# Two-sided Matching with Incomplete Information

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**Abstract**

Stability in a two-sided matching model with non-transferrable utility (NTU) and with incomplete information is investigated. Each agent has interdependent preferences which depend on his own type and on the (possibly unknown) types of agents on the other side of the market. Agents' utilities are increasing in types. First, a one-sided incomplete information model in which workers' types are private information is investigated. Firms react to their informational disadvantage with conservatism: a firm joins a worker in a block to a matching only if the firm is better off even with the lowest type of the worker interested in the potential block. A recursively-unblocked matching outcome is incomplete-information stable. With anonymous preferences, all strictly individually-rational matching outcomes are (one-sided) incomplete-information stable. Thus, in a positive assortative matching model all matching outcomes are incomplete-information stable including the negative assortative matching. An ex post incentive compatible mechanism exists. This mechanism implements the best complete-information stable matching for workers.

Extensions to two-sided incomplete information stability are investigated. Stable-matching outcomes with two-sided incomplete information are a superset of stable-matching outcomes with one-sided incomplete information, which in turn include complete-information stable matchings.

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# 1 Introduction

The concept of stability in two-sided matching was introduced by Gale and Shapley [10]. A matching is stable if there does not exist a blocking pair, i.e., no two agents prefer each other to their respective partners under the matching. Agents do not have any private information about their own types that might affect their own preferences or the preferences of other agents. Much of the subsequent literature makes the same complete information assumption – see Roth and Sotomayor [20].

In this paper, I investigate a two-sided matching model with incomplete information in which workers are matched with firms. Agents have interdependent preferences in the sense that the utility of an agent depends on the type of the agent it is matched with. While the model in this paper is of one-to-one matching, it is easily generalized to a many-to-one matching model.[1] Utility is non-transferrable and side payments between workers and firms are not possible. Once a matching prevails, a worker's type becomes known to its matched firm but not to other firms. The utility functions of a worker and the matched firm increase in their types. The restrictions imposed by stability on the set of matching outcomes under incomplete information are investigated.

Initially, I investigate a one-sided incomplete information model in which workers have private information about their types whereas firms' types are common knowledge. Firms respond to their lack of information by participating in a block only when they are certain that the block (to a matching) will increase their utility. In other words, a firm in a blocking coalition prefers, in comparison its currently matched worker, the lowest type of its blocking partner (worker) that would also be better off in the contemplated block. This notion of blocking was introduced by Liu, Mailath, Postlewaite, and Samuelson [13] in a matching model with one-sided incomplete-information and transferrable utility. Initially, all possible types of workers are considered. The set of possible worker types decreases recursively, with firms ruling out subsets of worker types from blocks that did not materialize. If a matching persists over time, it becomes known among agents that there are no blocking opportunities; the matching is stable. Liu et al. [13] point out that this is similar in spirit to Holmstrom and Myerson [12]'s notion of durable mechanisms.

I show that stable matching outcomes exist under one-sided incomplete information and non-transferrable utility. The set of stable matchings is larger under incomplete information than under complete information and, in a sense, too large:

[1] All the results presented here hold for a model in which firms employ more than one worker and each firm has responsive preferences.

if preferences are anonymous[2] then every strictly individually-rational matching outcome is incomplete-information stable. With anonymous preferences and one-sided incomplete information, stability is only slightly more restrictive than individual rationality.

In blocking coalitions of size greater than two, it is necessary to make assumptions about inferences drawn by coalition members. A firm $j$ in a larger coalition draws inferences about the type of worker $i$, its proposed match in the block, from worker $i$'s willingness to participate in the block. (Firm $j$ would draw the same inference if $i$ and $j$ formed a blocking pair.) In addition, if the firm currently matched to worker $i$ is also part of the coalition then that conveys additional information to firm $j$ about worker $i$'s type. It is shown that a two-agent blocking coalition exists whenever a blocking coalition of size greater than two exists. Thus, blocking coalitions of size greater than two do not decrease the size of incomplete-information stable matching outcomes.

A mechanism that is ex post incentive compatible under one-sided incomplete information model is presented. In this mechanism, workers reveal their types to the mechanism designer. The mechanism designer computes the complete-information stable matching (for the stated types) that is optimal for workers. The mechanism designer reveals the reported types of workers and implements the computed matching. If all agents report truthfully, then the resulting matching is complete-information stable. If any one agent lies, then either the implemented matching is (complete-information) unstable and agents settle on another stable matching or it is stable. In either event, the agent does not benefit by lying.

Next, a notion of stability for two-sided incomplete information is proposed. As in the one-sided incomplete information case, each agent in a potential blocking pair assumes that its blocking partner is of the lowest possible type that might be interested in a block. However, in determining this lowest possible type the conceptual problem is to fix the set of possible beliefs of agents in a blocking pair about the type of their partner. A firm does not know what the worker with which it is contemplating a block assumes about the firm's type, and vice versa. Therefore, in order to obtain the lowest type of worker $i$ that might wish to block with firm $j$, the firm $j$ assumes that worker $i$ presumes that firm $j$ is of the highest possible type that might be interested in a block. The worker $i$ makes a similar calculation to obtain the lowest possible type of firm $j$. This ensures that worker $i$ and firm $j$ participate in a block only if they are certain that their respective utilities will increase. However, this increases the conservatism of agents on both sides of the market resulting in fewer blocking opportunities. Consequently, the set of two-sided incomplete-information stable matching

[2] That is, if each agent's utility does *not* depend on the identity of the matched agent.

outcomes is a superset of the set of one-sided incomplete information stable matching outcomes.

In sum, the set of incomplete-information stable-matching outcomes is large. This has implications for matching markets. Usually, the absence of disorderly conduct, such as reneged contracts, market compression or exploding offers, leads to the conclusion that the status quo matching is complete-information stable and consequently it is also Pareto efficient. However, this need not be the case if, as seems plausible in some applications, there is incomplete-information among participants. An incomplete-information stable matching outcome need not be complete-information stable and, as an example shows, can be Pareto-dominated. Therefore, while orderly conduct is a necessary condition for a well-functioning matching market it is not sufficient. The mechanism by which the market reached an equilibrium must also be considered.

In addition to Liu et al. [13], Chakraborty, Citana, and Ostrovsky [2] and Dizdar and Moldovanu [4] are two recent papers on matching with incomplete information and interdependent preferences. Chakraborty et al. [2] investigate the incomplete-information stability of a mechanism, rather than of a matching, in a college-admissions model. Students' preferences are known but their quality is unknown. Preferences of colleges over students depend on student quality. Chakraborty et al. [2] show that stable mechanisms do not usually exist. Dizdar and Moldovanu [4] establish that (under transferrable utility) only fixed-proportion sharing rules are are compatible with efficiency.

There is an earlier literature on incomplete-information matching models with privately-known preferences. Dubins and Freedman [5] and Roth [16] independently show that in the deferred-acceptance mechanism it is a (weakly) dominant strategy for proposers to truthfully report their preferences. Moreover, every Nash equilibrium of the deferred-acceptance mechanism in which proposers follow their dominant strategy leads to a stable outcome (Roth [18]). However, Gale and Sotomayor [11] showed that by misreporting their preferences the non-proposing agents can achieve a stable outcome that is more favorable to them. Roth [16] established that there exists no stable mechanism in which it is a dominant strategy for all agents to truthfully reveal their preferences, while Roth [19] generalized this negative result under the weaker incentive constraints of Bayes Nash equilibrium. Ehlers and Masso [8] showed that in a matching model with two-sided incomplete information, an ordinally Bayesian incentive compatible mechanism exists if and only if there is exactly one stable matching at every state of the world. A related negative result was obtained by Majumdar [14]. Yenmez [23] investigates the existence of stable, efficient, and budget-balanced mechanisms in a model with transfers.

The literature on the core with incomplete information, surveyed in Forges et al. [9], is also relevant. Wilson [22], the first paper in the area, noted that information-sharing assumptions at the interim-stage were critical in determining blocks to a potential core allocation; minimal information-sharing leads to Wilson's concept of the coarse core while maximal information-sharing yields the fine core, which is a subset of the coarse core. Incentive-compatible information sharing within coalitions was introduced by Vohra [21], leading to refinements of Wilson's coarse core and fine core. Dutta and Vohra [6] proposed the credible core, in which members of a blocking coalition draw inferences from the nature of the contemplated objection. The credible core lies in between the (incentive compatible) fine core and the coarse core. In Liu et al. [13] and in this paper, incentive compatibility within a blocking coalition is satisfied for each state of nature rather than in expectation. Further, the absence of blocking implies that certain states of nature did not occur. This opens up the possibility of other potential blocks which, if they do not transpire, implies that some other states of nature did not occur. Thus, the persistence of a stable matching leads to the updating of beliefs in a recursive manner.

The rest of the paper is organized as follows. The basic model and results for one-sided uncertainty are presented in Section 2. Blocking coalitions larger than two are considered in Section 2.1 and an incentive compatible mechanism is presented in Section 2.3. As pointed out in Section 2.2, incomplete-information stability can be characterized as common knowledge of no blocking. Section 2.4 compares stability in the NTU model with stability in a TU model. Section 3 extends the results to two-sided incomplete information. Section 4 concludes.

