The Sequential Belief Representation of Harsanyi Type Spaces with Redundancy

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July 9, 2009

Abstract

In the epistemic analysis of games, the existence of *redundant types*, which are Harsanyi types which represent the same sequential beliefs over the basic uncertainty, has been an obstacle. To resolve the redundancy of types, we adopt an extended belief space. We consider sequential beliefs over not only the basic uncertainty but also a newly added *exogenous* payoff irrelevant parameter space C. We show that, when $C = \{0, 1\}$ or it has larger cardinality, any Harsanyi type space, even if it has redundant types, can be isomorphically embedded into the extended sequential belief space. Based on this result, we also show that there exists a *universal type space* where we can uniquely embed any Harsanyi type space. Finally, as an application, we define the *intrinsic correlation* by Brandenburger-Friedenberg [8] in terms of redundancy, and show that our result can be applied to obtain the same result as theirs.

1 Introduction

One difficulty in dealing with games of incomplete information is the infinite regress of uncertainty. Typically, an agent is uncertain about the payoff functions of the other agents. [1] In order to analyze an agent's decision under incomplete information, it is not enough to incorporate his belief over the basic uncertainty, that is, the uncertainty about the agents' payoffs. We

[1]The agents' uncertainty about action spaces can be represented as the uncertainty about payoff functions. See Hu-Stuart [21] for the details

have to incorporate what the agent believes about what his opponents believe about the basic uncertainty too. And next we have to consider the agent's belief about what his opponents believe about what he believes about the basic uncertainty, and so on *ad infinitum*. Therefore, to deal with games of incomplete information, we have to model this infinite regress of beliefs about beliefs, which we call *sequential beliefs*.

Since Harsanyi [18], we have been dealing with this difficulty by using the notion of *type* and the associated Bayesian game. We postulate that all the informational attributes of agents, including the sequential beliefs, can be reduced to one variable called the agent's "type". This postulation allows us to apply equilibrium concepts of games of complete information to games of incomplete information. In this paper, we say that the types defined by Harsanyi are *Harsanyi types* in order to distinguish them from *epistemic types* which we will define later.

Concerning individual informational attributes, we can conceive the information brought by private signals, predetermined personal conjectures (ex. personal characters, or habits in thinking), and so on. We can easily model these attributes with parameters. However, it is not clear that Harsanyi types correctly reflect the agents' sequential beliefs. This suspicion is cleared by Mertens-Zamir [24] and Brandenburger-Dekel [7]. They showed that, under reasonable conditions, the space of the sequential beliefs over the basic uncertainty forms a Harsanyi type space, and we can embed arbitrary Harsanyi type spaces into the space of the sequential beliefs. We say that this space of the sequential beliefs is the *universal type space* and sequential beliefs are *epistemic types*.

Still we have another difficulty about the sequential beliefs and Harsanyi types. Indeed Mertens-Zamir and Brandenburger-Dekel verified that we can represent sequential beliefs as Harsanyi types. However, there exists a difficulty with the specification of the model as shown in Ely-Peski [15]. The following example shows that several different Harsanyi type spaces represent the same sequential beliefs.

Example 1 (Ely-Peski (2006)): Consider the following two Harsanyi type spaces.

Type space A: The payoff parameter space is $S = \{-1, 1\}$, the set of agents is $N = \{1, 2\}$, the set of types is $T_i = \{-1, 1\}$ for $i = 1, 2$, and the belief structure is characterized by a common prior $\mu \in \Delta(S \times T)$ such that

$$\mu(s, t_i, t_{-i}) = \begin{cases} \frac{1}{4} & if\ s = t_i \cdot t_{-i} \\ 0 & otherwise \end{cases}$$

Let $h_i^k(t_i)$ be the kth order belief of the agent i associated with his type t_i. We can derive the sequential beliefs over S in the above structure as follows;

$$\begin{aligned} h_i^1(-1)[s] &= \begin{cases} \frac{1}{2} & \text{if } s = -1 \\ \frac{1}{2} & \text{if } s = -1 \end{cases} \\ h_i^2(-1)[s] &= \begin{cases} \frac{1}{2}h_j^1(-1)[-1] + \frac{1}{2}h_j^1(1)[-1] = \frac{1}{2} & \text{if } s = -1 \\ \frac{1}{2}h_j^1(-1)[1] + \frac{1}{2}h_j^1(1)[1] = \frac{1}{2} & \text{if } s = 1 \end{cases} \\ h_i^3(-1)[s] &= \begin{cases} \frac{1}{2}h_j^2(-1)[-1] + \frac{1}{2}h_j^2(1)[-1] = \frac{1}{2} & \text{if } s = -1 \\ \frac{1}{2}h_j^2(-1)[1] + \frac{1}{2}h_j^2(1)[1] = \frac{1}{2} & \text{if } s = 1 \end{cases} \\ &\vdots \end{aligned}$$

The resulting sequential belief of $t_i = -1$ is $\frac{1}{2}$ at each order to the infinite for $i = 1, 2$. In the same way, $h_i(1)$ is $\frac{1}{2}$ at each order for $i = 1, 2$.

Type space B: The payoff parameter is $S = \{-1, 1\}$, the set of agents is $N = \{1, 2\}$, the set of types is $T_i = \{0\}$ for $i = 1, 2$, and the belief structure is characterized by a common prior $\mu \in \Delta(S \times T)$ such that

$$\mu(s, 0, 0) = \begin{cases} \frac{1}{2} & \text{if } s = -1 \\ \frac{1}{2} & \text{if } s = 1 \end{cases}$$

In this case, both agents put probability $\frac{1}{2}$ on each element of S, and this is common knowledge between the agents. Therefore the resulting sequential belief of the type is $\frac{1}{2}, \frac{1}{2}, \ldots$ for $i = 1, 2$. Type space A and type space B have different type structures, but they result in the same sequential beliefs. It means that the representation of a sequential belief using a Harsanyi type is not unique.

Clearly, the type space A and the type space B in the example have different informational structures.[2] According to Mertens-Zamir, Harsanyi types

[2] Ely-Peski [15] showed that they have different sets of Bayesian equilibrium and rationalizable strategies.

which lead to the same sequential beliefs are called *redundant types*. In the example, the types $t_i = -1$ and $t_i' = 1$ in the type space A are redundant types. The existence of redundant types shows the difficulty in modeling games of incomplete information. But that is not all. The example also implies that the sequential belief over the payoff parameter is not enough to characterize the belief structure of agents. The universal type space does not allow redundancy of types. However, without redundant types, we cannot deal with an interesting class of games such as the type space A. In the type space A, redundancy happens due to the strong correlation of the agents' belief over the payoff parameter and their belief over the other agent's types. Such correlation is common in applications. In Morris-Shin (1996), the investors share the market information, such as the GDP report and personnel affairs in firms, with some private noises. In their model, the private signals are independent. But, if those private noises are correlated and every agent knows it, then we need redundant types to model it.

Example 2: Correlated public information with noise Let $N = \{1, \cdots, n\}$. There are two states $S = \{G, B\}$. The agents receive private signals X_i about the states from the government. The government tries to hide the state when the state is bad, but it cannot be completely hidden, because there is one agent that receives the true signal. Likewise, when the state is good, the government tries to make it public, but it cannot do so because one agents receives a wrong signal. The distribution of the signals and the states are given by a common prior μ such that

$$\forall i \in N,\ \mu(X_1 = G,\ \cdots,\ X_i = B,\ \cdots,\ X_n = G|s = G) = \frac{1}{n},$$

$$\text{Otherwise,}\ \ \mu(\ .\ |s = G) = 0.$$

$$\forall i \in N,\ \mu(X_1 = G,\ \cdots,\ X_i = B,\ \cdots,\ X_n = G|s = B) = \frac{1}{n},$$

$$\text{Otherwise,}\ \ \mu(\ .\ |s = B) = 0.$$

Then, each type X_i assigns probability $\frac{1}{2}$ to both states. Therefore, the resulting sequential belief is $\frac{1}{2}, \cdots$ at each type.

Ely-Peski (2006) constructed a different kind of sequential beliefs called Δ-*hierarchies*. That is, sequential beliefs over the space of probability distributions over the space of parameters. By using beliefs over beliefs as the first order belief, we can deal with the correlation between beliefs over the payoff parameter and beliefs over the other agents' types. And, in particular, some types that would be called redundant under standard sequential beliefs are mapped to different Δ-hierarchies. Δ-hierarchies can represent richer information about the belief structure of the agents than ordinary epistemic types, and give us a better foundation to work on the epistemic analysis of games. In Δ-hierarchies, however, we can only distinguish redundant types up to rationalizable actions. Harsanyi types which have different sets of rationalizable actions result in different Δ-hierarchies, but Harsanyi types which share the same set of rationalizable actions result in the same Δ-hierarchy. In the above example, the types $t_i = 1$ in the type space A and $t_i = 0$ in the type space B can be distinguished from each other in Δ-hierarchies, but $t_i = 1$ and $t'_i = -1$ in the type space A cannot be distinguished there. Therefore we cannot always map Harsanyi type spaces into the space of the Δ-hierarchies isomorphically.

