strategies are given by* (4.2)*.*

Under the assumptions of Theorem 4.2, there is a preemptive equilibrium, in which each of the firms may become the leader with a positive probability. This equilibrium takes place if the shock enters the preemption zone earlier than other zones. In this equilibrium, a weak form of rent equalization happens in the sense that the expected value of the player that enters with a positive probability is the value of the follower. If $h_{lp}(k) < x_L(k)$ exists, and the shock enters the interval $[h_{lp}(k), \min\{h_l, x_L(k)\}] \subset$LCZ earlier than other action regions, then there is another preemptive equilibrium, in which firm 1 is the leader, and firm 2 is the follower. If the interval $[x_H, h_f^2) \subseteq$LCZ or $[h_l, x_L(k)]$ (when this interval is non-empty) is reached earlier than any other action region, then firm 1 enters without a preemptive motive, and firm 2 follows. If the low cost firm is the first entrant, it extracts the monopolist's rents until the other firm enters. Notice that simultaneous investment may occur in the preemption zone as a coordination failure. If the shock enters the SEZ earlier than any other action region, then the simultaneous entry occurs as an equilibrium.

Theorem 4.3. *Let the density of positive jumps be non-trivial, let $k < k^*$, and let the equilibrium at points $x \in (x_L, x_H)$ be of type (2). Then* *(a)* *LCZ*$=[h_l, h_f^2) \setminus (x_L, x_H)$.

(b) *A perfect equilibrium in a subgame starting at* (t_0, X_{t_0}) *such that* $t_0 < \min\{\tau_{\rm lcz}, \tau_{\rm sez}, \tau_{\rm pz}\}$ *is given by the following pairs of closed loop strategies.*

- *For* $t < \min\{\tau_{\rm lcz}, \tau_{\rm sez}, \tau_{\rm pz}\}$, $G_i^{t_0,X_0}(t) = \alpha_i^{t_0,X_0}(t) = 0$ *(*$i = 1, 2$*).*
- *For* $t \geq \min\{\tau_{\rm lcz}, \tau_{\rm sez}, \tau_{\rm pz}\}$,

(i) if $\tau_{\rm pz} < \min\{\tau_{\rm lcz}, \tau_{\rm sez}\}$, *then* $G_1^{t_0,X_0}(t) = \alpha_1^{t_0,X_0}(t) = 1$ *for all* $t \geq \tau_{\rm pz}$. $G_2^{t_0,X_0}(\tau_{\rm pz}) = \alpha_2^{t_0,X_0}(\tau_{\rm pz}) = 0$. *If* $G_1^{t_0,X_0}(\tau_{\rm pz}) = 1$ *then*

$$G_2^{t_0,X_0}(t) = \alpha_2^{t_0,X_0}(t) = \begin{cases} 0 \text{ if} & t < \tau_f^2 \\ 1 \text{ if} & t \geq \tau_f^2; \end{cases}$$

If $G_1^{t_0,X_0}(\tau_{\rm pz}) = 0$, *then for all* $t > \tau_{\rm pz}$, $\alpha_i^{t_0,X_0}(t)$ *are given by the RHS of* (4.4)*, and* $G_i^{t_0,X_0}(t)$ *are given by the RHS of* (4.3) *with* $\alpha_i^{t_0,X_0}(\tau_{\rm pz})$ *and* $\alpha_j^{t_0,X_0}(\tau_{\rm pz})$ *being replaced by* $\alpha_i^{t_0,X_0}(\tau_{\rm pz}+)$ *and* $\alpha_j^{t_0,X_0}(\tau_{\rm pz}+)$, *respectively (*$i, j \in \{(1,2),(2,1)\}$*).*

(ii) if $\tau_{\rm lcz} < \min\{\tau_{\rm pz}, \tau_{\rm sez}\}$, *then the equilibrium strategies are given by* (4.1)*; if* $\tau_{\rm sez} < \min\{\tau_{\rm pz}, \tau_{\rm lcz}\}$, *then the equilibrium strategies are given by* (4.2)*.*

Hence, under the assumptions of Theorem 4.3, the roles of the firms are predetermined, unless the shock enters the SEZ region earlier than any other action regions: the low cost firm is the first to enter, and the high cost firm is the follower. In the preemption zone, the low cost firm preempts with probability one. In the LCZ, the low cost firm has no incentive to preempt. If the shock enters the SEZ earlier than any other action region, then the simultaneous entry occurs as an equilibrium.

Theorem 4.4. *Let the density of positive jumps be non-trivial, let* $k < k^*$, *and let the equilibrium at points* $x \in (x_L, x_H)$ *be of type (3). Then*

(a) LCZ is the same as in Theorem 4.2(a).

(b) A perfect equilibrium in a subgame starting at (t_0, X_{t_0}) *such that* $t_0 < \min\{\tau_{\rm lcz}, \tau_{\rm sez}, \tau_{\rm pz}\}$ *is given by the following pairs of closed loop strategies.*

- *For* $t < \min\{\tau_{\rm lcz}, \tau_{\rm sez}, \tau_{\rm pz}\}$, $G_i^{t_0,X_0}(t) = \alpha_i^{t_0,X_0}(t) = 0$ *(*$i = 1, 2$*).*
- *For* $t \geq \min\{\tau_{\rm lcz}, \tau_{\rm sez}, \tau_{\rm pz}\}$,

(i) if $\tau_{\rm pz} < \min\{\tau_{\rm lcz}, \tau_{\rm sez}\}$, *then the equilibrium strategies are the same as in Theorem 4.3 (b)(i), but subscripts 1 and 2 interchange throughout.*

(ii) if $\tau_{\rm lcz} < \min\{\tau_{\rm pz}, \tau_{\rm sez}\}$, *then the equilibrium strategies are given by* (4.1)*; if* $\tau_{\rm sez} < \min\{\tau_{\rm pz}, \tau_{\rm lcz}\}$, *then the equilibrium strategies are given by* (4.2)*.*

Given the assumptions of Theorem 4.4, there is a preemptive equilibrium, in which then firm 2 preempts firm 1 with probability one. This equilibrium occurs if $\tau_{\rm pz} < \min\{\tau_{\rm lcz}, \tau_{\rm sez}\}$. If $h_{lp}(k) < x_L(k)$ exists, then there is the second preemptive equilibrium, which happens when the shock enters the interval $[h_{lp}(k), \min\{h_l, x_L(k)\}] \subset$LCZ earlier than other action regions. In this equilibrium, firm 1 preempts firm 2 with probability one. If the interval $[x_H, h_f^2) \subseteq$LCZ or $[h_l, x_L(k)]$ (when this interval is non-empty) is reached earlier than any other action region, then firm 1 enters without a preemptive motive, and firm 2 follows. In either of these equilibria, the first mover enjoys the Stackelberg leader's advantage until the other firm follows. If the shock enters the SEZ earlier than any other action region, then the simultaneous entry occurs as an equilibrium.

5. Conclusion

We considered a stochastic version of Fudenberg and Tirole's (1985) preemption game where two firms contemplate entering a new market with stochastic demand that follows a jump-diffusion process. Firms differ is the sunk costs of entry. In the initial state, entry is optimal to none of the firms. We studied the effects of positive jumps and firms' asymmetry on equilibrium strategies. If the demand process admits positive jumps, then simultaneous entry can happen either as an equilibrium, or as a coordination failure with positive probability; sequential equilibrium may disappear; the high cost firm may be the first to enter. Assuming that the same type of equilibrium is played at each point of the preemption zone, we characterized strategies in subgame perfect equilibria in terms of stopping times and value functions. Analytical expressions for the value functions and thresholds that define stopping times were derived. The model may be used to calculate the probabilities of potentially observable outcomes of the preemption game as well as the expected waiting time until such outcomes may be observed, given the current state of the demand.

References

[1] V. Aguirregabiria, and P. Mira, "Sequential estimation of dynamic discrete games, Econometrica, 75:1 (2007): 1-53.

[2] L.H.R. Alvarez, "Minimum guaranteed payments and costly cancellation rights: a stopping game perspective," *Mathematical Finance*, 20:4 (2010), 733–751

[3] L.H.R. Alvarez, "A class of solvable stopping games," *Applied Mathematics & Optimization*, 58 (2008), 291–314

[4] R. Amir, "Continuous stochastic games of capital accumulation with convex transitions," *Games and Economic Behavior*, 15 (1996), 111–131

[5] R. Amir, and N. Lazzati, "Network effects, market structure and industry performance," *Journal of Economic Theory*, to appear, 2011

[6] K. Back, and D. Paulsen, "Open-Loop Equilibria and Perfect Competition in Option Exercise Games," *Review of Financial Studies*, 22:11 (2009), 4531–4552.

