Are incentives against justice?

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Abstract

This paper analyzes incentives for truthful revelation of preferences for the problem of fairly allocating a set of objects when monetary compensations are possible. An example is the allocation of the rooms and the rent among housemates. We investigate the manipulability of a family of solutions which are efficient, attain some intuitive form of distributive justice [Rawls J., 1972, A Theory of Justice, Harvard U. Press], and satisfy a strong form of solidarity under budget changes: the Generalized Money Rawlsian Fair (GMRF) correspondences [Alkan A., Demange G., Gale D., Fair allocation of indivisible goods and criteria of justice. Econometrica 59, 1023-1039]. A solution is strategy-proof if no agent can benefit by misrepresenting her preferences. (i) We show that even though no selection from these correspondences is strategy-proof, the Nash and strong Nash equilibrium outcomes of the "preference revelation game form" associated to each correspondence, retain the basic objectives of fairness and efficiency. Thus, even though each agent has an incentive to lie if the others truthfully report their preferences, in equilibrium, no agent prefers another agent's allotment to hers according to her true preferences; moreover, in equilibrium, efficiency is preserved according to agents' true preferences. (ii) As a corollary, we show that GMRF correspondences "naturally implement" the fair and efficient correspondence, in both Nash and strong Nash equilibria.

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1 Introduction

This paper analyzes incentives for truthful revelation of preferences for the problem of fairly allocating a set of objects when monetary compensations are possible. We investigate the manipulability of a family of correspondences which are efficient, attain some intuitive form of distributive justice (Rawls, 1972), and satisfy a strong form of solidarity when budget

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changes: the Generalized Money Rawlsian Fair correspondences (Alkan et al., 1991; Alkan, 1994). The following example motivates our study.

A group of housemates have to allocate the rooms in the house they share and divide the rent among them. An arbitrator is asked to propose an allocation protocol to guarantee the fair and efficient division of the house and the rent. The arbitrator knows that for each housemate money is desirable and no room is infinitely better than any other, but does not know the actual housemates' preferences. He makes the following proposal:

1. Ask each housemate to report her preferences. These preferences should satisfy the aforementioned properties.

2. Consider the allocations in which no one housemate would prefer to exchange her allotment with any other. Among these allocations, choose one that minimizes the maximal individual contribution to rent.

The arbitrator bases her proposal on the following considerations.

First, for each possible report, there is at least one allocation satisfying the desired properties (Alkan et al., 1991; Velez, 2008). Moreover, should multiple such allocations exist, then all the agents are indifferent among them (Alkan et al., 1991; Velez, 2007). Thus, there should be no controversy in making a selection from these allocations.

Second are fairness considerations. The proposed allocations are fair in the sense of Foley (1967), i.e., no one housemate prefers to exchange her allotment with any other. Thus, each housemate, according to her own preferences, gets the best deal.

Besides being fair, the selected allocations achieve some form of "Democratic Equality." In Rawls's Theory of Justice, this principle of distributive justice calls for the minimal welfare among the agents, measured by means of some index, to be maximized (Rawls, 1972). Here instead, the maximal contribution to rent is minimized among all the fair allocations. A formal connection exists between these seemingly independent objectives: There is a representation of preferences such that the allocations selected by the arbitrator for different rents, are the ones that maximize the utility of the agent whose utility is the smallest.

Third are efficiency considerations. Since the proposed allocations are fair and there are as many housemates as rooms, then these allocations are also efficient (Svensson, 1983).

The proposed protocol seemingly achieves the housemates' objective: A fair and efficient distribution. But, the housemates know each other well enough, and soon each realizes that she may gain by lying about her preferences if the others tell the truth (Tadenuma and Thomson, 1995). They accept the arbitrator's proposal. However, they prepare their best poker faces.

What outcomes would one expect when the housemates behave strategically? Previous literature shows how mechanisms, including the market mechanism, may lose efficiency when agents manipulate. Here, fairness is at stake too. The purpose of this paper is to determine to what extent incentives are against fairness and efficiency for a general family of allocation protocols of which the example above is a particular case.

1.1 The formal answer

We model the situation described above as the problem of allocating a social endowment of objects and an amount (possibly negative) of a perfectly divisible good, which we refer to as "money." There are as many agents as objects and each agent must receive an object. Agents consume bundles consisting of an object and an amount of money. Their preferences satisfy two properties: (i) no object is infinitely better than another, and (ii) money is desirable, i.e., for any two consumption bundles containing the same object, each agent prefers the one with the greater money component. Finally, agents' individual consumptions of money should add up to a given amount, which we refer to as the "budget."

A social choice correspondence (SCC) is a function that recommends a nonempty subset of allocations for each possible preference profile. We think of an SCC as representing an arbitrator's judgement of what the most desirable outcomes are. However, we assume that the arbitrator does not have information about preferences besides their satisfying the aforementioned properties. Hence, we are led to study whether agents have the incentive to truthfully reveal their preferences.

An allocation is fair if no agent prefers the allotment of any other agent to hers (Foley, 1967). We focus on a particular family of SCCs that select fair allocations: the Generalized Money-Rawlsian Fair (GMRF) correspondences (Alkan et al., 1991; Alkan, 1994). Consider a family of continuous and monotonically increasing real-valued functions defined on the real numbers and indexed by the set of objects A. The GMRF correspondence associated to such a family selects the allocations that maximize, among the fair allocations, the minimal consumption of money transformed by means of these functions (each function transforms the money component in the bundle containing the object that labels it). A prominent member of this family is the Money-Rawlsian Fair (MRF) correspondence. This is the GMRF correspondence associated to the family of identity functions, i.e., the correspondence that selects, among the fair allocations, the ones that maximize the minimal consumption of money.

GMRF correspondences share several desirable properties. For each problem, they select a non-empty set of allocations and they are essentially single-valued, i.e., for each problem, each agent is indifferent among all the recommended allocations. They satisfy a strong form of solidarity when budget changes: Consider a GMRF correspondence, S, and a given preference profile; the welfare of each agent attained in the S-optimal allocations for different budgets is an increasing function of the budget (Alkan et al., 1991). In our interpretation of the model as the allocation of rooms and rent among housemates, this means that each agent benefits if the rent decreases; analogously, each agent contributes in welfare terms if the rent increases.

Besides being fair, GMRF allocations capture some form of the aforementioned Democratic Equality: For each GMRF correspondence, there is a representation of preferences for which the recommended allocations for all possible budgets are the fair allocations that maximize the minimal utility across agents with respect to it (Velez, 2007).

We investigate how manipulable the GMRF correspondences are. A single-valued correspondence is strategy-proof if no agent can benefit by misreporting her preferences. A first question is whether there is a strategy-proof selection from each GMRF correspondence. Unfortunately there are no such selections. In fact, there is no strategy-proof selection from the fair correspondence, i.e., the correspondence that recommends, for each problem, its set of fair allocations (Tadenuma and Thomson, 1995).

The non-existence of a strategy-proof selection from some SCC of interest is not the end of the road, however. It only means that an agent may have the incentive to lie if all the other agents report their true preferences. The next step is to determine which outcomes result from the manipulations of the allocation protocol induced by S. Formally, this allocation protocol is seen as a game form in which each agent's strategy space is the domain of admissible preferences and the outcome function is S (an adjustment is necessary here, for GMRF correspondences are not single-valued; see Subsection 2.2 for details). Our prediction for the allocation process is the set of Nash-equilibrium outcomes of the game obtained by augmenting this game form with the agents' true preferences.

Our main result is the characterization of the Nash-equilibrium and strong Nash-equilibrium outcome correspondences of the game form induced by the GMRF correspondences. We show that for each GMRF correspondence, S, these outcome correspondences coincide with the fair correspondence: For each preference profile, each outcome of the allocation process induced by S is fair, and thus, efficient for the true preferences. Moreover, each fair allocation for a given preference profile is an outcome of the game form induced by S played at these preferences. Even though the allocations selected by S are not necessarily attained, the basic objectives of fairness and efficiency survive. Thus, incentives go against truthful revelation of preferences, but not against fairness and efficiency.

Our results have consequences for the implementation of the fair correspondence (see Jackson, 2001, for a survey on the implementation literature). Our main theorem implies that the game form induced by each GMRF correspondence implements the fair correspondence both in Nash and strong Nash equilibria.

If agents' reports are further restricted to be quasi-linear preferences, our implementation results still hold. Consider the game form in which each agent's strategy space is the domain of quasi-linear preferences and the outcome function is given by a GMRF correspondence. We show that the Nash-equilibrium outcome correspondence of such a game form coincides with the fair correspondence. We also provide abstract sufficient conditions under which this strategy space reduction is possible.

The analysis of game forms in which preference reports are restricted has two purposes. First, one may think that a game form with simple strategy spaces is more practical. For instance, it is more realistic to imagine agents reporting preferences in a finite dimensional space (e.g. the quasi-linear domain). Second, the simplification of the agents' strategy spaces may reduce the "complexity" of the game form associated to an SCC. For instance, there is a polynomial algorithm to calculate MRF allocations on the quasi-linear domain (Aragones, 1995). Nevertheless, we do not emphasize these results, because the restriction of strategy spaces precludes the possibility that agents report their true preferences. Since agents do not have the opportunity to report their true preferences, the outcomes of these game forms can not be interpreted as the outcomes resulting from agents' manipulation.

1.2 Related literature

The study of the manipulation of SCCs was initiated by Hurwicz (1972). Hurwicz introduces the property of strategy-proofness in the context of classical exchange economies and shows that the Walrasian correspondence is not strategy-proof.[1] Moreover, he shows, in the two-agent-two-good case, that no strategy-proof single-valued correspondence is efficient and individually rational, i.e., each agent finds her allotment at least as desirable as her endowment. Further research extended this impossibility to the n-agent and l-good case (Serizawa, 2002).

The aforementioned impossibility raised a natural question: How manipulable are SCCs? In order to asses the manipulability of an SCC, one has to determine what allocations ensue from its manipulation. Only then can one compare how different the real outcomes and the allocations selected by the SCC are.

This type of question was first asked in the context of classical economies. This literature concluded that the Nash-equilibrium outcome correspondence of the game form associated to each selection from the individually rational and efficient correspondence contains the Walrasian correspondence (Lindahl correspondence in the public good case), but is not efficient (Hurwicz, 1972; Thomson, 1979; Otani and Sicilian, 1982; Thomson, 1984). Similar results hold for the manipulation of the Shapley value (Thomson, 1988) and the manipulation of cooperative bargaining solutions in private and public good economies (Sobel, 1981, 2001; Kıbrıs, 2002).

For the distribution of a collectively owned bundle of infinitely divisible commodities among a group of agents, the conclusions parallel the ones for exchange economies. Here a notion of fairness plays the role of individual rationality and the Walrasian correspondence operated from equal endowments plays the role of the Walrasian correspondence (Thomson, 1987). Our results relate to these in that in our model the Fair correspondence coincides with the Walrasian correspondence operated from equal endowments (Svensson, 1983). Nevertheless, our results differ in that in the games we analyze, equilibrium outcomes are always Fair and thus, efficient.