Liu (2006) took a different approach from Ely-Peski. He augmented the universal type space by adding an additional parameter space, which he called the *payoff irrelevant parameter space*. He showed that any Harsanyi type space, even if it has redundant types, has its isomorphic image in the space of the sequential beliefs over the payoff parameter S and the payoff irrelevant parameter C. However, the payoff irrelevant parameter space that Liu used was the agents' type space T. Therefore the resulting epistemic types space vary depending on Harsanyi type spaces to be studied. Since we cannot compare Harsanyi type spaces on one epistemic space, topological arguments such as Fudenberg-Dekel-Morris [12] and Ely-Peski [16] are not possible here. In this sense, the space that Liu constructed is different from the universal type space that Mertens-Zamir and Brandenburger-Dekel did. Besides, from the philosophical perspective, we cannot obtain any insight into what kind of information beyond the universal type space is needed to deal with the redundancy of types.

In this paper, we offer a solution by adopting an *exogenous* payoff irrelevant parameter space. Moreover we show that it is enough that the exogenous parameter space has two elements. To get an intuition of our argument, consider a two person game. Let us make on the agents' Harsanyi type spaces a partition of equivalence classes whose elements have the same sequential belief over the payoff parameter. Equivalently, we sort Harsanyi types into classes of redundant types. In type spaces with redundant types, the beliefs of redundant types have the same probability distribution over the equivalence classes of the other agent's type space although they are different within each equivalence class. This means that even if redundant types have different conjectures over the payoff parameter and the other agents's type, they are different just within each equivalence class of the other agent, not across equivalence classes. Since the members of each equivalence class of the other agent's types cannot be distinguished by their sequential beliefs, the agent's redundant types also result in the same sequential belief. Our method to deal with redundancy is to distinguish the members of each equivalence class by attaching to each type of an agent a different conjecture over a newly added payoff irrelevant parameter. As a result, those redundant types have different first order beliefs over the payoff parameter and the payoff irrelevant parameter. It enables us to distinguish the redundant types of the other agent by their second order beliefs because we can distinguish their different conjectures within each equivalence class. We have a further result when we assume that S and T_i, for all $i \in N$, are countable or uncountable Polish. Then it is sufficient to distinguish the redundant types if the payoff irrelevant parameter space has two elements. Here is a brief explanation of it. All spaces are infinite and Polish, and so the type space of the agent 1 is Borel equivalent to the closed interval $[0, 1]$. On the other hand, the space of probability measures over $\{0, 1\}$ is homeomorphic to $[0, 1]$. Thus we can assign Borel equivalent different first order beliefs over the set $\{0, 1\}$ to all the types of the player 1. By doing this, we can distinguish the members of each equivalence class of the player 1's redundant types by their first order belief, and so the player 2's redundant types are distinguishable by their second order belief whenever they have different conjectures over the payoff parameter and the player 1's type space.

This result is strong for two reasons. First, we can completely represent any Harsanyi type space as a subspace of the sequential beliefs over $S \times \{0, 1\}$. One interpretation is that any correlation of the beliefs of agents which is not captured by the sequential belief over the payoff parameter can be recovered just by introducing a coin flip as a moderator across agents. Here is another interpretation. The hidden information behind the sequential belief over the payoff parameter is just the uncertainty about one agent's personality. For example, whether he believes in God or not. The sequential conjecture over an agent's personality generates the correlation of beliefs over the payoff parameter and agents' types. Second, the payoff irrelevant parameter $\{0,1\}$ is exogenous and we do not have to change the payoff irrelevant parameter as Liu did. It enables us to make topological arguments of types on the space of the sequential beliefs over $S \times \{0,1\}$.

Based on this exogenous parameter space C, we make another theoretical contribution to an argument of the existence of the universal type space. As we explained, we can embed Harsanyi type spaces isomorphically into the space of the sequential belief over $S \times C$, $U(S \times C)$. However $U(S \times C)$ is larger than needed. In this space, a Harsanyi type can have several isomorphic images. According to Mertens-Zamir [24] and Heifetz-Samet [20], the unique embedding constitutes a part of the universal type space. Therefore, we want to find a space where we can uniquely embed Harsanyi types. In fact it exists. We will show that we can pick up a subspace of $U(S \times C)$ so that we can embed Harsanyi type spaces uniquely. Mertens-Zamir showed the existence in the case of non-redundancy, and Heifetz-Samet showed it in the topology-free case. Our finding is a generalization of both works above and can be viewed as a final remark in the literature on the existence of the universal type space.

In the final section, we apply our main result to the argument of the *intrinsic correlation* which was initiated by Brandenburger-Friedenberg [8]. They showed that the set of the correlated rationalizable actions are strictly larger than the set of the intrinsic correlated rationalizable actions which only allow the correlation across beliefs but not the direct correlation of actions. Aumann [3] and Brandenburger-Dekel [6] showed that any correlated rationalizable actions can be realized as an equilibrium of some Bayesian game. We show that the inequality of these rationalizability occurs due to the redundancy of types in the Bayesian game. Therefore, by applying our main result, we can also show that the equality between the intrinsic and the usual notion of correlated rationalizability is achieved by adding a parameter $\{0,1\}$ to the basic uncertainty. It is the same result shown by Brandenburger-Friedenberg [8] in a different way.

This paper is organized as follows. Section 2 is an informal explanation of our proof using a countable example. In Section 3, we present the formal model and the proof of our main result: the elimination of redundancy by adding the payoff irrelevant parameter space $C = \{0,1\}$. In Section 4, we show the existence of the universal type space. In Section 5, we discuss the intrinsic correlation in terms of redundant types. In the Appendix, we give the detailed proofs about measurability.

2 Preliminaries

Let X be an arbitrary set. We use $\Delta(X)$ to denote the space of the probability measures over X. When X is equipped with a topology, we use $\Sigma(X)$ to denote the Borel σ-algebra on X.

Let N be a finite set, and $(Y_i)_{i\in N}$ be a family of sets. Then, for any $i \in N$, we use Y_{-i} to denote the product space $\Pi_{j\in N\setminus\{i\}}Y_j$.

2.1 Harsanyi type space

We consider a finite set of agents $N = \{1, \dots n\}$. All the agents face the same basic uncertainty about their payoffs. It can be represented by a parameter space S.[3] We call this S the *payoff parameter space*. A *Harsanyi type space* is a sequence $\langle S,\ (T_i)_{i\in N},\ (\lambda_i)_{i\in N}\rangle$, where, for each $i \in N$, λ_i is a function from T_i to $\Delta(S \times T_{-i})$. We call each element in T_i a *Harsanyi type*. By the function λ_i, each type stands for a belief over the payoff parameter and the other players' types. Hereafter we make some assumptions on Harsanyi type spaces.

Assumption 1: The parameter space S and the each agent's type space T_i are uncountable Polish spaces.

Let $T \equiv \Pi_{i\in N}T_i$. Then, as it is known, the product type space T is also a Polish space.

In many works such as Mertens-Zamir [24] *etc.*, the belief mapping λ_i is assumed to be homeomorphism. Here we relax this usual assumption slightly.

Definition: A function $f : X \to Y$ is bimeasurable if f is measurable and , for each measurable set $E \subset X$, $f(E)$ is also measurable.

[3]See Mertens-Zamir [24], and Hu-Stuart [21].

Assumption 2: For each $i \in N$, the function λ_i is a *bimeasurable* injection.

This assumption also precludes *purely redundant* types, which are Harsanyi types $t_i, t'_i \in T_i$ such that $t_i \neq t'_i$ and $\lambda_i(t_i) = \lambda_i(t'_i)$.

2.2 Universal type spaces

The universal type space was introduced by Mertens-Zamir[24]. It is the space of the sequential beliefs over S which satisfy some coherency conditions. They showed that the space is also a Harsanyi type space and any Harsanyi type spaces is embedded there. To define the universal type space, we have to define the space of the sequential beliefs first. Let a family of spaces $(Z^k)_{k \in \{1, \cdots\}}$ be such that

$$Z^1 \equiv S$$
$$For\ k > 1,\ Z^k \equiv Z^{k-1} \times \Delta(Z^{k-1}).$$

The space Z^k is the set of the kth order belief over S. We say that $\Pi_{k=1}^{\infty} Z^k$ is the *sequential belief space* and each element of it is the *sequential belief*. Under the coherency of beliefs, we can consider each element in $\Pi_{k=1}^{\infty} Z^k$ as its projective limit. Let the set of the projective limits be Z^{∞}. We say that each $e_i \in Z^{\infty}$ is an *epistemic type*. The *universal type space* is the subset of the product space $\Pi_{i \in N} Z^{\infty}$ which satisfies the coherency of beliefs. We denote it as $U(S)$. Mertens-Zamir showed the next strong theorem about the universal type space.

Theorem 2.1. *(Mertens-Zamir [24]) The universal type space $U(S)$ and its associated natural homeomorphism constitutes a Harsanyi type space.*

Then we can define the function which maps Harsanyi types onto the sequential belief space. Let the first order mapping $h_i^1 : T_i \to \Delta(S)$ be such that

$$h_i^1(t_i) = Marg_{(S)}\lambda_i(t_i).$$

For $k > 1$, let the kth order mapping $h_i^k : T_i \to \Delta(Z^k)$ be such that

$$h_i^k(t_i) = \lambda_i(t_i) \circ [Id_S, (h_j^{k-1})_{j \in N \setminus \{i\}}]^{-1},$$

where Id_S is an identical function from S to S.