[7] S. Berry, and E. Tamer, "Identification in models of oligopoly entry," in *Advances in Economics and Econometrics: Theory and Applications, Ninth World Congress, vol. 2,* R. Blundell, W.K. Newey and T. Persson, eds., Cambridge Univ. Press, 2007.

[8] P. Bolton and C. Harris, "Strategic expirementation," *Econometrica*, 67:2 (1999), pp. 349–374.

[9] A.N. Borodin and P. Salminen, *Handbook on Brownian Motion – Facts and Formulae,* 2nd edition, Birkhäuser, Basel, 2002

[10] R. Bouis, K.J.M. Huisman, and P.M. Kort, "Investment in oligopoly under uncertainty: The accordion effect," *International Journal of Industrial Organization* 27 (2009), pp. 320–331.

[11] M. Boyarchenko and S.Z. Levendorskiĭ, "Prices and sensitivities of barrier and first-touch digital options in Lévy-driven models", *International Journal of Theoretical and Applied Finance* 12 (2009), pp. 1125–1170

[12] S.I. Boyarchenko and S.Z. Levendorksiĭ, *Non-Gaussian Merton-Black-Scholes theory.* Singapore: World Scientific 2002.

[13] S.I. Boyarchenko and S.Z. Levendorksiĭ, "American options: the EPV pricing model", *Annals of Finance* 1:3 (2005), 267–292.

[14] S.I. Boyarchenko and S.Z. Levendorksiĭ, "General Option Exercise Rules, with Applications to Embedded Options and Monopolistic Expansion", *Contributions to Theoretical Economics,* 6:1 (2006), Article 2

[15] S.I. Boyarchenko and S.Z. Levendorksiĭ, *Irreversible Decisions Under Uncertainty (Optimal Stopping Made Easy)*, Springer, Berlin, 2007.

[16] S.I. Boyarchenko and S.Z. Levendorksiĭ, "Optimal stopping in Lévy models, for non-monotone discontinuous payoffs," *SIAM Journal on Control and Optimization*, 49:5 (2011), 2062–2082.

[17] S.I. Boyarchenko and S.Z. Levendorksiĭ, "Preemption Games under Levy Uncertainty," Working Paper (May 14, 2011). Available at SSRN: http://ssrn.com/abstract=1841823

[18] A. Dixit, and R. Pindyck, *Investment under uncertainty,* Princeton University Press, Princeton, 1994

[19] U. Doraszelski, and J.F. Escobar, "A Theory of Regular Markov Perfect Equilibria in Dynamic Stochastic Games: Genericity, Stability, and Purification," *Theoretical Economics*, 5 (2010), 369-402.

[20] U. Doraszelski, and M. Satterthwaite, "Computable Markov-Perfect Industry Dynamics," *RAND Journal of Economics*, 41:2 (2010), 215–243.

[21] P.K. Dutta, and A. Rustichini "A Theory of Stopping Time Games With Applications to Product Innovations and Asset Sales, *Economic Theory*, 3 (1993), 743-763.

[22] P.K. Dutta, S. Lach, and A. Rustichini "Better Late than Early," *Journal of Economics and Management Strategy*, 4 (1995), 563–589.

[23] E.B. Dynkin "Game Variant of a Problem on Optimal Stopping," *Soviet Math. Dokl.*, 10, (1969), 270-274.

[24] E. Ekström, and S. Villeneuve, "On the Value of Optimal Stopping Games," *The Annals of Applied Probability*, 16:3 (2006), 1576-1596.

[25] E.Z. Ferenstein, "Randomized Stopping Games and Markov Market Games,“ *Mathematical Methods of Operations Research*, 66 (2007), 531-544.

[26] D. Fudenberg and J. Tirole, "Preemption and Rent Equalization in the Adoption of New Technology," *Review of Economics Studies*, 52 (1985), 383–401.

[27] P. Ghemawat and B. Nalebuff, "Exit," *Rand Journal of Economics* 16 (1985), 184–194.

[28] S. Grenadier, *Game Choices: The Intersection of Real Options and Game Theory,* Risk Books, 2000

[29] S. Grenadier, "Option Exercise Games: An Application to the Equilibrium Investment Strategies of Firms," *Review of Financial Studies*, 2002

[30] S.R. Grenadier, and N. Wang "Investment under uncertainty and time inconsistent preferences," *Journal of Financial Economics*, 84 (2007), 2–39.

[31] H. Hoppe, and U. Lehmann-Grube, "Innovation timing games: a general framework with applications," *Journal of Economic Theory*, 121 (2005), 30–50.

[32] Y.-W. Hsu and B.M. Lambrecht, "Preemptive patenting under uncertainty and asymmetric information," *Annals of Operations Research*, 151 (2007), 5–28.

[33] G. Keller, S. Rady, and M. Cripps, "Strategic Experimentation with Exponential Bandits," *Econometrica*, 73:1 (2005), 39–68.

[34] Y. Kifer, "Game Options," *Finance and Stochastics*, 4 (2000), 443-463.

[35] Kijima, M., M. Egami, K. Tanaka, and Y. Muromachi, *Recent Advances in Financial Engineering. Proceedings of the 2008 Daiwa International Workshop on Financial Engineering Otemachi Sankei Plaza, Tokyo, Japan, 4 5 August 2008,* World Scientific 2009
[36] A.E. Kyprianou, "Some calculations for Israeli options," *Finance and Stochastics*, 8 (2004), 73–86.
[37] B.M. Lambrecht and W. Perraudin, "Real options and preemption under incomplete information," *Journal of Economic Dynamics & Control* 27 (2003), 619–643.
[38] R. Laraki, E. Solan, and N. Vieille, "Continuous-time Games of Timing," *Journal of Economic Theory*, 120 (2005), 206–238.
[39] G. Mankiw, "Real Business Cycles: A New Keynesian Perspective," *Journal of Economic Perspectives*, 3 (1989), 79–90.
[40] P. Murto and J. Välimäki , "Learning and Information Aggregation in an Exit Game," (2010) *Review of Economic Studies*, forthcoming
[41] P. Murto and J. Välimäki , "Delay and Information Aggregation in Stopping Games with Private Information," (2011), Working paper
[42] G. Pawlina and P.M. Kort, "Real Options in Asymmetric Duopoly: Who Benefits from Your Comparative Disadvantage?" *Journal of Economics & Management Strategy*, 15:1 (2006), 1–35.
[43] A. Park and L. Smith, "Caller Number Five and Related Timing Games," *Theoretical Economics*, 3 (2008), 231–256.
[44] M. Pesendorfer, and P. Schmidt-Dengler, "Asymptotic least square estimators for dynamic games," *Review of Economic Studies*, 75:3 (2008), 901–928.
[45] P. Protter, *Stochastic Integration and Differential Equations.* Berlin: Springer-Verlag 1992
[46] M. Sîrbu, and S.E. Shreve, "A two person game for pricing convertible bonds," *SIAM Journal on Control and Optimization*, 45:4 (2006), 1508–1539.
[47] Sato, Ken-Iti, *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press 1999.
[48] E. Solan, and N. Vieille, "Stopping Games - Recent Results," in *Advances in Dynamic Games Applications to Economics, Finance, Optimi.zation, and Stochastic Control,* (eds.: Novak, A.S.and K. Szajowski) Birkhauser Boston, vol. 7 (2002), 235–245.
[49] J.-H. Steg, "Irreversible investment in oligopoly," *Finance and Stochastics,* 16:2 (2012), 207–224.
[50] J.J.J. Thijssen, "Preemption in a real option game with a first mover advantage and player-specific uncertainty," *Journal of Economic Theory*, 145 (2010), 2448–2462.
[51] J.J.J. Thijssen, K.J.M. Huisman, and P.M. Kort, "Symmetric Equilibrium Strategies in Game Theoretical Real Option Models," *Journal of Mathematical Economics*, 48 (2002), 219–225.
[52] J.J.J. Thijssen, K.J.M. Huisman, and P.M. Kort, "The Effects of Information on Strategic Investment and Welfare," *Economic Theory*, 28 (2006), 399–424.
[53] N. Touzi, and N. Vieille, "Continuous Time Dynkin Games with Mixed Strategies," *SIAM Journal on Control and Optimization*, 41:4 (2002), 1073–1088.
[54] H. Weeds, "Strategic Delay in a Real Options Model of R&D Competition," *Review of Economic Studies*, 69 (2002), 729–747.