However, efficiency is not always compromised by misrepresentation of preferences. Velez and Thomson (2008) propose to measure an agent's sacrifice at an allocation by the size of the set of feasible bundles that the agent prefers to her consumption. They define the Equal Sacrifice correspondence as the SCC that selects the allocations at which sacrifices

[1] Hurwicz (1972) introduces strategy-proofness under the name of incentive compatibility.

are equal across agents and this common sacrifice is minimal. Their main result is that, if all goods are normal, the outcomes from the manipulation of these correspondence coincide with the ones selected by the constrained Walrasian correspondence operated from equal division.

Our work is related to previous literature on the manipulation of selections from the fair correspondence in the allocation of objects when monetary compensations are possible. Tadenuma and Thomson (1995) study the allocation of one object and an amount of money among n agents ($n-1$ agents receive a null object). They conclude that the outcomes from the manipulation of each selection from the fair correspondence satisfying a mild condition coincide with the ones selected by the fair correspondence. Our main theorem restricted to the two-agent case is a consequence of this result.

Beviá (2001) studies the allocation of n objects among n agents when monetary compensations are possible and preferences are quasi-linear. Her main conclusion is that the outcomes from the manipulation of each selection from the fair correspondence satisfying a mild condition coincides with the ones selected by the fair correspondence. More recently, and again under the restriction of quasi-linearity of preferences, Āzacis (2008) analyzes the game form associated to a particular selection from the fair correspondence. This correspondence is defined as the solution to a certain linear program introduced by Abdulkadiroğlu et al. (2004). His main result is that this game form implements the fair correspondence in Nash and strong Nash equilibria.[2]

The aforementioned results hinge on one of two assumptions: (i) there is only one object, or (ii) preferences are quasi-linear. Our results show that the game form associated to a GMRF correspondence can be analyzed without any of these assumptions (see Section 4 for details).

The remainder of this paper is organized as follows. Section 2 presents the model and some preliminary results. Section 3 introduces and studies the GMRF correspondences. Section 4 analyzes the manipulability of the GMRF correspondences. Section 5 presents our results concerning implementation of the fair correspondence. Section 6 discusses the extension of our results to situations in which individual consumptions of money are bounded.

2 The Model

2.1 Environment and axioms

Let N be a finite set of agents and A be a finite set of objects such that $|A| = |N|$. Generic objects are α and β. Agents consume bundles in $\mathbb{R} \times A$.[3] The generic consumption bundle is

[2]See Thomson (2005) for the implementation of the fair correspondence in the context of the allocation of a bundle of infinitely divisible commodities.

[3]See Section 6 for an extension of our results when consumption bundles are limited to $\mathbb{R}_- \times A$, i.e., individual consumptions of money are non-positive.

$(x_\alpha, \alpha) \in \mathbb{R} \times A$. We consider the problem of allocating the set of objects A and an amount $M \in \mathbb{R}$ of an infinitely divisible good to which we refer as money, among the members of N. Even though resources and population are fixed in the problems we study, we embed these problems in a variable population and variable resource environment in order to facilitate the presentation of our results.

2.1.1 Variable population and variable resource environment

For each $B \subseteq A$ the domain of preferences on $\mathbb{R} \times B$ is $\boldsymbol{\mathcal{R}(B)}$. The generic preference is $R_0 \in \mathcal{R}(B)$. As usual, I_0 and P_0 are the symmetric and asymmetric parts of R_0. We make two standard assumptions on preferences in $\mathcal{R}(B)$.

- **Money-monotonicity**, i.e., for each $R_0 \in \mathcal{R}(B)$, each $\beta \in B$, and each $\{x_\beta, x'_\beta\} \subset \mathbb{R}$ such that $x'_\beta > x_\beta$, $(x'_\beta, \beta)\, P_0\, (x_\beta, \beta)$.[4]
- **No object is infinitely better than any other**, i.e., for each R_0, each $\{\beta, \delta\} \subset B$, and each $x_\delta \in \mathbb{R}$, there exists $x_\beta \in \mathbb{R}$ such that $(x_\delta, \delta)\, I_0\, (x_\beta, \beta)$.

Let $\boldsymbol{\mathcal{R}} \equiv \mathcal{R}(A)$. We consider a special subdomain of $\mathcal{R}$, the subdomain of quasi-linear preferences. A preference $R_0 \in \mathcal{R}$ is **quasi-linear** if for each $\{\alpha, \beta\} \subseteq A$, each $\{x_\alpha, x_\beta\} \subset \mathbb{R}$ such that $(x_\alpha, \alpha)\, R_0\, (x_\beta, \beta)$, and each $\Delta \in \mathbb{R}$, $(x_\alpha + \Delta, \alpha)\, R_0\, (x_\beta + \Delta, \beta)$. We denote this subdomain, $\boldsymbol{\mathcal{Q}}$.

Let $K \subseteq N$. An **economy** with agent set K is a triple $e \equiv (B, R, m)$, where $B \subseteq A$ is such that $|B| = |K|$, $R \equiv (R_i)_{i \in K} \in \mathcal{R}(B)^K$, and $m \in \mathbb{R}$ is an amount of money (possibly negative) to distribute among the members of K. The set of economies with agent set K is $\mathcal{E}^K$. Let $\boldsymbol{\mathcal{E}} \equiv \bigcup_{K \subseteq N} \mathcal{E}^K$ be the set of all economies.

Let $e \equiv (B, R, m) \in \mathcal{E}^K$. An **allocation for $\boldsymbol{e}$** is a pair $z \equiv (x, \mu) \in \mathbb{R}^B \times A^K$ such that $\sum_{\beta \in B} x_\beta = m$ and $\mu : K \to B$ is a bijection. The consumption of money associated with object α at z is x_α. **Agent $\boldsymbol{i}$'s allotment at $\boldsymbol{z}$** is $z_i \equiv (x_{\mu(i)}, \mu(i))$. Let $Z(e)$ be the **set of allocations for $\boldsymbol{e}$**. Agent i's preferences R_i induce preferences on $Z(e)$, which for convenience we also denote R_i, as follows: for each $\{z, z'\} \subseteq Z(e)$, $z'\, R_i\, z$ if and only if $z'_i\, R_i\, z_i$.

Let $K \subseteq N$, $B \subseteq A$ such that $|B| = |K|$, and $R \in \mathcal{R}(B)^K$. For each $i \in K$ and each $R_i \in \mathcal{R}(B)$, the profile $(R_{-i}, R'_i) \in \mathcal{R}(B)^K$ is obtained from R by replacing R_i by R'_i. For each $e \equiv (B, R, m) \in \mathcal{E}^K$, the profile R induces an incomplete ordering on $Z(e)$, which for convenience we also denote R, as follows: for each $\{z, z'\} \subset Z(e)$, $z'\, R\, z$ if and only if for each $i \in K$, $z'\, R_i\, z$.

[4] Money-monotonicity implies continuity, i.e., weak upper and lower contour sets are closed in the product topology on $\mathbb{R} \times B$ induced by the Euclidean and discrete topologies.

2.1.2 Social choice correspondences

A **social choice correspondence** associates to each $R \in \mathcal{R}^N$ a non-empty subset, $S(R)$, of $Z(A, R, M)$. The generic social choice correspondence is S. A **selection**, s, from a correspondence S is a function that associates to each $R \in \mathcal{R}^N$ an element of $S(R)$. We write $s \in S$.

2.1.3 Fairness and efficiency

Let $K \subseteq N$ and $e \in \mathcal{E}^K$. We consider two properties of allocations in $Z(e)$. First is fairness. An allocation $z \in Z(e)$ is **fair for *e*** if for each $\{i, j\} \subseteq K$, $z_i \, R_i \, z_j$. This formulation, which goes back to Foley (1967), provides an ordinal and operational notion of fairness. Second is efficiency. As usual, an allocation $z \in Z(e)$ is **efficient** if there is no $z' \in Z(e)$ such that $z' \, R \, z$ and for at least one $j \in K$, $z' \, P_j \, z$.

For each $e \in \mathcal{E}$, let $F(e)$ and $P(e)$ be the sets of *fair* and *efficient* allocations for e, respectively. It is well known that for each $e \in \mathcal{E}$, the set $F(e)$ is non-empty (Alkan et al., 1991; Velez, 2008); moreover, since there are as many agents as objects, then $F(e) \subseteq P(e)$ (Svensson, 1983).

The social choice correspondence that associates to each $R \in \mathcal{R}^N$ the set of *fair* allocations for (A, R, M), is F.

2.2 Manipulation of a social choice correspondence

Let S be an SCC. Suppose that agents are asked to report preferences with the proviso that one of the S-optimal allocations for the reported preferences will be chosen. What are the outcomes that ensue if agents engage in the manipulation of S? If S is single-valued, it is straightforward to answer this question: it induces a game form, whose Nash-equilibrium outcome correspondence constitutes the outcome correspondence of the allocation protocol. However, if S is not single-valued, no such prediction is directly available.

The main issue here is that if a correspondence is not single-valued, it is not clear when an agent will want to change her report given the other agents' reports. This is so even if the correspondence is *essentially single-valued* (Thomson, 1979, 1984). When an agent envisions the outcomes attainable by a change in her report, these outcomes are welfare-equivalent according to her reported preferences. However, these outcomes may not be welfare-equivalent according to her true preferences.

Two approaches to this problem have been studied in the literature. A first approach develops a solution concept that parallels Nash equilibrium and which applies to "games" with multiple outcomes (Thomson, 1979, 1984, 1987, 1988, Tadenuma and Thomson, 1995, and Beviá, 2001). A second approach completes the description of the allocation protocol by assuming that if S is not single-valued, then a selection from it is used if agents do not "agree" on the allocation that should result from the allocation process (Velez and

Thomson, 2008). This augmented protocol defines a game form whose Nash-equilibrium outcome correspondence can be evaluated. If the resulting outcome correspondence does not depend on the selection used to determine the outcome when the agents do not agree, then one can regard these outcomes as the ones resulting from the manipulation of S.

We follow Velez and Thomson (2008) and complete the description of the allocation protocol as follows. We assume that agents report not only preferences but also a bundle, which can be interpreted as the bundle they request. If the reported list of bundles is one of the S-optimal allocations for the reported preference profile, then it is the outcome of the allocation process. Otherwise, there is a selection $s \in S$ that determines this outcome.