We say that the function $(h_i^k)_{k=1}^{\infty} : T_i \to \Pi_{k=1}^{\infty} Z^k$ is the *hierarchy mapping.* Let $h \equiv (h_i)_{i \in N}$. Then, this h enables us to map any Harsanyi type space to the sequential belief space. Also you can see that these sequential beliefs derived in this way satisfy the coherency condition.

3 An extended sequential belief space

In this section, we extend the universal type space by adding another parameter space C which is irrelevant to payoffs. And we show that we can isomorphically embed Harsanyi type spaces there even if they have *redundant types.*

3.1 Redundant types

Let $\Lambda = \langle S, T, (\lambda)_{i \in N} \rangle$ be a Harsanyi type space. Mertens-Zamir showed that Harsanyi type spaces can be embedded as a subspace of $U(S)$ homeomorphically only if they have redundant types. To discuss the matter, we have to define redundant types first.

Definition: In a Harsanyi type space Λ, two Harsanyi types t_i and $t'_i \in T_i$ are *redundant* if $h_i(t_i) = h_i(t'_i)$.

We say that the Harsanyi types which are not redundant are *non-redundant types.* Then we can formally state what Mertens-Zamir showed.

Proposition 3.1. *(Mertens-Zamir [24]) Any Harsanyi type space without redundant types can be embedded onto $U(S)$ homeomorphically. And the hierarchy mapping h is the unique embedding.*

3.2 Extension with a payoff irrelevant parameter space

Now we construct an extended space of sequential beliefs so that we can embed Harsanyi type spaces there even if they have redundant types. We introduce a parameter space $C = \{0, 1\}$ and consider the sequential belief space over $S \times C$ instead of S. In the rest of this section, we assume that $N = \{1, 2\}$.

Let $C \equiv \{0, 1\}$. We assume that any element does not affect the payoffs of the agents. Therefore we call C the *payoff irrelevant parameter space*. We define sequential beliefs over $S \times C$ and construct the coherent sequential belief space over $U(S \times C)$ in the same way as we did over S.

Let

$$Z_1 \equiv S \times C,$$
$$\forall k \geq 2,\ Z_k \equiv \Delta(\Pi_{n=1}^{n=k-1} Z_n).$$

And let

$$\begin{aligned} H^k(S \times C) &\equiv \Delta(\Pi_{n=1}^{n=k} Z_k) \\ &= Z_{k+1}. \end{aligned}$$

and

$$\begin{aligned} H(S \times C) &\equiv \Pi_{k=1}^{k=\infty} H^k(S \times C) \\ &= \Pi_{k=1}^{k=\infty} \Delta(Z_k). \end{aligned}$$

For each k, $H^k(S \times C)$ is the set of the kth order belief over $S \times C$. Let $U(S \times C) \subset \Pi_{i \in N} H(S \times C)$ be the product space of the coherent sequential beliefs.

We also define Harsanyi type spaces based on $S \times C$ by the sequence $\Phi = \langle S \times C, V, (\phi_i)_{i \in N} \rangle$ where ϕ_i is a bimeasurable injection from V_i to $\Delta(S \times C \times V_{-i})$.

Before we embed a Harsanyi type space onto $U(S \times C)$, we extend it to a Harsanyi type space on $S \times C$. To do that, we should clarify what is "isomorphism" between Harsanyi type spaces.

Definition (Liu [22]) : Let $X = \langle S, T, \lambda \rangle$ and $Y = \langle S \times C, V, \phi \rangle$ be Harsanyi type spaces on S and $S \times C$ respectively. Then, X and Y are *S-isomorphic* to each other if there exists a $g = (g_i)_{i \in \{0\} \cup N}$ such that (1) $g_0 : S \to S$ is an identity function, (2) $g_i : T_i \to V_i$ is Borel equivalence for all $i \in N$, and (3) $Marg_{S \times V} \phi_i(v_i) = \lambda_i(t_i) \circ g^{-1} \circ Proj_{S \times V}$.

Hereafter, when Harsanyi type spaces X and Y are S-isomorphic, we use $X \sim_S Y$. And when both spaces are defined on S, we use $X \sim Y$.

Next, we want to construct a Harsanyi type space on $S \times C$ which is S-isomorphic to the original type space on S. For the construction, we need the next well-known theorem.[4]

Theorem 3.2. *Let X be an uncountable Polish space. Then X is Borel equivalent to the closed interval* $[0, 1]$.

Let $\Lambda \equiv \langle S, (T_i)_{i \in \{1,2\}}, (\lambda_i)_{i \in \{1,2\}} \rangle$ be a Harsanyi type space. Since T_i is an uncountable Polish space, there exists a Borel equivalence from T_i to $[0, 1]$. Let this equivalence be $p_i : T_i \to [0, 1]$. Using p_i, we define a Harsanyi type space $\Phi = \langle S \times C, (V_i)_{i \in \{1,2\}}, (\phi_i)_{i \in N} \rangle$ so that

1. For all $i \in \{1, 2\}$, $V_i = [0, 1]$.

2. For all $i \in \{1, 2\}$, $\phi_i : V_i \to \Delta(S \times C \times V_{-i})$ satisfies the next property;

   For the agent 1,

[4]See Royden [28] for the detailed argument.

$$\forall v_1 \in V_1, \quad Marg_{(S\times V_2)}\phi_1(v_1) = \lambda_1(p_1^{-1}(v_1)) \circ [id_S,\ p_2]^{-1},$$
$$\forall E \in \Sigma(S\times V_2), \quad \phi_1(v_1)[E\times\{0\}] = v_1\lambda_1(p_1^{-1}(v_1)) \circ [id_S,\ p_2]^{-1}[E].$$

For the agent 2,

$$\forall v_2 \in V_2, \quad Marg_{(S\times V_1)}\phi_2[v_2] = \lambda_2[p_2^{-1}(v_2)] \circ [id_S,\ p_1]^{-1},$$
$$Marg_{(C)}\phi_2(\{0\}) = 1.$$

The bimeasurability of $(\phi_i)_{i\in\{1,2\}}$ is given in the appendix. Then, you can see that Φ is a well defined Harsanyi type space. Concerning this Harsanyi type space Φ, we have the next fundamental lemma.

Lemma 3.3. *The above type space Φ is* S-isomorphic *to Λ.*

Proof. Let $Id_S : S \to S$ be identity function. Then, (Id_S, p_1, p_2) is S-isomorphism from Λ to Φ by construction. □

3.3 S-isomorphic embedding onto $U(S\times C)$

We go to the main part of this paper. We show that, in the Harsanyi type space Φ defined above, all elements of V_i correspond to different sequential beliefs over $S\times C$.

Theorem 3.4. *Let Λ and Φ be Harsanyi type spaces defined above. Then, for each $i\in\{1,2\}$, the agent i's hierarchy mapping induced by Φ, $h_i : V_i \to H(S\times C)$, is injection.*

Proof. Let $h_i^k : V_i \to H_i^k(S\times C)$ be the agent i's kth order belief mapping on $S\times C$ induced by Φ, and let $g_i^k : T_i \to H_i^k(S)$ be the agent i's kth order belief mapping onto S induced by Λ.

(Step 1: For the agent 1)

Let $v_1, v_1' \in V_i$ be such that $v_1 \neq v_1'$. His first order belief of v_1 is

$$\begin{aligned}\forall E \in \Sigma(S),\ h_1^1(v_1)[E \times \{0\}] &= \phi_1(v_1)[E \times \{0\} \times V_2] \\ &= v_1 \lambda_1(p_1^{-1}(v_1)) \circ [id_S,\ p_2]^{-1}[E \times V_2] \\ &= v_1 g_1^1(p_1^{-1}(v_1))[E].\end{aligned}$$

By the symmetric argument,

$$\forall E \in \Sigma(S),\ h_1^1(v_1')[E \times \{0\}] = v_1' g_1^1(p_1^{-1}(v_1'))[E].$$

(Case 1:) Suppose that $v_1 g_1^1[p_1^{-1}(v_1)](E) = v_1' g_1^1[p_1^{-1}(v_1')](E)$. Then, since $v_1 \neq v_1'$, $g_1^1[p_1^{-1}(v_1)](E) \neq g_1^1[p_1^{-1}(v_1')](E)$.

On the other hand,

$$\begin{aligned}h_1^1(v_1)[E \times C] &= \phi_1(v_1)[E \times C \times V_2] \\ &= Marg_{(S \times V_2)} \phi_1(v_1)[E \times V_2] \\ &= \lambda_1(p_1^{-1}(v_1)) \circ [id_S,\ p_2]^{-1}[E \times V_2] \\ &= g_1^1(p_1^{-1}(v_1))[E].\end{aligned}$$

From these results, we have

$$\begin{aligned}h_1^1(v_1)[E \times \{1\}] &= h_1^1(v_1)[E \times C] - h_1^1(v_1)[E \times \{0\}] \\ &= g_1^1(p_1^{-1}(v_1))[E] - v_1 g_1^1(p_1^{-1}(v_1))[E] \\ &= g_1^1(p_1^{-1}(v_1))[E] - v_1' g_1^1(p_1^{-1}(v_1'))[E] \\ &\neq g_1^1(p_1^{-1}(v_1'))[E] - v_1' g_1^1(p_1^{-1}(v_1'))[E] \\ &= h_1^1(v_1')[E \times \{1\}].\end{aligned}$$

Thus h_1 is injection.

