Efficient Division Given Private Preferences: Using the Expected Externality Mechanism

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Abstract

We study the problem of allocating n items to two agents whose cardinal preferences are private information. If money is available as a medium of exchange, Bayesian incentive compatibility and ex-ante efficiency can be achieved, thus implying ex-post efficiency. If money is not available as a medium of exchange, ex-ante efficiency is lost, though Bayesian incentive compatibility and ex-post efficiency are achievable, under certain reasonable conditions, using a variation of the Expected Externality Mechanism. That mechanism uses one of the goods as a numeraire good in lieu of money.

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1. Introduction

The problem of how to divide a set of goods between individuals has been studied in a variety of settings. For instance, the goods may be homogeneous or heterogenous, divisible or indivible; the individuals' preferences may be ordinal or cardinal; and money may or may not be available as a medium of exchange. We study the problem of allocating a set of n items to two individuals whose cardinal preferences are private information. Cardinal preferences are important for both inter-personal and intra-personal comparisons — especially if individuals do not know their preferences at the time that the allocation mechanism is chosen. We consider the case where money is available as a medium of exchange, but focus on the more difficult case where money is not available. Our results apply to both divisible and indivisible goods.

When dividing a set of goods between individuals, usually the goal is to achieve efficiency and fairness while taking into account the individuals' incentives. A variety of simple methods, such as divide-and-choose (where one person divides the set in two parts and the other person chooses which part he wants) or strict alternation (where individuals take turns in selecting a good from the set that has not been claimed yet), fail to achieve efficiency or incentive compatibility. The adjusted winner mechanism (Brams and Taylor, 1999) for dividing a set of goods between two people has several desirable properties assuming that cardinal preferences are reported truthfully; however, the mechanism is not incentive compatible.

One strand of the literature on allocating indivisible goods to agents focuses on single-unit assignment problems, that is, every agent is allocated exactly one object (e.g., see Zhou, 1990; Bogomolnaia and Moulin, 2001; Manea, 2009; Sethuraman and Katta, 2006; Hylland and Zeckhauser, 1979). Typically, these analyses consider a setting with n agents and n items. In this setting, it is not possible to achieve ex-ante efficiency, symmetry and strategy-proofness; however, it is possible to achieve ex-post efficiency, symmetry and strategy-proofness (Zhou, 1990). In contrast, we consider two agents and n items; as a result, each agent will often be allocated multiple items and strategyproofness cannot be achieved. Allocation mechanisms, such as the probabilistic serial (Bogomolnaia and Moulin, 2001) and random priority (Abdukadiroglu and Snmez, 1998), that are efficient for single-unit assignment problems do not yield efficient outcomes in the setting we study in this paper.

Some recent literature considers settings where multiple items are allocated to a single agent. Kojima (2009) studies a model where every agent has a quota representing the number of objects he will receive. Budish, Che, Kojima, and Milgrom (2013) consider an expanded class of problems, including many-to-one and many-to-many matchings, and problems with certain auxiliary constraints. In these papers, the number of items that will be allocated to a given individual is predetermined. In contrast, in our setting the number of items allocated to each agent will arise endogenously from the allocation mechanism and will depend on the agents' cardinal preferences.

In our setting, a set of n goods is divided between two individuals who are privately informed about their cardinal preferences. If money is available as a medium of exchange, Bayesian incentive compatibility and ex-ante efficiency can be achieved, thus implying ex-post efficiency. If money is not available as a medium of exchange, ex-ante efficiency is lost. However, we show that even without money, it is possible to achieve Bayesian incentive compatibility and ex-post efficiency using a variation of the Expected Externality Mechanism (under certain reasonable conditions). That mechanism uses one of the goods as a numeraire good in lieu of money.

The remainder of the paper is organized as follows. The problem is formulated in section 2. Section 3 provides results related to ex-ante efficiency, and section 4 focuses on ex-post efficiency. Section 5 concludes.

2. Model

There are two agents, A and B, and n goods. Denote by s_j the available quantity of good j. Let $s \equiv (s_1, s_2, ..., s_n)$. The agents' preferences across goods are additively separable. Denote the valuation of agent i for item j by v_{ij} and let $v_i \equiv (v_{i1}, v_{i2}, ..., v_{in})$. In what follows we consider the cases where money is and is not available as a medium of exchange. If money is available as a medium exchange, we assume agents' utilities are linear in monetary payments.

The valuations of each agent are drawn from the distributions, F_A and F_B, which are common knowledge. Each agent observes his valuations, but not the valuations of the other agent. We denote by F_{ij} the marginal distribution of the valuation of good j for agent i.