Appendix A. Auxiliary results and technical details of optimal stopping results

A.1. **Probability of crossing.**

Lemma A.1. *a) Let* Ψ *admit the representation*

$$\Psi(\beta) = \frac{m_2}{2}\beta^2 + m_1\beta + O(|\beta|^3), \quad \beta \to 0, \tag{A.1}$$

and let $\{X_t\}$ starts at $x < h$. Then, if $m_1 \ge 0$, $\{X_t\}$ reaches or crosses h, a.s., and if $m_1 < 0$, then, with a positive probability, $\{X_t\}$ remains below h.

b) Let $m_1 < 0$, and, in addition, let (2.6) hold. Then the probability that $\{X_t\}$ reaches or crosses h ever equals

$$\lim_{T\to\infty} P(\overline{X}_T \ge h \mid X_0 = x) = \int d\mu_0(\beta) e^{\beta(x-h)}, \tag{A.2}$$

where $\mu_0(d\beta)$ is a measure of the total unit mass supported on $(1, +\infty)$.

We see that the probability that the level h will be reached decays exponentially with the distance $h - x$.

In full generality, part a) can be proved using the general definitions and facts from [47] (Definitions 24.13, 35.1; Theorems 35.4, 35.8; Corollary 37.6). We will give a simple proof of both parts a) and b) in the case of BM, DEJD and HEJD models. The probability that a trajectory starting from x reaches h or crosses x equals

$$\lim_{T\to\infty} \mathbb{E}^x \left[\mathbb{1}_{\overline{X}_T \ge h}\right] = \lim_{q\downarrow 0} \mathbb{E}^x \left[e^{-q\tau_h^+}\right]$$

where τ_h^+ is the first entrance time by $\{X_t\}$ into $[h, +\infty)$. We have $\mathbb{E}^x \left[e^{-q\tau_h^+}\right] = \mathcal{E}_q^+ \mathbb{1}_{[h,+\infty)}(x)$. Under condition (2.6), $\mathcal{E}_q^+ = \int d\mu_q(\beta) I_\beta^+$, therefore

$$\mathcal{E}_q^+ \mathbb{1}_{[h,+\infty)}(x) \;=\; \int \mu_q(d\beta) \int_0^{+\infty} \beta e^{-\beta y} \mathbb{1}_{[h,+\infty)}(x+y)dy = \int d\mu_q(\beta) e^{\beta(x-h)}$$

In the case of BM, DEJD and HEJD models,

$$\int d\mu_q(\beta) e^{\beta(x-h)} = \sum_{j=1}^m a_{q,j}^+ e^{\beta_{q,j}^+(x-h)}, \tag{A.3}$$

where $0 < \beta_{q,1}^+ < \beta_{q,2}^+ < \cdots$ are positive solutions of the equation $q - \Psi(\beta) = 0$, and $a_{q,j}^+$ are positive constants. In the BM case, there is only one positive root, and $a_{q,1}^+ = 1$; in the DEJD model, there are two roots, and $a_{q,j}^+$ are given by (2.9). Formulas in the HEJD model are similar. It is easy to see that, as $q \downarrow 0$,

(1) if $m_1 \ge 0$, then $\beta_{q,1}^+ \to 0$, $\beta_{q,j}^+, j \ge 2$, have finite limits, and $a_{q,1}^+ \to 1$, hence, the RHS in (A.3) tends to 1;

(2) if $m_1 < 0$, then all $\beta_{q,j}^+$ have positive limits, hence, the limit of the LHS in (A.3) is as stated in b), and it is less than 1.

A.2. **Main properties of EPV-operators.** In the proofs of optimal stopping results, we systematically use the following properties of the EPV-operators.

Lemma A.2. *a) EPV-operators $\mathcal{E}_q$ and $\mathcal{E}_q^\pm$ are positive.*

b) If $u(x) = 0 \;\; \forall\, x \in (-\infty, h)$, then $\mathcal{E}_q^- u(x) = 0 \;\; \forall\, x \in (-\infty, h)$, and the same statement holds with $(-\infty, h]$ instead of $(-\infty, h)$.

c) If $u(x) = 0 \;\; \forall\, x \in (h, +\infty)$, then $\mathcal{E}_q^+ u(x) = 0 \;\; \forall\, x \in (h, +\infty)$, and the same statement holds with $[h, +\infty)$ instead of $(h, +\infty)$.

d) Statements b) and c) hold for the inverses $(\mathcal{E}_q^-)^{-1}$ and $(\mathcal{E}_q^+)^{-1}$, respectively, if u is continuous at h and piece-wise differentiable. If the process has no BM component, then the continuity condition at h can be relaxed.

Note that in all cases, which we will consider, functions are sufficiently regular so that d) is applicable. For details, see [12, 14, 15].

A.3. **General theorem on optimal stopping.** For a stopping time τ and measurable f and g, define

$$V_{\mathrm{en}}(\tau; f; x) = \mathbb{E}\left[\int_\tau^{+\infty} e^{-qt} f(x+X_t)dt\right], \tag{A.4}$$

$$V_{\mathrm{inst}}(\tau; g; x) = \mathbb{E}^x[e^{-q\tau} g(X_\tau)]. \tag{A.5}$$

The following general formulas are derived in [15, eqn.(11.53), (11.56)] under an additional restriction on X, and proved in [11] for an arbitrary Lévy process.

Theorem A.3. *a) Let $h \in \mathbb{R}$ and let f be a measurable function, which is either semi-bounded or satisfies the no-bubble condition $\mathcal{E}_q|f| < +\infty$. Then*

$$V_{\mathrm{en}}(\tau_h^+; f; x) = q^{-1}\mathcal{E}_q^+ \mathbb{1}_{[h,+\infty)} \mathcal{E}_q^- f(x). \tag{A.6}$$

b) Let there exists a measurable f satisfying the no-bubble condition $\mathcal{E}_q|f| < +\infty$ such that the instantaneous payoff in (A.5) *is given by $g = q^{-1}\mathcal{E}_q f$. Then*

$$V_{\mathrm{inst}}(\tau_h^+; g; x) = \mathcal{E}_q^+ \mathbb{1}_{[h,+\infty)} (\mathcal{E}_q^+)^{-1} g(x). \tag{A.7}$$

Under weak regularity conditions on g and X, one can define $w(x) = (\mathcal{E}_q^+)^{-1} g(x)$ without resorting to f and prove (A.7) (see [12, 14, 15] for a detailed analysis).

Expressions (A.6) and (A.7) are convenient for the theoretical analysis because EPV-operators are positive. In addition, if we know that a function vanishes above a certain point, then $\mathcal{E}_q^+ f(x)$ also vanishes above this point. Similarly, if a function vanishes below a certain point, then $\mathcal{E}_q^- f(x)$ also vanishes below this point. The proofs of all main optimal stopping results are based, ultimately, on these two crucial properties.

Here we present the main optimal stopping theorems which are used in the leader-follower game. The detailed proofs of all these theorems can be found in [16, 17]. Assume that X is a Lévy process satisfying (ACP)-property[4], with non-trivial supremum and infimum processes.

Theorem A.4. *Let there exist $h \in \mathbb{R}$ such that*

$$\mathcal{E}_q^- f(x) \le 0, \quad x \le h, \quad \text{and} \quad \mathcal{E}_q^- f(x) \ge 0, \quad x \ge h. \tag{A.8}$$

Then

(a) τ_h^+ maximizes $V_{\mathrm{en}}(\tau; f; x)$ in the class of stopping times of the threshold type.

[4]In [47, pp. 288-289], the reader can find several equivalent definitions of (ACP)-property. One of these is: for any $f \in L_\infty(\mathbb{R})$, $\mathcal{E}_q f$ is continuous. A sufficient condition is: for some $t > 0$, the transition measure $\mathbb{P}_{X_t}$ is absolutely continuous.