(Case 2:) Suppose that $v_1 g_1^1[p_1^{-1}(v_1)](E) \neq v_1' g_1^1[p_1^{-1}(v_1')](E)$. It means that $h_1^1(v_1)[E \times \{0\}] \neq h_1^1(v_1')[E \times \{0\}]$. Thus h_1 is injection.

(Step 2: For the agent 2)
Let $v_2, v_2' \in V_2$ be such that $v_2 \neq v_2'$. Concerning his first order belief, by construction,

$$\begin{aligned} &\forall E \in \Sigma(S),\\ &\quad h_2^1(v_2)[E \times \{0\}] = g_2^1(p_2^{-1}(v_2))[E].\\ &\quad h_2^1(v_2)[E \times \{1\}] = 0. \end{aligned}$$

Let, for each $\mu_1 \in \Delta(S \times C)$, $h_1^{-1}(\mu_1) \equiv \{v_1 \in V_1 : h_1^1(v_1) = \mu_1\}$. As we have shown, the function $h_1^1 : V_1 \to \Delta(S \times C)$ is injection. Therefore $h_1^{-1} : h_1^1(V_1) \to V_1$ is the well defined inverse bijection.

Then we can derive the agent 2's second order belief over $S \times C$.[5]

Note that

$$\begin{aligned} &\forall E \in \Sigma(S),\ \forall Q \in \Sigma(\Delta(S \times C))\\ &\qquad h_2^2[v_2](E \times \{0\} \times Q) = \phi_2(v_2)[E \times \{0\} \times h_1^{-1}(Q)]\\ &\qquad\qquad = \lambda_2(p_2^{-1}(v_2)) \circ [id_S,\ p_1]^{-1}[E \times h_1^{-1}(Q)]. \end{aligned}$$

Since $\lambda_2 : T_2 \to \Delta(S \times T_1)$ is a bimeasurable injection, $\lambda_2(p_2^{-1}(v_2)) \neq \lambda_2(p_2^{-1}(v_2'))$. By Dynkin's lemma[6], there exists a rectangle $F \equiv \hat{S} \times \hat{T}_1$ such that $\hat{S} \in \Sigma(S)$, $\hat{T}_1 \in \Sigma(T_1)$, and $\lambda_2(p_2^{-1}(v_2))[F] \neq \lambda_2(p_2^{-1}(v_2'))[F]$. Let $\hat{V}_1 \equiv p_1(\hat{T}_1)$. Then, $\hat{V}_1 \in \Sigma(V_1)$ and $h_1^1(\hat{V}_1) \in \Sigma(\Delta(S \times C))$. Therefore

$$\begin{aligned} h_2^2(v_2)(\hat{S} \times \{0\} \times h_1^1(\hat{V}_1)) &= \phi_2(v_2)[\hat{S} \times \{0\} \times h_1^{-1}(h_1^1(\hat{V}_1))]\\ &= \phi_2(v_2)[\hat{S} \times \{0\} \times \hat{V}_1]\\ &= \lambda_2[p_2^{-1}(v_2)] \circ [id_S, p_1^{-1}](\hat{S} \times \hat{V}_1)\\ &= \lambda_2(p_2^{-1}(v_2))(\hat{S} \times \hat{T}_1) = \lambda_2(p_2^{-1}(v_2))[F]\\ &\neq \lambda_2(p_2^{-1}(v_2'))[F]\\ &= h_2^2(v_2')[\hat{S} \times \{0\} \times h_1^1(\hat{V}_1)]. \end{aligned}$$

It means that $h_2(v_2) \neq h_2(v_2')$. Therefore, h_2 is injection. □

[5]Concerning the bimeasurablity of h_1^1, see appendix.
[6]See Theorem 10-10 in [1]

So far we did not consider topological structures of Harsanyi type spaces except that they are Polish. As it plays a crucial role in Weistein-Yildiz [29], *etc*, it is widely considered to be nice if each agent's type space V_i is homeomorphic to the belief space $\Delta(S \times C \times V_{-i})$.

Definition : A Harsanyi type space $\mathcal{X} = \langle X, T, (x_i)_{i \in N} \rangle$ is a *continuous type space* if, for all $i \in N$, $x_i : T_i \to \Delta(X \times T_{-i})$ is homeomorphic embedding.

Next we show that the embedded image on $U(S \times C)$ of each Harsanyi type by the hierarchy mapping is a continuous Harsanyi type space.

Lemma 3.5. *Let Φ be a type space and $H(S \times C)$ be the space of sequential belief over $S \times C$ as we defined before. The function $h_i : V_i \to H(S \times C)$ is the full hierarchy mapping. Now let $f : h_i(V_i) \to \Delta(S \times C \times h_i(V_{-i}))$ be such that $f(h_i(v_i)) \equiv \phi(v_i) \circ [id_{(S \times C)}, h_{-i}]^{-1}$. Then, f is homeomorphism.*

Proof. Since $S \times C$ is a Polish space, there exists a unique homeomorphism $\psi : H(S \times C) \to \Delta(S \times C \times H(S \times C))$ such that, for each $m \in H(S \times C)$, $\psi(m)$ is the Kolmogorov extension of m.[7]. So it is enough to show that, for all $i \in \{1, 2\}$ and $v_i \in V_i$, $f_i(h_i(v_i))$ is the Kolmogorov extension of $h_i(v_i)$

Let $m_i \in h_i(V_i)$. First, for all $E \in \Sigma(S \times C \times H(S \times C))$, by letting $f_i(m_i)(E) \equiv f_i(m_i)[E \bigcap (S \times C \times h_{-i}(V_{-i}))]$, we can extend $f_i(m_i)$ so that $f_i(m_i) \in \Delta(S \times C \times H(S \times C))$. And as we defined before,

$$Z_1 \equiv S \times C,$$
$$\forall k \geq 2,\ Z_k \equiv \Delta(\Pi_{n=1}^{n=k-1} Z_n),$$

$$\begin{aligned} H^k(S \times C) &= \Delta(\Pi_{n=1}^{n=k} Z_k) \\ &= Z_{k+1}, \end{aligned}$$

[7] See Prop 1 and Prop 2 in Brandenburger-Dekel [7]

and

$$
\begin{aligned}
H(S \times C) &= \Pi_{k=1}^{k=\infty} H^k(S \times C) \\
&= \Pi_{k=1}^{k=\infty} \Delta(Z_k).
\end{aligned}
$$

The equations above also implies that

$$H(S \times C) = \Pi_{n=2}^{n=\infty} Z_n.$$

Therefore,

$$S \times C \times H(S \times C) = \Pi_{n=1}^{n=\infty} Z_n.$$

From these equalities, we have $f_i(m_i) \in \Delta(\Pi_{n=1}^{n=\infty} Z_n)$, and $m_i \in H(S \times C) = \Pi_{k=1}^{k=\infty} \Delta(Z_k)$.

To show that $f_i(m_i)$ is the Kolmogorov extension of m_i, it is enough to show that the next property holds:

$$\forall k,\ Marg_{(\Pi_{n=1}^{n=k} Z_n)} f_i(m_i) = Proj_k m_i.$$

Let $E \in \Sigma(\Pi_{n=1}^{n=k} Z_n)$ and $\hat{E} = E \times \Pi_{n=k+1}^{n=\infty} Z_n$. Then,

$$
\begin{aligned}
f(m_i)(\hat{E}) &= \phi[v_i] \circ [id_{(S \times C)}, h_{-i}]^{-1}(\hat{E} \bigcap h_{-i}(V_{-i})) \\
&= \phi(v_i)(E_1 \times \hat{V}_{-i}^k), \\
where\ \ \hat{V}_{-i}^k &= \{v_{-i} \in V_{-i} : h_{-i}^k(v_{-i}) \in \Pi_{n=2}^{n=k} E_n\}.
\end{aligned}
$$

On the other hand, from the kth order belief of the agent i,

$$
\begin{aligned}
\exists v_i \in V_i,\ Proj_k m_i[E] &= h_i^k[v_i](E) \\
&= \phi_i[v_i](E_1 \times \hat{V}_{-i}).
\end{aligned}
$$

This equation means that $f_i(m_i)[\hat{E}] = Proj_k(m_i)(E)$. Consequently, $f_i(m_i)$ is the Kolmogorov extension of m_i. □

As a consequence, we have the next theorem.

Theorem 3.6. *For any Harsanyi type space* $\Lambda = \langle S, T, (\lambda_i)_{i\in\{1,2\}} \rangle$, *there exists a* continuous BL-subspace *in* $U(S \times C)$ *which is S-isomorphic to* Λ.

Proof. Let a Harsanyi type space $\Phi = \langle S \times C, V, \phi \rangle$ be an S-isomorphic extension of Λ, and let $E_i = h_i(V_i)$ for all $i \in N$. Let $\mathcal{E} = \langle S \times C, E, (f_i)_{i\in N} \rangle$, where f_i is defined as in the lemma. Since h_i is bimeasurable injection, $\mathcal{E}$ is S-isomorphic to Φ by construction. As a direct result of the lemma, $\mathcal{E}$ is a continuous Harsanyi type space. $\square$

3.4 Extension to the N-person game

We can extend the above theorems to the N-person game. Let N be the finite set of the agents and $|N| = n$. Consider an N-person Harsanyi type space $\Lambda \equiv \langle S, T, (\lambda_i)_{i\in N} \rangle$ as before. We maintain the same assumptions on S, T, C and λ_i.