Since the valuations of each agent are private information to him, we consider direct mechanisms where each agent i observes his valuations v_i and reports them to the mechanism. Denote the reported vectors by $\hat{v}_A$ and $\hat{v}_B$. Each report $\hat{v}_i$ has to be in the support of the distribution F_i. The mechanism uses the reported valuations ($\hat{v}_A$ and $\hat{v}_B$) as well as the known priors (F_A and F_B) to allocate the goods to the two agents.

The mechanism outputs two non-negative vectors, $q_A = q_A(\hat{v}_A, \hat{v}_B)$ and $q_B = q_B(\hat{v}_A, \hat{v}_B)$; they represent the assignment of each agent. In particular, q_{ij} represents the quantity of good j that is assigned to agent i.[1]

We are interested in mechanisms that are Bayesian incentive compatible. These are mechanisms with a Bayesian equilibrium where agents truthfully report their valuations.[2]

We are interested in mechanisms that yield efficient allocations in equilibrium. The following definitions give the two notions of efficiency that we study.[3]

Definition 1. *A mechanism is* ex-ante efficient *if before agents learn their valuations the expected utility vector is Pareto efficient.*

Definition 2. *A mechanism is* ex-post efficient *if after agents learn their valuations there are no mutually beneficial trades available.*

Ex-ante efficiency implies ex-post efficiency.

If the agents' valuations are known, the set of ex-post efficient allocations can be found in the following way. First, reorder the goods so that v_{Aj}/v_{Bj} is decreasing in j. Then, for any $k \in \{1, .., n\}$ and $x \in [0, s_j]$, the allocation where $q_{Aj} = s_j$ for $j < k$, $q_{Ak} = x$, $q_{Aj} = 0$ for $j > k$, and $q_B = s - q_A$ is ex-post efficient.

3. Ex-ante Efficiency

If money is available as a medium of exchange, then it is possible to achieve both ex-ante efficiency and Bayesian incentive compatibility. These desirable

[1] Our analysis and results also apply to the case that each unit of each good is indivisible if we allow for potentially probabilistic assignments. In that case, if q_{ij} is an integer, it represents the number of units of good j that are (deterministically) assigned to agent i. If q_{ij} is not an integer, agent i gets $\lfloor q_{ij} \rfloor$ units of good j with probability 1; he then gets an additional unit of good j with probability $q_{ij} - \lfloor q_{ij} \rfloor$.

[2] Strategyproofness cannot be achieved in this setting (Mas-Colell, Whinston, and Green, 1995).

[3] Ordinal efficiency (Bogomolnaia and Moulin, 2001) is another notion of efficiency that is used for probabilistic assignements and only takes into account ordinal preferences. We do not study ordinal efficiency here, because we consider a setting with cardinal utilities.

properties are achieved with the Expected Externality Mechanism (EEM), e.g., see Pratt and Zeckhauser (1987).

With the EEM, each good is allocated to the agent who values it the most and each agent pays to the other agent an amount equal to the externality that his reported valuations impose to the other agent.

Given the agents' reports $\hat{v}_A$ and $\hat{v}_B$, the allocation is detemined as follows: If $\hat{v}_{Aj} > \hat{v}_{Bj}$ (resp. $\hat{v}_{Aj} < \hat{v}_{Bj}$) then $q_{Aj} = s_j$ and $q_{Bj} = 0$ (resp. $q_{Aj} = 0$ and $q_{Bj} = s_j$), that is, all s_j units of good j are assigned to agent A (resp. B). If $\hat{v}_{Aj} = \hat{v}_{Bj}$ then $q_{Aj} = q_{Bj} = s_j/2$. In words, if the two agents report the same valuation for good j, the s_j units of good j are split evenly between the two agents.[4]

Agent A pays agent B the following amount:

$$\sum_{j=1}^{n} s_j \int_0^{\hat{v}_{Aj}} \left(1 - 1_{\{y=\hat{v}_{Aj}\}}/2\right) y dF_{Bj}(y),$$

where $1_{\{.\}}$ denotes the indicator function. The amount above represents the expected externality that A's report $\hat{v}_A$ imposes on B. In particular, whenever $v_{Bj} < \hat{v}_{Aj}$, the externality that A's report imposes on B is equal to $s_j v_{Bj}$, because all s_j units of good j are allocated to agent A. If $v_{Bj} = \hat{v}_{Aj}$, the externality is equal to $(s_j/2)v_{Bj}$, because in this case the s_j units of good j are split evenly between the two agents.
Similarly, Agent B pays agent A:

$$\sum_{j=1}^{n} s_j \int_0^{\hat{v}_{Bj}} \left(1 - 1_{\{y=\hat{v}_{Bj}\}}/2\right) y dF_{Aj}(y).$$