# 2 One-sided Uncertainty

There are $i = 1, 2, \ldots, n$ workers and $j = 1, 2, \ldots, m$ firms. Each worker has private information about his own type.[3] Worker $i$'s type, $w_i$, is in the interval $[\underline{w}, \overline{w}]$. If worker $i$ and firm $j$ are matched together then their respective utilities are:

$$u_i(w_i, j) \quad \text{and} \quad v_j(w_i, i) \tag{1}$$

The utility of a worker $i$, $u_i$, depends on the worker's own type and the identity of the firm it is matched with. The utility of firm $j$, $v_j$, depends on the unknown type of the worker it is matched with and the worker's identity. The type of the worker, $w_i$, may be thought of as representing the productivity of the worker, which is unknown to the firm. There may be other characteristics of the worker, such as education and

[3] The results in this section hold under the alternative assumption that workers' types are common knowledge among workers but are not known to the firms.

experience, that are observed by the firm and these are captured by the dependence of $v_j$ on $i$. If worker $i$ and firm $j$ are matched, then firm $j$ learns worker $i$'s type, $w_i$.

A dummy worker is indexed $i = 0$ with type $w_0$ and a dummy firm is indexed $j = 0$. An unmatched worker (firm) is matched to the dummy firm (worker). The utility of an unmatched worker or firm is normalized to zero: $u_i(w_i, 0) = 0$ and $v_j(w_0, 0) = 0$.

As mentioned earlier, there are no side payments between matched workers and firms in this model. This may appear counter-factual as firms pay wages to workers. The important assumption is that if side payments are present then there is a standard payment (wage) over which there is little or no bargaining. This is the case with medical residents, law clerks, or college interns who view the job as building their human capital (see Roth and Sotomayor [20], p. 125). If a firm makes the same payment to any worker it might hire, the side payment need not be explicitly modeled and is reflected in the utilities of the matched firm and worker. The model here may also be appropriate for matching students to schools or colleges.

The following assumption is made throughout the paper.

Increasing and continuous utility: *The utility functions $u_i(w_i, j)$ and $v_j(w_i, i)$ are strictly increasing and continuous in $w_i$ for all $i$ and $j$.*

A matching is a function $\mu : \{1, 2, \dots, n\} \to \{0, 1, 2, \dots, m\}$, where $\mu(i)$ is the firm that worker $i$ is matched to. If $\mu(i) = 0$ then worker $i$ is unmatched and if $\mu(i) = \mu(\hat{i})$ then $\mu(i) = 0$. For any $j \in \{1, 2, \dots, m\}$,

$$\nu(j) \;\equiv\; \mu^{-1}(j) \;=\; \begin{cases} i, & \text{if there is an } i \in \{1,2,\dots,n\} \text{ s.t. } \mu(i) = j, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbf{w} = (w_1, w_2, \dots, w_n)$ be the type vector of the workers. A *matching outcome* is a matching together with a worker type vector: $(\mu, \mathbf{w})$. The matching outcome $(\mu, \mathbf{w})$ is *individually rational* if

$$\begin{aligned} u_i(w_i, \mu(i)) &\geq 0, \qquad \forall i \\ v_j(w_{\nu(j)}, \nu(j)) &\geq 0, \qquad \forall j \end{aligned}$$

Let $\Sigma^0$ be the set of individually-rational matching outcomes. As utility is increasing in worker type, if $(\mu, \mathbf{w}) \in \Sigma^0$ then $(\mu, \mathbf{w}') \in \Sigma^0$ for all $\mathbf{w}' \geq \mathbf{w}$.

The matching outcome $(\mu, \mathbf{w})$ is *strictly individually rational* if it is individually rational and

$$\begin{aligned} &\text{either} \quad u_i(w_i, \mu(i)) > 0 \quad \text{or} \quad v_{\mu(i)}(w_i, i) > 0, \qquad \forall i \text{ s.t. } \mu(i) \neq 0 \\ &\text{either} \quad u_i(w_i, j) \leq 0 \quad \text{or} \quad v_j(w_i, i) \leq 0, \qquad \forall i, j \text{ s.t. } \mu(i) = \nu(j) = 0 \end{aligned}$$

That is, at least one agent in each matched pair at $\mu$ obtains strictly positive utility and if worker $i$ and firm $j$ are unmatched at $\mu$ then at least one of the two must obtain non-positive utility from the match $(i, j)$. Let $\Sigma^0_+$ denote the set of strictly individually-rational matching outcomes.

It is useful to state the definition of complete-information stability before defining incomplete-information stability.

An individually-rational matching outcome $(\mu, \mathbf{w}) \in \Sigma^0$, where $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, is *blocked* if there is a worker-firm pair $(i, j)$ such that

$$u_i(w_i, j) \;>\; u_i(w_i, \mu(i)) \quad \text{and} \quad v_j(w_i, i) \;>\; v_j(w_{\nu(j)}, \nu(j)) \tag{2}$$

If there does not exist an $(i, j)$ satisfying (2), then $(\mu, \mathbf{w})$ is *complete-information stable*. Gale and Shapley [10] provide a constructive proof of existence of a complete-information stable matching; their deferred-acceptance algorithm stops at a stable matching.

Complete-information stability is satisfied by a matching if no worker-firm pair can improve their respective utilities by matching together rather than with their partners in the matching. Stability under incomplete information satisfies a similar requirement with the additional proviso that the two participants in a potential block should not be disappointed after they match together. The additional proviso is automatically satisfied under complete information.

An individually-rational matching outcome $(\mu, \mathbf{w}) \in A \subseteq \Sigma^0$, where $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, is $A$-*blocked* if there is a worker-firm pair $(i, j)$ satisfying

$$u_i(w_i, j) \;>\; u_i(w_i, \mu(i)) \tag{3}$$

$$\text{and for all } (\mu, \mathbf{w}') \in A \;\; \text{s.t.} \;\; w'_{\nu(j)} = w_{\nu(j)},$$

$$\text{if} \quad u_i(w'_i, j) \;>\; u_i(w'_i, \mu(i)) \;\geq\; 0 \tag{4}$$

$$\text{then} \quad v_j(w'_i, i) \;>\; v_j(w_{\nu(j)}, \nu(j)) \tag{5}$$

Individual rationality implies that $u_i(w_i, \mu(i)) \geq 0$. Therefore, if (3) is satisfied then (4) is also satisfied with $w'_i = w_i$ (and possibly with other $w'_i$ as well).

Inequality (3) requires that worker $i$ is better off in the potential block. The corresponding condition for firm $j$ is more complicated. Firm $j$ knows only that worker $\nu(j)$'s type is $w_{\nu(j)}$, but does not know worker $i$'s type. Therefore, in order to participate in the block, firm $j$ should be better off with any worker $i$ type $w'_i$ (in $A$ at a worker type vector $\mathbf{w}'$ that is consistent with firm $j$'s knowledge about worker $\nu(j)$ and the fact that $(\mu, \mathbf{w}')$ is individually rational) that is better off in the block. If whenever (4) is satisfied, (5) is satisfied then the pair $(i, j)$ blocks $(\mu, \mathbf{w})$ in $A$.

Suppose that the worker type vector $\mathbf{w}$ is common knowledge among all workers. Further, suppose that it is common knowledge among workers and firms that $\mathbf{w}$ is such that $(\mu, \mathbf{w}) \in A$. If $(i, j)$ agree to form a coalition to $A$-block $\mu$ then it becomes common knowledge between $i$ and $j$ that they are better off by blocking $\mu$.

The matching outcome $(\mu, \mathbf{w}) \in A$ is $A$-*stable* if it is not $A$-blocked by any firm-worker pair. The set $A$ is *self-stabilizing* if every $(\mu, \mathbf{w}) \in A$ is $A$-stable.

If the set $\{(\mu, \mathbf{w})\}$ is self-stabilizing then, by definition, $\mu$ is a complete-information stable matching at $\mathbf{w}$. Additional results are gathered in the following lemma, where $A$ and $B$ are sets of matching outcomes.

**Lemma 1**

*(i) Suppose that $B \subset A$. If $(\mu, \mathbf{w}) \in B$ is $B$-stable then it is $A$-stable.*

*(ii) Let $\mu$ be a complete-information stable matching at $\mathbf{w}$. Then, $(\mu, \mathbf{w})$ is $A$-stable for any $A$ such that $(\mu, \mathbf{w}) \in A$.*

**Proof:** (i) It follows directly from (4) and (5) that if $(\mu, \mathbf{w})$ is $A$-blocked then it is also $B$-blocked.

(ii) Let $B = \{(\mu, \mathbf{w})\}$. As $\mu$ is complete-information stable at $\mathbf{w}$, $(\mu, \mathbf{w})$ is $B$-stable. Therefore, by (i), $(\mu, \mathbf{w})$ is $A$-stable for any $A \supset B$. ∎

The next assumption states that preferences depend only on the types of agents and not on their identity.