We can define an extension of Λ on $S \times C$, $\Phi = \langle S \times C, V, (\phi_i)_{i\in N} \rangle$, as follows. For all $i \in N$, let $p_i : T_i \to [0, 1]$ be a Borel equivalence. Let Φ be such that

$$
\begin{aligned}
&\forall i \in N, \quad V_i = [0, 1]. \\
&\forall i \in N\backslash\{1\},\ \forall v_i \in V_i,\ \forall E \in \Sigma(S \times V_{-i}), \\
&\qquad \phi_i(v_i)[E \times \{0\}] = v_i\{\lambda_i(p_i^{-1}(v_i)) \circ [id_s, p_{-i}^{-1}](E)\}. \\
&\qquad Marg_(S \times V_{-i})\phi_i(v_i) = \lambda_i(p_i^{-1}(v_i)) \circ [id_s, p_{-i}^{-1}].
\end{aligned}
$$

And,

$$
\begin{aligned}
&\forall v_1 \in V_1,\ \forall E \in \Sigma(S \times V_{-1}), \\
&\qquad \phi_1(v_1)[E \times \{0\}] = \lambda_1(p_1^{-1}(v_1)) \circ [id_s, p_{-1}^{-1}](E). \\
&\qquad Marg_(S \times V_{-1})\phi_1(v_1) = \lambda_1(p_1^{-1}(v_1)) \circ [id_s, p_{-1}^{-1}].
\end{aligned}
$$

In the same way as we did above, we can show that Φ is S-isomorphic to Λ and the resulting hierarchy mapping is injection.

4 Universal type space on $S \times C$

In the above sections, we have shown that Harsanyi type spaces can be embedded isomorphically to a continuous type space on $U(S \times C)$ even if they include redundant types. However, the embedding is not unique. For example, we can embed a Harsanyi type space to a different subspace of $U(S \times C)$ by changing the roles of the agent 1 and 2. It means that $U(S \times C)$ is larger than we needs. According to Heifetz-Samet [20], the unique embedding is one of the axioms characterizing the universal type space. Therefore we try to find an appropriate subspace of $U(S \times C)$ which is small enough to embed any Harsanyi type space uniquely.

For this purpose, we deal with a kind of "proper" Harsanyi type spaces. We give the formal definition of the "proper" type space below.

Based on the previous sections, we can focus on the subspaces of $U(S \times C)$. We define a *sub type space* of $U(S \times C)$ formally.

Definition: A Harsanyi type space $\Lambda = \langle S \times C, T, \lambda : S \times C \times T \to \Delta(S \times C \times T) \rangle$ is a *sub type space* of $U(S \times C)$ if (1), for each $i \in N$, $T_i \subset H(S \times C)$, (2) T_i is endowed with the relative Borel σ-algebra, and (3) λ_i is the natural homeomorphism and, for each $t_i \in T_i$, $\lambda_i(t_i)[S \times C \times \Pi_{i \neq j} T_i] = 1$.

Let $\mathcal{T} = \{\Lambda : \Lambda \text{ is a sub type space of } U(S \times C)\}$. Hereafter we use T^{Λ} to represent the product space of the agents' type spaces of an arbitrary $\Lambda \in \mathcal{T}$.

Definition : For any $\Lambda, \Lambda' \in \mathcal{T}$, $\Lambda \supset_H \Lambda'$ if $T^{\Lambda} \supset T^{\Lambda'}$.

Definition : A sub type space $\Lambda \in \mathcal{T}$ is *proper* if there are no *measurable sub-type spaces* $E, E' \subset_H \Lambda$ such that $E \neq E'$ and $E \sim E'$.

We define a class of sub type spaces as follows.

$$\begin{aligned}\mathcal{M} =&\{\Lambda \in \mathcal{T} : h_1^1(t_1) = \mu_1(t_1) \times \tilde{p}(t_1), \text{and } h_2^1(t_2) = \mu_2(t_2) \times \nu\},\\ &\text{where } \mu_i : T_i \to \Delta(S \times T_{-i}), \text{ and } \mu_2 \in \Delta(C) \ s.t.\ \mu_2(\{0\}) = 1,\\ &\text{and } \tilde{p}(t_1) : T_1 \to \Delta(C) \text{ is injection.}\end{aligned}$$

And then,

$$\mathcal{L} \equiv \{\Lambda \in \mathcal{T} : \Lambda \text{ is a proper sub type space.}\} \cap \mathcal{M}$$

We postulate that if $\mathcal{L}$ has a maximal element, it would be a smallest Harsanyi type space where any proper Harsanyi type space has its isomorphic image as a subset. To show the existence of such a maximal element, we apply Zorn's lemma. Let $\mathcal{C} \subset \mathcal{L}$ be a chain. Then we have the next lemma.

Lemma 4.1. *Any chain $\mathcal{C} \subset \mathcal{L}$ have a upper bound in $\mathcal{L}$.*

Proof. Let $\mathcal{C} \subset \mathcal{L}$ be a chain. Let $\Lambda^* \equiv \langle S \times C, T^*, (\lambda_i)_{i \in N} \rangle$, where $T^* = \bigcup_{\Lambda \in \mathcal{C}} T^\Lambda$. We show that Λ^* is an upper bound of $\mathcal{C}$. All we have to show is that Λ^* is proper.

Suppose that Λ^* is not proper. Then there exists at least two measurable sub-type spaces $E \neq E' \subset \Lambda^*$ such that $E \sim E'$. Let the isomorphism be $h : E \to E'$. Since $E \neq E'$, there exists $t_i \in E_i$ such that $t_i \notin E'$. We know that $t_i \in \Lambda^*$ and so there exists $\Lambda \in \mathcal{C}$ such that $t_i \in \Lambda_i$.

Let $E \cap \Lambda \equiv \langle S \times C, T^E \cap T^\Lambda, \lambda \rangle$. By the same way, there exists $\Lambda' \in \mathcal{C}$ such that $h(t_i) \in \Lambda'_i$, and $E' \cap \Lambda'$ is a sub-type space. We define two sub-type spaces such that

$$\begin{aligned}H &\equiv E \cap \Lambda \cap h^{-1}(\Lambda' \cap E'),\\ H' &\equiv E' \cap \Lambda' \cap h(\Lambda \cap E).\end{aligned}$$

Then we have that $H \subset \Lambda$ and $H' \subset \Lambda'$, and they are sub-type spaces. Also we can easily show that $H \sim H'$. Since $t_i \in H$ and $t_i \notin H'$, $H \neq H'$. Since $\mathcal{C}$ is a chain, without loss of generality, $\Lambda \subset \Lambda'$. Then, $H' \subset \Lambda$. It contradicts the fact that Λ is proper. Thus, Λ^* is a proper and an upper bound of $\mathcal{C}$. $\square$

Lemma 4.2. *There exists a maximal Harsanyi type space in $\mathcal{L}$.*

Proof. It is shown by the previous lemma and Zorn's lemma. $\square$

In the next proposition, we state that this maximal element is the universal type space where any Harsanyi type space is uniquely embedded.

Proposition 4.3. *There exists a proper Harsanyi type space M^* such that, for any proper Harsanyi type space $\Lambda \in \mathcal{L}$, M^* has an unique sub-type space which is isomorphic to Λ.*

Note: The following proof is under revision and subject to change.

Proof. Suppose that there is no such proper space. Let M^* be the maximal proper type space derived by the preceding lemma. Then there exists $\Lambda \in \mathcal{L}$ and there is no $E \subset M^*$ such that $E \sim \Lambda$. Let

$$\begin{aligned} \mathcal{R} \equiv \{E \subset M^* : E \text{ is a sub type space s.t. } \exists E' \subset \Lambda \\ \text{and } E' \sim E.\} \\ \mathcal{R}' \equiv \{E' \subset \Lambda : E' \text{ is a sub type space s.t. } \exists E \subset M^* \\ \text{and } E' \sim E.\} \end{aligned}$$

Let $V \equiv \cup_{E \in \mathcal{R}} E$. Since $V \subset M^*$ and M^* is proper, V is also proper and the largest sub type space which has the isomorphic image in Λ. We can pick $\Lambda' \in \mathcal{L}$ such that (1) $\Lambda' \sim \Lambda$, and (2) $\Lambda' \supset V$. By the construction of Λ', $M^* \cup \Lambda'$ is proper. Since V is not isomorphic to Λ, we have $\Lambda' \neq V$. Therefore $M^* \cup \Lambda' \supsetneq M^*$. It contradicts with the fact that M^* is a maximal element in $\mathcal{L}$. $\square$

By the theorem in the previous section and this proposition, we can show the existence of the universal type space over $S \times C$.

Theorem 4.4. *There exists a type space $U^*(S \times C) \subset U(S \times C)$ where any proper Harsanyi type space on S can be S-isomorphically embedded in a unique way.*

5 Application to intrinsic correlation

We have shown that any Harsanyi type space can be mapped isomorphically to a sub-space of $U(S \times C)$. One application of this theorem is the *intrinsic correlation* of beliefs proposed by Brandenburger-Friedenberg [8]. They showed that, in some complete information games, we cannot achieve all correlated rationalizable actions without any external mediator. They also showed that we can achieve all correlated rationalizable actions as intrinsic ones by adding a coin-flip to the basic uncertainty. In fact, their results are closely related to redundant types. In this section, we show the results of Brandenburger-Friedenberg in a different way; using redundant types and our theorems above.