(b) If, in addition,

$$-f(x)+\int_{-\infty}^{h-x} V_{\mathrm{en}}(\tau;-f;x+y)F(dy)\le 0,\quad x>h,\ \text{a.e.,} \tag{A.9}$$

then τ_h^+ maximizes $V_{\mathrm{en}}(\tau;f;x)$ in the class of all stopping times.

Observe that (a) follows immediately from (A.6) and properties of EPV-operators: since operator $\mathcal{E}_q^+$ is monotone, an optimal choice of h must replace all negative values of $\mathcal{E}_q^- f$ with zeroes and leave positive ones as they are. Hence h that satisfies (A.8) is the only optimal threshold.

To formulate the next theorem, we need the definition of the infinitesimal operator L of the Lévy process X ([47, Thm. 31.5]). If u is sufficiently regular, then

$$Lu(x)=\frac{\sigma^2}{2}u''(x)+bu'(x)+\int_{\mathbb{R}\setminus 0}(u(x+y)-u(x))F(dy). \tag{A.10}$$

An important general relation between Ψ and L, the infinitesimal generator of X, is $Le^{\beta x}=\Psi(\beta)e^{\beta x}$. Also, $\mathcal{E}_q=q(q-L)^{-1}$ as operators in appropriate function spaces.

Theorem A.5. *Let there exist $h\in\mathbb{R}$ such that*

$$(\mathcal{E}_q^+)^{-1}g(x)\le 0,\quad x\le h,\quad \text{and}\quad (\mathcal{E}_q^+)^{-1}g(x)\ge 0,\quad x\ge h, \tag{A.11}$$

Then
(a) τ_h^+ maximizes $V_{\mathrm{inst}}(\tau;g;x)$ in the class of stopping times of the threshold type.
(b) If, in addition,

$$(q-L)g(x)+\int_{-\infty}^{h-x}(\mathcal{E}_q^+\mathbb{1}_{(-\infty,h)}(\mathcal{E}_q^+)^{-1}g)(x+y)F(dy)\ge 0,\quad x>h,\ \text{a.e.,} \tag{A.12}$$

then τ_h^+ maximizes $V_{\mathrm{inst}}(\tau;g;x)$ in the class of all stopping times.

Notice that if $g=q^{-1}\mathcal{E}_q f$, then Theorems A.4 and A.5 are equivalent. The next two theorems give sets of conditions on f, $(q-L)G$ and $F(dy)$, which imply that (A.9) and (A.12) hold (see [16, 17] for other simple sufficient conditions).

Theorem A.6. *Let f be a non-decreasing function, which changes sign. Then (i) there exists h such that (A.8) holds, and (ii) τ_h^+ maximizes $V_{\mathrm{en}}(\tau;f;x)$ in the class of all stopping times.*

Theorem A.7. *Let the following three conditions hold (i) there exists $h\in\mathbb{R}$ such that (A.11) holds; (ii) $(q-L)g$ is non-decreasing on $(h,+\infty)$; (iii) measure $F(dy)$ is non-decreasing on $(-\infty,0)$. Then τ_h^+ maximizes $V_{\mathrm{inst}}(\tau;g;x)$ in the class of all stopping times.*

A.4. **Proof of Theorem 3.3.** We apply Theorem A.7 with $g = V^1_{\text{lead}}$, with the following adjustment. In the case under consideration, the entry at $x \ge h^2_f$ is mandatory, and, therefore, the optimizing decision can be made only while $X_t \le h^2_f$. The straightforward analysis of the proof shows that it suffices to verify the conditions (i)-(iii) of Theorem A.7 for $x < h^2_f$ only. Part (iii) holds by Assumption $\mathbf{X}$(ii). To verify (i), recall that, $V^1_{\text{lead}}(x) = q^{-1}(\mathcal{E}_q f^2_1)(x)$, for $x \ge h^2_f$, and, for $x < h^2_f$,

$$V^1_{\text{lead}}(x) = q^{-1}\mathcal{E}_q f^1_1(x) - q^{-1}\mathcal{E}^+_q \mathbb{1}_{[h^2_f,+\infty)}\mathcal{E}^-_q(f^1_1 - f^2_1)(x) \tag{A.13}$$

(see (3.10), (3.11), and (3.12)). Therefore,

$$\begin{aligned} qV^1_{\text{lead}}(x) &= \mathbb{1}_{(-\infty,h^2_f)}\left[\mathcal{E}_q f^1_1(x) - \mathcal{E}^+_q \mathbb{1}_{[h^2_f,+\infty)}\mathcal{E}^-_q(f^1_1 - f^2_1)(x)\right] + \mathbb{1}_{[h^2_f,+\infty)}\mathcal{E}_q f^2_1(x) \\ &= \mathcal{E}_q f^1_1(x) - \mathcal{E}^+_q \mathbb{1}_{[h^2_f,+\infty)}\mathcal{E}^-_q(f^1_1 - f^2_1)(x) \\ &\quad +\mathbb{1}_{[h^2_f,+\infty)}\left[\mathcal{E}_q(f^2_1 - f^1_1)(x) + \mathcal{E}^+_q \mathbb{1}_{[h^2_f,+\infty)}\mathcal{E}^-_q(f^1_1 - f^2_1)(x)\right] \\ &= \mathcal{E}_q f^1_1(x) - \mathcal{E}^+_q \mathbb{1}_{[h^2_f,+\infty)}\mathcal{E}^-_q(f^1_1 - f^2_1)(x) - \mathbb{1}_{[h^2_f,+\infty)}\mathcal{E}^+_q \mathbb{1}_{(-\infty,h^2_f]}\mathcal{E}^-_q(f^1_1 - f^2_1)(x). \end{aligned} \tag{A.14}$$

For $x < h^2_f$, using (A.13), we get

$$(\mathcal{E}^+_q)^{-1}V^1_{\text{lead}}(x) = (\mathcal{E}^+_q)^{-1}q^{-1}\mathcal{E}_q f^1_1(x) = q^{-1}(\mathcal{E}^-_q)f^1_1(x) = q^{-1}(D(1)\kappa^-_q(1)e^x - qI)$$

which is monotone on $(-\infty, h^2_f)$ and changes sign; h_l is the only zero. Moreover,

$$e^{h_l} = \frac{qI}{D(1)\kappa^-_q(1)} < \frac{qkI}{D(2)\kappa^-_q(1)} = e^{h^2_f}.$$

To verify part (ii), we apply $q-L$ to (A.14). Using equalities $(q-L)q^{-1}\mathcal{E}_q f^1_1(x) = f^1_1(x)$ and $(q-L)q^{-1}\mathcal{E}^+_q = (\mathcal{E}^-_q)^{-1}$, we obtain

$$\begin{aligned} (q-L)V^1_{\text{lead}}(x) &= f^1_1(x) - (\mathcal{E}^-_q)^{-1}\mathbb{1}_{[h^2_f,+\infty)}\mathcal{E}^-_q(f^1_1 - f^2_1)(x) \\ &\quad + \int_0^{+\infty} q^{-1}\mathbb{1}_{[h^2_f,+\infty)}\mathcal{E}^+_q \mathbb{1}_{(-\infty,h^2_f)}\mathcal{E}^-_q(f^1_1 - f^2_1)(x+y)F(dy) \end{aligned}$$

The first term is non-decreasing since f^1_1 is non-decreasing. The third term is also non-decreasing since $f^1_1 - f^2_1$ is non-decreasing. Since $\mathbb{1}_{[h^2_f,+\infty)}\mathcal{E}^-_q(f^1_1 - f^2_1)$ vanishes on $(-\infty, h^2_f)$, $(\mathcal{E}^-_q)^{-1}\mathbb{1}_{[h^2_f,+\infty)}\mathcal{E}^-_q(f^1_1 - f^2_1)$ also vanishes on $(-\infty, h^2_f)$ (see Lemma A.2). We conclude that $(q-L)V^1_{\text{lead}}$ is non-decreasing on $(-\infty, h^2_f)$, which finishes the proof.