Anonymous Preferences: *For any $i, \hat{i} \in \{1, 2, \ldots, n\}$ and $j, \hat{j} \in \{1, 2, \ldots, m\}$ and $w_i, w_i' \in (\underline{w}, \overline{w}]$,*

$$\begin{array}{rcl} v_j(w_i, i) & = & v_j(w_i, \hat{i}) \\ \text{If } \; u_i(w_i, j) \; > \; u_i(w_i, \hat{j}) \; \text{ then } \; u_i(w_i', j) & > & u_i(w_i', \hat{j}) \end{array}$$

The condition of anonymous preferences is satisfied whenever the utility of firms (workers) over workers (firms) depends only on the type and not on the identity of the workers (firms). The above equation states that firms' preferences over workers depend only on workers' types and not on their identities. The if-then statement above requires that if worker $i$ of type $w_i > \underline{w}$ prefers firm $j$ to $\hat{j}$ then worker $i$ of any other type $w_i' > \underline{w}$ prefers firm $j$ to $\hat{j}$. Essentially, this also states that workers' preferences over firms depend only on firms' "types" and not on their identities. To see this, assume that (i) firms also have types and that these types are common knowledge and (ii) the underlying utility functions of workers and firms, $U_i$ and $V_j$, depend on the agent's own type and the type but not the identity of the matched agent. The utility functions $u_i$ and $v_j$ are obtained from $U_i$ and $V_j$ by replacing firm types $f_j$ and $f_{\hat{j}}$ with firm identities $j$ and $\hat{j}$:

$$u_i(w_i, j) \equiv U_i(w_i, f_j), \qquad u_i(w_i, \hat{j}) \equiv U_i(w_i, f_{\hat{j}}) \tag{6}$$

$$v_j(w_i, i) \equiv V_j(w_i, f_j) \tag{7}$$

Assuming that worker utility increases with firm types, the hypothesis of the if statement, $u_i(w_i, f_j) > u_i(w_i, f_{\hat{j}})$ implies that $f_j > f_{\hat{j}}$, which in turn implies $u_i(w_i', f_j) > u_i(w_i', f_{\hat{j}})$.[4]

The next result says that if preferences are anonymous and any matching is strictly preferred to being unmatched by a worker or a firm (except when the worker's type is the lowest), then every strictly IR matching outcome is stable.

**Proposition 1** *Suppose that $u_i(\underline{w}, j) = 0$, $v_j(\underline{w}, i) = 0$ for all $i, j$. Then, under anonymous preferences the set of strictly individually-rational matching outcomes is self-stabilizing.*

**Proof:** If $w_i = \underline{w}$ then, as $u_i(\underline{w}, j) = 0$, $v_j(\underline{w}, i) = 0$ for all $j$, worker $i$ is not matched at any strictly individual-rational matching. We can eliminate worker $i$ from consideration. Therefore, assume that $w_i > \underline{w}$ for all $i$. Then, the set of strictly individual-rational matchings is

$$\Sigma_+^0 = \{(\mu, \mathbf{w}), \ \mathbf{w} \in (\underline{w}, \overline{w}]^n\}$$

where $\mu$ is any matching in which all agents on the shorter side of the market are matched.

Suppose that (3) is satisfied by worker-firm pair $(i, j)$ at $(\mu, \mathbf{w}) \in \Sigma_+^0$. Then by anonymity, (3) is satisfied for all $w_i' \in (\underline{w}, \overline{w}]$. Because $v_j(w_{\nu(j)}, \nu(j)) > 0$ by strict individual rationality, and $v_j(\underline{w}, \nu(j)) = 0$, we have $w_{\nu(j)} > \underline{w}$ as utility is increasing. Select $w_i' \in (\underline{w}, w_{\nu(j)}]$. Now, for $(\mu, \mathbf{w}')$, where $\mathbf{w}' = (w_{-i}, w_i')$, (4) is satisfied but, by increasing utility and anonymity, (5) is not satisfied. Hence, $(\mu, \mathbf{w})$ is not $\Sigma_+^0$-blocked.

If, instead, there does not exist a worker-firm pair $(i, j)$ that satisfies (3) at any $\mathbf{w}$, where $(\mu, \mathbf{w}) \in \Sigma_+^0$, then again $(\mu, \mathbf{w})$ is not $\Sigma_+^0$-blocked. ∎

[4] Note that the Anonymous Preferences assumption is more general than (6) and (7). It is possible that $u_i(w, j) > u_i(w, \hat{j})$ and $u_{\hat{i}}(w, j) < u_{\hat{i}}(w, \hat{j})$ under the anonymous preferences but not under (6) and (7).

Remark: Observe that when $\overline{w} = \underline{w}$ there is no incomplete information and therefore incomplete-information stability coincides with complete information stability. But if $\overline{w} = \underline{w} + \epsilon$ for any $\epsilon > 0$ then the set of incomplete-information stable matches is much larger. That is, at $\overline{w} = \underline{w}$, there is a discontinuity in the set of incomplete-information stable matchings outcomes.

Thus, under the assumptions of Proposition 1, stability provides no restrictions on matching outcomes. In particular, consider the following example which satisfies the hypothesis of Proposition 1.

Example 1: (Assortative Matching)

Let the number of workers equal the number of firms, $n = m$. The utility functions of workers and firms are

$$u_i(w_i, j) \;=\; v_j(w_i, i) \;=\; jw_i$$

Then, with $0 < w_{i_1} \leq w_{i_2} \leq \ldots \leq w_{i_n}$, the unique complete-information stable matching pairs worker $i_j$ with firms $j$, the positive assortative matching. But Proposition 1 implies that all matchings are incomplete-information stable.

To see this directly, consider the negative assortative matching where worker $i_n$ is matched to firm 1, worker $i_{n-1}$ is matched to firm 2, etc. Firm $n$, who is matched to the lowest type worker $i_1$ but does not know that $i_1$ has the lowest type, will reject a blocking proposal from all other workers, including worker $i_n$. This is because types $w'_{i_n} < w_{i_1}$ would also do better by matching with firm $n$ than with their current match firm 1.

Similarly, every firm $j \geq 2$ will reject a blocking proposal from any other worker. No worker will propose a block with firm 1. □

Remark: The difference between an NTU model and a TU model is stark. In a TU model with side payments, Proposition 3 of Liu et al. [13] implies that in any positive assortative matching model, only the efficient matching is incomplete information stable.

In the sequel, I *do not* assume either anonymous preferences or that any firm is indifferent between being matched to a lowest type worker and remaining unmatched. Before proceeding, I define the notion of incomplete-information stability. As in Liu et al. [13], this definition calls for an iterative elimination of unstable matching outcomes from the beliefs of firms. The idea is that if a matching persists, then it is reasonable to conclude that the worker types are not in a subset that would block the matching.

Recall that $\Sigma^0$ is the set of individually-rational matching outcomes. Define

$$\Sigma^k = \{(\mu, \mathbf{w}) \in \Sigma^{k-1} \,\Big|\, (\mu, \mathbf{w}) \text{ is not } \Sigma^{k-1}\text{-blocked}\} \quad (8)$$

Then

$$\Sigma^* \equiv \bigcap_{k=0}^{\infty} \Sigma^k$$

is the set of *incomplete-information stable-matching outcomes.*

For any $\mathbf{w}$, let $\mu$ be a matching that is complete-information stable at $\mathbf{w}$. As $\mu$ is individually rational at $\mathbf{w}$, $\{\mu, \mathbf{w}\} \in \Sigma^0$. By repeated application of Lemma 1(ii) it follows that $\{\mu, \mathbf{w}\} \in \Sigma^k$, for all $k$ and hence $\Sigma^*$ is non-empty.

It is clear from (8) that $\Sigma^k \subseteq \Sigma^{k-1}$ and that $\Sigma^*$ is self-stabilizing. The next result shows that $\Sigma^*$ is the largest self-stabilizing set.

**Lemma 2** *If $A$ is a self-stabilizing set then $A \subseteq \Sigma^*$.*

**Proof:** If $A$ is self-stabilizing then $A \subseteq \Sigma^0$. Let $k \geq 0$ be such that $A \subseteq \Sigma^k$. As every $(\mu, \mathbf{w}) \in A$ is $A$-stable, it is also $\Sigma^k$-stable by Lemma 1(i). Therefore, $A \subseteq \Sigma^{k+1}$. ∎

The set of incomplete-information matching outcomes can be quite large even when preferences are not anonymous. The following example demonstrates that even a Pareto-dominated matching outcome can be incomplete information stable.

Example 2: Pareto-dominated stable matching outcome

There are two firms and two workers. The workers' types lie in the interval $[0, 2]$. The utility functions of workers are in the table below:

| Worker utility | Firm 1 | Firm 2 |
|---|---|---|
| $u_1(w_1, j)$ | $w_1 + 1$ | $2w_1$ |
| $u_2(w_2, j)$ | $w_2 + 1$ | $2w_2$ |

| Firm utility | Worker 1 | Worker 2 |
|---|---|---|
| $v_1(w_i, i)$ | $w_1 + 0.5$ | $w_2$ |
| $v_2(w_i, i)$ | $w_1 + 0.1$ | $w_2$ |

Consider the following matchings:

$$\mu_1 = \{(W_1, F_1), (W_2, F_2)\}$$

$$\mu_2 = \{(W_1, F_2), (W_2, F_1)\}$$

Of these two matchings, $(\mu_1, (w_1, w_2))$ is $\Sigma$-blocked by $(W_1, F_2)$ when $w_1 > 1$ and $w_2 < 1.1$. Next, $(\mu_2, (w_1, w_2))$ is $\Sigma$-blocked by $(W_1, F_1)$ if $w_1 < 1$ and $w_2 < 0.5$. And $(\mu_2, (w_1, w_2))$ is $\Sigma$-blocked by $(W_2, F_2)$ if $w_1 < 0.9$ and $w_2 > 1$.

The stable-matching outcomes in the table below are also depicted in a figure at the end of the paper.