5.1 Bayesian representation of correlated equilibrium

Consider a complete information game. Let $G \equiv \langle (S_i)_{i \in N}, (\pi_i)_{i \in N} \rangle$ be a game, where S_i is the strategy space of the agent i and $\pi_i : S \to \mathbb{R}_+$ is a payoff function. We assume that, for all $i \in N$, S_i is finite.[8] To define the correlated equilibrium of the game G, we introduce the Bayesian framework *a la* Aumann. Let the basic uncertainty space be Ω, the information partition of the agent be $\mathscr{H}_i$, and the interim belief systems be $P(.|\mathscr{H}_i) \in \Delta(\Omega)$. Since S is finite, we can assume that Ω is finite without loss of generality.

Definition (Aumann [2]): For all $i \in N$, Let $f_i : \Omega \to S_i$ be measurable with regard to $\mathscr{H}_i$. Then $f \equiv (f_i)_{i \in N}$ is an *a posteriori equilibrium*

[8] This is the same assumption as Brandenburger-Friedenberg.

iff

$\forall i \in N$, $\forall \omega \in \Omega, \forall s_i \in S_i$,

$$\sum_{\omega \in \Omega} \pi_i(f_i(\omega), f_{-i}(\omega)) \cdot P(\omega|\mathscr{H}_i(\omega)) \geq \sum_{\omega \in \Omega} \pi_i(s_i, f_{-i}(\omega)) \cdot P(\omega|\mathscr{H}_i(\omega)).$$

Definition (Bernheim [5], Pearce [25]): A set of strategies $R^\infty \subset \Pi_{i\in N} S_i$ is the set of the *correlated rationalizable actions* if (1)for each $i \in N$ and each $s_i \in R_i^\infty$, there exists $\mu \in \Delta(R_{-i}^\infty)$ such that s_i is a best response to μ, and (2) there is no set $F \subset \Pi_{i\in N} S_i$ such that it satisfies (1) and $F \supsetneq R^\infty$.

Concerning a posteriori equilibria and correlated rationalizable actions, we have the next equivalence result.

Proposition 5.1. *(Epstein [17] [9]) For any $s^* \in R^\infty$, there exists a posterior equilibrium $\langle S, (\mathscr{H}_i)_{i\in N}, (P(.|\mathscr{H}_i))_{i\in N}, f\rangle$ such that, for all $i \in N$, $\mathscr{H}_i = S_i$, for all $s \in S$, $f(s) = s$, and $f(s^*) = s^*$.*

From $\langle S, (\mathscr{H}_i)_{i\in N}, (P(.|\mathscr{H}_i))_{i\in N}, f\rangle$, where $\mathscr{H}_i = S_i$, we can construct a Harsanyi type space on S. For all $i \in N$, let $T_i \equiv \mathscr{H}_i$. and $\lambda_i : T_i \to \Delta(S \times T)$ be as follows;

$$\lambda_i(s_i)[(s_{-i}, s_{-i})] = \begin{cases} P(s_{-i}|s_i) & \text{if } s_{-i} = f_{-i}(s_{-i}) \\ 0 & \textit{otherwise.} \end{cases}$$

Let $\Lambda \equiv \langle S, T, \lambda\rangle$. We can easily confirm that Λ is a Harsanyi type space on S.

Let $G' \equiv \langle \pi, \Lambda\rangle$ be a Bayesian game. For $i \in N$, let a strategy $\beta_i : T_i \to S_i$ be such that $\beta_i(t_i) = f_i(\omega)$ where $\omega \in t_i$. Then $\beta \equiv (\beta)_{i\in N}$ becomes a Bayesian Nash equilibrium of the game G'. The a posterior equilibrium of the original game G is a Bayesian Nash equilibrium of G'.

[9] Aumann [3] and Brandenburger-Dekel [6] showed the same result.

5.2 Conditional independence and rationality and common certainty of rationality

Brandenburger-Friedenberg characterized intrinsic correlation by two conditions on Harsanyi types.

Definition: A Harsanyi type $t_i \in T_i$ satisfies *conditional independence* if $\lambda_i(t_i)[s_{-i}|h(t_{-i})] = \Pi_{j \in N \setminus \{i\}} \lambda_i(t_i)[s_j|h(t_{-i})]$, where h is the hierarchy mapping from $T \to U(S)$.

For the definition of another condition, *rationality and common certainty of rationality*, we need some preliminary definitions.

Definition: For each $i \in N$, a pair $(s_i, t_i) \in S_i \times T_i$ satisfies *rationality* if s_i is a best response to $Marg_{(S_{-i})} \lambda_i(t_i)$.

We use R_i to denote the set of the pairs that satisfies rationality.

Definition: For any $E \subset S_{-i} \times T_{-i}$, $t_i \in K_i(E)$ if $\lambda_i(t_i)[E] = 1$.

Definition: For each $i \in N$, $t_i \in T_i$ satisfies *rationality and common certainty of rationality* if $t_i \in R_i \cap \bigcap_{k=1}^{\infty} K^k(R)$, where K^k is the kth iteration of the operator K.

Since β is a BNE, it is almost clear that, for all $t_i \in T_i$, $(t_i, \beta_i(t_i))_{i \in N}$ satisfies RCBR. Polak [26] showed that RCBR is not sufficient condition for Nash equilibrium as shown in Aumann-Brandenburger [4], but Nash equilibrium satisfies RCBR under complete information about the payoffs. And he showed that the same thing applies to BNE. Here is a brief sketch of the proof. By construction, it is clear that, for all $i \in N$, $(t_i, \beta_i(t_i)) \in R_i^1$. Suppose that, for all $i \in N$ and $t_i \in T_i$, $(t_i, \beta_i(t_i)) \in R_i^k$. Then, since $\lambda_i(t_i)[\{(t_{-i}, s_{-i}) : s_{-i} = \beta_{-i}(t_{-i})\}] = 1$ and $\{(t_{-i}, s_{-i}) : s_{-i} = \beta_{-i}(t_{-i})\} \subset R_i^k$, we have $t_i \in B(R_{-i}^k)$. By the induction hypothesis, $(t_i, \beta_i(t_i)) \in R_i^k \cap [S_i \times B(R_{-i}^k)]$. Thus, $(t_i, \beta_i(t_i)) \in R_i^{\infty}$.

5.3 Conditional independence and redundancy

Note that conditional independence defined above is conditional on the sequential beliefs of the other agents' types. Therefore, when there are redundant types in Λ, it is hard for CI to be satisfied. However, the results that we have shown allows us to get rid of redundant types without affecting resulting equilibria.

In this section, we show first that, for any $s^* \in R^\infty$, there exists a Harsanyi type space Φ such that s^* is a realization of a BNE, and Φ has no *purely redundant types* except for one agent. As a result, we get the result that, for any $s^* \in R^\infty$, there exists a Bayesian formulation where s^* satisfies RCBR at a type which satisfies CI.

Proposition 5.2. *For any $s^* \in R^\infty$, there exists a posterior equilibrium such that, for some $\omega \in \Omega$, $f(\omega) = s^*$, and, for all $i \neq 1$, if $H_i \neq H_i'$, $P([s_j]_{j\neq i}|H_i) \neq P([s_j]_{j\neq i}|H_i')$ for some s_{-i}.*

Proof. By the proposition above, there exists a posterior equilibrium such that $\Omega = S$, for all $i \in N$, $\mathscr{H}_i = \{s_i \times S_{-i} : s_i \in S_i\}$ and $f_i(s) = s_i$. Let this a posteriori equilibrium be $\mathcal{F}$ and $[s_i] \equiv s_i \times S_{-i}$. For notational convenience, we denote each class in the agent i's information partition as $[s_i]$. Now it is possible that there exists $[s_i] \neq [s_i']$ such that, for all H_{-i}, $P(H_{-i}|[s_i]) = P(H_{-i}|[s_i'])$. Then we can duplicate the agent 1's information partition.

Suppose that, for a $a_1 \in S_1$, $P([a_1]|[s_i]) > 0$. We add another set of states so that the states of the world $\hat{\Omega} = (S_1 \cup \{s_1^2\}) \times S_{-1}$ and associate another information partition $[s_1^2]$ to the agent 1. We define a new a posterior equilibrium $\hat{\mathscr{F}} \equiv \langle S, \hat{\Omega}, \hat{P}, \hat{f}\rangle$ such that

$$\begin{aligned} For\ j = 1, \quad & \hat{f}_1([s_1^2]) = a_1 \\ \forall s_1 \in S_1,\ & \hat{f}_1([s_1]) = f_1([s_1]). \\ \forall j \neq 1, \forall s_j \in S_j,\ & \hat{f}_j([s_j]) = f_j([s_j]). \end{aligned}$$

The new sequence of conditional beliefs is defined in the following way;

$$\begin{aligned}
&For\ j = 1, & \hat{P}(.|[s_1^2]) = P(.|[s_1]) \\
& & \hat{P}(.|[s_1]) = P(.|[s_1])\ otherwise. \\
&\forall\ j \neq 1, \forall s_j \neq [s_i'], & \hat{P}(.|[s_j]) = P(.|[s_j]) \\
&For\ j = i\ and\ s_i',\ \forall s_{-1,i} \in S_{-1,i}, & \hat{P}((s_1^2, s_{-1,i})|[s_i']) = P((a_1, s_{-1,i})|[s_i']) \\
& & \hat{P}(.|[s_i']) = P(.|[s_i'])\ otherwise.
\end{aligned}$$

It is easy to show that $\hat{\mathscr{F}} \equiv \langle S, \hat{\Omega}, \hat{P}, \hat{f} \rangle$ is an a posterior equilibrium, and $\hat{f}(s^*) = s^*$. Note that, in this a posterior equilibrium, $\hat{P}(.|[s_i])$ and $\hat{P}(.|[s_i'])$ are distinguishable at the event $[a_i]$.