A.5. **Proof of Theorem 3.4.** If the constraint in (3.19) does not bind, then, by (3.20) and Theorem 3.3, h^l is the optimal entry threshold for firm 1. If the constraint in (3.19) binds, then x_L is the optimal entry threshold. To prove that it is non-optimal for the low cost firm to enter earlier than at x_L, consider the value function of the firm, if it chooses $h < x_L$ as the entry threshold:

$$\mathcal{V}^1(x) = q^{-1}\mathcal{E}^+\mathbb{1}_{[h,+\infty)}(\mathcal{E}^+)^{-1}V^1_{\text{lead}}(x).$$

In the proof of optimality of h_l, it was shown that $(\mathcal{E}^+)^{-1}V^1_{\text{lead}}(x) < 0$ for $x < h_l$, hence, for $x < x_L$. Moreover, $(\mathcal{E}^+)^{-1}V^1_{\text{lead}}(x) < 0$ is monotone for $x < x_L$. Therefore any choice $h < x_L$ decreases the value function $\mathcal{V}^1(x)$.

A.6. **Proof of Theorem 3.5.** (a) For $x < x_L$, consider

$$
\begin{aligned}
(\mathcal{E}^+_q)^{-1}V(x) &= (\mathcal{E}^+_q)^{-1}V^1_{\text{lead}}(x) - (\mathcal{E}^+_q)^{-1}\mathbb{1}_{(x_L,x_H)}(x)(V^1_{\text{lead}} - V^1_f)(x) \\
&= (\mathcal{E}^+_q)^{-1}q^{-1}\mathcal{E}_q f^1_1(x) + \int_0^{+\infty} k^{+-}_q(y)\mathbb{1}_{(x_L,x_H)}(x+y)(V^1_{\text{lead}} - V^1_f)(x+y)dy \qquad \text{(A.15)} \\
&= q^{-1}(\mathcal{E}^-_q)f^1_1(x) + \int_{x_L}^{x_H} k^{+-}_q(y-x)(V^1_{\text{lead}} - V^1_f)(y)dy. \qquad \text{(A.16)}
\end{aligned}
$$

The second term in (A.15) is obtained with the help of (2.7). By assumption, k^{+-}_q is a non-negative, non-increasing on $(0,+\infty)$ function, and $q^{-1}(\mathcal{E}^-_q)f^1_1(x) = q^{-1}(D(1)\kappa^-_q(1)e^x - qI)$ is increasing on $(-\infty, x_L]$. Moreover,
$\lim_{x\to-\infty} k^{+-}_q(y-x) = 0$,[5] hence $\lim_{x\to-\infty}(\mathcal{E}^+_q)^{-1}V(x) = -I < 0$. Therefore, if $(\mathcal{E}^+_q)^{-1}V(x_L) > 0$, there exists a unique h_{lp} s.t. $(\mathcal{E}^+_q)^{-1}V(h_{lp}) = 0$.
(c) Let $h_l \le x_L$, then at $x = h_l$, the first term on the RHS of (A.16) is zero (see the proof of Theorem 3.4) and the second term is positive. Therefore $(\mathcal{E}^+_q)^{-1}V(h_l) > 0$, hence there exists a unique $h_{lp} < h_l$ s.t. $(\mathcal{E}^+_q)^{-1}V(h_{lp}) = 0$.

(b) Let $(\mathcal{E}^+_q)^{-1}V(x_L) < 0$. We proved in (a) that $(\mathcal{E}^+_q)^{-1}V(x)$ is monotone on $(-\infty, x_L]$, therefore stopping earlier than at x_L will decrease the value function $\mathcal{V}^1(x)$. Let $(\mathcal{E}^+_q)^{-1}V(x_L) > 0$. We have to prove that $\tau^+_{h_{lp}}$ is the optimal entry time in the class of all stopping times. We need to verify condition (A.12) in Theorem A.5 for the pair (V, h_{lp}). The analysis of the proof of Theorem A.5 (see [16, 17] for details) shows that it suffices to require that condition (A.12) holds only in the region $x < x_L$, where firm 1 can optimize. By Assumption **X**(ii), the Lévy density of negative jumps is monotone. It follows from Theorem A.7 that it suffices to verify that $(q-L)V(x)$ is non-decreasing on (h_{lp}, x_L). We have

$$
(q-L)V(x) \;=\; (q-L)V^1_{\text{lead}}(x) - \int_0^{\infty} F(dy)(\mathbb{1}_{(x_L,x_H)}(V^1_{\text{lead}} - V^1_f))(x+y).
$$

The monotonicity of $(q-L)V^1_{\text{lead}}(x)$ on $(-\infty, h^2_f)$ is demonstrated in the proof of Theorem 3.4. For $x < x_L$, the second term can rewritten in the form $-\int_{x_L}^{x_H} F(dy - x)(V^1_{\text{lead}} - V^1_f)(y)$. By Assumption **X**(ii), the density of positive jumps is monotone. Taking into account that function $(V^1_{\text{lead}} - V^1_f)$ is positive on (x_L, x_H), we conclude that the second term is non-decreasing. This finishes the proof of optimality of $\tau^+_{h_{lp}}$. □

[5] If $\lim_{x\to+\infty} k^{+-}_q(x) \neq 0$, then it follows from (2.7) that $(\mathcal{E}^+_q)^{-1}e^x$ is not well-defined. But $(\mathcal{E}^+_q)^{-1}e^x = (\kappa^+_q(1))^{-1}e^x$, where $\kappa^+_q(1) \in (0,+\infty)$ by the no-bubble condition.

Appendix B. Preemption zone: a detailed study

The results below demonstrate that, depending on the cost differential and realization of the demand shock, the high cost firm may or may not have an incentive to become the leader, but the low cost firm will find it optimal to be the leader at any k, if the realization of the demand shock is sufficiently high. Notice that, since $V^i_{\rm lead}(x) = V^i_f(x) = P_i(x)$ for $x \ge h^2_f$, $DV^i_f(k,x) = 0$ for $x \ge h^2_f$. Also, $V^2_{\rm lead}(x) = P_2(x)$ for $x \ge h^1_f < h^2_f$; therefore (3.7) implies that

$$DV^2_f(k,x) > 0 \quad \text{for } x \in (h^1_f, h^2_f), \tag{B.1}$$

with the equality at $x = h^1_f$. Let $\mu_q(d\beta)$ be the measure on $(1,+\infty)$ from the condition (2.6), of unit mass, and set $Z = D(1)/D(2)$, $Y = e^{x-h^1_f}$. Direct calculations at the end of this section (which, for the reader's convenience, we give separately in the jump-diffusion case and the BM case) give

Lemma B.1. *For $x \le h^1_f$, $DV^i_f(k,x) = I \cdot g_i(k,Y)$, where*

$$g_1(k,Y) \;=\; 1 - \kappa^+_q(1)ZY + \int \mu_q(d\beta)Y^\beta \frac{1}{\beta-1}\left(1+\beta k^{1-\beta}(Z-1)\right) \tag{B.2}$$

$$g_2(k,Y) \;=\; k - \kappa^+_q(1)ZY + \int \mu_q(d\beta)Y^\beta \frac{1}{\beta-1}\left(k^{1-\beta}+\beta(Z-1)\right) \tag{B.3}$$

Recall that in the BM case, $\mu_q(d\beta)$ is an atom at $\beta^+ = \beta^+_q$, where β^+_q is the positive root of the fundamental quadratic $q - \Psi(\beta) = 0$, and $\kappa^+_q(1) = \beta^+/(\beta^+ - 1)$.

Lemma B.2. *As functions on $[1,+\infty)\times[0,1]$, $g_j(k,Y), j = 1,2,$ are continuous, and convex in Y; $g_1(k,Y)$ is decreasing in k and $g_2(k,Y)$ is increasing in k.*

Proof. Continuity is evident. Since $\operatorname{supp}\mu_q \in (1,+\infty)$, both functions are convex in Y, and $g_1(k,Y)$ is decreasing in k. Finally, since $\int \mu_q(d\beta) = 1$, $k \ge 1$ and $Y \le 1$,

$$\partial_k g_2(k,Y) = 1 - \int \mu_q(d\beta)Y^\beta k^{-\beta} \ge 0,$$

with the strict inequality if $k > 1$. □

Lemma B.3. *There exists $k^* > 1$ such that*

(a) if $k > k^*$, then $g_2(k,Y) > 0$ for all $Y \in [0,1]$;
(b) if $k = k^*$, then $g_2(k,Y) \ge 0$ for all $Y \in [0,1]$, and there exists a unique $Y^* \in (0,1)$ such that $g_2(k,Y^*) = 0$;
(c) if $k \in (1,k^*)$, then there exist $0 < Y_L(k) < Y_H(k) < 1$ such that
- *(i) $g_2(k,Y) > 0$ for all $0 < Y < Y_L(k)$ and $Y \in (Y_H(k),1]$;*
- *(ii) $g_2(k,Y) < 0$ for all $Y \in (Y_L(k),Y_H(k))$;*
- *(iii) $g_2(k,Y) = 0$ for $Y = Y_L(k)$ and $Y = Y_H(k)$.*

(d) $Y_L(k)$ is a continuous increasing function, and $Y_H(k)$ is a continuous decreasing function of k on $[1,k^*]$.