Let $\mathcal{D} \subseteq \mathcal{R}$ be a sub-domain of preferences, S be an SCC, and $s \in S$. The **game form $\langle \mathcal{D} \times (\mathbb{R} \times A), S, s\rangle$** is defined as follows:

- Each agent's strategy space is $\mathcal{D} \times (\mathbb{R} \times A)$.
- Given strategy profile $(R, z) \equiv (R_i, z_i)_{i\in N} \in (\mathcal{D} \times (\mathbb{R} \times A))^N$, the outcome is

$$O\langle \mathcal{D} \times (\mathbb{R} \times A), S, s\rangle(R, z) \equiv \begin{cases} z & \text{if } z \in S(R) \\ s(R) & \text{otherwise.} \end{cases}$$

For each $R^0 \in \mathcal{R}^N$, the **game $\langle \mathcal{D} \times (\mathbb{R} \times A), S, s, R^0\rangle$** is obtained by augmenting the game form $\langle \mathcal{D} \times (\mathbb{R} \times A), S, s\rangle$ by the preference profile R^0. Since the consumption space is fixed throughout, we simplify the notation for $\langle \mathcal{D} \times (\mathbb{R} \times A), S, s\rangle$ to $\langle \mathcal{D}, S, s\rangle$; likewise, we simplify the notation for $\langle \mathcal{D} \times (\mathbb{R} \times A), S, s, R^0\rangle$ to $\langle \mathcal{D}, S, s, R^0\rangle$

A **Nash equilibrium of $\langle \mathcal{D}, S, s, R^0\rangle$** is a strategy profile $(R, z) \in (\mathcal{D} \times (\mathbb{R} \times A))^N$, such that for each $i \in N$ and each $(R'_i, z'_i) \in \mathcal{D} \times (\mathbb{R} \times A)$,

$$O\langle \mathcal{D}, S, s\rangle(R, z)\, R^0_i\, O\langle \mathcal{D}, S, s\rangle(R_{-i}, R'_i, z_{-i}, z'_i).$$

For each game $\langle \mathcal{D}, S, s, R^0\rangle$ the set of **Nash equilibria** is $\boldsymbol{\mathcal{N}\langle \mathcal{D}, S, s, R^0\rangle}$ and the set of **Nash equilibrium outcomes** is $\boldsymbol{\mathcal{O}\langle \mathcal{D}, S, s, R^0\rangle}$.

If for any two selections of S, s and s', $\mathcal{O}\langle \mathcal{D}, S, s, R^0\rangle = \mathcal{O}\langle \mathcal{D}, S, s', R^0\rangle$, we denote this common set by $\boldsymbol{\mathcal{O}\langle \mathcal{D}, S, R^0\rangle}$.

2.3 Implementation of a social choice correspondence

Let S and S' be two SCCs, $s' \in S'$, $\mathcal{D} \subseteq \mathcal{R}$, and $\mathcal{D}' \subseteq \mathcal{R}$. The pair **$\langle \mathcal{D}', S'\rangle$ implements S on $\mathcal{D}$** if for each $R^0 \in \mathcal{D}$, $\mathcal{O}\langle \mathcal{D}', S', R^0\rangle = S(R^0)$. The correspondence **$S'$ naturally implements S on $\mathcal{D}$** if $\langle \mathcal{D}, S'\rangle$ implements S on $\mathcal{D}$.

We consider also a stronger form of implementation. Let $\mathcal{D} \subset \mathcal{R}$ be a sub-domain of preferences, S be an SCC, $s \in S$, and $R^0 \in \mathcal{R}^N$. A **strong Nash equilibrium of $\langle \mathcal{D}, S, s, R^0\rangle$** is a strategy profile $(R, z) \in (\mathcal{D} \times (\mathbb{R} \times A))^N$, such that for each $N' \subseteq N$, each $(R'_{N'}, z'_{N'}) \in (\mathcal{D} \times (\mathbb{R} \times A))^{N'}$, if there is $i \in N'$ such that

$$O\langle \mathcal{D}, S, s\rangle(R_{-N'}, R'_{N'}, z_{-N'}, z'_{N'})\, P_i^0\, O\langle \mathcal{D}, S, s\rangle(R, z),$$

then there is $j \in N'$ such that

$$O\langle \mathcal{D}, S, s\rangle(R, z)\, P_i^0\, O\langle \mathcal{D}, S, s\rangle(R_{-N'}, R'_{N'}, z_{-N'}, z'_{N'}).$$

For each game $\langle \mathcal{D}, S, s, R^0\rangle$ the set of **strong Nash equilibria** is $\boldsymbol{\mathcal{N}^*\langle \mathcal{D}, S, s, R^0\rangle}$ and the set of **strong Nash equilibrium outcomes** is $\boldsymbol{\mathcal{O}^*\langle \mathcal{D}, S, s, R^0\rangle}$.

If for any two selections of S, s and s', $\mathcal{O}^*\langle \mathcal{D}, S, s, R^0\rangle = \mathcal{O}^*\langle \mathcal{D}, S, s', R^0\rangle$, we denote this common set by $\boldsymbol{\mathcal{O}^*\langle \mathcal{D}, S, R^0\rangle}$.

Let S and S' be two SCCs, $s' \in S'$, $\mathcal{D} \subseteq \mathcal{R}$, and $\mathcal{D}' \subseteq \mathcal{R}$. The pair $\boldsymbol{\langle \mathcal{D}', S'\rangle}$ **implements** $\boldsymbol{S}$ **in strong Nash equilibria on** $\boldsymbol{\mathcal{D}}$ if for each $R^0 \in \mathcal{D}$, $\mathcal{O}^*\langle \mathcal{D}', S', R^0\rangle = S(R^0)$. The correspondence $\boldsymbol{S'}$ **naturally implements** $\boldsymbol{S}$ **in strong Nash equilibria on** $\boldsymbol{\mathcal{D}}$ if $\langle \mathcal{D}, S'\rangle$ implements S in strong Nash equilibria on $\mathcal{D}$.

2.4 Decomposition and perturbation lemmas

In this section we state three lemmas which we use repeatedly in our proofs.

First, let us introduce some definitions. Let $K \subseteq N$ and $B \subseteq A$ be such that $|B| = |K|$. For each $R \in \mathcal{R}(B)^K$, each $\{m, m'\} \subset \mathbb{R}$, each $z \equiv (x, \mu) \in Z(B, R, m)$, and each $z' \equiv (x', \mu') \in Z(B, R, m')$, let $K_z^{z'} \equiv \{i \in K : z'\, P_i\, z\}$, $K_{z'z} \equiv \{i \in K : z'\, I_i\, z\}$, $B_z^{z'} \equiv \{\alpha \in B : x'_\alpha > x_\alpha\}$, and $B_{zz'} \equiv \{\alpha \in B : x'_\alpha = x_\alpha\}$. The following lemma states that the set B can be partitioned into three subsets: the objects received, both at z and z', by agents who prefer z' to z; the objects received, both at z and z', by agents who prefer z to z'; and the objects received, both at z and z', by agents who are indifferent between z and z'. Moreover, these sets are $B_z^{z'}$, $B_{z'}^{z}$, and $B_{z'z}$, respectively.

Lemma 1 (Decomposition lemma)**.** Let $K \subseteq N$, $B \subseteq A$ be such that $|B| = |K|$, $R \in \mathcal{R}(B)^K$, $\{m, m'\} \subset \mathbb{R}$, $z \equiv (x, \mu) \in F(B, R, m)$, and $z' \equiv (x', \mu') \in F(B, R, m)$. The functions μ and μ' are bijections between $K_z^{z'}$ and $B_z^{z'}$, between $K_{z'z}$ and $B_{z'z}$, and between $K_{z'}^{z}$ and $B_{z'}^{z}$.

We refer the reader to Alkan et al. (1991, Lemma 3) for a proof of Lemma 1.

The next lemma states that starting from a *fair* allocation for some economy e, for each positive amount of money, ε, one can find *fair* allocations for the economy obtained from e by adding ε to its budget, and at which each agent is better off.

Lemma 2 (Right perturbation lemma)**.** Let $K \subseteq N$, $B \subseteq A$ be such that $|B| = |K|$, $R \in \mathcal{R}(B)^K$, $m \in \mathbb{R}$, $z \equiv (x, \mu) \in F(B, R, m)$. For each $\varepsilon \in \mathbb{R}_{++}$ there is $z(\varepsilon) \in F(B, R, m + \varepsilon)$ such that for each $i \in K$, $z(\varepsilon)\, P_i\, z$.

We refer the reader to Alkan et al. (1991, Theorem 4) for a proof of Lemma 2.[5]

The next lemma states that starting from a *fair* allocation for some economy e, for each positive amount of money, ε, one can find *fair* allocations for the economy obtained from e by subtracting ε from its budget, and at which each agent is worse off.

Lemma 3 (Left perturbation lemma)**.** Let $K \subseteq N$, $B \subseteq A$ be such that $|B| = |K|$, $R \in \mathcal{R}(B)^K$, $m \in \mathbb{R}$, $z \equiv (x, \mu) \in F(B, R, m)$. For each $\varepsilon \in \mathbb{R}_{++}$ there is $z^\varepsilon \in F(B, R, m - \varepsilon)$ such that for each $i \in K$, $z \, P_i \, z(\varepsilon)$.

We refer the reader to Velez (2007) for a proof of Lemma 3. Let us emphasize that even though Lemmas 2 and 3 are symmetric statements, there is an asymmetry in the techniques that successfully have established them.

3 Generalized Money Rawlsian Fair correspondences

A Money Rawlsian Fair allocation is a *fair* allocation that maximizes the minimal consumption of money among the set of *fair* allocations given the resources available. Let us formalize this definition. Let $K \subseteq N$ and $e \equiv (B, R, m) \in \mathcal{E}^K$. The set of **Money Rawlsian Fair (MRF) allocations for $\boldsymbol{e}$** is:

$$\boldsymbol{\Re(e)} \equiv \underset{(x,\mu)\in F(e)}{\arg\max} \left\{ \min_{\beta\in B} x_\beta \right\}.$$

Since $F(e)$ is compact, then $\Re(e)$ is non-empty.[6]

The **MRF correspondence, $\boldsymbol{\Re}$**, associates to each $R \in \mathcal{R}^N$ the set $\Re(A, R, M)$.

Selecting a money Rawlsian allocation one guarantees that the minimal consumption of money at the allocation is maximal among all the available fair allocations given agents' preferences and resources in the economy. More generally, one may want to assign different importance to the consumption of money associated to each object and maximize the minimal adjusted consumption of money. To formalize this idea, let $K \subseteq N$, $B \subseteq A$ be such that $|K| = |B|$, $\mathcal{I}$ be the space of real-valued, continuous, and monotone increasing functions defined on $\mathbb{R}$, $f \equiv (f_\beta)_{\beta\in B} \in \mathcal{I}^B$, and $e \equiv (B, R, m) \in \mathcal{E}^K$. The set of **Generalized Money Rawlsian Fair (GMRF) allocations with respect to $\boldsymbol{f}$ for $\boldsymbol{e}$** is:

$$\boldsymbol{\Re^f(e)} \equiv \underset{(x,\mu)\in F(e)}{\arg\max} \left\{ \min_{\beta\in B} f_\beta(x_\beta) \right\}.$$

Since $F(e)$ is compact, then $\Re^f(e)$ is non-empty.[7]

[5]Velez (2007) provides an alternative proof of Lemma 2.

