Stag Hunt Contests and the Alliance Formation Puzzle

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Abstract: This study introduces the concept of a stag hunt contest game and uses the concept to present an alternative solution to the alliance formation puzzle. A stag hunt contest can evolve from any Tullock contest of three or more parties. In a stag hunt contest, efforts from the respective groups within an alliance interact as complements (rather than as substitutes) within the contest success function. As in a standard stag hunt game, efforts within an alliance are treated as complements because they are coordinated and targeted toward non-allied parties. A given party of the alliance is more effective against a given opponent as its coordinated ally presents a greater challenge to the same opponent. In an armed conflict, a rebel group's ground attack against an incumbent army is expected to be more effective in the presence of coordinated NATO air strikes against the same incumbent army. Conversely, NATO air strikes are expected to be more effective (e.g., less likely to meet with sustained anti-aircraft missile fire) as the rebel ground attack intensifies. On the more primitive level of a fistfight, one's punches are expected to be more effective as his or her friend's effort to restrain the opponent increases. Conversely, the friend's effectiveness in restraining the opponent improves when one is able to land punches vigorously. Therefore, the value of alliance formation may lie in the complementarity of *coordinated* efforts. Within a stag hunt contest, we find conditions by which alliance formation improves the expected payoff of each allied party. These conditions are found to exist whether an alliance divides the contest prize exogenously (via an agreed upon sharing rule) or endogenously (via intra-alliance contest) in the event of victory. The model provides an explanation of alliance-formation in contest and conflict that is complementary to existing explanations. The model also generates conditions that are conducive to the formation of alliance.

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I. Introduction

Alliance formation is puzzling within a standard, three-party model of conflict. Esteban and Sakovics (2003) find that (two-party) alliance formation within a three-party conflict (contest) unambiguously decreases the expected payoff of the allied parties. Konrad (2009), Kovenock and Roberson (2008), Konrad and Kovenock (2009), and Ke and Konrad (2013) further document (and provide solutions to) this puzzle. Decreased expected payoff under alliance is found to emerge for two potential reasons. Firstly, there is a bi-lateral "cheap ridership" problem within a two-party contest alliance, whereby a party's marginal benefit of allocating arms decreases in the number of allied arms allocated. This problem causes an under-allocation of arms by an alliance that, in turn, creates a loss of expected payoff among allied parties. Under-allocation of arms is the sole source of payoff loss for an alliance that engages in exogenous prize sharing. Parties that engage in endogenous prize sharing (i.e., hold an intra-alliance (subgame) prize-division contest in the case of victory) face an additional source of payoff loss in the form of iterated rent dissipation.

Konrad (2009), Kovenock and Roberson (2008), Konrad and Kovenock (2009), and Ke and Konrad (2013) provide potential solutions to the alliance formation puzzle. These include the presence of capacity constraints and the presence of resource transfers in multi-front battle. Moreover, Konrad (2009) suggests that complementarities between ally inputs may explain the choice of alliance formation. The concept of stag hunt contest incorporates the complementarity of allied inputs into the technology of contest. Herein, we construct a model that considers a conflict involving three parties. These parties can engage in non-coalitional conflict, form a grand coalition and divide the prize between all parties, or form a two-party alliance that targets its conflict efforts against the opposing party.

In the case of two-party alliance, we consider a stag hunt alliance formation, whereby the marginal productivity of one ally's inputs increases in the other ally's input allocation, within the contest technology. A given party of the alliance is more effective against a given opponent as its coordinated ally presents a greater challenge to the same opponent. In an armed conflict, a rebel group's ground attack against an incumbent army is expected to be more effective in the presence of coordinated NATO air strikes against the same incumbent army. Conversely, NATO air strikes are expected to be more effective (e.g., less likely to meet with sustained anti-aircraft missile fire) as the rebel ground attack intensifies. Hence, the value of alliance formation may lie in the complementarity of *coordinated* efforts. In Section II of the paper, we lay out a baseline (three-party) model of alliance formation under stag hunt contest. We consider the case of exogenous prize division and endogenous prize division and find conditions by which alliance formation improves the expected payoff of each allied party. These conditions are found to exist whether an alliance divides the contest prize exogenously (via an agreed upon sharing rule) or endogenously (via intra-alliance contest) in the event of victory. The model provides an explanation of alliance-formation in contest and conflict that is complementary to existing explanations. The model also generates conditions that are conducive to the formation of an alliance. The authors will generalize upon this baseline model (to n-players) in a future draft. Section III of the paper concludes.