Proof. Since $\kappa_q^+(1) = \int \mu_q(d\beta)\frac{\beta}{\beta-1}$ and $Z = D(1)/D(2) > 1$,

$$\begin{aligned} g_2(1,1) &= \int \mu_q(d\beta) - \kappa_q^+(1)Z + \int \mu_q(d\beta)\frac{1}{\beta-1}\left(1+\beta(Z-1)\right) \\ &= -\kappa_q^+(1)Z + Z\int \mu_q(d\beta)\frac{\beta}{\beta-1} = -\kappa_q^+(1)Z + Z\kappa_q^+(1) = 0; \end{aligned} \tag{B.4}$$

$$\begin{aligned} \frac{\partial g_2(1,1)}{\partial Y} &= -Z\int \mu_q(d\beta)\frac{\beta}{\beta-1} + \int \mu_q(d\beta)\frac{\beta}{\beta-1}\left(1+\beta(Z-1)\right) \\ &= (Z-1)\int \mu_q(d\beta)\beta > 0. \end{aligned} \tag{B.5}$$

By Lemma B.2, $g_2(k,Y)$ is convex in Y. Convexity of $g_2(k,Y)$ in Y, (B.4)–(B.5) and equality $g_2(k,0) = k(>0)$ imply together that equation $g_2(1,Y) = 0$ has two solutions: $Y_L(1) < 1$ and $Y_H(1) = 1$. Since $g_2(k,Y)$ is increasing in k, $g_2(k,1) > 0$ for $k > 1$. By continuity in Y, (B.4) and (B.5), $g_2(1,Y) < 0$ in a left neighborhood of Y. By continuity in k, for k sufficiently close to 1, there exists $Y < 1$ such that $g_2(k,Y) < 0$. Hence, equation $g_2(k,Y) = 0$ has two solutions: $Y_L(k) < Y_H(k) < 1$ if k is sufficiently close to 1.

Since $g_2(k,Y)$ is convex in Y and increasing in k, $Y_L(k)$ is a continuous increasing function, and $Y_H(k)$ is a continuous decreasing function of k. As k increases, eventually, $Y_L(k)$ and $Y_H(k)$ collide, and then vanish. The values $Y_L(k^*) = Y_H(k^*) = Y^*$ and k^* are defined by $g_2(k^*,Y^*) = 0$, and $\partial g_2(k^*,Y^*)/\partial Y = 0$. For $k > k^*$, $g_2(k,Y) > 0$ for all Y.

□

Lemma B.4. *Let $g_1(k,Y)$ be given by* (B.2). *For any $k > 1$,*

(a) there exists a unique $Y_*(k) \in (0,1)$ s.t. $g_1(k,Y_*(k)) = 0$, $g_1(k,Y) > 0$ for all $Y < Y_*(k)$, and $g_1(k,Y) < 0$ for all $Y \in (Y_*(k),1)$;
(b) $Y_*(k)$ is a decreasing function of k on $(1,+\infty)$;
(c) $Y_*(k) < Y_L(k)$ for all $1 < k \le k^*$.

Proof. Observe that $g_1(k,0) = 1 > 0$, and, for $k > 1$,

$$\begin{aligned} g_1(k,1) &= \int \mu_q(d\beta) - Z\int \mu_q(d\beta)\frac{\beta}{\beta-1} + \int \mu_q(d\beta)\frac{1}{\beta-1}\left(1+\beta k^{1-\beta}(Z-1)\right) \\ &= (Z-1)\int \mu_q(d\beta)\frac{1}{\beta-1}(k^{1-\beta}-1) < 0; \end{aligned}$$

since $g_1(k,Y)$ is convex in Y, (a) follows. Since $g_1(k,Y)$ is decreasing in k, (b) holds.

To prove (c), we calculate the difference

$$\begin{aligned} \Delta g(k,Y) &= g_2(k,Y) - g_1(k,Y) \\ &= k - 1 + \int \mu_q(d\beta)Y^\beta \frac{1-k^{1-\beta}}{\beta-1}\left(\beta(Z-1)-1\right). \end{aligned}$$

We have $\Delta g(1,Y)=0$, and

$$\begin{aligned}\frac{\partial \Delta g(k,Y)}{\partial k} &= 1+\int \mu_q(d\beta)Y^\beta k^{-\beta}\left(\beta(Z-1)-1\right)\\ &= (Z-1)\int \mu_q(d\beta)Y^\beta k^{-\beta}\beta+\int \mu_q(d\beta)\left(1-Y^\beta k^{-\beta}\right)>0\end{aligned}$$

for $Y\le 1<k$. Hence, for $k>1$ and $Y\in(0,1]$, the graph of $g_2(k,Y)$ is above the graph of $g_1(k,Y)$. Hence $Y_*(k)$, the solution of equation $g_1(k,Y)=0$, is to the left of $Y_L(k)$, the left solution of equation $g_2(k,Y)=0$. □

Lemmata B.3 and B.4 establish existence of the preemption zone for $k<k^*$ with the boundaries $x_L(k)=\ln Y_L(k)+h^1_f$ and $x_H(k)=\ln Y_H(k)+h^1_f$.

Denote by $Y_*(k)$ the solution of equation $g_1(k,Y)=0$, and set $x_*(k)=\ln Y_*(k)+h^1_f$. Let h_l denote the optimal entry threshold of firm 1 if this firm were a monopolist.

Lemma B.5. *Let $k\in[1,k^*)$. Then $x_*(k)<h_l<x_H(k)$.*

Proof. Assuming that the trajectories of X_t may move up with non-negative probability (equivalently, $X_t\neq \underline{X}_t$), it must be that $x_*(k)<h_l$. Indeed, firm 1's problem is equivalent to timing the exchange of the follower's value $V^1_f(X_t)$ for the value of leader at entry $V^1_{\mathrm{lead}}(X_t)$. The choice of $x_*(k)$ as the exercise boundary corresponds for the naive present rule; due to the positive value of waiting, $x_*(k)<h_l$.

By (3.2) and (3.17), $e^{h_l-h^1_f}=D(2)/D(1)=Z^{-1}$. Therefore in order to prove that $h_l<x_H(k)$, we have to prove that $Z^{-1}<Y_H(k)$. Recall that $g_2(k,Y)$ is convex in Y and $Y_L(k)<Y_H(k)$ are solutions of the equation $g_2(k,Y)=0$. Therefore, there exists a unique $Y^*(k)\in(Y_L(k),Y_H(k))$ such that $\partial_Y g_2(k,Y)(k,Y)<0$ iff $Y<Y^*(k)$, and >0 iff $Y>Y^*(k)$. Hence, once we prove that $\partial_Y g_2(k,Y)|_{Y=1/Z}<0$, we can conclude that $Z^{-1}<Y_H(k)$, and $h_l<x_H(k)$.