As a result, the net payment to agent A is:

$$\sum_{j=1}^{n} s_j \int_0^{\hat{v}_{Bj}} \left(1 - 1_{\{y=\hat{v}_{Bj}\}}/2\right) y dF_{Aj}(y) - \sum_{j=1}^{n} s_j \int_0^{\hat{v}_{Aj}} \left(1 - 1_{\{y=\hat{v}_{Aj}\}}/2\right) y dF_{Bj}(y)$$

The following proposition follows from standard results on the EEM, e.g., see Mas-Colell, Whinston, and Green (1995).

Proposition 1. *When money is available as a medium of exchange, the EEM satisfies Bayesian incentive compatibility and ex-ante efficiency.*

In the setting that we are considering here, the EEM satisfies both ex-ante and ex-interim individual rationality.[5] In other words, an individual wants

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[4]In fact, any way of dividing a good j for which $\hat{v}_{Aj} = \hat{v}_{Bj}$ between the two individuals will yield an efficient allocation. We assume that such goods are divided evenly for fairness and simplicity.

[5]In the case of indivisible goods, where the allocation is probabilistc (bacause fractional allocations are not possible), ex-post individual rationality may fail under the EEM in a setting with money. However, the EEM in a setting without money (described in Section 4) always satisfies ex-post individual rationality — regardless of whether goods are divisible or indivisible.

to participate both before and after learning his valuation. In particular, an individual has a non-negative expected utility by reporting $\hat{v} = 0$; if he reports his true valuation vector, his expected utility will be even higher.

We next show that it is not possible to achieve both ex-ante efficiency and Bayesian incentive compability in a setting without money.

Proposition 2. *When money is not available as a medium of exchange, there does not a mechanism that satisfies ex-ante efficiency and Bayesian incentive compatibility.*

Proof. Suppose $n = 1$ and both agents' valuations are uniformly distributed on $[0, 1]$. Ex-ante efficiency would give the good to the person who values it most in each case. Therefore, in a direct mechanism, each agent would report his valuation, and the good (in its entirety) would be allocated to the agent that reports the highest value. But independent of his valuation, each agent would use the strategy that maximizes his probability of getting the good. In other words, both agents would announce a valuation of 1, independent of their actual draws. The mechanism would have no way to determine to whom the good should be allocated. Thus, any mechanism would have to risk giving the good (or part of it) to an individual who values it less, implying the loss of ex-ante efficiency. □

4. Ex-post Efficiency

In this section we show that even without money, in the special case where there is one good with sufficiently many available units, ex-post efficiency is achievable with a variation of the EEM.

When considering ex-post efficiency, we restrict attention to mechanisms where each agent gets a strictly positive amount of some goods. This implies that information elicitation is necessary.[6] Formally, we require that the following condition holds.

Condition 1. *Whenever $\hat{v}_A \neq 0$ and $\hat{v}_B \neq 0$, the allocation is such that $q_A(\hat{v}_A, \hat{v}_B) \neq 0$ and $q_B(\hat{v}_A, \hat{v}_B) \neq 0$.*

This EEM in the setting without money uses relative valuations with respect to a numeraire good. Suppose that n is the numeraire good. Let G_i denote the cumulative distribution function of the ratios $(v_{i1}/v_{in}, v_{i2}/v_{in}, ..., v_{i,n-1}/v_{in})$. We use G_{ij} to denote the marginal distributions and Φ_i to denote the support of G_i. For $j = 1, ..., n-1$, let $\hat{r}_{ij} = \hat{v}_{ij}/\hat{v}_{in}$ be the ratio of agent i's reported valuation for j over his reported valuation for the numeraire good n; define $\hat{r}_i \equiv (\hat{r}_{i1}, \hat{r}_{i2}, ..., \hat{r}_{i,n-1})$. In this notation, we have suppresed the dependence on n for simplicity.

[6] Otherwise, one would trivially achieve ex-post efficiency simply by alocating all goods to one agent.