I. Baseline Model

Three-Player Contest without Alliance Formation

Consider a three-player contest (Tullock lottery contest). In the case that no alliances are formed, each player's likelihood of victory is given by the following set of contest success functions.

$$p_1 = \frac{s_1}{s_1 + s_2 + s_3}$$

$$p_2 = \frac{s_2}{s_1 + s_2 + s_3}$$

$$p_3 = \frac{s_3}{s_1 + s_2 + s_3}$$

where p_1 represents the likelihood that Party $i \epsilon \{1,2,3\}$ will win the contest prize and s_i represents units of contest input spending allocated by Party i. The objective functions of the three parties to contest are given as follows.

$$Max_{\{s_1\}}\, U_1 = p_1 V - s_1$$

$$Max_{\{s_2\}}\, U_2 = p_2 V - s_2$$

$$Max_{\{s_3\}}\, U_3 = p_3 V - s_3$$

We develop first order conditions and solve for optimal arms allocations.

$$\frac{s_2 + s_3}{(s_1 + s_2 + s_3)^2} V = 1$$

$$\frac{s_1 + s_3}{(s_1 + s_2 + s_3)^2} V = 1$$

$$\frac{s_1 + s_2}{(s_1 + s_2 + s_3)^2} V = 1$$

$$s^*_{1,no\ alliance} = s^*_{2,no\ alliance} = s^*_{3,no\ alliance} = \frac{2}{9} V$$

Lastly, we find the expected payoff for each party to conflict.

$$U^*_{1,no\ alliance} = U^*_{2,no\ alliance} = U^*_{3,no\ alliance} = \frac{1}{9}V$$

Next, we analyze the case of three-player contest with stag hunt alliance formation.

Three-Player Nested Contest with Stag Hunt Alliance Formation and Endogenous Intra-Alliance Prize Division

We now consider a three-player contest in which Players 1 and 2 have formed an alliance. In the first stage, allied parties contest with Party 3 for resource (prize) control. Should the alliance win, allied parties then engage in a second stage (intra-group) contest for prize division in stage 2. We backwards induct toward an equilibrium solution beginning from the second-stage subgame that occurs when the alliance has won in first-stage contest.

Second Stage Intra-Group Contest subgame:

In the intra-alliance subgame, contest success function for parties 1 and 2 are represented as follows.

$$p_{1,intra} = \frac{s_{1,intra}}{s_{1,intra} + s_{2,intra}}$$

$$p_{2,intra} = \frac{s_{2,intra}}{s_{1,intra} + s_{2,intra}}$$

The objective functions of the two parties to intra-alliance contest are given as follows.

$$Max_{\{s_{1,intra}\}}\, p_{1,intra}V - s_{1,intra}$$

$$Max_{\{s_{2,intra}\}}\, p_{2,intra}V - s_{2,intra}$$

We develop first order conditions and solve for optimal inputs for the intra-alliance contest.

$$\frac{s_{2,intra}}{(s_{1,intra}+s_{2,intra})^2}V = 1$$

$$\frac{s_{1,intra}}{(s_{1,intra}+s_{2,intra})^2}V = 1$$

$$s^*_{1,intra} = s^*_{2,intra} = \frac{V}{4}$$

In the subgame, expected payoffs for Parties 1 and are given as follows.

$$U^*_{1,intra} = U^*_{2,intra} = \frac{V}{4}$$

In the event of first-stage victory, each party to alliance expects a payoff of $\frac{V}{4}$ given endogenous division by intra-alliance contest. Half of the prize value is dissipated in this sub-game contest. We now consider first-stage contest.

First Stage Contest

In first-stage contest, Parties 1 and 2 align against Party 3. Under the case of stag hunt alliance, Party 1 efforts and Party 2 efforts are taken to be complements (rather than substitutes) within the technology of the contest success function. As in the original version of the stag hunt game, two allied parties to contest who are coordinated against a common adversary may overwhelm said adversary. Moreover, such an effect may not be possible without coordination of efforts. Therefore, the value of alliance formation may lie in the complementarity of *coordinated* efforts. In the first stage of contest, allied party likelihood of victory is represented as follows.