Since N and S are finite, we can iterating this process until every pair of each agent's, except for the agent 1, information states $[s_j] \neq [s_j']$ have different conditional beliefs over the other players' information states. □

Corollary 5.3. *For any $s^* \in R^\infty$, there exists a Harsanyi type space $\Lambda = \langle S, T, \lambda \rangle$ and a pair of Bayesian equilibrium strategy $\beta = (\beta_i)_{i \in N}$ such that $s^* = \beta(t)$, and, for all $i \neq 1$, T_i has no purely redundant types.*

Then we can apply the theorem to find an S-isomorphic Harsanyi type space Φ on $S \times C$ which has no redundant types. And, in Φ, no types result in the same sequential belief. Therefore, each type and its action associated by the equilibrium strategy β satisfy CI. Therefore we have the next result, which is the same result shown in a different way by Brandenburger-Friedenberg.

Theorem 5.4. *For any $s^* \in R^\infty$, there exists a Harsanyi type space $\Phi = \langle S \times C, V, \phi \rangle$ such that s^* satisfies RCBR at some state $v \in V$ which satisfies CI.*

6 Conclusion

In this paper, we showed it possible to embed Harsanyi type spaces isomorphically onto the space of sequential beliefs over an augmented uncertainty, even if they have redundant types. The technique to introduce a payoff irrelevant parameter is an extension of Liu. However distinctions are; (1) our payoff irrelevant parameter is an exogenous, and (2) it is enough that the parameter has at least two possible values. The latter finding is remarkable. Any correlation of types in Bayesian frameworks which cannot be explained by the basic uncertainty is resolved by adding a coin flip to the uncertainty. Concerning the first finding, the exogeneity of the parameter allowed us to show the existence of the universal type space where the vast majority of Harsanyi type spaces are uniquely embedded.

The result provided here has many contributions. One of them is an interpretation of the intrinsic correlation that we showed. Another is to enable us to include the redundant types to the topological arguments on the universal type space. The existence of redundancy has been an obstacle to the topological approaches. Dekel-Fudenberg-Morris [13] had to adopt interim correlated rationalizability (ICR) instead of interim independent rationalizability (IIR) when they defined the strategic topology. Ely-Peski [15] could identify Harsanyi types only up to IIR in the Δ-hierarchies. Our finding would eliminate the obstacle and allow us to argue more complicated solution concepts such as Bayesian equilibrium on the "universal type space".

7 Appendix

7.1 Bimeasurability of the function ϕ

Let $\Phi = \langle S \times C, V_1 \times V_2, (\phi_i)_{i\in\{1,2\}} \rangle$ be a Harsanyi type space, $V_1 = V_2 = [0,1]$, and $C = \{0,1\}$ as defined in the section 3. First we show that $\phi_1 : V_1 \to \Delta(S \times V_2 \times C)$ is bimeasurable. It is worth while to notice that ϕ_1 maps each element in V_1 to a product measure on the measurable space ($S \times V_2 \times C$, $\Sigma(S \times V_2 \times C)$).

We define the following functions.

$$f_1 : V_1 \to \Delta(S \times V_2) \text{ such that } f_1(v_1) = \lambda_1(p_1^{-1}) \circ [Id_S, p_2]^{-1}.$$
$$g_1 : V_1 \to \Delta(C) \text{ such that } g_1(v_1)(0) = v_1.$$

You can see that both f_1 and g_1 are bimeasurable functions. Then we have that $\phi_1(v_1) = f_1(v_1) \times g_1(v_1)$, where $f_1(v_1) \times g_1(v_1)$ is the product measure on the Borel measure space ($S \times V_2 \times C,\ \Sigma(S \times V_2 \times C)$).[10] Since $S \times V_2$ and C are both second countable, $\Sigma(S \times V_2) \times \Sigma(C) = \Sigma(S \times V_2 \times C)$.

Lemma 7.1. *The Borel σ-algebra* $\Sigma(S \times V_2 \times C) = \{E : \exists A \in \Sigma(S \times V_2),\ \exists B \in \Sigma(C),\ E = A \times B\}$.

Proof. Let $\hat{\Sigma} \equiv \{E : \exists A \in \Sigma(S \times V_2),\ \exists B \in \Sigma(C),\ E = A \times B\}$. We only have to show that $\hat{\Sigma}$ is a σ-algebra. It is clear that $\varnothing,\ S \times V_2 \times C \in \hat{\Sigma}$. Let $E \in \hat{\Sigma}$. Then there exists $A \in \Sigma(S \times V_2)$ and $B \in \Sigma(C)$ such that $E = A \times B$. Therefore $E^c = A^c \times C \cup A \times B^c$.

Let $\Delta^P(S \times V_2 \times C) \subset \Delta(S \times V_2 \times C)$ be the set of the product measures over $S \times V_2$ and C.

Lemma 7.2. *The subspace $\Delta^P(S \times V_2 \times C)$ is homeomorphic to the product space $\Delta(S \times V_2) \times \Delta(C)$.*

Proof. By Caratheodory's extension theorem, the function $d : \Delta(S \times V_2) \times \Delta(C) \to \Delta^P(S \times V_2 \times C)$ such that $(\eta, \mu) \mapsto \eta \times \mu$ is bijection.

First we want to show that d is a continuous function. The topological base of $S \times V_2 \times C$ is $t = \{G \times a : G \text{ is an open subset of } S \times V_2, \text{and } a \in 2^C\}$. Therefore any open set $G' \subset S \times V_2 \times C$ takes the form of

$$G' = \tilde{G}_1 \times \{0\} \cup \tilde{G}_2 \times \{1\} \cup \tilde{G}_3 \times \{0, 1\},$$

[10] By Caratheodory's extension theorem, the product measure is uniquely determined.

where, for $i = 1, 2, 3$, $\tilde{G}_i$ is an open set in $S \times V_2$. It is reduced to

$$G' = G_1 \times \{0\} \cup G_2 \times \{1\},$$

where, for $i = 1, 2$, G_i is an open set in $S \times V_2$.

Let $\{\eta_\alpha\}$ be a net in $\Delta(S \times V_2)$ such that $\eta_\alpha \to \eta$. And let $\{\mu_\alpha\}$ be a net in $\Delta(C)$ such that $\mu_\alpha \to \mu$. Then,

$$\begin{aligned} &\forall G: \text{ open}, \ \mathit{lim\ inf}\ \eta_\alpha(G) \geq \eta(G), \\ &\forall a \in 2^C, \ \mathit{lim\ inf}\ \mu_\alpha(a) \geq \mu(a). \end{aligned}$$

Let $\nu_\alpha \equiv \eta_\alpha \times \mu_\alpha$, and $\nu = \eta \times \mu$. Then, for each open set $G' \subset S \times V_2 \times C$,

$$\begin{aligned} \nu_\alpha(G') &= \nu_\alpha(G_1 \times \{0\}) + \nu_\alpha(G_2 \times \{1\}) \\ &= \eta_\alpha(G_1)\mu_\alpha(\{0\}) + \eta_\alpha(G_2)\mu_\alpha(\{1\}). \end{aligned}$$

In the same way,

$$\nu(G') = \eta(G_1)\mu(\{0\}) + \eta(G_2)\mu(\{1\}).$$

Since $\eta_\alpha \to \eta$ and $\mu_\alpha \to \mu$,

$$\begin{aligned} &\mathit{lim\ inf}\ \eta_\alpha(G_1)\mu_\alpha(\{0\}) \geq \eta(G_1)\mu(\{0\}). \\ &\mathit{lim\ inf}\ \eta_\alpha(G_2)\mu_\alpha(\{1\}) \geq \eta(G_2)\mu(\{1\}). \end{aligned}$$

And,

$$\begin{aligned} \mathit{lim\ inf}\ \nu_\alpha(G') &= \mathit{lim\ inf}\{\eta_\alpha(G_1)\mu_\alpha(\{0\}) + \eta_\alpha(G_2)\mu_\alpha(\{1\})\} \\ &\geq \mathit{lim\ inf}\ \eta_\alpha(G_1)\mu_\alpha(\{0\}) + \mathit{lim\ inf}\ \eta_\alpha(G_2)\mu_\alpha(\{1\}) \\ &\geq \eta(G_1)\mu(\{0\}) + \eta(G_2)\mu(\{1\}) \\ &= \nu(G'). \end{aligned}$$

Therefore, $\nu_\alpha \to \nu$. Therefore d is a continuous function.