We now specify how q_A is determined as a function of the reported valuations $\hat{v}_A$, $\hat{v}_B$ and the distribution functions G_A and G_B; q_B is defined symmetrically. For $j < n$, $q_{Aj} = s_j$ if $\hat{r}_{Aj} > \hat{r}_{Bj}$; $q_{Aj} = s_j/2$ if $\hat{r}_{Aj} = \hat{r}_{Bj}$; and $q_{Aj} = 0$ if $\hat{r}_{Aj} < \hat{r}_{Bj}$. The following formula gives q_{An}, using $1_{\{\cdot\}}$ to denote the indicator function.

$$\begin{aligned} q_{An} = s_n/2 - &\sum_{j=1}^{n-1} \int_0^{\hat{r}_{Aj}} s_j \left(1 - 1_{\{x=\hat{r}_{Aj}\}}/2\right) x dG_{Bj}(x) \\ + &\sum_{j=1}^{n-1} \int_0^{\hat{r}_{Bj}} s_j \left(1 - 1_{\{x=\hat{r}_{Bj}\}}/2\right) x dG_{Aj}(x) \end{aligned} \tag{1}$$

The first summation in (1) represents the expected externality that A's report imposes on B per unit of good n because of the way that goods $j = 1, ..., n-1$ are allocated. Whenever $v_{Bj}/v_{Bn} < \hat{r}_{Aj}$, this externality is equal to $s_j v_{Bj}/v_{Bn}$, because all s_j units of good j are allocated to agent A. If $v_{Bj}/v_{Bn} = \hat{r}_{Aj}$, the externality is equal to $(s_j/2)v_{Bj}/v_{Bn}$, because in this case the s_j units of good j are split evenly between the two agents. The second summation in (1) represents the expected externality that B's report imposes on A. Neither agent's expected externality imposition depends on the report of the other player. This property is critical to assure honest reporting.

It is as if we initially give each agent $s_n/2$ units of good n and then use the EEM described in Section 3, but replace valuations with ratios. Instead of payments, the mechanism identifies quantities of good n that each agent will give the other agent.

We next define M_A (resp. m_A) as the maximum (resp. minimum) value of the expected externality that A's report can impose on B among all valuation ratios that arise with positive probability:

$$M_A \equiv \max_{\hat{r}_A \in \Phi_A} \left\{ \sum_{j=1}^{n-1} \int_0^{\hat{r}_{Aj}} s_j \left(1 - 1_{\{x=\hat{r}_{Aj}\}}/2\right) x dG_{Bj}(x) \right\} \tag{2}$$

$$m_A \equiv \min_{\hat{r}_A \in \Phi_A} \left\{ \sum_{j=1}^{n-1} \int_0^{\hat{r}_{Aj}} s_j \left(1 - 1_{\{x=\hat{r}_{Aj}\}}/2\right) x dG_{Bj}(x) \right\} \tag{3}$$

Similarly, we define M_B and m_B.

In order to be able to use the EEM, it is necessary that $q_{An} \geq 0$ and $q_{Bn} \geq 0$ for all possible values of $\hat{r}_A$ and $\hat{r}_B$. In other words, we want the right hand side of (1) to be non-negative for any posible reports of the agents. If $\max\{M_A - m_B, M_B - m_A\} \leq s_n/2$, then q_{An}, $q_{Bn} \geq 0$ and as a result it is possible to use the EEM.[7]

[7] It is possible to use a variation of the EEM described above under a weaker condition, namely that $M_A - m_B + M_B - m_A \leq s_n$. In particular, we would first identify values x_{An}

Example 1. *Suppose that* $n = 2$ *and* v_{A1}, v_{A2}, v_{B1} *and* v_{B2} *are (independently) uniformly distributed on* $\{1, 2\}$. *Thus, the ratio* v_{A1}/v_{A2} *is equal to 1/2 with probability 1/4, equal to 1 with probability 1/2, and equal to 2 with probability 1/4. Then* $M_A = 7/8$ *and* $m_A = 1/16$*; these are the values of the expected externality for* $\hat{r}_{A1} = 2$ *and* $\hat{r}_{A2} = 1/2$ *respectively. By symmetry,* $M_B = 7/8$ *and* $M_B = 1/16$. *We conclude that it is possible to use the EEM if* $8s_2 \geq 13s_1$.

Proposition 3. *Suppose that money is not available as a medium of exchange and* $\max\{M_A - m_B, M_B - m_A\} \leq s_n/2$. *Then, the EEM satisfies ex-post efficiency and Bayesian incentive compatibility.*

Proof. Suppose that agents report their valuations truthfully. Then all goods j for which $v_{Aj}/v_{Bj} > v_{An}/v_{Bn}$ are allocated to agent A in their entirety; all goods j for which $v_{Aj}/v_{Bj} < v_{An}/v_{Bn}$ are allocated to agent B in their entirety; and all goods j for which $v_{Aj}/v_{Bj} = v_{An}/v_{Bn}$ (including good n) are split between the two agents. Such allocations are ex-post efficient.