Incomplete-information Stable Matching Outcomes

| Matching | Worker types |
|---|---|
| $\mu_1$ | $w_1 \leq 1$ or $w_2 \geq 1.1$ |
| $\mu_2$ | $(w_1 \geq 1)$ or $(w_1 \in [0.9, 1] \cap w_2 \geq 0.5)$ or $(w_1 \in [0, 0.9] \cap w_2 \in [0.5, 1])$ |

However, note that an incomplete-information stable matching outcome need not be a complete-information stable matching. For example, $\mu_1, \mu_2 \in \Sigma^*$ at $w_1 = 0.95,\ w_2 = 1.1$ even though $\mu_1$ is the unique complete-information stable matching (as $\mu_2$ is complete-information blocked by $(W_1, F_1)$).

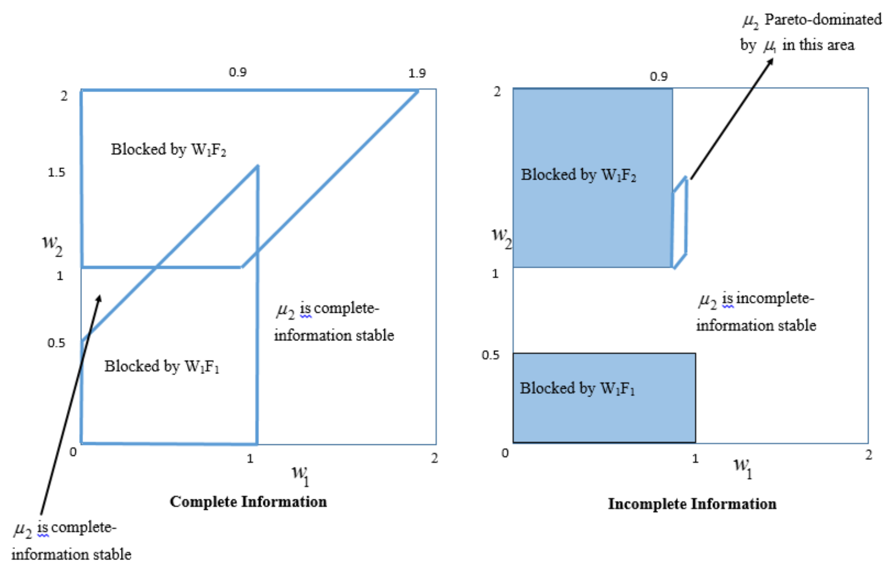


Figure 1: Stability of matching $\mu_2 = \{W_1F_2, W_iF_2\}$

This example also shows that an incomplete-information stable matching outcome need not be Pareto-efficient. At $w_1 = 0.95, w_2 = 1.1$, $\mu_2$ is strongly Pareto-dominated by $\mu_1$. That is, under complete information about worker types, if matching $\mu_2$ were proposed then each worker and each firm would be part of a blocking pair. Under incomplete information, however, it is not common knowledge among workers and firms that a blocking pair exists. See Figure 1. □

Before concluding this section, a characterization of blocking is provided. To this end, let

$$P^i_{j\,\hat{j}} \;\equiv\; \{\, w_i \in [\underline{w}, \overline{w}] \,\Big|\, u_i(w_i, j) > u_i(w_i, \hat{j}) \geq 0 \,\}$$

be the set of types at which worker $i$ strictly prefers firm $j$ to firm $\hat{j}$. Note that if $\mu$ is $A$-blocked at $(w_i, w_{-i}) \in A$ by $(i, j)$, then $w_i \in P^i_{j\,\mu(i)}$. As $u_i$ is continuous on a closed interval, $P^i_{j\,\hat{j}}$ is a finite union of intervals.

Suppose that at worker types $w_i$, $w_{\nu(j)}$ the matching $\mu$ is [not] $A$-blocked by the pair $(i, j)$. The next lemma establishes the types of workers $i$ and $\nu(j)$ that are more [less] favorable than $w_i$, $w_{\nu(j)}$ to an $A$-block of the matching $\mu$ by the pair $(i, j)$. The types of workers other than $i$ and $\nu(j)$ do not matter.

**Lemma 3** *Let $\mu$ be an individually-rational matching.*

*(i) If $(\mu, (w_{-i,\nu(j)}, w_i, w_{\nu(j)})) \in A$ is $A$-blocked by $(i, j)$, then $(\mu, (w^b_{-i,\nu(j)}, w^b_i, w^b_{\nu(j)})) \in A$, with $w^b_{\nu(j)} \leq w_{\nu(j)}$, $w^b_i \in P^i_{j\,\mu(i)}$, and any $w^b_{-i,\nu(j)}$, also is $A$-blocked.*

*(ii) For any $w_i \in P^i_{j\,\mu(i)}$, if $(\mu, (w_{-i,\nu(j)}, w_i, w_{\nu(j)})) \in A$ is not $A$-blocked by $(i, j)$, then $(\mu, (w^{nb}_{-i,\nu(j)}, w^{nb}_i, w^{nb}_{\nu(j)})) \in A$, with $w^{nb}_{\nu(j)} \geq w_{\nu(j)}$ and any $w^{nb}_i, w^{nb}_{-i,\nu(j)}$, also is not $A$-blocked.*

**Proof:** (i) Suppose that $(\mu, (w_{-i,\nu(j)}, w_i, w_{\nu(j)})) \in A$ is $A$-blocked by $(i, j)$. That is, $u_i(w_i, j) > u_i(w_i, \mu(i))$ and for all $(\mu, (w'_{-i,\nu(j)}, w'_i, w_{\nu(j)})) \in A$

$$\text{if} \quad u_i(w'_i, j) \;>\; u_i(w'_i, \mu(i)) \;\geq\; 0 \quad \text{then} \quad v_j(w'_i, i) \;>\; v_j(w_{\nu(j)}, \nu(j)) \tag{9}$$

Consider any $(\mu, (w^b_{-i,\nu(j)}, w^b_i, w^b_{\nu(j)})) \in A$ with $w^b_{\nu(j)} \leq w_{\nu(j)}$ and $w^b_i \in P^i_{j\,\mu(i)}$. As $w^b_i \in P^i_{j\,\mu(i)}$, we have $u_i(w^b_i, j) > u_i(w^b_i, \mu(i)) \geq 0$. By increasing utility and (9), we have for all $(\mu, (w'_{-i,\nu(j)}, w'_i, w^b_{\nu(j)})) \in A$

$$\text{if} \quad u_i(w'_i, j) \;>\; u_i(w'_i, \mu(i)) \;\geq\; 0 \quad \text{then} \quad v_j(w'_i, i) \;>\; v_j(w^b_{\nu(j)}, \nu(j))$$

Therefore, $(\mu, (w^b_{-i,\nu(j)}, w_i, w^b_{\nu(j)}))$ is $A$-blocked by $(i, j)$.

(ii) Next, suppose that $(\mu, (w_{-i,\nu(j)}, w_i, w_{\nu(j)})) \in A$ is not $A$-blocked by $(i, j)$. As $w_i \in P^i_{j\;\mu(i)}$, firm $j$ must not be interested in a block. Thus, there exists $w'_i$ such that

$$u_i(w'_i, j) > u_i(w'_i, \mu(i)) \;\geq\; 0 \qquad \text{and} \qquad v_j(w'_i, i) \leq v_j(w_{\nu(j)}, \nu(j))$$

Then, by increasing utility, we have $v_j(w'_i, i) \leq v_j(w_{\nu(j)}, \nu(j)) \leq v_j(w^{nb}_{\nu(j)}, \nu(j))$ for any $w^{nb}_{\nu(j)} \geq w_{\nu(j)}$. Thus $(\mu, (w^{nb}_{-i,\nu(j)}, w^{nb}_i, w^{nb}_{\nu(j)}))$ is not $A$-blocked by $(i, j)$ for any $w^{nb}_i$, $w^{nb}_{-i,\nu(j)}$. ∎

Remark: Another implication of Lemma 3 is that no inference can be drawn about the types of workers other than $i$ and $\nu(j)$ from the fact that $\mu$ is blocked (or not blocked) by $(i,j)$. Further, Lemma 3(i) implies that if $(i,j)$ constitutes a blocking pair, then for all $w_i$ satisfying $u_i(w_i',j) > u_i(w_i',\mu(i)) \geq 0$, we have $v_j(w_i',i) > v_j(w_{\nu(j)},\nu(j))$. Therefore, after observing the pair $(i,j)$ $\Sigma^0$-block the matching $\mu$, an uninformed observer concludes that $w_{\nu(j)} < w^*_{\nu(j)}$ where $w^*_{\nu(j)}$ is the largest worker $\nu(j)$ type such that $v_j(w_i',i) \geq v_j(w^*_{\nu(j)},\nu(j))$ for all $w_i' \in P^i_{j\,\mu(i)}$.