$$p_{1\cup 2} = \frac{s_1 s_2}{s_1 s_2 + s_3}$$

Party 3 likelihood of victory, p_3, equals $(1 - p_{1\cup 2})$. That is,

$$p_3 = \frac{s_3}{s_1 s_2 + s_3}$$

Objective functions under stag hunt contest (with alliance formation and endogenous prize division) are represented as follows.

$$Max_{\{s_1\}}\, p_{1\cup 2} * \frac{V}{4} - s_1$$

$$Max_{\{s_2\}}\, p_{1\cup 2} * \frac{V}{4} - s_2$$

$$Max_{\{s_3\}}\, p_3 * V - s_3$$

The following first order conditions are then derived.

$$\frac{s_2 s_3}{(s_1 s_2 + s_3)^2} * \frac{V}{4} = 1$$

$$\frac{s_1 s_3}{(s_1 s_2 + s_3)^2} * \frac{V}{4} = 1$$

$$\frac{s_1 s_2}{(s_1 s_2 + s_3)^2} * V = 1$$

From the set of first order conditions, we have that $s_1 = s_2$ and $s_3 = 4s_1 = 4s_2$. As such, we find the following equilibrium allocations and outcomes.

$$s_1^* = s_2^* = \sqrt{V} - 4$$

$$s_3^* = 4\left(\sqrt{V} - 4\right)$$

and

$$p_{1\cup 2}^* = \frac{\sqrt{V} - 4}{\sqrt{V}}$$

$$p_3^* = \frac{4}{\sqrt{V}}$$

Moreover, we derive expected payoffs under alliance as follows.

$$U_{1,alliance}^* = U_{2,alliance}^* = \frac{\left(\sqrt{V} - 4\right)^2}{4}$$

Recall that expected payoffs for parties 1 and 2 in the absence of alliance were given as follows.

$$U_{1,no\ alliance}^* = U_{2,no\ alliance}^* = \frac{1}{9}V$$

We find, then, that $U_{1,alliance}^* > U_{1,no\ alliance}^*$ $\left(U_{2,alliance}^* > U_{2,no\ alliance}^*\right)$ iff:

$$\frac{\left(\sqrt{V} - 4\right)^2}{4} > \frac{1}{9}V \quad \text{and} \quad V > 16$$

These conditions simultaneously hold for $V > 144$. That is, if the value of the contest prize is more than 144 times the unit cost of contest inputs (e.g., arms), then Parties 1 and 2 will raise their expected payoff through alliance. We next consider the case of stag hunt alliance formation and exogenous intra-alliance prize division.

Three-Player Nested Contest with Stag Hunt Alliance Formation and Exogenous Intra-Alliance Prize Division

In this case, Parties 1 and 2 act as allies who divide the contest prize according to a pre-determined (exogenous) division rule. Namely, Parties 1 and 2 divide the prize evenly in the absence of an intra-alliance sub-game contest. Such a division rule might be characterized as a fair division rule given that Parties 1 and 2 are symmetric (and therefore allocate an equal number of contest input units in equilibrium). In the case of exogenous prize division, the game becomes a one-stage contest between the allied parties and Party 3. Objective functions under stag hunt contest with exogenous prize division are represented as follows.

$$Max_{\{s_1\}}\, p_{1\cup 2} * \frac{V}{2} - s_1$$

$$Max_{\{s_2\}}\, p_{1\cup 2} * \frac{V}{2} - s_2$$

$$Max_{\{s_3\}}\, p_3 * V - s_3$$

The following set of first order conditions is then derived.

$$\frac{s_2 s_3}{(s_1 s_2 + s_3)^2} * \frac{V}{2} = 1$$

$$\frac{s_1 s_3}{(s_1 s_2 + s_3)^2} * \frac{V}{2} = 1$$

$$\frac{s_1 s_2}{(s_1 s_2 + s_3)^2} * V = 1$$

From the set of first order conditions, we have that $s_1 = s_2$ and $s_3 = 2s_1 = 2s_2$. As such, we find the following equilibrium allocations and outcomes.

$$s_1^* = s_2^* = \sqrt{V} - 2$$

$$s_3^* = 2\left(\sqrt{V} - 2\right)$$

$$p_{1\cup 2}^* = \frac{\sqrt{V} - 2}{\sqrt{V}}$$

$$p_3^* = \frac{2}{\sqrt{V}}$$

Moreover, we derive expected payoffs under alliance as follows.

$$U_{1,alliance}^* = U_{2,alliance}^* = \frac{\left(\sqrt{V} - 2\right)^2}{2}$$

Recall, once again, that expected payoffs for parties 1 and 2 in the absence of alliance were given as follows.