[6]Since preferences are continuous, then $F(e)$ is closed. Now, since at each $z \in Z(B, R, m)$, at least, one agent consumes no less than $\frac{m}{|K|}$, then $F(e)$ is bounded, and thus compact.

[7]See footnote 6.

Let $f \in \mathcal{I}^A$. The **GMRF correspondence with respect to $\boldsymbol{f}$, $\boldsymbol{\Re^f}$,** associates to each $R \in \mathcal{R}^N$ the set $\Re^f(A, R, M)$.

Money Rawlsian allocations are GMRF allocations with respect to the family of identity functions indexed by the set of objects in the economy.

The following example illustrates that, in some cases, depending on the structure of the family of functions f, the description of GMRF allocations with respect to f can be simplified.

Example 1. Let $K \subseteq N$, $B \subseteq A$ be such that $|K| = |B|$, $\beta \in B$, and $e \equiv (B, R, m) \in \mathcal{E}^K$. The set of **$\boldsymbol{\beta}$-money maximal fair allocations for $\boldsymbol{e}$** is:

$$\boldsymbol{\Re^{\beta}(e)} \equiv \underset{(x,\mu)\in F(e)}{\arg\max} \{x_\beta\}.$$

One can easily see that for each family of functions $f \in \mathcal{I}^B$ such that for each $\delta \in B \setminus \{\beta\}$, $\sup \text{Range}(f_\beta) \leq \inf \text{Range}(f_\delta)$, $\Re^\beta(e)$ coincides with the set $\Re^f(e)$.[8]

3.1 Characterization of Generalized Money Rawlsian Fair allocations

In this subsection we develop a mathematical characterization of GMRF allocations, which allows us to understand, in the next section, how agents manipulate generalized Money Rawlsian correspondences.

Let $K \subseteq N$, $e \equiv (B, R, m) \in \mathcal{E}^K$, and $z \equiv (x, \mu) \in Z(e)$. A binary relation $\succeq(R, z)$ on B is defined as follows: for each $\{\alpha, \beta\} \subseteq B$, $\alpha \succeq(R, z)\, \beta$ if and only if there are $\{\beta_0, \ldots, \beta_T\} \subseteq B$ such that $\beta_0 = \alpha$, $\beta_T = \beta$, and

$$(x_{\beta_0}, \beta_0)\, I_{\mu^{-1}(\beta_0)}\, (x_{\beta_1}, \beta_1) \ldots (x_{\beta_{T-1}}, \beta_{T-1})\, I_{\mu^{-1}(\beta_{T-1})}\, (x_{\beta_T}, \beta_T).$$

The next lemma states two basic properties of $\succeq(R, z)$. This binary relation is transitive. Let $i \in N$. If an object dominates the object received by agent i at z, then the domination relation is preserved if agent i's preferences are replaced by some arbitrary preferences.

Lemma 4. For each $K \subseteq N$, each $e \equiv (B, R, m) \in \mathcal{E}^K$, and each $z \equiv (x, \mu) \in Z(e)$, $\succeq(R, z)$ is transitive. Moreover, for each $i \in K$ and each $R'_i \in K$, if $\alpha \in B$ is such that $\alpha \succeq(R, z)\, \mu(i)$, then $\alpha \succeq(R_{-i}, R'_i, z)\, \mu(i)$.

We omit the straightforward proof.

Let $K \subseteq N$, $B \subseteq A$ be such that $|B| = |K|$, $R \in \mathcal{R}(B)^K$, and $\{m, m'\} \subseteq \mathbb{R}$. Suppose that $z \equiv (x, \mu) \in F(B, R, m)$. The following lemma states that for each allocation $z' \equiv (x', \mu') \in F(B, R, m')$ and each $\beta \in B$, if $x'_\beta \geq x_\beta$, then for each other object $\alpha \in B$ that dominates β with respect to $\succeq(R, z)$, $x'_\alpha \geq x_\alpha$. Moreover, the statement is also true for

[8] For instance, $f_\beta : \mathbb{R} \to]-\infty, 0[$ and for each $\delta \in B \setminus \{\beta\}$, $f_\delta : \mathbb{R} \to]0, +\infty[$.

strict inequalities, i.e., if $x'_\beta > x_\beta$ and $\alpha \succeq\!\!\!\!\!\!\prec(R,z)\beta$, then $x'_\alpha > x_\alpha$. Thus, the consumptions of money in fair allocations are "increasing with respect to the relation $\succeq\!\!\!\!\!\!\prec(R,z)$".

Lemma 5. Let $K \subseteq N$, $B \subseteq A$ be such that $|B| = |K|$, $R \in \mathcal{R}(B)^K$, $\{m, m'\} \subseteq \mathbb{R}$, $z \equiv (x,\mu) \in F(B,R,m)$, $z' \equiv (x',\mu') \in F(B,R,m')$, and $\beta \in B$. If $x'_\beta \geq x_\beta$ and $\alpha \in B$ is such that $\alpha \succeq(R,z)\,\beta$, then $x'_\alpha \geq x_\alpha$. Moreover, if $x'_\beta > x_\beta$ and $\alpha \in B$ is such that $\alpha \succeq\!\!\!\!\!\!\prec(R,z)\,\beta$, then $x'_\alpha > x_\alpha$.

Proof. Let K, B, R, $\{m,m'\}$, z, z', and β be as in the statement of the lemma. Suppose that $x'_\beta \geq x_\beta$ and $\alpha \in B$ is such that $\alpha \succeq(R,z)\,\beta$. We want to prove that $x'_\alpha \geq x_\alpha$. Since $\alpha \succeq\!\!\!\!\!\!\prec(R,z)\,\beta$, then there are $\{\beta_0,\ldots,\beta_T\} \subseteq B$ such that $\beta_0\, I_{\mu^{-1}(\beta_0)}\, \beta_1 \ldots I_{\mu^{-1}(\beta_{T-1})}\, \beta_T$, $\beta_0 = \alpha$, and $\beta_T = \beta$. Since $x'_\beta \geq x_\beta$, then $\beta_T \in B_z^{z'} \cup B_{z'z}$. Let $1 \leq t \leq T$. Suppose that $\{\beta_t, \beta_{t+1}, \ldots, \beta_T\} \subseteq B_z^{z'} \cup B_{z'z}$. We claim that $\beta_{t-1} \in B_z^{z'} \cup B_{z'z}$. Suppose w.l.o.g. that $i = \mu^{-1}(\beta_{t-1})$. Since $z' \in F(B,R,m')$ and $x'_{\beta_t} \geq x_{\beta_t}$, then $z'_i\, R_i\, (x'_{\beta_t}, \beta_t)\, R_i\, (x_{\beta_t}, \beta_t)\, I_i\, z_i$. Thus, $i \in K_z^{z'} \cup K_{z'z}$. By Lemma 1, $\beta_{t-1} = \mu(i) \in B_z^{z'} \cup B_{z'z}$. We conclude from the recursive argument that $\beta_0 \in B_z^{z'} \cup B_{z'z}$. Thus, $x'_\alpha \geq x_\alpha$.

Finally, a similar argument shows that if $x'_\beta > x_\beta$ and $\alpha \in B$ is such that $\alpha \succeq\!\!\!\!\!\!\prec(R,z)\,\beta$, then $x'_\alpha > x_\alpha$. □

We now characterize the GMRF allocations with respect to a family of functions f. These allocations are the *fair* allocations for which each object dominates, in terms of the binary relation induced by the allocation, one of the objects that is received by an agent whose f-adjusted consumption of money is minimal.

Proposition 1. Let $K \subseteq N$, $B \subseteq A$ be such that $|B| = |K|$, $f \in \mathcal{I}^B$, $e \equiv (B,R,m) \in \mathcal{E}^K$, and $z \in Z(e)$. Then $z \in \Re^f(e)$ if and only if $z \in F(e)$ and for each $\alpha \in B$, there is $\beta \in \arg\min_{\delta\in B} f_\delta(x_\delta)$ such that $\alpha \succeq\!\!\!\!\!\!\prec(R,z)\,\beta$.

Proof. Let K, B, f, R, e, and $z \equiv (x,\mu)$ be as in the statement of the lemma.

($\Rightarrow$) We prove that if $z \in \Re^f(e)$ and $\alpha \in B$, then there is $\beta \in \arg\min_{\delta\in B} f_\delta(x_\delta)$ such that $\alpha \succeq\!\!\!\!\!\!\prec(R,z)\,\beta$. Let $C \equiv \{\alpha \in B :$ there is $\beta \in \arg\min_{\delta\in B} x_\delta,$ s.t. $\alpha \succeq\!\!\!\!\!\!\prec(R,z)\,\beta\}$. We want to prove that $C = B$. Suppose by contradiction that $B \setminus C \neq \emptyset$.

Let $K(C) \equiv \mu^{-1}(C)$ and $K(B\setminus C) \equiv \mu^{-1}(B \setminus C)$. We claim that for each $i \in K(B\setminus C)$ and each $\beta \in C$, $z_i\, P_i\, (x_\beta, \beta)$. Suppose by contradiction that there is $\alpha \in B$ such that $(x_\alpha, \alpha)\, R_i\, z_i$. Since $\Re(e) \subseteq F(e)$, then $z_i\, I_i\, (x_\alpha, \alpha)$. Thus, $\mu(i) \succeq(R,z)\,\alpha$. Now, since there is $\beta \in \arg\min_{\delta\in B} f_\alpha(x_\delta)$ such that $\alpha \succeq(R,z)\,\beta$, then by Lemma 4, $\mu(i) \succeq(R,z)\,\beta$. Consequently, $\mu(i) \in C$. This is a contradiction.

Since preferences are continuous, then there is $\varepsilon \in \mathbb{R}_{++}$ such that for each $i \in K(B\setminus C)$ and each $j \in K(C)$, $(x_{\mu(i)} - \varepsilon, \mu(i))\, P_i\, (x_{\mu(j)} + \varepsilon, \mu(j))$. Since the relation $\succeq\!\!\!\!\!\!\prec(R,z)$ is reflexive, then $\arg\min_{\delta\in B} f_\delta(x_\delta) \subseteq C$. Thus, since f is a family of continuous functions, then ε can be chosen small enough that $\min_{\delta\in B\setminus C} f_\delta(x_\delta - \varepsilon) > \min_{\delta\in C} f_\delta(x_\delta + \varepsilon)$. Since $z \in F(e)$, then $z|_C \equiv (x|_C, \mu|_{K(C)}) \in F(C, R|_{K(C)}, \sum_{\delta\in C} x_\delta)$.[9] From Lemma 2, there exists $v \equiv (w,\lambda) \in$

[9] $R|_{K(C)} \equiv (R_i|_{\mathbb{R}\times C})_{i\in K(C)}$.