For any set of matching outcomes $A$, let

$$A_i(\mu) \;\equiv\; \{w_i \in [\underline{w},\overline{w}] \,|\, (\mu,(w_i,w_{-i})) \in A \text{ for some } w_{-i}\}$$

As $\Sigma_i^0(\mu)$ is an interval, Lemma 3 implies that for any $k$, $\Sigma_i^k(\mu)$ is a finite union of intervals. Let

$$\underline{P}^i_j(\mu,A) \;\equiv\; \begin{cases} \inf\{\, w_i \in P^i_{j\,\mu(i)} \cap A_i(\mu)\,\}, & \text{if } P^i_{j\,\mu(i)} \cap A_i(\mu) \neq \emptyset \\ \\ \overline{w}, & \text{otherwise.} \end{cases}$$

Any worker $i$ in $A$ who prefers $j$ to $\mu(i)$ has type of $w_i \geq \underline{P}^i_j(\mu,A)$. Moreover, because $A_i(\mu)$ is a collection of intervals, if $P^i_{j\,\mu(i)} < \overline{w}$ then any $w_i' \equiv \underline{P}^i_j(\mu,A) + \epsilon \in A_i(\mu)$ for small enough $\epsilon > 0$.

Next, let

$$Q^j_{i\,\hat{i}}(w_{\hat{i}}) \;\equiv\; \left\{\, w_i \in [\underline{w},\overline{w}] \;\middle|\; v_j(w_i,i) > v_j(w_{\hat{i}},\hat{i}) \geq 0 \,\right\}$$

be the set of worker $i$ types that firm $j$ strictly prefers to worker $\hat{i}$ of type $w_{\hat{i}}$. As $v_j$ is increasing, $Q^j_{i\,\hat{i}}$ is an interval. Similarly, define

$$\underline{Q}^j_i(w_{\nu(j)},\mu,A) \;\equiv\; \begin{cases} \inf\{\, w_i \in Q^j_{i\,\nu(j)}(w_{\nu(j)}) \cap A_i(\mu)\,\}, & \text{if } Q^j_{i\,\nu(j)}(w_{\nu(j)}) \cap A_i(\mu) \neq \emptyset \\ \\ \overline{w}, & \text{otherwise.} \end{cases}$$

**Proposition 2** *A matching outcome $(\mu,\mathbf{w}) \in A$ is $A$-blocked by $(i,j)$ if and only if $w_i \in P^i_{j\,\mu(i)}$ and $\underline{Q}^j_i(w_{\nu(j)},\mu,A) \leq \underline{P}^i_j(\mu,A)$.*

**Proof:** *Sufficiency.* If $w_i \in P^i_{j\,\mu(i)}$ then (3) is satisfied. If $w_i' \in A_i(\mu)$ satisfies (4) then $w_i' > \underline{P}^i_j(\mu,A)$ by definition. Each $w_i' > \underline{Q}^j_i(w_{\nu(j)},\mu,A)$ satisfies (5). As $\underline{Q}^j_i(w_{\nu(j)},\mu,A) \leq \underline{P}^i_j(\mu,A)$, each $w_i' \in A_i(\mu)$ that satisfies (4) also satisfies (5).

*Necessity.* If $w_i \notin P^i_{j\,\mu(i)}$ then $i$ will not participate in a block. Therefore assume that $w_i \in P^i_{j\,\mu(i)}$ but $\underline{Q}^j_i(w_{\nu(j)},\mu,A) > \underline{P}^i_j(\mu,A)$. As $w_i \in A_i(\mu)$, $P^i_{j\,\mu(i)} \cap A_i(\mu) \neq \emptyset$ and $\underline{P}^i_j(\mu,A) < \overline{w}$. Then, for small enough $\epsilon > 0$, any $w_i' = \underline{P}^i_j(\mu,A) + \epsilon < \underline{Q}^j_i(w_{\nu(j)},\mu,A)$ satisfies (4) but not (5). Taking $\epsilon > 0$ small enough such that $w_i' \in A_i(\mu)$ implies that $(\mu,\mathbf{w}) \in A$ is not $A$-blocked by $(i,j)$. ∎

The figures below illustrate the proof of Proposition 2. For simplicity, assume that $A_i(\mu) = [\underline{w}, \overline{w}]$. In both figures, the gain to worker $i$ in switching from firm $\mu(i)$ to firm $j$ as a function of $w_i'$ is $u_i(w_i', j) - u_i(w_i', \mu(i))$. This is the blue curve and its smallest intersection of the horizontal axis is from below; this point is $\underline{P}_j^i(\mu, A)$. The two intervals indicated by the green broken arrows is the set $P_{j\mu(i)}^i$. The gain to firm $j$ in switching from worker $\nu(j)$ to worker $i$ is an increasing function of $w_i'$. It intersects the horizontal axis at $\underline{Q}_i^j(w_{\nu(j)}, \mu, A)$. In Figure 2, $\underline{Q}_i^j(w_{\nu(j)}, \mu, A) \leq \underline{P}_j^i(\mu, A)$ and if worker $i$'s type is in the set $P_{j\mu(i)}^i$, then $(i, j)$ blocks $\mu$. In Figure 3, $\underline{Q}_i^j(w_{\nu(j)}, \mu, A) > \underline{P}_j^i(\mu, A)$ and even if worker $i$'s type is in the set $P_{j\mu(i)}^i$ firm $j$ will not participate in a block. This is because if worker $i$'s type is in the region indicated by the red broken arrow, then firm $j$ is worse off with worker $j$ than with worker $\nu(j)$.

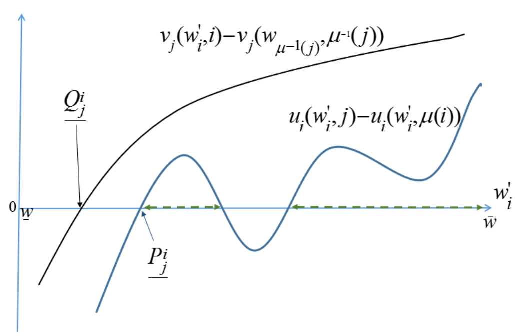


Figure 2: $\underline{Q}_i^j(w_{\nu(j)}, \mu, A) \leq \underline{P}_j^i(\mu, A)$

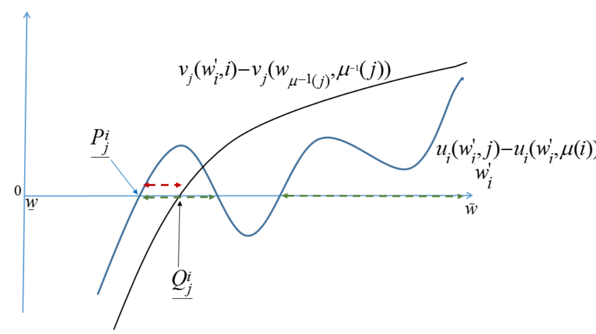


Figure 3: $\underline{Q}_i^j(w_{\nu(j)}, \mu, A) > \underline{P}_j^i(\mu, A)$

In Figures 2 and 3, it is assumed that $u_i(w_i', j) - u_i(w_i', \mu(i))$ intersects the horizontal axis and the smallest intersection is from below. If, instead, the smallest intersection is from above or if $u_i(w_i', j) - u_i(w_i', \mu(i)) > 0$ for all $w_i'$ then $\underline{P}_j^i(\mu, A) = \underline{w}$. In this case, firm $j$ would participate in a block only if $\underline{Q}_i^j(w_{\nu(j)}, \mu, A) = \underline{w}$, i.e., only if firm $j$ prefers even the lowest type of worker $i$ to its current match at $(\mu, \mathbf{w})$. On the other hand, if $u_i(w_i', j) - u_i(w_i', \mu(i)) < 0$ for all $w_i'$ then $\underline{P}_j^i(\mu, A) = \overline{w}$ and worker $i$ will not participate in a block with $j$.

## 2.1 Blocking Coalitions with Multiple Agents

In a complete-information model, if a matching is not blocked by a worker-firm pair then it is not blocked by a larger coalition. In an incomplete-information setting, depending on the information-sharing assumptions within coalitions it may be possible that a larger coalition blocks a matching that is unblocked by any two-agent coalition. Paralleling the definition of a two-person blocking coalition, I assume that only information that is inferred from the fact that they are willing to participate in a block is available to coalition members. As in a two-person blocking pair with worker $i$ and firm $j$, the participation of worker $i$ in a coalition to block matching $\mu$ conveys some information to firm $j$ about the possible values of $w_i$. Moreover, if firm $\mu(i)$ is also a member of the larger coalition, then an additional inference is possible about $w_i$ – that its value could not be too high.

A coalition blocks a matching if each coalition member is strictly better off in any state of the world, consistent with the member's beliefs, under which the block might be contemplated. One immediate conclusion is that a blocking coalition must have equal numbers of workers and firms because an unmatched worker (or firm) cannot be strictly better off. Thus, let $(i_1, j_1), \ldots, (i_k, j_k)$ be a blocking coalition, where each pair $(i_\ell, j_\ell)$, $\ell = 1, 2, \ldots, k$ are matched together in the proposed blocking coalition.

An individually-rational matching outcome $(\mu, \mathbf{w}) \in A$ is *A-blocked by a coalition* $(i_1, j_1), \ldots, (i_k, j_k)$ if for all $\ell = 1, \ldots, k$,

$$u_{i_\ell}(w_{i_\ell}, j_\ell) \quad > \quad u_{i_\ell}(w_{i_\ell}, \mu(i_\ell)), \tag{10}$$

$$\text{and for all } (\mu, w') \in A \quad \text{s.t.} \quad w'_{\nu(j_\ell)} = w_{\nu(j_\ell)}, \quad w'_{i_\ell} \leq w^*_{i_\ell},$$

$$\text{if} \quad u_{i_\ell}(w'_{i_\ell}, j_\ell) \quad > \quad \max[\, u_{i_\ell}(w'_{i_\ell}, \mu(i_\ell)), 0\,] \tag{11}$$

$$\text{then} \quad v_{j_\ell}(w'_{i_\ell}, i_\ell) \quad > \quad v_{j_\ell}(w_{\nu(j_\ell)}, \nu(j_\ell)) \tag{12}$$

where if $i_\ell \neq \nu(j_{\hat{\ell}})$ for any $\hat{\ell} = 1, \ldots k$ then $w^*_{i_\ell} = \overline{w}$; that is, when the firm matched with worker $i_\ell$ is not in the coalition, no additional restriction is imposed on $w'_{i_\ell}$ than is implied by (11) and (12).