Using (B.3), we derive

$$\begin{aligned}\partial_Y g_2(k,Y)|_{Y=1/Z} &= Z\left[-\kappa^+_q(1)+\int \mu_q(d\beta)\frac{\beta}{\beta-1}\left(k^{1-\beta}+\beta(Z-1)\right)Z^{-\beta}\right]\\ &= Z^{1-\beta}\int \mu_q(d\beta)\frac{\beta}{\beta-1}\left[-Z^\beta+k^{1-\beta}+\beta(Z-1)\right]\end{aligned}$$

The integrand is strictly decreasing in k, since $\beta>1$ on the support of $\mu(d\beta)$, therefore, it suffices to prove that, if $\beta>1$, then $F(Z):=-Z^\beta+1+\beta(Z-1)$ is decreasing on $[1,+\infty)$. But $F'(Z)=\beta(1-Z^{\beta-1})<0$ if $Z>1$. □

B.1. **Proof of Lemma B.1: the jump-diffusion case.** Using the Wiener-Hopf factorization formula in the analytical form

$$\frac{q}{q-\Psi(\beta)}=\kappa^-_q(\beta)\kappa^+_q(\beta), \tag{B.6}$$

where $\kappa^{\pm}_q(\beta)=\left(\mathcal{E}^{\pm}_q e^{\beta x}\right)\big|_{x=0}$, we rewrite equation (3.2) for the optimal entry threshold of the follower as

$$\frac{D(2)}{q-\Psi(1)}e^{h^i_f}=\kappa^+_q(1)I_i. \tag{B.7}$$

It follows immediately from (B.7) that $e^{h^2_f-h^1_f}=k$. Under condition (2.6), $\mathcal{E}^+_q=\int\mu(d\beta)I^+_\beta$, where I^+_β is the convolution operator given by (2.8). Let $\beta^+>1$ be the lowest point in the support of $\mu(d\beta)$. If $z<\beta^+$, then, on $(-\infty,h)$, we have

$$I^+_\beta\mathbb{1}_{[h,+\infty)}(x)e^{zx}=\beta\int_0^{+\infty}e^{-\beta y}\mathbb{1}_{[h,+\infty)}(x+y)e^{z(x+y)}dy=e^{\beta(x-h)}\frac{\beta}{\beta-z}e^{zh}. \tag{B.8}$$

Using (3.5), (3.11), and (3.12), we derive

$$\begin{aligned} DV^i_f(k,x) &= V^i_f(x)-V^i_{\text{lead}}(x)\\ &= q^{-1}\mathcal{E}^+_q\left(\mathbb{1}_{[h^i_f,+\infty)}f^2_i(x)-\mathbb{1}_{[h^j_f,+\infty)}\mathcal{E}^-_q\left(f^2_i(x)-f^1_i(x)\right)\right)-V^i_m(x). \end{aligned}\tag{B.9}$$

Substituting for $V^i_m(x)$ and $f^k_i(x), k=1,2$, their definitions and using the formula $\mathcal{E}^-e^x=\kappa^-_q(1)e^x$, we write (B.9) as

$$\begin{aligned} DV^i_f(k,x) &= I_i-\frac{D(1)e^x}{q-\Psi(1)}+q^{-1}(\mathcal{E}^+_q\mathbb{1}_{[h^i_f,+\infty)}(\kappa^-_q(1)D(2)e^{\cdot}-qI_i))(x)\\ &\quad+q^{-1}\left(D(1)-D(2)\right)\kappa^-_q(1)(\mathcal{E}^+_q\mathbb{1}_{[h^j_f,+\infty)}e^{\cdot})(x). \end{aligned}\tag{B.10}$$

Applying $\mathcal{E}^+_q=\int\mu_q(d\beta)I^+_\beta$ to functions $\mathbb{1}_{[h,+\infty)}(x)e^{zx}$, $z=1$ and $z=0$, and substituting the results into (B.10) with $i=1$, $j=2$, we obtain

$$DV^1_f(k,x)=I-\frac{D(1)e^x}{q-\Psi(1)}+q^{-1}\int\mu_q(d\beta)e^{\beta(x-h^1_f)}C_1(\beta;k), \tag{B.11}$$

where

$$\begin{aligned} C_1(\beta;k) &= \kappa^-_q(1)D(2)e^{h^1_f}\frac{\beta}{\beta-1}-I+(Z-1)\kappa^-_q(1)D(2)e^{\beta(h^1_f-h^2_f)}\frac{\beta}{\beta-1}e^{h^2_f}\\ &= qI\left(\frac{\beta}{\beta-1}-1+k^{1-\beta}(Z-1)\frac{\beta}{\beta-1}\right)=\frac{qI}{\beta-1}\left(1+\beta k^{1-\beta}(Z-1)\right). \end{aligned}$$

This gives (B.2). Similarly,

$$DV^2_f(k,x) = kI-\frac{D(1)e^x}{q-\Psi(1)}+q^{-1}\int\mu_q(d\beta)e^{\beta(x-h^1_f)}C_2(\beta;k), \tag{B.12}$$

where

$$\begin{aligned}C_2(\beta;k) &= \kappa_q^-(1)D(2)\frac{\beta}{\beta-1}e^{h_f^2+\beta(h_f^1-h_f^2)} - kqIe^{\beta(h_f^1-h_f^2)} + (Z-1)D(2)\kappa_q^-(1)\frac{\beta}{\beta-1}e^{h_f^1}\\ &= qI\left(\frac{\beta}{\beta-1}k^{1-\beta} - k^{1-\beta} + (Z-1)\frac{\beta}{\beta-1}\right)\\ &= \frac{qI}{\beta-1}\left(k^{1-\beta}+\beta(Z-1)\right),\end{aligned}$$

and (B.3) follows.

B.2. **Proof of Lemma B.1: the BM case.** We use the following standard results from [18]. For firm i, the value of entry when the other firm is on the market and the value of the simultaneous entry equal $P_i(x) = D(2)e^x/(q-\Psi(1)) - I_i$. In the inaction region $x < h_f^i$, $j \neq i$, the value of follower is of the form $V_f^i(x) = \alpha e^{\beta^+(x-h_f^i)}$, where $\beta^+ > 1$ is the positive root of the fundamental quadratic $q - \Psi(\beta) = 0$, and α is a constant. Using the continuous pasting condition at h_f^i and equation (B.7) for $e^{h_f^i}$, we find

$$V_f^i(x) = (D(2)e^x/(q-\Psi(1))-I_i)e^{\beta^+(x-h_f^i)} = (\kappa_q^+(1)-1)I_ie^{\beta^+(x-h_f^i)} = \frac{I_i}{\beta^+-1}e^{\beta^+(x-h_f^i)}$$

In the region $x < h_f^j$, where $j \neq i$, the value of the leader is of the form

$$V_{\text{lead}}^i(x) = \frac{D(1)e^x}{q-\Psi(1)} - I_i + \alpha_1 e^{\beta^+(x-h_f^j)}.$$

Using the continuous pasting condition at h_f^j, we find

$$\begin{aligned}V_{\text{lead}}^i(x) &= \frac{D(1)e^x}{q-\Psi(1)} - I_i + \frac{(D(2)-D(1))e^x}{q-\Psi(1)}e^{\beta^+(x-h_f^j)}\\ &= \kappa_q^+(1)I_ie^{x-h_f^i}\left(Z-(Z-1)e^{\beta^+(x-h_f^j)}\right)\end{aligned}$$

Finally, calculating the differences $DV^i = V_f^i - V_{\text{lead}}^i$ and using the equalities $e^{h_f^2} = ke^{h_f^1}$ and $\kappa_q^+(1) = \beta^+/(\beta^+-1)$, we obtain (B.2), (B.3).

Appendix C. Proofs of the main theorems

In the statements of Theorems 3.2 – 4.2, by inspection, all strategies are adapted to the filtration $(\mathcal{F}_t)_{t_0\le t<\infty}$; at each point, they are either right continuous with left limits or left continuous with right limits. Functions $G_i^{t_0,X_0}(\cdot,\omega)$ are non-decreasing by construction. In addition, all strategies are α-consistent and satisfy intertemporal consistency conditions of Definition 2.4. Therefore, we need to check only optimality of strategies.