Next, we show that d^{-1} is a continuous function. Let $\{\nu_\alpha\} \equiv \{\eta_\alpha \times \mu_\alpha\}$ be a net of product measures such that $\nu_\alpha \to \nu = \eta \times \mu$. Then, $\nu_\alpha(S \times V_2 \times a) = \mu_\alpha(a)$, and $\nu(S \times V_2 \times a) = \mu(a)$. Since $\mathit{lim\ inf}\ \nu_\alpha(S \times V_2 \times a) \geq \nu(S \times V_2 \times a)$, *lim inf* $\mu_\alpha(a) \geq \mu(a)$. In the symmetric way, *lim inf* $\eta_\alpha(G) \geq \eta(G)$. It means that $(\eta_\alpha,\ \mu_\alpha) \to (\eta,\ \mu)$. Therefore d^{-1} is a continuous function. $\square$

Corollary 7.3. *The subspace* $\Delta^P(S \times V_2 \times C)$ *is closed.*

Since $\Delta(S \times V_2) \times \Delta(C)$ is second countable, $\Sigma(\Delta(S \times V_2) \times \Delta(C)) = \Sigma(\Delta(S \times V_2)) \times \Sigma(\Delta(C))$

Proposition 7.4. *The function* $\phi_1 : V_1 \to \Delta(S \times V_2) \times \Delta(C)$ *is a bimeasurable function.*

Proof. (*Inverse measurability*) Let $\phi_1 = (f_1, g_1)$. The space of the probability measures $\Delta(C)$ is homeomorphic to V_1, and g_1 is its homeomorphism. We consider that $\phi_1 : V_1 \to \Delta(S \times V_2) \times V_1$ and $\phi_1 = (f_1, Id)$. It allows us to consider that $\phi_1(V_1) \subset \Delta(S \times V_2) \times V_1$ is a graph of the function f_1^{-1}. Since f_1^{-1} is a measurable function, the graph $\phi_1(V_1)$ is a Borel set in the product measure space $\Delta(S \times V_2) \times V_1$.[11] For each $E \in \Sigma(V_1)$, $\phi_1(E) = f_1(E) \times E \cap \phi(V_1)$. We know that both $f_1(E) \times E$ and $\phi(V_1)$ are measurable. Therefore, $\phi_1(E)$ is measurable.

(*Measurablity*) Let $E \subset \Delta(S \times V_2) \times V_1$ be a rectangle. Let π_1 and π_2 be the projection onto V_1 and $\Delta(S \times V_2)$ respectively. Let $F_2 \equiv f_1 \circ \pi_1(E) \subset \Delta(S \times V_2)$. Since f_1 is bimeasurable, F_2 is also Borel. For each $y \in \pi_2(E)$, $f_1^{-1}(y) \in \pi_1(E)$ if and only if $y \in \pi_2(E) \cap F_2$. Let Therefore, the intersection of the rectangle E and the entire graph $\phi_1(V_1) \equiv \{(x, f_1(x)) : x \in V_1\}$ becomes $G \equiv \{(f_1^{-1}(y), y) : y \in \pi_2(E) \cap F_2\}$. Since $\pi_2(E)$ and F_2 are both Borel, $\pi_2(E) \cap F_2$ is also Borel. We can see that $\phi_1^{-1}(E) = \pi_1(G)$. Since f_1 is measurable, $\pi_1(G)$ is also Borel. Therefore $\phi_1^{-1}(E)$ is Borel. $\square$

[11] See Halmos [19] pp143.

Proposition 7.5. *The function $\phi_2 : V_2 \to \Delta(S \times V_1 \times \Delta(C))$ is a bimeasurable function.*

Proof. Let

$$\Delta_0 \equiv \{\mu \in \Delta(S \times V_1 \times C) : \forall E \in \Sigma(S \times V_1),\ \mu(E \times \{1\}) = 0\}.$$

Let $f : S \times V_1 \times C \to \mathbb{R}$ such that

$$\begin{aligned} \forall e \in S \times V_1,\ f(e, 0) &= a, \\ f(e, 1) &= b. \end{aligned}$$

Then, $f \in C_b(S \times V_1 \times C)$. Therefore, when a net $\{\mu_\alpha\}$ converges to some probability measure, it must be in Δ_0. Therefore, Δ_0 is a closed set. Since λ_2 is bimeasurable between V_2 and $\Delta(S \times V_1)$ and $\Delta(S \times V_1)$ is homeomorphic to Δ_0, ϕ_2 is bimeasurable between V_2 and Δ_0. □

I use the term "bimeasurable" in a slightly different way.

Definition: A function $f : X \to Y$ is bimeasurable if f is measurable and , for each measurable set $E \subset X$, $f(E)$ is also measurable.

Lemma 7.6. *Let X and Y be Polish, and $f_1 : X \to \Delta(X)$ and $g_1 : X \to Y$ be both bimeasurable. Let $f_2 : X \to \Delta(Y)$ be such that, for each $x \in X$, $f_2(x) = f_1(x) \circ g_1^{-1}$. Then, the function f_2 is bimeasurable.*

Proof. The measurability of f_2 is shown by Liu.[12] We only show that, for all $E \in \Sigma(X),\ \ f_2(E) \in \Sigma(\Delta(Y))$.

Let $g_2 : \Delta(X) \to \Delta(Y)$ such that, for all $\mu \in \Delta(X),\ \ g_2 : \mu \mapsto \mu \circ g_1^{-1}$. Let $A \equiv \{\mu \in \Delta(X) : \mu(E) \geq p\}$, where $E \in \Sigma(X)$ and $p \in [0, 1]$. Then, $g_2(A) = \{\nu \in \Delta(Y) : v(g_1(E)) \geq p\} \cap \{\nu \in \Delta(Y) : \nu(g_1(X)) = 1\}$. Notice

[12]See Lemma 5 in Liu [22]

that $g_1(X) \in \Sigma(Y)$. The both sets in the right hand side of the equation are measurable. Therefore $g_2(A) \in \Sigma(Y)$.

Since $f_2 = g_2 \circ f_1$, we have that, for each $E \in \Sigma(X)$, $f_2(E) \in \Sigma(\Delta(Y))$. $\square$

Lemma 7.7. *For each $k \geq 1$, the kth order hierarchy mapping h_i^k is bimeasurable.*

Proof. Without loss of generality, we only have to show that h_1^k is bimeasurable.

For $k = 1$, let $X \equiv S \times V \times C$, $Y \equiv S \times C$, $f_1 \equiv \tilde{\phi}_i$, and $g_1 \equiv proj_{(S\times C)}$, where, for all $(s, c, v_1, v_2) \in S \times V \times C$, $\tilde{\phi}_i(s, c, v_1, v_2) \equiv \phi_1(v_1)$. It is easy to see that f_1 and g_1 are bimeasurable. By the lemma, $\tilde{h}_1^1 : S \times V \times C \to \Delta(S \times C)$ is bimeasurable. We can just restrict the domain from $S \times V \times C$ to V_1 to get the h_1^1 which is measurable.

For $k \geq 2$, we assume that, for $i = 1, 2$, h_i^{k-1} is bimeasurable as the induction hypothesis. Let $X \equiv S \times V \times C$, $Y \equiv S \times C \times H^k(S \times C)$, $f_1 \equiv \tilde{\phi}_1$, and $g_1 \equiv \tilde{h}_2^{k-1}$, where $\tilde{h}_2^{k-1}(s, c, v_1, v_2) \equiv h_2^{k-1}(v_2)$. By the lemma, $\tilde{h}_1^k : S \times V \times C \to \Delta(S \times C)$ is bimeasurable. We can just restrict the domain from $S \times V \times C$ to V_1 to get the h_1^k which is measurable. $\square$

Proposition 7.8. *The full hierarchy mapping h_i is bimeasurable.*

Proof. First, we show that h_i is measurable. The σ-algebra on $\Pi_{k=1}^{\infty} H^k$ is the σ-algebra generated by

$$\mathcal{F} \equiv \{F = \Pi_{k \notin I} E_k \times \Pi_{k \in I} H^k : I \subset \mathbb{N} \text{ is finite, and } E_k \in \Sigma(H^k)\}.$$

Since $h_i \equiv (h_i^1, \ \ldots, \ )$, for each $F \in \mathcal{F}$, $h_i^{-1}(F) \in \Sigma(V_i)$. By Theorem 4-1-6 in Dudley [14], h_i is measurable.

Next, we show that, for each $E \in \Sigma(V_i)$, $h_i(E)$ is measurable. Let $E \in \Sigma(V_i)$.

Since h_i^1 and h_i^2 are bimeasurable injection, $h_i^2 \circ (h_i^1)^{-1}$ is bimeasurable bijection from $h_i^1(E)$ to $h_i^2(E)$. Let the image of E by (h_i^1, h_i^2) be $\Gamma_2(E)$. It means $\Gamma_2(E) \equiv \{(h_i^1(v_i), h_i^2(v_i)) \in H^1 \times H^2 : v_i \in E\}$. We can see that it is the graph of $h_i^2 \circ (h_i^1)^{-1}$. Therefore, $\Gamma_2(E)$ is measurable in the product measurable space $H^1 \times H^2$. By the mathematical induction, for each $k \geq 1$, $\Gamma_k(E) \subset \Pi_{l=1}^k H^l$, the image of E by $(h_i^1, \cdots, h_i^k)$, is measurable. The image of the full hierarchy $h_i(E)$ is the projective limit of $(\Gamma_k(E))_{k\in\mathbb{N}}$, and as we saw, each $\Gamma_k(E)$ is measurable. Therefore $h_i(E)$ is measurable. □

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