This definition is similar to the earlier definition of a blocking pair except for the possible restrictions $w'_{i_1} \leq w^*_{i_1}$, $w'_{i_2} \leq w^*_{i_2}$, etc. If worker $i_1 \neq \nu(j_\ell)$ for any $\ell = 2, \ldots, k$ then firm $j_1$ learns nothing from the fact that $(i_2, j_2), \ldots, (i_k, j_k)$ are in the blocking coalition and we have $w^*_{i_1} = \overline{w}$. Therefore, (11) is similar to (4). If, instead, $i_1 = \nu(j_\ell)$ for some $\ell \geq 2$, then an upper bound on $w_{\nu(j_\ell)}$, that is on $w_{i_1}$, is implied by Lemma 3(i) (see the remark immediately after the lemma); that is, from the fact that $j_\ell$ will be better off with all possible types of $i_\ell$ that are "reasonable" than by staying with $\nu(j_\ell)$. Hence, the restriction $w'_{i_1} \leq w^*_{i_1}$. The exact value of $w^*_{i_1}$ is unimportant for the next result.

**Proposition 3** *If a matching outcome is blocked by a coalition then each matched worker-firm pair within the coalition constitutes a blocking pair.*

**Proof:** Let $(i_1, j_1), \ldots, (i_k, j_k)$ be a blocking coalition.

Suppose that $i_1 \neq \nu(j_\ell)$ for any $\ell \in \{2, \ldots, k\}$. Then $w^*_{i_1} = \overline{w}$ and for the pair $(i_1, j_1)$, (11) implies (12) if and only if (4) implies (5). Therefore, $(i_1, j_1)$ is a blocking pair.

Next, suppose that $i_1 = \nu(j_\ell)$ for some $\ell \in \{2, \ldots, k\}$ and $w^*_{i_1} < \overline{w}$. Then any $w'_{i_1} \leq w^*_{i_1}$ that satisfies (11) also satisfies (12). By increasing utility, (12) implies $v_{j_1}(w''_{i_1}, i_1) > v_{j_\ell}(w_{\nu(j_1)}, \nu(j_1))$ for all $w''_{i_1} > w'_{i_1}$, thus (5) is true for all $w''_i > w^*_{i_1}$ including $w''_i$ that satisfy (4). Hence, $(i_1, j_1)$ is a blocking pair. ∎

Thus, allowing for blocks by larger coalitions does not reduce the size of $\Sigma^*$.

## 2.2 Common Knowledge at Self-Stabilizing Sets

Self-stabilizing sets may be characterized by common knowledge of no blocking among workers and firms. It is convenient to illustrate this with $\Sigma^*$, the largest self-stabilizing set. Consider any $\mu$ such that there exists $(\mu, \mathbf{w}) \in \Sigma^*$. Let

$$\Sigma^k(\mu) \;=\; \{\mathbf{w} \,|\, (\mu, \mathbf{w}) \in \Sigma^k\}$$

As $\Sigma^* \subseteq \Sigma^k$, $\Sigma^k(\mu)$ is nonempty. Let $B^k(\mu) \equiv \Sigma^k(\mu) \backslash \Sigma^{k+1}(\mu)$. If $\mathbf{w} \in B^k(\mu) \subset \Sigma^k(\mu)$, then a worker-firm pair $\Sigma^k$-blocks $\mu$. If $B^k(\mu) = \emptyset$ then $\Sigma^*(\mu) = \Sigma^k(\mu)$.

An uninformed observer conducts the following process which attains common knowledge of $\Sigma^*(\mu)$ when $\mathbf{w} \in \Sigma^*(\mu)$. Initially, it is common knowledge among workers, firms, and the observer that $\mathbf{w} \in \Sigma^0(\mu)$. At each stage $k = 0, 1, 2, 3, \ldots$, workers and firms are asked to privately report to the observer (i) whether they are interested in blocking $\mu$ and (ii) the identities of all agents with whom they are interested in participating in a blocking pair. A blocking pair $(i, j)$ is formed if worker $i$ lists firm $j$ as a blocking partner and firm $j$ lists worker $i$ as a blocking partner. If a blocking pair is formed then the process stops and the matching $\mu$ is abandoned; otherwise, the process continues to the next stage without disclosing the reports of workers and firms, if any. If the process reaches stage $k$ (i.e., no blocking pair is formed up till stage $k$), then the uninformed observer knows that $\mathbf{w} \in \Sigma^k(\mu)$.

If $\mathbf{w} \in \Sigma^*$ then the process stops if there exists $\hat{k}$ such that $\Sigma^{\hat{k}} = \Sigma^{\hat{k}+1}$, in which case $\Sigma^* = \Sigma^{\hat{k}}$; otherwise the process continues forever, with the observer's state of knowledge approaching $\Sigma^*$.

At each stage $k$, worker $i$ and firm $\mu(i)$ know the value of $w_i$ and know that $\mathbf{w} \in \Sigma^k(\mu)$. Thus, the event $\{\mathbf{w} \in \Sigma^k(\mu)\}$ is *self-evident* to workers and firms, as defined in Osborne and Rubinstein [15]. Therefore, Proposition 74.2 in Osborne and Rubinstein implies that $\{\mathbf{w} \in \Sigma^k(\mu)\}$ is common knowledge between workers and firms at stage $k$. Hence, $\Sigma^*(\mu)$ also becomes common knowledge between workers and firms.

## 2.3 An Ex Post Incentive-Compatible Mechanism

Properties of the set of complete-information stable matchings can be exploited to construct an efficient, ex post incentive-compatible mechanism. For almost all worker types $\mathbf{w}$, this mechanism implements the worker-optimal complete-information stable matching.

I assume that the mechanism designer knows the utility functions of the workers and the firms. The rules of the mechanism are as follows. Workers report their types to the mechanism designer. The mechanism designer computes the worker-optimal complete-information stable matching (if one exists) for the reported worker types, pairs the workers and firms according to this stable matching, and reveals the reported worker types to all firms. If a worker-optimal matching does not exist then the mechanism designer picks an arbitrary complete-information stable matching. As before, after the matching is implemented each firm will learn its matched worker's type and, in particular, will find out whether it is different from the worker type revealed by the mechanism designer. Call this mechanism the *worker-optimal mechanism.*

**Proposition 4** *For almost all worker types, truthful reporting of types is an ex post equilibrium for workers in the worker-optimal mechanism.*

**Proof:** Consider a vector of worker types, $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, at which all agents have strict preference. (As utility is increasing, this holds for almost all type vectors.) Therefore, from Gale and Shapley [10] we know that a worker-optimal mechanism exists at $\mathbf{w}$; call it $\mu_w$.

Suppose that worker 1 reports a type $w_1' \neq w_1$. Let $\mu'$ be the matching implemented at $(w_1', w_{-1})$. If $\mu'(1) = \mu_w(1)$, then worker 1 does not benefit from this deviation. Therefore, suppose that $\mu'(1) \neq \mu_w(1)$ and that $u_1(w_1, \mu'(1)) > u_1(w_1, \mu_w(1))$. But then, firm $\mu'(1)$ is not achievable for worker 1 at any complete-information stable matching at $\mathbf{w}$. After the matching $\mu'$ is implemented, firm $\mu'(1)$ learns that worker 1's type is $w_1$ and not $w_1'$. Therefore, as $\mu'(1)$ is not achievable for worker 1 at $\mathbf{w}$, there exists a worker $\hat{\imath}$ that, together with firm $\mu'(1)$, complete-information blocks matching $\mu'$. Hence, worker 1 does not profit from the deviation.

Next, suppose that after firms learn the types of the workers they are matched with at $\mu_w$, firm $j$ incorrectly claims that the worker $\nu_w(j)$ lied about his type. But this does not change the utility that any other worker derives from matching with firm $j$ and therefore does not lead to a blocking pair with firm $j$ and some other worker. Hence, firm $j$ cannot benefit by his misreport. ∎

Observe that increasing utility is used in the proof only in so far as it implies that for almost all worker types there is strict preference.

## 2.4 Comparison with a TU Model

In this section, the previous NTU model is compared to a TU model with quasilinear utility in which side payments between matched agents are allowed. If worker $i$ and firm $j$ are matched and $j$ pays $i$ an amount $p$, then the utilities of the worker and firm are:

$$u_i(w_i, j) + p \quad \text{and} \quad v_j(w_i, i) - p$$

(If $p < 0$ then the worker pays the firm an amount $-p$.) A matching outcome is a $(\mu, \mathbf{w}, \mathbf{p})$ where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is the vector of payments received by the workers from their matched firms, i.e., worker $i$ receives $p_i$ from firm $\mu(i)$ at the matching outcome $(\mu, \mathbf{w}, \mathbf{p})$. If $\mu(i) = 0$ then $p_i = 0$. This is the model of Liu et al. (2013). The impact of side payments on the size of the self-stabilizing set is not immediately clear. On the one hand, each matching could potentially be stabilized by one of many side payments, increasing the size of self-stabilizing sets. On the other hand, the number of potential blocking contracts is also much larger, reducing the size of self-stabilizing sets. It turns out, however, that the second effect dominates and the self-stabilizing set is smaller in the TU model.