$$U_{1,no\ alliance}^* = U_{2,no\ alliance}^* = \frac{1}{9}V$$

We find, then, that $U_{1,alliance}^* > U_{1,no\ alliance}^*$ $\left(U_{2,alliance}^* > U_{2,no\ alliance}^*\right)$ iff:

$$\frac{\left(\sqrt{V} - 2\right)^2}{2} > \frac{1}{9}V \qquad and \qquad V > 4$$

Roughly, these conditions simultaneously hold for $V > 14.316$. That is, if the value of the contest prize is more than 14.316 times larger than the unit cost of contest inputs (e.g., arms), then Parties 1 and 2 raise their expected payoff through alliance.

Allowing for an Exhaustive Coalition with Exogenous Sharing

We next consider a scenario in which it is possible to divide the prize via a) contest without alliance, b) contest with an alliance that exogenously shares the prize, or c) exogenous, equal distribution among all three parties in lieu of any level of contest. As parties are symmetric in ability and (potential) contest input allocation, equal distribution (in lieu of contest) represents a fair distribution of the prize. We first note that, if available, the formation of an exhaustive coalition that fairly divides the prize Pareto dominates contest without alliance in that it avoids rent dissipation. Of the three choices above, then, contest without alliance can immediately be eliminated as a possible allocation method. We can further explore conditions under which contest with an alliance that exogenously shares the prize is chosen over exogenous, equal distribution among all three parties in lieu of any level of contest. We first note that an exhaustive coalition causes the following payoffs.

$$U^*_{1,exhaustive\ coalition} = U^*_{2,exhaustive\ coalition} = U^*_{3,exhaustive\ coalition} = \frac{1}{3}V$$

We further find that $U^*_{1,alliance} > U^*_{1,exhaustive\ coalition}$ $\left(U^*_{2,alliance} > U^*_{2,exhaustive\ coalition}\right)$ iff:

$$\frac{\left(\sqrt{V}-2\right)^2}{2} > \frac{1}{3}V \quad and \quad V > 4$$

Roughly, this condition holds for $V > 118.79$. That is, if the value of the contest prize is more than 118.79 times larger than the unit cost of contest inputs (e.g., arms), then Parties 1 and 2 raise their expected payoff through two-party alliance with contest as opposed to three-party settlement. Let us then fully consider alliance formation outcomes in the case of exogenous prize division.

If $0 < V \leq 118.79$, then $U^*_{1,alliance} > U^*_{1,exhaustive\ coalition} > U^*_{1,no\ alliance}$

If $V > 118.79$, then $U^*_{1,exhaustive\ coalition} > U^*_{1,alliance} > U^*_{1,no\ alliance}$

If an exhaustive alliance with exogenous sharing rule is possible, however, it remains a puzzle why conflict without alliance would ever be chosen. Perhaps the exogenous sharing rule lacks credibility in many settings such that this case is often only a theoretical consideration. On the other hand, it may require transaction costs to formulate an exogenous sharing rule. This may be especially true in contests for which players are not completely informed regarding the prize valuations of other players.

We avoid comparing the case of exhaustive coalition with that of two-party alliance with endogenous prize division. The formation of an exhaustive coalition implies that an exogenous sharing rule is possible for three parties. If an exogenous sharing rule is possible for three parties, then it is possible for two parties. Moreover, we know from the welfare analysis that members of an alliance prefer exogenous sharing when it is available (credible). It follows, therefore, that two-party alliance with endogenous prize division will never be chosen when it is possible to form an exhaustive coalition. In such a case, it is either true that an exhaustive coalition will form or that a two-party alliance with exogenous prize division will form.

II. Conclusion

This study has introduced the notion of stag hunt alliance formation in contest. Under stag hunt alliance formation, allied parties allocate inputs that are complementary within the contest technology. Given this form of alliance formation, we find conditions under which the formation of a two-party alliance raises the expected payoff of the allied parties (as compared to the case of contest without alliance or the case of an exhaustive coalition). Conditions for the formation of a two-party alliance exist in the case of exogenous prize division, as well as in the case of endogenous prize division. Stag hunt alliance formation complements capacity constraints and resource transfers in multi-front battle contests in explaining the observation of alliances in contest. It remains to be determined how allies optimally match with one another in contest. This question is particularly crucial in the case of asymmetric contest.