We now show that the EEM satisfies Bayesian incentive compatibility. Given his valuations v_A, agent A will report the valuations $\hat{v}_A$, or equivalently the ratios $\hat{r}_A$, that maximize his expected utility assuming that agent B is reporting his valuations truthfully. In particular, agent A aims to maximize

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\begin{aligned}
&E_{r_B}\left[\sum_{j=1}^{n} v_{Aj}\cdot q_{Aj}(\hat{v}_A, v_B)\right] = \\
&\sum_{j=1}^{n-1} v_{Aj}\int_0^{\hat{r}_{Aj}} s_j\left(1 - 1_{\{x=\hat{r}_{Aj}\}}/2\right) dG_{Bj}(x) + v_{An}E_{r_B}[q_{An}(\hat{v}_A, v_B)] = \\
&\sum_{j=1}^{n-1} s_j \int_0^{\hat{r}_{Aj}} \left(1 - 1_{\{x=\hat{r}_{Aj}\}}/2\right)(v_{Aj} - v_{An}x) dG_{Bj}(x) \\
&+ v_{An}\left(E_{r_B}\left[\sum_{j=1}^{n-1}\int_0^{r_{Bj}} s_j\left(1 - 1_{\{x=r_{Bj}\}}/2\right) x dG_{Aj}(x)\right] + s_n/2\right) v_{An}
\end{aligned}
$$

This is increasing in $\hat{r}_{Aj}$ when $\hat{r}_{Aj} < v_{Aj}/v_{An}$ and decreasing in $\hat{r}_{Aj}$ when $\hat{r}_{Aj} > v_{Aj}/v_{An}$. We conclude that it is optimal for agent A to report his valuation ratios r_{Aj} truthfully, if agent B is reporting his valuations truthfully. Similarily, truthful reporting is optimal for agent B, if agent A is truthful. We conclude that the EEM satisfies Bayesian incentive compatibility. □

The following example illustrates how the EEM works.

Example 2. *Suppose that* $n = 2$ *and for each agent* i, $v_i = (1, 10)$ *with probability 1/2 and* $v_i = (4, 8)$ *with probability 1/2. We choose good 2 as the numeraire*

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and x_{Bn} such that $x_{An} \geq M_A - m_B$, $x_{Bn} \geq M_B - m_A$ and $x_{An} + x_{Bn} = s_n$. Then, in (1) we would replace $s_n/2$ by x_{An}; we would also replace $s_n/2$ by x_{Bn} in the corresponding equation for agent B.

good; therefore, the mechanism will use the ratios $\hat{v}_{i1}/\hat{v}_{i2}$ of the reported valuations for $i \in \{A, B\}$. Each agent's true ratio is uniformly distributed on $\{1/10, 1/2\}$. The expected externality that an agent imposes to the other agent by reporting 1/10 (resp. 1/2) is equal to 1/40 (resp. 9/40) of good 2. This implies that $M_A = M_B = 9/40$ and $m_A = m_B = 1/40$, so the condition of Proposition 3 is satisfied. If agent A reports 1/2 and agent B reports 1/10, then $q_A = (1, 3/10)$ and $q_B = (0, 7/10)$. If both agents report the same ratio, then $q_A = q_B = (1/2, 1/2)$. In equilibrium, the agents report their valuation ratios truthfully and the resulting allocation is ex-post efficient. Notice that the allocation is not ex-ante efficient; for instance, ex-ante agent A would be better off if (i) $q_A = (0, 1)$ when $v_A = (1, 10)$ and $v_B = (4, 8)$ and (ii) $q_A = (1, 0)$ when $v_A = (4, 8)$ and $v_B = (1, 10)$.

5. Conclusion

We study the problem of allocating n goods to two agents whose cardinal preferences are private information. If money is available as a medium of exchange, Bayesian incentive compatibility and ex-ante efficiency can be achieved, thus implying ex-post efficiency.

If money is not available as a medium of exchange, Bayesian incentive compatibility and ex-post efficiency are achievable using a variation of the Expected Externality Mechanism under certain reasonable conditions. That mechanism uses one of the goods as a numeraire good instead of money. Each agent is allocated the goods that he values more than the numeraire good; as a result, there are no mutually beneficial trades available and ex-post efficiency is achieved. However, each good is not necessarily allocated to the agent who values it the most; in other words, ex-ante efficiency is not achieved.

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