An individually-rational matching outcome $(\mu, \mathbf{w}, \mathbf{p})$ is *A-blocked* if there is a worker-firm pair $(i, j)$ and a payment $p$ satisfying

$$u_i(w_i, j) + p \quad > \quad u_i(w_i, \mu(i)) + p_i \tag{13}$$

$$\text{and for all } (\mu, \mathbf{w}') \in A \quad \text{s.t.} \quad w'_{\nu(j)} = w_{\nu(j)},$$

$$\text{if} \quad u_i(w'_i, j) + p \quad > \quad u_i(w'_i, \mu(i)) + p_i \ \geq \ 0 \tag{14}$$

$$\text{then} \quad v_j(w'_i, i) - p \quad > \quad v_j(w_{\nu(j)}, \nu(j)) - p_i \tag{15}$$

As $(\mu, \mathbf{w}, \mathbf{p}) \in A$, if (13) is satisfied then (14) is also satisfied with $w'_i = w_i$ (and possibly with other $w'_i$ as well).

**Proposition 5** *Let $(\mu, \mathbf{w})$ and $(\mu, \mathbf{w}, \mathbf{p})$ be individually-rational matching outcomes in the NTU and TU games, respectively. If $(\mu, \mathbf{w})$ is A-blocked in the NTU game then $(\mu, \mathbf{w}, \mathbf{p})$ is A-blocked in the TU game.*

**Proof:** Suppose that $(\mu, \mathbf{w}) \in A$ is $A$-blocked by $(i, j)$ in the NTU game. Then, the definitions of blocking in the NTU and TU games imply that for any side payments $\mathbf{p} = (p_1, p_2, \ldots, p_i, \ldots, p_n)$ such that $(\mu, \mathbf{w}, \mathbf{p})$ is an individually-rational matching outcome in the TU game, $(\mu, \mathbf{w}, \mathbf{p})$ is $A$-blocked by $(i, j)$ with $i$ receiving payment $p_i$ from $j$. ∎

Thus, the set of incomplete-information stable-matching outcomes is at least as large in the NTU model as it is in the TU model. That the converse of Proposition 5 is not true follows from Example 1 of Section 2, where it is shown that every strictly individually rational matching is incomplete-information stable in the NTU model, and Liu et al.'s Proposition 3, which shows that only efficient matchings are stable under assortative matching in a TU model.

# 3 Two-sided Uncertainty

Returning to the NTU model, consider incomplete information on both sides of the market. Worker $i$'s type, $w_i$, is in the interval $[\underline{w}, \overline{w}]$. In addition, firm $j$'s type, $f_j$, is in the interval $[\underline{f}, \overline{f}]$. If the worker-firm pair $(i, j)$ is matched to each other then the two agents' utilities are:

$$u_i(w_i, f_j, j) \text{ and } v_j(w_i, f_j, i) \tag{16}$$

As before, the utility of an unmatched worker or firm is normalized to zero. Further, it is common knowledge that each worker knows the types of all workers and each firm knows the the types of all firms.[5] At any matching, each agent knows the type of the agent it is matched with, if any. The next assumption is similar to that made in the one-sided uncertainty model.

Increasing and Continuous Utility: *The utility functions $u_i(w_i, f_j, j)$ and $v_j(w_i, f_j, i)$ are strictly increasing in $w_i$ and $f_j$, for all $i$ and $j$.*

The definition of a matching is the same as earlier. Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ and $\mathbf{f} = (f_1, f_2, \ldots, f_m)$ be the type vectors of workers and firms, respectively. A *matching outcome* is a matching function together with type vectors: $(\mu, \mathbf{w}, \mathbf{f})$. The matching outcome $(\mu, \mathbf{w}, \mathbf{f})$ is *individually rational* if

$$\begin{aligned} u_i(w_i, f_{\mu(i)}, \mu(i)) &\geq 0, \qquad \forall i \\ v_j(w_{\nu(j)}, f_j, \nu(j)) &\geq 0, \qquad \forall j \end{aligned}$$

Let $\Sigma^0$ be the set of individually-rational matching outcomes.

[5] As noted in footnote 3, the results of the one-sided uncertainty model hold under the alternative assumption that $\mathbf{w}$ is common knowledge to workers but is not known to firms.

In order to simplify the exposition, only $\Sigma^0$-blocks are considered. The definitions and results for $\Sigma^k$ and $\Sigma^*$ are similar to those in the one-sided incomplete information model.

Suppose that worker $i$ and firm $j$ are contemplating a block to matching $\mu$. As there is asymmetric information on both sides, both $i$ and $j$ must consider the lowest possible type of the other, $\underline{f}_j^*$ and $\underline{w}_i^*$ respectively, that might be interested in participating in a block. To determine $\underline{f}_j^*$, worker $i$ needs to ascertain the highest possible value of his own type, $\overline{w}_i^*$, that firm $j$ thinks might be interested in participating in a block with it. Similarly, in order to determine $\underline{w}_i^*$ firm $j$ needs to figure out $\overline{f}_j^*$, the highest possible value of firm $j$'s type that worker $i$ thinks might participate in a block. To this end, define the gains to worker $i$ and firm $j$, as functions of $w_i'$ and $f_j'$ respectively, from participating in a block to matching $\mu$

$$\begin{aligned} \Delta U_i(w_i', \hat{f}_j, f_{\mu(i)}, \mu) &\equiv u_i(w_i', \hat{f}_j, j) - u_i(w_i', f_{\mu(i)}, \mu(i)) \\ \Delta V_j(f_j', \hat{w}_i, w_{\nu(j)}, \mu) &\equiv v_j(\hat{w}_i, f_j', i) - v_j(w_{\nu(j)}, f_j', \nu(j)), \end{aligned}$$

where $i$ assumes that $j$'s type is $\hat{f}_j$ and $j$ assumes that $i$'s type is $\hat{w}_i$. Define,

$$\begin{aligned} W_i(\hat{f}_j) &\equiv \begin{cases} \sup\{w_i' \in [\underline{w}, \overline{w}] \mid \Delta U_i(w_i', \hat{f}_j, f_{\mu(i)}, \mu) > 0\}, & \text{if the set if non-empty} \\ \underline{w}, & \text{otherwise} \end{cases} \\ F_j(\hat{w}_i) &\equiv \begin{cases} \sup\{f_j' \in [\underline{f}, \overline{f}] \mid \Delta V_j(f_j', \hat{w}_i, w_{\nu(j)}, \mu) > 0\}, & \text{if the set is non-empty} \\ \underline{f}, & \text{otherwise} \end{cases} \end{aligned}$$

where the dependence of $W_i$ and $F_j$ on $\mu$, $f_{\mu(i)}$ or $w_{\nu(j)}$ is suppressed in the notation. As $\Delta U_i$ and $\Delta V_j$ are increasing in their respective second arguments, $W_i(\cdot) : [\underline{f}, \overline{f}] \to [\underline{w}, \overline{w}]$ and $F_j(\cdot) : [\underline{w}, \overline{w}] \to [\underline{f}, \overline{f}]$ are weakly increasing functions. Therefore, by Tarski's fixed-point theorem, there exists a greatest fixed point to the mapping $T_{ij}(\hat{w}_i, \hat{f}_j) \equiv (W_i(\hat{f}_j), F_j(\hat{w}_i))$. Let $(\overline{w}_i^*, \overline{f}_j^*) \equiv (W_i(\overline{f}_j^*), F_j(\overline{w}_i^*))$ be this greatest fixed point. Thus, $\overline{w}_i^*$ is the most optimistic assumption, consistent with common knowledge of rationality, that firm $j$ can make about worker $i$'s type. Similarly, $\overline{f}_j^*$ is the most optimistic assumption about firm $j$'s type consistent with common knowledge of rationality.