$F(C,R|_{K(C)},\sum_{\delta\in C}x_\delta+\varepsilon)$ such that for each $i\in K(C)$, $v_i\,P_i\,z_i$. Thus, from Lemma 1, $w\gg x|_C$.[10] Consequently, for each $\alpha\in C$, $x_\alpha+\varepsilon>w_\alpha$. Since $z\in F(e)$, then $z|_{B\setminus C}\equiv(x|_{B\setminus C},\mu|_{K(B\setminus C)})\in F(B\setminus C,R|_{K(B\setminus C)},\sum_{\delta\in B\setminus C}x_\delta)$. Let $m^*\equiv\sum_{\delta\in B\setminus C}x_\delta-\varepsilon$. From Lemma 3, there exists $y\equiv(u,\sigma)\in F(B\setminus C,R|_{K(B\setminus C)},m^*)$ such that for each $i\in K(B\setminus C)$, $z_i\,P_i\,y_i$. Thus, from Lemma 1, $x|_{B\setminus C}\gg u$. Consequently, for each $\alpha\in B\setminus C$, $u_\alpha>x_\alpha-\varepsilon$.

Let $x^\varepsilon\in\mathbb{R}^B$ be the vector obtained by concatenating w and u. Let $\mu^\varepsilon\in B^K$ be the bijection that coincides with λ on $K(C)$ and with σ on $K(B\setminus C)$. Let $z^\varepsilon\equiv(x^\varepsilon,\mu^\varepsilon)$. Since $\sum_B x^\varepsilon_\alpha=\sum_{\delta\in C}x_\delta+\varepsilon+\sum_{\delta\in B\setminus C}x_\delta-\varepsilon=\sum_{\delta\in B}x_\delta$, then $z^\varepsilon\in Z(e)$. We claim that $z^\varepsilon\in F(e)$, i.e., for each $\{i,j\}\subseteq K$, $z^\varepsilon_i\,R_i\,z^\varepsilon_j$. There are three cases.

Case 1: $\{i,j\}\subseteq K(C)$ or $\{i,j\}\subseteq K(B\setminus C)$. This case is trivial because $z^\varepsilon|_C=v\in F(C,R|_{K(C)},\sum_{\delta\in C}x_\delta+\varepsilon)$ and $z^\varepsilon|_{B\setminus C}=y\in F(B\setminus C,R|_{K(B\setminus C)},\sum_{\delta\in B\setminus C}x_\delta-\varepsilon)$.

Case 2: $i\in K(C)$ and $j\in K(B\setminus C)$. Since, $z^\varepsilon_i=v_i\,P_i\,z_j$ and $z\in F(e)$, then $z^\varepsilon_i\,P_i\,z_j$. Now, since $x|_{B\setminus C}\gg u$, then $z_j\,P_i\,z^\varepsilon_j$. Consequently, $z^\varepsilon_i\,P_i\,z^\varepsilon_j$.

Case 3: $j\in K(C)$ and $i\in K(B\setminus C)$. Since $z^\varepsilon|_{B\setminus C}=y\in F(B\setminus C,R|_{K(B\setminus C)},\sum_{\delta\in B\setminus C}x_\delta-\varepsilon)$ and for each $\alpha\in B\setminus C$, $u_\alpha>x_\alpha-\varepsilon$, then $z^\varepsilon_i\,R_i\,(x^\varepsilon_{\mu(i)},\mu(i))\,P_i\,(x_{\mu(i)}-\varepsilon,\mu(i))$. Recall that $(x_{\mu(i)}-\varepsilon,\mu(i))\,P_i\,(x_{\mu(j)}+\varepsilon,\mu(j))$. Thus, $z^\varepsilon_i\,P_i\,(x_{\mu(j)}+\varepsilon,\mu(j))$. Now, since for each $\alpha\in C$, $x_\alpha+\varepsilon>w_\alpha$, then $(x_{\mu(j)}+\varepsilon,\mu(j))\,P_i\,z^\varepsilon_j$. Thus, $z^\varepsilon_i\,P_i\,z^\varepsilon_j$.

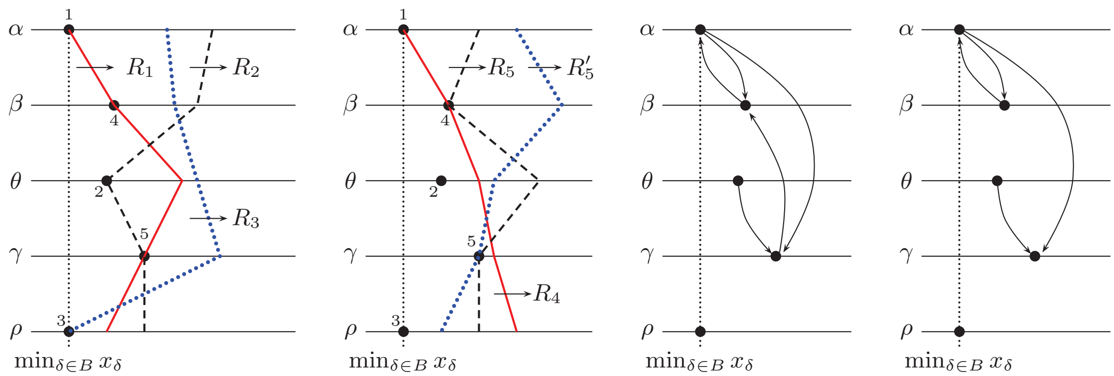
Now, we claim that $\min_{\delta\in B}f_\delta(x^\varepsilon_\delta)>\min_{\delta\in B}f_\delta(x^\varepsilon_\delta)$. Since $\min_{\delta\in B\setminus C}f_\delta(x_\delta-\varepsilon)>\min_{\delta\in C}f_\delta(x_\delta+\varepsilon)$, then $\arg\min_{\delta\in B}f_\delta(x^\varepsilon_\delta)\subset C$. Since $x^\varepsilon|_C=w\gg x|_C$, then $\min_{\delta\in B}f_\delta(x^\varepsilon_\delta)>\min_{\delta\in B}f_\delta(x_\delta)$. Thus, $z\notin\Re(e)$. This is a contradiction.

($\Leftarrow$) Suppose that $z\in F(e)$ and for each $\alpha\in B$ there is $\beta\in\arg\min_{\delta\in B}f_\delta(x_\delta)$ such that $\alpha\succeq(R,z)\,\beta$. We want to prove that $z\in\Re^f(e)$. Suppose by contradiction that there is $z'\equiv(x',\mu')\in F(e)$ such that $\min_{\delta\in B}f_\delta(x'_\delta)>\min_{\delta\in B}f_\delta(x_\delta)$. Thus, since f is a family of monotone increasing functions, then for each $\beta\in\arg\min_{\delta\in B}f_\delta(x_\delta)$, $x'_\beta>x_\beta$. Now, let $\alpha\in B$. By the hypothesis there is $\beta\in\arg\min_{\delta\in B}f_\delta(x_\delta)$ such that $\alpha\succeq(R,z)\,\beta$. By Lemma 5 and since $x'_\beta>x_\beta$ and $\{z,z'\}\in F(e)$, then $x'_\alpha>x_\alpha$. Thus, $\sum_{\delta\in B}x'_\delta>\sum_{\delta\in B}x_\delta$. This is a contradiction. □

Let $e\equiv(B,R,m)\in\mathcal{E}$, $z\in F(e)$, and $f\in\mathcal{I}^B$. Proposition 1 provides a simple test to verify whether $z\in\Re^f(e)$ or $z\notin\Re^f(e)$. Figure 1 illustrates it for money Rawlsian allocations.

Let $e\equiv(B,R,m)\in\mathcal{E}$, $f\in\mathcal{I}^B$, and $\varepsilon\in\mathbb{R}_{++}$. The following corollary establishes properties of GMRF corresponcences. First, they are essentially single-valued. Second, they satisfy a strong form of solidarity when budget changes: Let R be a preference profile and S a GMRF correspondence. At the S-optimal allocations for R, the welfare of each

[10] We follow the convention of vector inequalities: for each $B\subset A$ and each $\{x,x'\}\subset\mathbb{R}^B$, $x'\gg x$ if and only if for each $\beta\in B$, $x'_\beta>x_\beta$, and $x'\geq x$ if and only if for each $\beta\in B$, $x'_\beta\geq x_\beta$.



(a) (b) (c) (d)

Figure 1: Verifying whether $z \in \Re(e)$ or $z \notin \Re(e)$. Let $K \equiv \{1,2,3,4,5\}$ and $B \equiv \{\alpha,\beta,\theta,\gamma,\rho\}$. Panels (a) to (d) display the the consumption space $\mathbb{R}\times B$ (for some range of consumptions of money); each point x_α on the axis corresponding to object α, represents bundle (x_α,α). Let $R \in \mathcal{R}(B)^K$, $m \in \mathbb{R}$, and $z \equiv (x,\mu) \in Z(B,R,m)$. Panels (a) and (b) display the consumption of each agent at z as a black dot with the identity of the agent next to it. Panel (a) also displays agents 1, 2, and 3's "indifference curves" through their respective allotment at z, i.e., bundles that are indifferent for the agent are joined by a line. Panel (b) displays agents 4 and 5's "indifference curves" at their respective allotment at z. Panel (b) also displays alternative preferences for agent 5, i.e., R'_5. To check if $z \in \Re(e)$ one has to: (*i*) verify that $z \in F(e)$; and (*ii*) construct "arrows" from each agent's allotment to the other bundles at the allocation for which the agent is indifferent to her own consumption; then, verify that from each consumption bundle at the allocation there is a "path of arrows" which "flows" from the reference bundle to one of the bundles with minimal consumption of money. If at least one of these two tests fail, then $z \notin \Re(e)$. Panel (c) displays this construction for z at profile R. Observe that z passes both tests (*i*) and (*ii*). Thus, $z \in \Re(e)$. Panel (d) displays this construction for (R_{-5}, R'_5). Observe that z passes test (*i*), but not (*ii*): there is no path of arrows flowing from the bundles with objects θ and γ to one of the bundles with minimal consumption of money. Thus, $z \notin \Re(B, R_{-5}, R'_5, m)$.

agent is an increasing function of the budget. In our interpretation of the model as the allocation of rooms and rent among housemates, this means that each agent benefits if the rent decreases; analogously, each agent contributes (in welfare terms) if the rent increases.