C.1. **Proof of Theorem 3.2.** Consider a subgame that starts at $t = \tau_{\mathrm{pz}}$ and $X_t = X_{\tau_{\mathrm{pz}}}$. Then $G_i^{t,X_t}(t-) = \alpha_i^{t,X_t}(t-) = 0$ for $i = 1, 2$. Let $s^{t,X_t} = (s_1^{t,X_t}, s_2^{t,X_t})$, where $s_i^{t,X_t} = (G_i^{t,X_t}, \alpha_i^{t,X_t})$ for $i = 1, 2$, be a strategy profile prescribed by Theorem 3.2. Consider equilibrium (1). If both players play the profile s^{t,X_t}, then the probability that the players enter simultaneously is (see [51] for details)

$$\mathcal{G}^{t,X_t}(t) = \frac{\alpha_1^{t,X_t}(t)\alpha_2^{t,X_t}(t)}{\alpha_1^{t,X_t}(t) + \alpha_2^{t,X_t}(t) - \alpha_1^{t,X_t}(t)\alpha_2^{t,X_t}(t)},$$

and the probability that firm i enters earlier than firm j is

$$G_i^{t,X_t}(t) - \mathcal{G}^{t,X_t}(t) = \frac{\alpha_i^{t,X_t}(t)(1 - \alpha_j^{t,X_t}(t))}{\alpha_i^{t,X_t}(t) + \alpha_j^{t,X_t}(t) - \alpha_i^{t,X_t}(t)\alpha_j^{t,X_t}(t)}.$$

Therefore the expected value of player i is

$$\begin{aligned}W_t^i(s^{t,X_t}) &= \frac{\alpha_i^{t,X_t}(t)(1 - \alpha_j^{t,X_t}(t))V_{\mathrm{lead}}^i(X_t)}{\alpha_i^{t,X_t}(t) + \alpha_j^{t,X_t}(t) - \alpha_i^{t,X_t}(t)\alpha_j^{t,X_t}(t)}\\ &+ \frac{(1 - \alpha_i^{t,X_t}(t))\alpha_j^{t,X_t}(t)V_f^i(X_t)}{\alpha_i^{t,X_t}(t) + \alpha_j^{t,X_t}(t) - \alpha_i^{t,X_t}(t)\alpha_j^{t,X_t}(t)} + \frac{\alpha_i^{t,X_t}(t)\alpha_j^{t,X_t}(t)P_i(X_t)}{\alpha_i^{t,X_t}(t) + \alpha_j^{t,X_t}(t) - \alpha_i^{t,X_t}(t)\alpha_j^{t,X_t}(t)}.\end{aligned}$$

Straightforward algebraic manipulations allow one to rewrite the value function as

$$\begin{aligned}W_t^i(s^{t,X_t}) &= V_f^i(X_t)\\ &+ \frac{\alpha_i^{t,X_t}(t)}{\alpha_i^{t,X_t}(t) + \alpha_j^{t,X_t}(t)(1 - \alpha_i^{t,X_t}(t))}\left[V_{\mathrm{lead}}^i(X_t) - V_f^i(X_t) - \alpha_j^{t,X_t}(t)\left(V_{\mathrm{lead}}^i(X_t) - P_i(X_t)\right)\right].\end{aligned}$$

Substituting (3.15) for $\alpha_j^{t,X_t}(t)$, we arrive at

$$W_t^i(s^{t,X_t}) = V_f^i(X_t), \tag{C.1}$$

and this value is independent of $\alpha_i^{t,X_t}(t)$.

Let player j play the equilibrium strategy s_j^{t,X_t}, and player i ($(i,j) \in \{(1,2),(2,1)\}$) play some other strategy $\tilde{s}_i^{t,X_t} = (\tilde{G}_i^{t,X_t}, \tilde{\alpha}_i^{t,X_t})$. If $\tilde{\alpha}_i^{t,X_t}(t) > 0$, then $\tilde{G}_i^{t,X_t}$ is predetermined by α-consistency condition, and $(G_j^{t,X_t}(t), \alpha_j^{t,X_t}(t))$ becomes $(\hat{G}_j^{t,X_t}(t), \alpha_j^{t,X_t}(t))$, where $\hat{G}_j^{t,X_t}(t)$ also adjusts according to α-consistency condition. Since the value of player i is given by (C.1) for any $\tilde{\alpha}_i^{t,X_t}(t) > 0$ as long as $\alpha_j^{t,X_t}(t)$ is given by (3.15), player i has no profitable deviations. If $\tilde{\alpha}_i^{t,X_t}(t) = 0$, by α-consistency condition, $\hat{G}_j^{t,X_t}(t) = 1$. In this case, $W_t^i(\tilde{s}_i^{t,X_t}, s_j^{t,X_t}) \le V_f^i(X_t)$. Hence player i has no profitable deviations. Hence (1) is a subgame perfect equilibrium.

Consider next equilibrium (2). If the players are playing the strategy profile prescribed by equilibrium (2), the values are $W_t^1(s^{t,X_t}) = V_{\mathrm{lead}}^1(X_t)$, and $W_t^2(s^{t,X_t}) = V_f^2(X_t)$. First, we check if player 1 has profitable deviations. Let player 2 play the equilibrium strategy s_2^{t,X_t}, and player 1 play some other strategy $\tilde{s}_1^{t,X_t} = (\tilde{G}_1^{t,X_t}, \tilde{\alpha}_1^{t,X_t})$.

If $\tilde{G}_1^{t,X_t} = 0$, then by stochastic continuity of X, at $t+0$, the players play perfect equilibrium of type (1) with probability 1. Therefore $W_t^1(\tilde{s}_1^{t,X_t}, s_2^{t,X_t}) = V_f^1(X_{t+0})$. For X_t in the preemption zone, $V_{\text{lead}}^1(X_t) > V_f^1(X_t)$. By continuity of V_f^1 and stochastic continuity of X, $V_f^1(X_t) = V_f^1(X_{t+0})$. Hence

$$W_t^1(\tilde{s}_1^{t,X_t}, s_2^{t,X_t}) = V_f^1(X_{t+0}) < V_{\text{lead}}^1(X_t) = W_t^1(s^{t,X_t}).$$

Hence, player 1 has no profitable deviations.

Now let player 1 play the equilibrium strategy s_1^{t,X_t}, and player 2 play some other strategy $\tilde{s}_2^{t,X_t} = (\tilde{G}_2^{t,X_t}, \tilde{\alpha}_2^{t,X_t})$. Let $\tilde{s}_2^{t,X_t}$ be such that $\tilde{\alpha}_2^{t,X_t}(t) > 0$, hence $\tilde{G}_2^{t,X_t}(t) > 0$ by α-consistency condition; or $\tilde{\alpha}_2^{t,X_t}(t) = 0$, and $\tilde{G}_2^{t,X_t}(t) > 0$.

Then

$$W_t^2(s_1^{t,X_t}, \tilde{s}_2^{t,X_t}) = \tilde{G}_2^{t,X_t}(t) P_2(X_t) + \left(1 - \tilde{G}_2^{t,X_t}(t)\right) V_f^2(X_t) < V_f^2(X_t).$$

If $\tilde{s}_2^{t,X_t}$ is such $\tilde{\alpha}_2^{t,X_t}(t) = 0$, $\tilde{G}_2^{t,X_t}(t) = 0$, then player 2 has no profitable deviations on the strength of optimal stopping results in Appendix A. Hence (2) is a subgame perfect equilibrium. Symmetric argument argument shows that (3) is also a subgame perfect equilibrium.

□

C.2. **Proof of Theorem 4.1.** By general results on optimal stopping in Appendix A, entry is not optimal for any of the firms until the stochastic demand reaches the level given by (3.17). It is optimal for the high cost firm to invest the first time X_t enters the interval SEZ= $[h_f^2, +\infty)$, and it is optimal for the low cost firm to invest the first time X_t enters the interval $[h_l, +\infty) = [h_l, h_f^2) \cup [h_f^2, +\infty) = \text{LCZ} \cup \text{SEZ}$. Whence the statements of Theorem 4.1 follow. □

C.3. **Proof of Theorems 4.2, 4.3, and 4.4.** It follows from the optimal stopping results in Appendix A and Theorem 3.2 that the strategic waiting region of firm 2 is the region $(-\infty, x_L] \cup [x_H, h_f^2)$, and the strategic waiting region of firm 1 is the interval $(-\infty, h_{lp})$, if there exists $h_{lp} < x_L$, or the interval $(-\infty, x_L)$, otherwise. Therefore, $G_i^{t_0,X_0}(t) = \alpha_i^{t_0,X_0}(t) = 0$ for $t < \min\{\tau_{\text{lcz}}, \tau_{\text{sez}}, \tau_{\text{pz}}\}$. If the shock first enters the low cost entry zone or the simultaneous entry zone ($\tau_{\text{pz}} > \max\{\tau_{\text{lcz}}, \tau_{\text{sez}}\}$), then strategies in the statement of the theorems is a perfect Nash equilibrium by the same argument as in the proof of Theorem 4.1. If the shock enters the preemption zone earlier than other zones ($\tau_{\text{pz}} < \min\{\tau_{\text{lcz}}, \tau_{\text{sez}}\}$), then strategies in the statement of the theorems is a perfect Nash equilibrium by Theorem 3.2. □