An individually-rational matching outcome $(\mu, \mathbf{w}, \mathbf{f}) \in \Sigma^0$ is *blocked*[6] if there is a worker-firm pair $(i, j)$ satisfying

$$u_i(w_i, f_j, j) > u_i(w_i, f_{\mu(i)}, \mu(i)), \qquad v_j(w_i, f_j, i) > v_j(w_{\nu(j)}, f_j, \nu(j)) \tag{17}$$

$$\text{and for all } (\mu, w', f') \in \Sigma^0 \text{ s.t. } w'_{\nu(j)} = w_{\nu(j)},\ f'_{\mu(i)} = f_{\mu(i)},\ f_j' = \overline{f}_j^*$$

$$\text{if} \quad u_i(w_i', \overline{f}_j^*, j) > u_i(w_i', f_{\mu(i)}, \mu(i)) \geq 0 \tag{18}$$

[6]Recall that only $\Sigma^0$-blocks are considered in this section.

$$\text{then} \quad v_j(w'_i, f_j, i) \quad > \quad v_j(w_{\nu(j)}, f_j, \nu(j)), \tag{19}$$

$$\text{and for all } (\mu, w', f') \in \Sigma^0 \quad \text{s.t.} \quad w'_{\nu(j)} = w_{\nu(j)},\ w'_j = \overline{w}^*_j,\ f'_{\mu(i)} = f_{\mu(i)}$$

$$\text{if} \quad v_j(\overline{w}^*_i, f'_j, i) \quad > \quad v_j(w_{\nu(j)}, f'_j, \nu(j)) \ \geq \ 0 \tag{20}$$

$$\text{then} \quad u_i(w_i, f'_j, j) \quad > \quad u_i(w_i, f_{\mu(i)}, \mu(i)) \tag{21}$$

Condition (17) requires that worker $i$ and firm $j$ are better off in the potential block. The remaining conditions require that firm $j$ and worker $i$ know that they are better off in the potential block. Inequality (19) states that firm $j$ is better off with all possible types $w'_i$ of worker $i$ that, from firm $j$'s state of knowledge, might benefit from blocking with firm $j$; these types $w'_i$ satisfy (18) where worker $i$ makes the most optimistic assumption (consistent with rationality) about firm $j$'s type, i.e., $f'_j = f^*_j$. Inequalities (18) and (19) assume only that which is common knowledge between worker $i$ and firm $j$ about the types of worker $\nu(j)$ and firm $\mu(i)$: that $(\mu, w', f') \in \Sigma^0$ satisfies $w'_{\nu(j)} = w_{\nu(j)}$ and $f'_{\mu(i)} = f_{\mu(i)}$. The restriction $f'_{\mu(i)} = f_{\mu(i)}$, for instance, is imposed because it is common knowledge that worker $i$ knows that $f'_{\mu(i)} = f_{\mu(i)}$ and that firm $j$ know the types of all firms, including firm $\mu(i)$. Without the assumption that is common knowledge among all firms, (19) would have to satisfied for a larger set of $w'_i$ making blocking more difficult to achieve. That worker $i$ knows that it is better off in the potential block follows by a similar argument from (20) and (21).

In the one-sided incomplete information model, firm $j$ responds to its informational disadvantage by participating in a block with worker $i$ only if firm $j$'s utility increases even with the lowest type of worker $i$ interested in blocking with firm $j$. In the two-sided incomplete information model, firm $j$ becomes even more conservative because worker $i$ does not know firm $j$'s type. Firm $j$'s assessment of the lowest type of worker $i$ interested in blocking with firm $j$ depends on what firm $j$ assumes worker $i$ thinks about firm $j$'s type. In effect, firm $j$ assumes that worker $i$ is as optimistic as possible about firm $j$'s type, within the constraints imposed by rationality and the state of knowledge of $i$ and $j$. Note that the assumption $f'_j = \overline{f}^*_j$ in (18), together with increasing utility, yields a worst-case assessment by firm $j$ about the type of worker $i$. Thus, if (18) and (19) are satisfied, firm $j$ will not regret participating in a block no matter what worker $i$ may have assumed about firm $f_j$'s type. Consequently, conservative behavior by firms is intensified in the two-sided incomplete information model. Similarly, workers are also equally conservative.

The matching outcome $(\mu, \mathbf{w}, \mathbf{f}) \in \Sigma^0$ is $\Sigma^0$-*stable* if it is not $\Sigma^0$-blocked by any firm-worker pair.

As the next proposition implies, the set of two-sided incomplete-information stable matchings is larger than the set of one-sided incomplete-information stable matchings.

Let $\Sigma^0_{\mathbf{f}}$ be the projection of a set of matching outcomes $\Sigma_0$ at $\mathbf{f}$. That is,

$$\Sigma^0_{\mathbf{f}} \;=\; \{(\mu, \mathbf{w}) \,|\, (\mu, \mathbf{w}, \mathbf{f}) \in \Sigma^0\,\}$$

Thus, $\Sigma^0_{\mathbf{f}}$ is a set of matching outcomes in the one-sided uncertainty model at which it is common knowledge that the firm type vector is $\mathbf{f}$.

**Proposition 6** *Let $(\mu, \mathbf{w}, \mathbf{f}) \in \Sigma^0$. If $(\mu, \mathbf{w})$ is $\Sigma^0_{\mathbf{f}}$-stable in the one-sided uncertainty model then $(\mu, \mathbf{w}, \mathbf{f})$ is $\Sigma^0$-stable in the two-sided uncertainty model.*

**Proof:** Suppose that $(\mu, \mathbf{w})$ is $\Sigma^0_{\mathbf{f}}$-stable in the one-sided uncertainty model. Therefore, no worker $i$ and firm $j$ $\Sigma^0_{\mathbf{f}}$-blocks $(\mu, \mathbf{w})$. That is, with $\mathbf{f} = (f_j, f_{\mu(i)}, f_{-j\,\mu(i)})$, either (i) $u_i(w_i, f_j, j) \leq u_i(w_i, f_{\mu(i)}, \mu(i))$ or (ii) (19) is not satisfied for some $w''_i$ such that $u_i(w''_i, f_j, j) > u_i(w''_i, f_{\mu(i)}, \mu(i)) \geq 0$.

If (i) then $(\mu, \mathbf{w}, \mathbf{f})$ is not $\Sigma^0$-blocked by $(i, j)$. Therefore, suppose (ii). If (17) is not satisfied at $(\mathbf{w}, \mathbf{f})$ then $(\mu, \mathbf{w}, \mathbf{f})$ is not $\Sigma^0$-blocked by $(i, j)$. Hence, suppose also that (17) holds. Therefore, $\overline{f}^*_j > \underline{f}$ and $\overline{w}^*_i > \underline{w}$. If $\overline{f}^*_j \geq f_j$ then $w''_i$ satisfies (18) but not (19) and $(\mu, \mathbf{w}, \mathbf{f})$ is not $\Sigma^0$-blocked by $(i, j)$. If, instead, $\overline{f}^*_j < f_j$ then we must have $\overline{w}^*_i = \overline{w}$. To see this, note that $\overline{f}^*_j \in (\underline{f}, \overline{f})$ and $\overline{w}^*_i \in (\underline{w}, \overline{w})$, together with increasing utility, implies that $W_i(\overline{f}^*_j + \epsilon) > \overline{w}^*_i$ and $F_j(\overline{w}^*_i + \epsilon) > \overline{f}^*_j$, contradicting the fact that $(\overline{w}^*_i, \overline{f}^*_j)$ is the greatest fixed point of the mapping $T_{ij}(\hat{w}_i, \hat{f}_j)$. Finally, note that $\overline{w}^*_i = \overline{w}$ implies that any $w'_i \in (\overline{w} - \epsilon, \overline{w})$ satisfies (18) but not (19) and once again $(\mu, \mathbf{w}, \mathbf{f})$ is not $\Sigma^0$-blocked by $(i, j)$. ∎

It is assumed in the definition of stability above that each agent's type is common knowledge on their side of the market but is not known to agents on the other side with whom they are not matched. If, instead, each agent's type is known only to the matched pair of agents, the definition of blocking needs to be changed with any $f'_{\mu(i)}$ replacing $f_{\mu(i)}$ in (18) and any $w'_{\nu(j)}$ replacing $w_{\nu(j)}$ in (20). This change would make blocking more difficult and, consequently, increase the set of incomplete-information stable-matching outcomes.

Using the argument in the proof of Proposition 4, it can be shown that an ex post incentive compatible mechanism exists if there exists exactly one (though not necessarily the same one) complete-information stable matching for almost all worker type and firm type vectors. However, conditions under there is a unique stable matching are restrictive – see Eeckhout [7] and Clark [3].

# 4 Concluding Remarks

Roth [18] and Avery et al. [1] document the consequences of instability in the labor markets for medical residents and for law clerks, respectively. Conduct such as early hiring, market compression, or outright cheating is commonplace in such markets. Stable matchings are associated with the absence of such behavior. Under complete information, stable matchings have the additional desirable property of efficiency.

The preceding results suggest that under incomplete information, stability does not provide much restriction on the observed matching outcomes. In particular, under anonymous preferences stability provides no restriction at all. Therefore, in a matching environment with substantial incomplete information, the absence of signs of instability, such as attempts by workers and firms to change their matches, need not imply that the market is at a desirable outcome. Matching outcomes that make some, or perhaps all, participants better off may exist but cannot be reached because of a lack of information. Therefore, in such environments the process by which a matching outcome is reached is important. An efficient incentive-compatible stable-matching mechanism exists when there is one-sided incomplete information.

Incomplete-information stable matching outcomes are larger in NTU models than in TU models. The difference is stark in assortative matching where any strictly individually-rational matching outcome is stable in a NTU model while, as Liu et al. [13] show, in a TU model only efficient matching outcomes are stable. In TU models, prices facilitate mutually-beneficial deviations from inefficient matching outcomes.

A natural question is whether the definition of incomplete-information blocking used in this paper is too strong. If, for instance, the firm in a potential blocking pair is required to do better in expectation only rather than for all reasonable worker types, then it becomes easier to block a matching outcome, thereby decreasing the set of stable matching outcomes. However, observe that a Bayesian notion of blocking assumes an ability to commit to binding contracts. Suppose that a firm agrees to join with a worker to form a blocking pair (to a status quo matching) because the firm's expected utility will increase. Later, after learning the type of the matched worker, the firm may find that its utility in fact decreased and then it may wish to return to the status quo in the absence of a binding contract. But if a binding contract is feasible in the blocking pair, then it ought to be feasible in the status quo matching as well, rendering a potential block to the status quo infeasible.

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