Corollary 1. (Alkan et al., 1991, Theorem 6) Let $K \subseteq N$, $e \equiv (B,R,m) \in \mathcal{E}^K$. For each $z \equiv (x,\mu) \in \Re^f(e)$ and each $z \equiv (x',\mu') \in \Re^f(e)$, $x = x'$ and for each $i \in K$, $z'\, I_i\, z$. For each $\varepsilon \in \mathbb{R}_{++}$, each $z \equiv (x,\mu) \in \Re^f(e)$, and each $z' \equiv (x',\mu') \in \Re^f(B,R,m+\varepsilon)$, $x' \gg x$ and for each $i \in K$, $z'\, P_i\, z$.

Proof. Let K, e, and ε be as in the statement of the corollary. Suppose that $z \equiv (x,\mu) \in \Re^f(e)$ and $z \equiv (x',\mu') \in \Re(e)$. We will prove that $x = x'$. Since $\min_{\delta\in B} f_\delta(x_\delta) = \min_{\delta\in B} f_\delta(x'_\delta)$, then for each $\beta \in \arg\min_{\delta\in B} f_\delta(x_\delta)$, $x'_\beta \geq x_\beta$. Since $z \in \Re^f(e)$, then by Proposition 1, for each $\alpha \in B$, there is $\beta \in \arg\min_{\delta\in B} f_\delta(x_\delta)$ such that $\alpha \succeq(R,z)\, \beta$. Thus, by Lemma 5, $x'_\alpha \geq x_\alpha$. Consequently, $x' \geq x$. Since $\{z,z'\} \subset Z(e)$, then $x = x'$ and thus, $B_{z'z} = B$. By Lemma 1, $K_{z'z} = K$. Thus, for each $i \in K$, $z'\, I_i\, z$.

Now, let $z \equiv (x,\mu) \in \Re^f(e)$, $z' \equiv (x',\mu') \in \Re^f(B,R,m+\varepsilon)$. By Lemma 2 there is $z^\varepsilon \in F(B,R,m+\varepsilon)$ such that for each $j \in K$, $z^\varepsilon\, P_i\, z$. Thus, $\min_{\delta\in B} f_\delta(x'_\delta) > \min_{\delta\in B} f_\delta(x_\delta)$. Let $\alpha \in B$. By Proposition 1, there is $\beta \in \arg\min_{\delta\in B} f_\delta(x_\delta)$ such that $\alpha \succeq(R,z)\, \beta$. Now, since $x'_\beta > x_\beta$, then by Lemma 5, $x'_\alpha > x_\alpha$. Thus, $B_z^{z'} = B$ and by Lemma 1, $K_z^{z'} = K$. Thus, for each $i \in K$, $z' \, P_i \, z$. □

Let $e \equiv (B, R, m) \in \mathcal{E}^K$. The following corollary characterizes the set $\Re(e)$: An allocation $z \in Z(e)$ is a MRF allocation for e if and only if it is a *fair* allocation for e for which each object dominates, in terms of the binary relation associated to (R, z), one of the objects that is consumed by an agent whose consumption of money is minimal.

Corollary 2. (Alkan et al., 1991; Alkan, 1994) Let $K \subseteq N$, $B \subseteq A$ be such that $|B| = |K|$, $e \equiv (B, R, m) \in \mathcal{E}^K$, and $z \in Z(e)$. Then $z \in \Re(e)$ if and only if $z \in F(e)$ and for each $\alpha \in B$ there is $\beta \in \arg\min_{\delta \in B} x_\delta$ such that $\alpha \succeq(R, z)\, \beta$.

We omit the straightforward proof.

Let $e \equiv (B, R, m) \in \mathcal{E}^K$ and $\beta \in B$. The following corollary characterizes the set $\Re^\beta(e)$: An allocation $z \in Z(e)$ is an element of $\Re^\beta(e)$ if and only if it is a *fair* allocation for e for which each object dominates object β, in terms of the binary relation associated to (R, z).

Corollary 3. Let $K \subseteq N$, $B \subseteq A$ be such that $|B| = |K|$, $\beta \in B$, $e \equiv (B, R, m) \in \mathcal{E}^K$, and $z \in Z(e)$. Then $z \in \Re^\beta(e)$ if and only if $z \in F(e)$ and for each $\alpha \in B$, $\alpha \succeq(R, z)\, \beta$.

We omit the straightforward proof.

3.2 How Rawlsian are Generalized Money Rawlsian Fair allocations?

This subsection investigates the sense in which GMRF correspondences achieve some form of "Democratic Equality." In Rawls's Theory of Justice, this principle of distributive justice calls for the minimal welfare among the agents, measured by means of some index, to be maximized (Rawls, 1972). GMRF allocations instead, maximize among the fair allocations, the minimal individual adjusted consumption of money. Both objectives are seemingly independent, but there is a formal connection between them.

Fix $R \in \mathcal{R}^N$. Let Λ be a function that associates to each $m \in \mathbb{R}$ a subset of allocations $\Lambda(m) \subseteq F(A, R, m)$. Then, Λ has the **Maxmin property** if there exists a continuous representation of R, u, such that for each $m \in \mathbb{R}$,

$$\Lambda(m) \equiv \underset{z \in F(A,R,m)}{\arg\max} \left\{ \min_{i \in N} u_i(z) \right\}.$$

Suppose that a function Λ has the Maxmin property and let $m \in \mathbb{R}$. The set $\Lambda(m)$ contains the fair allocations for (A, R, m) that maximize the agents' individual minimum welfare with respect to a representation of preferences. So the sets selected by Λ can be interpreted in the following way. These are the allocations selected by an arbitrator who, based on the democratic equality principle, fixes a representation for R, u, and selects the fair allocations that maximize the agents' individual minimum welfare with respect to u.

The following theorem states that the functions induced by GMRF correspondences have the Maxmin property.

Theorem 1. Let $f \in \mathcal{I}^A$. The function that associates to each $m \in \mathbb{R}$ the set $\Re^f(A, R, m)$ has the Maxmin property.

We refer the reader to Velez (2007) for the proof.

Consider the problem of fairly allocating the objects A and an amount of money m among a group of agents with preferences R. If one were to include the democratic equality principle as a consideration in the solution to this problem, it would be necessary to select a utility representation with respect to which the agents' minimal welfare is maximized. Theorem 1 states that by following the recommendations of a GMRF correspondence this task is performed implicitly.

4 Manipulation of Generalized Money Rawlsian Fair correspondences

In this section we study the manipulation of GMRF correspondences. Our main theorem characterizes the Nash-equilibrium-outcome correspondence of the game form associated to each $\Re^f$ and each of its selections in the domain $\mathcal{R}$. It is the correspondence, F, which associates to each economy its set of *fair* allocations.

Theorem 2. Let $f \in \mathcal{I}^A$. For each $R^0 \in \mathcal{R}^N$, $\mathcal{O}\langle \mathcal{R}, \Re^f, R^0 \rangle = F(R^0)$.

The proof of Theorem 2 follows from four lemmas.

The key to understanding the extent to which agents can manipulate GMRF correspondences is to establish under what conditions an agent is able to unequivocally gain by changing her report given the others' reports. The following lemma states such conditions.

Lemma 6. Let $K \subseteq N$ $e \equiv (B, R, m) \in \mathcal{E}^K$, and $z \equiv (x, \mu) \in F(e)$. If $i \in K$ is such that for each $\alpha \in B$, $\alpha \succeq(R, z)\, \mu(i)$, then for each $\varepsilon \in \mathbb{R}_{++}$, there is an allocation $z^\varepsilon \equiv (x^\varepsilon, \mu^\varepsilon) \in Z(B, R, m + \varepsilon)$ such that:

1. $x^\varepsilon \gg x$,
2. for each $j \in K \setminus \{i\}$ and each $l \in K$, $z^\varepsilon_j \, R_j \, z^\varepsilon_l$,
3. $\mu^\varepsilon(i) = \mu(i)$, and
4. for each $\alpha \in B$, $\alpha \succeq(R, z^\varepsilon)\, \mu^\varepsilon(i)$.

Proof. Let K, e, $z \equiv (x, \mu)$, and ε be as in the statement of the lemma. Let $\widehat{R}_i$ be a preference relation such that for each $\alpha \in B \setminus \{\mu(i)\}$, $(x_{\mu(i)} - \varepsilon, \mu(i))\, \widehat{P}_i\, (x_\alpha + 2\varepsilon, \alpha)$. Since $z \in F(e)$, then $z \in F(B, R_{-i}, \widehat{R}_i, m)$. Observe that since for each $\alpha \in B$, $\alpha \succeq(R, z)\, \mu(i)$, then for each $\alpha \in B$, $\alpha \succeq(R_{-i}, \widehat{R}_i, z)\, \mu(i)$. Thus, $z \in \Re^{\mu(i)}(B, R_{-i}, \widehat{R}_i, m)$. Let $z^\varepsilon \equiv (x^\varepsilon, \mu^\varepsilon) \in \Re^{\mu(i)}(B, R_{-i}, \widehat{R}_i, m+\varepsilon)$. By Corollary 1, $x^\varepsilon \gg x$. Since $\Re^{\mu(i)}(B, R_{-i}, \widehat{R}_i, m) \subseteq F(B, R_{-i}, \widehat{R}_i, m)$, then for each $j \in K \setminus \{i\}$ and each $l \in K$, $z^\varepsilon_j\, R_j\, z^\varepsilon_l$. Since for each $j \in K \setminus \{i\}$, $z_i\, \widehat{R}_i\, z_j$, for each $\alpha \in B \setminus \{\mu(i)\}$, $(x_{\mu(i)} - \varepsilon, \mu(i))\, \widehat{P}_i\, (x_\alpha + 2\varepsilon, \alpha)$, and $x^\varepsilon \gg x$, then $\mu(i)^\varepsilon = \mu(i)$. Now, since $z^\varepsilon \in \Re^{\mu(i)}(B, R_{-i}, \widehat{R}_i, m)$, then by Corollary 3, for each $\alpha \in B$, $\alpha \succeq(R_{-i}, \widehat{R}_i, z^\varepsilon)\, \mu^\varepsilon(i)$. Thus, for each $\alpha \in B$, $\alpha \succeq(R, z^\varepsilon)\, \mu^\varepsilon(i)$. □

The following lemma states that at equilibrium the binary relation induced by the reported preferences and the equilibrium outcome, is maximal (with respect to inclusion). That is, a necessary condition at equilibrium is that each object dominates each other object with respect to the binary relation induced by the reported preferences and the equilibrium outcome.

Lemma 7. Let $\mathcal{D} \subseteq \mathcal{R}$ be such that $\mathcal{D} \supseteq \mathcal{Q}$, $f \in \mathcal{I}^A$, and $r \in \Re^f$. For each $R^0 \in \mathcal{R}^N$, if $(R, z) \in \mathcal{N}\langle \mathcal{D}, \Re^f, r, R^0\rangle$ then for each $\{\alpha, \beta\} \in A$, $\alpha \succeq(R, O\langle \mathcal{D}, \Re^f, r\rangle(R, z))\, \beta$.

Proof. Let $\mathcal{D}$ and r be as in the statement of the lemma. Let $R^0 \in \mathcal{R}^N$. Suppose that $(R, z) \in \mathcal{N}\langle \mathcal{D}, \Re^f, r, R^0\rangle$ and assume w.l.o.g. that $z \equiv (x, \mu) = O\langle \mathcal{D}, \Re^f, r\rangle(R, z)$.[11] We will prove that for each $\{\alpha, \beta\} \in A$, $\alpha \succeq(R, z)\, \beta$. Suppose by contradiction that there are $\{\alpha, \beta\} \in A$, such that $\neg(\alpha \succeq(R, z)\, \beta)$. Let $C \equiv \{\delta \in B : \delta \succeq(R, z)\, \beta\}$. Since $\alpha \notin C$, then $B \setminus C \neq \emptyset$. We claim that for each $\gamma \in B \setminus C$ and each $\delta \in C$, $(x_\gamma, \gamma)\, P_{\mu^{-1}(\gamma)}\, (x_\delta, \delta)$. Suppose by contradiction that there are $\gamma \in B \setminus C$ and $\delta \in C$, such that $(x_\delta, \delta)\, R_{\mu^{-1}(\gamma)}\, (x_\gamma, \gamma)$. Since $z \in \Re^f(R) \subseteq F(R)$, then $(x_\gamma, \gamma)\, R_{\mu^{-1}(\gamma)}\, (x_\delta, \delta)$. Thus, $(x_\gamma, \gamma)\, I_{\mu^{-1}(\gamma)}\, (x_\delta, \delta)$ and consequently, $\gamma \succeq(R, z)\, \delta$. Now, since $\delta \in C$, then $\delta \succeq(R, z)\, \beta$. By Lemma 4, $\gamma \succeq(R, z)\, \beta$. This is a contradiction because $\gamma \in B \setminus C$.

We claim that for each $\gamma \in B \setminus C$, there is $\rho \in (B \setminus C) \cap \arg\min_{\delta \in B} f_\delta(x_\delta)$ such that $\gamma \succeq(R, z)\, \rho$.[12] Since $z \in \Re^f(R)$, then by Proposition 1, there is $\rho \in \arg\min_{\delta \in B} f_\alpha(x_\delta)$ such that $\gamma \succeq(R, z)\, \rho$. We claim that $\rho \in B \setminus C$. To prove this, suppose by contradiction that $\rho \in C$. Thus, $\rho \succeq(R, z)\, \beta$. By Lemma 4, $\gamma \succeq(R, z)\, \beta$. This is a contradiction because $\gamma \in B \setminus C$.

Let $N(C) \equiv \mu^{-1}(C)$, $N(B \setminus C) \equiv \mu^{-1}(B \setminus C)$, and $z|_{B \setminus C} \equiv (x|_{B \setminus C}, \mu|_{N(B \setminus C)})$. Since $z \in F(R)$, then $z|_{B \setminus C} \in F(B \setminus C, R|_{N(B \setminus C)}, \sum_{\gamma \in B \setminus C} x_\gamma)$. Since for each $\gamma \in B \setminus C$, there is $\rho \in (B \setminus C) \cap \arg\min_{\delta \in B} f_\delta(x_\delta)$ such that $\gamma \succeq(R, z)\, \rho$, then by Proposition 1, $z|_{B \setminus C} \in \Re^f(B \setminus C, R|_{N(B \setminus C)}, \sum_{\gamma \in B \setminus C} x_\gamma)$. Recall that for each $\gamma \in B \setminus C$ and each $\delta \in C$, $(x_\gamma, \gamma)\, P_{\mu^{-1}(\gamma)}\, (x_\delta, \delta)$. Thus, there is $\varepsilon \in \mathbb{R}_{++}$ such that for each $i \in N(B \setminus C)$ and each $j \in N(C)$, $(x_{\mu(i)} - \varepsilon, \mu(i))\, P_i\, (x_{\mu(j)} + \varepsilon, \mu(j))$. Let $e|^\varepsilon_{B \setminus C} \equiv (B \setminus C, R|_{N(B \setminus C)}, \sum_{\gamma \in B \setminus C} x_\gamma - \varepsilon)$ and $y \equiv (u, \sigma) \in \Re^f(e|^\varepsilon_{B \setminus C})$. By Corollary 1, $x|_{B \setminus C} \gg u$. Let $z|_C \equiv (x|_C, \mu|_{N(C)})$. Since $z \in F(R)$, then that $z|_C \in F(C, R|_{N(C)}, \sum_{\delta \in C} x_\delta)$. Let $k \equiv \mu^{-1}(\beta)$.

Let $j \in N(C)$. Since $\mu(j) \succeq(R, z)\, \mu(k)$, then $\mu(j) \succeq(R|_{N(C)}, z)\, \mu(k)$, for otherwise there is $\alpha \in B \setminus C$ such that $\alpha \succeq(R|_{N(B \setminus C)}, z)\, \mu(k) = \beta$. By Lemma 6, there is $v \equiv (w, \lambda) \in$

[11] If $z \neq O\langle \mathcal{D}, \Re^f, r\rangle(R, z)$, the same argument applies. Just let $z \equiv O\langle \mathcal{D}, \Re^f, r\rangle(R, z)$.

[12] This proves in particular that $(B \setminus C) \cap \arg\min_{\delta \in B} f_\delta(x_\delta) \neq \emptyset$.

$Z(C, R_{N(C)}, \sum_{\delta\in C} x_\delta + \varepsilon)$ such that: (1) $w \gg x|_C$; (2) for each $j \in N(C) \setminus \{k\}$ and each $l \in N(C)$, $v_j \, R_j \, v_l$; (3) $\lambda(k) = \mu|_C(k)$; and (4) for each $\delta \in C$, $\delta \succeq(R|_{N(C)}, v)\, \lambda(k)$.

We claim that there exists $R'_i \in \mathcal{Q}$ such that

$$O\langle \mathcal{D}, \Re, r\rangle(R_{-i}, R'_i, z)\, P_i^0\, O\langle \mathcal{D}, \Re, r\rangle(R, z).$$

Let $\rho \in \arg\min_{\gamma\in B\setminus C} f_\gamma(w_\gamma)$ and $R'_k \in \mathcal{Q}$ be such that $(w_\beta, \beta)\, I'_k\, (w_\rho, \rho)$ and for each $\delta \in B \setminus \{\beta, \rho\}$, $(w_\beta, \beta)\, P'_k\, (x_\delta + \varepsilon, \delta)$.

Let x^ε be the vector obtained by concatenating u and w, and let μ^ε be the bijection that coincides with σ on $B \setminus C$ and with λ on C. Let $z^\varepsilon \equiv (x^\varepsilon, \mu^\varepsilon)$. We claim that $z^\varepsilon \in F(R_{-k}, R'_k)$. Since $x + 1\varepsilon \geq x^\varepsilon$, then for each $j \in K$, $z^\varepsilon_k\, R'_k\, z^\varepsilon_j$.[13] Thus, it remains to prove that for each $i \subseteq K \setminus \{k\}$ and each $j \in K$, $z^\varepsilon_i\, R_i\, z^\varepsilon_j$. There are four cases.

Case 1: $\{i, j\} \subseteq N(C)$. Since v satisfies property (2) stated above and $i \neq k$, then $z^\varepsilon_i = v_i\, R_i\, v_j = z^\varepsilon_j$.

Case 2: $\{i, j\} \subseteq N(B \setminus C)$. Let $\widehat{e} = (B \setminus C, R|_{N(B\setminus C)}, \sum_{\gamma\in B\setminus C} x_\gamma - \varepsilon)$. Then $y \in \Re^f(\widehat{e}) \subseteq F(\widehat{e})$. Thus, $z^\varepsilon_i = y_i\, R_i\, y_j = z^\varepsilon_j$.

Case 3: $i \in N(B \setminus C)$ and $j \subseteq N(C)$. Since $z \in \Re^f(R) \subseteq F(R)$, then $z^\varepsilon_i = y_i\, R_i\, (u_{\mu(i)}, \mu(i))$. Since $x|_{B\setminus C} \gg u$ and $\sum_{\gamma\in B\setminus C} x_\gamma - \varepsilon = \sum_{\gamma\in B\setminus C} u_\gamma$, then $u_{\mu(i)} > x_{\mu(i)} - \varepsilon$. Thus, $(u_{\mu(i)}, \mu(i))\, P_i\, (x_{\mu(i)} - \varepsilon, \mu(i))$. Recall that $(x_{\mu(i)} - \varepsilon, \mu(i))\, P_i\, (x_{\mu^\varepsilon(j)} + \varepsilon, \mu^\varepsilon(j))$. Now, since $w \gg x|_C$ and $\sum_{\delta\in C} w_\gamma = \sum_{\delta\in C} x_\delta + \varepsilon$, then $x_{\mu^\varepsilon(j)} + \varepsilon > w_{\mu^\varepsilon(j)}$. Thus, $(x_{\mu^\varepsilon(j)} + \varepsilon, \mu^\varepsilon(j))\, P_i\, (w_{\mu^\varepsilon(j)}, \mu^\varepsilon(j)) = z^\varepsilon_j$. Consequently, $z^\varepsilon_i\, P_i\, z^\varepsilon_j$.

Case 4: $i \subseteq N(C)\setminus\{k\}$ and $j \in N(B\setminus C)$. Since v satisfies property (2) stated above and $i \neq k$, then $z^\varepsilon_i = v_i\, R_i\, (w_{\mu(i)}, \mu(i))$. Since $w \gg x|_C$, then $(w_{\mu(i)}, \mu(i))\, P_i\, (x_{\mu(i)}, \mu(i))$. Since $z \in \Re^f(R)$, then $(x_{\mu(i)}, \mu(i))\, R_i\, (x_{\mu^\varepsilon(j)}, \mu^\varepsilon(j))$. Now, since $x|_{B\setminus C} \gg u$, then $(x_{\mu^\varepsilon(j)}, \mu^\varepsilon(j))\, P_i\, y_j = z^\varepsilon_j$. Thus, $z^\varepsilon_i\, P_i\, z^\varepsilon_j$.

Now, we claim that for each $\alpha \in B$, there is $\delta \in \arg\min_{\delta\in B} f_\delta(x^\varepsilon_\delta)$, such that $\alpha \succeq (R_{-k}, R'_k, z^\varepsilon)\, \delta$. First, observe that since $(B\setminus C) \cap \arg\min_{\delta\in B} x_\delta \neq \emptyset$, $x|_{B\setminus C} \gg u$, $w \gg x|_C$, and $f \in \mathcal{I}^B$, then $\arg\min_{\delta\in B\setminus C} f_\delta(u_\delta) = \arg\min_{\delta\in B} f_\delta(x^\varepsilon_\delta)$. There are two cases.

Case 1: $\alpha \in B \setminus C$. Since $y \in \Re^f(B \setminus C, R|_{N(B\setminus C)}, \sum_{\gamma\in B\setminus C} x_\gamma - \varepsilon)$, then there exists $\delta \in \arg\min_{\delta\in B\setminus C} f_\delta(u_\delta)$ such that $\alpha \succeq(R_{-k}, R'_k, z^\varepsilon)\, \delta$.

Case 1: $\alpha \in C$. Since v satisfies property (4) stated above, then $\alpha \succeq(R|_C, v)\, \lambda(k)$. Thus, $\alpha \succeq(R, z^\varepsilon)\, \lambda(k)$ and consequently, $\alpha \succeq(R_{-k}, R'_k, z^\varepsilon)\, \lambda(k)$. Now, since $z^\varepsilon_k\, I_k\, (x^\varepsilon_\rho, \rho)$ and $\rho \in \arg\min_{\delta\in B\setminus C} f_\delta(u_\delta)$, then $\lambda(k) \succeq(R_{-k}, R'_k, z^\varepsilon)\, \rho$. By Lemma 4, $\alpha \succeq(R_{-k}, R'_k, z^\varepsilon)\, \rho$.

Since $z^\varepsilon \in F(R_{-k}, R'_k)$ and for each $\alpha \in B$ there is $\delta \in \arg\min_{\delta\in B} f_\delta(x^\varepsilon_\delta)$ such that $\alpha \succeq(R_{-k}, R'_k, z^\varepsilon)\, \delta$, then by Proposition 1, $z^\varepsilon \in \Re^f(R_{-k}, R'_k)$.

We claim that for each $\widehat{z} \in \Re^f(R_{-k}, R'_k)$, $\widehat{z}_k = z^\varepsilon_k$. Let $\widehat{z} \equiv (\widehat{x}, \widehat{\mu}) \in \Re^f(R_{-k}, R'_k)$. By Corollary 1, $N_{\widehat{z}z} = N$ and thus, $\widehat{z}_k\, I'_k\, z^\varepsilon_k$. Since, $\Re^f(R_{-k}, R'_k) \subset F(R_{-k}, R'_k)$, then either $\widehat{z}_l = z^\varepsilon_k$ or $\widehat{z}_i = (x^\varepsilon_\rho, \rho)$. Recall from the proof that $z^\varepsilon \in F(R_{-k}, R'_k)$, Case 3, that for each $i \in N(B \setminus C)$ and each $j \subseteq N(C)$, $z^\varepsilon_i\, P_i\, z^\varepsilon_j$. Since $N_{\widehat{z}z} = N$, then for each

[13] Here 1ε is the vector in $\mathbb{R}^A$ with all components equal to ε.

$i \in N(B \setminus C)$, $\widehat{\mu}(i) \in B \setminus C$. Now, since $\widehat{\mu}$ is a bijection and $|N(B \setminus C)| = |B \setminus C|$, then $\widehat{\mu}(N(B \setminus C)) = B \setminus C$. Since $k \in N(C)$ and $\rho \in B \setminus C$, then $\widehat{\mu}(k) \neq \rho$ and thus, $\widehat{z}_k = z^\varepsilon_k$. Now, since $z^\varepsilon_k = (x^\varepsilon_{\mu(k)}, \mu(k))$ and $x^\varepsilon_{\mu(k)} = w_{\mu(k)} > x_{\mu(k)}$, then for each $\widehat{z} \in \Re^f(R_{-k}, R'_k)$, $\widehat{z}\, P^0_k\, z$. Thus,

$$O\langle \mathcal{D}, \Re^f, r\rangle(R_{-k}, R'_k, z)\, P^0_k\, O\langle \mathcal{D}, \Re^f, r\rangle(R, z).$$

Consequently, $(R, z) \notin \mathcal{N}\langle \mathcal{D}, \Re^f, r, R^0\rangle$. This is a contradiction. □

A symmetric statement to Lemma 6 also holds.

Lemma 8. Let $K \subseteq N$, $e \equiv (B, R, m) \in \mathcal{E}^K$, $z \equiv (x, \mu) \in F(e)$, and $\varepsilon \in \mathbb{R}_{++}$. If $i \in K$ is such that for each $\alpha \in B$, $\alpha \succeq(R, z)\, \mu(i)$, then there is an allocation $z^\varepsilon \equiv (x^\varepsilon, \mu^\varepsilon) \in Z(B, R, m - \varepsilon)$ such that:

1. $x \gg x^\varepsilon$,
2. for each $j \in K \setminus \{i\}$ and each $l \in K$, $z^\varepsilon_j\, R_j\, z^\varepsilon_l$,
3. $\mu^\varepsilon(i) = \mu(i)$, and
4. for each $\alpha \in B$, $\alpha \preceq(R, z^\varepsilon)\, \mu^\varepsilon(i)$.

Proof. Let K, e, $z \equiv (x, \mu)$, and ε be as in the statement of the lemma. Let $\widehat{R}_i$ be a preferences such that for each $\alpha \in B \setminus \{\mu(i)\}$, $(x_{\mu(i)} - \varepsilon, \mu(i))\, \widehat{P}_i\, (x_\alpha + 2\varepsilon, \alpha)$. Since $z \in F(e)$, then $z \in F(B, R_{-i}, \widehat{R}_i, m)$. Observe that since for each $\alpha \in B$, $\alpha \succeq(R, z)\, \mu(i)$, then for each $\alpha \in B$, $\alpha \succeq(R_{-i}, \widehat{R}_i, z)\, \mu(i)$. Thus, $z \in \Re^{\mu(i)}(B, R_{-i}, \widehat{R}_i, m)$. Let $z^\varepsilon \in \Re^{\mu(i)}(B, R_{-i}, \widehat{R}_i, m - \varepsilon)$. A similar argument to the one in Lemma 8 shows that z^ε satisfies conditions 1 to 4. □

The following lemma states that if the agents' strategy space contains the quasi-linear preferences, then the Nash-equilibrium outcome correspondence of the game form associated to each GMRF correspondence coincides with the fair correspondence.

Lemma 9. Let $\mathcal{D} \subseteq \mathcal{R}$ be such that $\mathcal{D} \supseteq \mathcal{Q}$, $f \in \mathcal{I}^A$, and $r \in \Re^f$. For each $R^0 \in \mathcal{R}^N$, $\mathcal{O}\langle \mathcal{D}, \Re^f, r, R^0\rangle = F(R^0)$.

Proof. Let $\mathcal{D}$ and r be as in the statement of the lemma. Let $R^0 \in \mathcal{R}^N$. We will prove that $\mathcal{O}\langle \mathcal{D}, \Re^f, r, R^0\rangle = F(R^0)$.

First, we prove that $\mathcal{O}\langle \mathcal{D}, \Re^f, r, R^0\rangle \supseteq F(R^0)$. Let $z \equiv (x, \mu) \in F(R^0)$. Let $R \in \mathcal{Q}^N$ be such that for each $\{i, j\} \subseteq N$, $z_i\, I_i\, z_j$. We claim that $(R, z) \in \mathcal{N}\langle \mathcal{D}, \Re^f, r, R^0\rangle$. Suppose by contradiction that there is $i \in N$ and $(R'_i, z'_i) \in \mathcal{D} \times (\mathbb{R} \times A)$ such that

$$O\langle \mathcal{D}, \Re^f, r\rangle(R_{-i}, R'_i, z_{-i}, z'_i)\, P^0_i\, O\langle \mathcal{D}, \Re^f, r\rangle(R, z).$$

Let $y \equiv (u, \sigma) \equiv O\langle \mathcal{D}, \Re^f, r\rangle(R_{-i}, R'_i, z_{-i}, z'_i)$. Since, $z \in F(R^0)$, then $u_{\sigma(i)} > x_{\sigma(i)}$. Since $y \in \Re^f(R_{-i}, R'_i) \subset F(R_{-i}, R'_i)$, then for each $j \in N$, $u_{\sigma(j)} > x_{\sigma(j)}$. This is a contradiction because $\sum_{\alpha \in A} u_\alpha = \sum_{\alpha \in A} x_\alpha$.

Second, we prove that $\mathcal{O}\langle \mathcal{D}, \Re^f, r, R^0\rangle \subseteq F(R^0)$. Let $(R, z) \in \mathcal{N}\langle \mathcal{D}, \Re^f, r, R^0\rangle$. We claim that $O\langle \mathcal{D}, \Re^f, r\rangle(R, z) \in F(R^0)$. Suppose by contradiction that $O\langle \mathcal{D}, \Re^f, r\rangle(R, z) \notin F(R^0)$. Suppose w.l.o.g. that $z = O\langle \mathcal{D}, \Re^f, r\rangle(R, z)$. Since $z \notin F(R^0)$, then there are $\{i, j\} \subseteq N$ such that $z_j \, P^0_i \, z_i$. By Lemma 7, $\mu(j) \succeq(R, z) \, \mu(i)$. Thus, there are $\{\alpha_t\}_{t=0}^T \subseteq A$ such that $\alpha_0 = \mu(j)$, $\alpha_T = \mu(i)$, and $(x_{\alpha_0}, \alpha_0) \, I_{\mu^{-1}(\alpha_0)} \ldots I_{\mu^{-1}(\alpha_{T-1})} \, (x_{\alpha_T}, \alpha_T)$. Let $C \equiv \{\alpha_0, \ldots, \alpha_T\}$. Let $z' = (x', \mu') \in Z(A, R, M)$ be the allocation defined as follows: (1) for each $k \in N$ such that $\mu(k) \in A \setminus C$, $z'_k \equiv z_k$; (2) for each $k \in N$ such that there is $t \in \{0, \ldots, T-1\}$, such that $\mu(k) = \alpha_t$, let $z'_k \equiv (x_{\alpha_{t+1}}, \alpha_{t+1})$; and (3) let $z'_i \equiv (x_{\alpha_0}, \alpha_0)$. Observe that $x' = x$.

Let $R'_i \in \mathcal{R}$. We claim that for each $\alpha \in A$, $\alpha \succeq(R_{-i}, R'_i, z') \, \alpha_0$. By the definition of z', $(x_{\alpha_T}, \alpha_T) \, I_{(\mu')^{-1}(\alpha_T)} \ldots I_{(\mu')^{-1}(\alpha_1)} \, (x_{\alpha_0}, \alpha_0)$. Thus, for each $\alpha \in C$, $\alpha \succeq (R_{-i}, R'_i, z') \, \alpha_0$. Let $\alpha \in A \setminus C$. We claim that $\alpha \succeq(R_{-i}, R'_i, z') \, \alpha_0$. Since $\alpha \succeq(R, z)\alpha_0$, then there are $\{\beta_t\}_{t=0}^{T'} \subseteq A$ such that $\beta_0 = \alpha$, $\beta_{T'} = \mu(i)$, and $(x_{\beta_0}, \beta_0) \, I_{\mu^{-1}(\beta_0)} \ldots I_{\mu^{-1}(\beta_{T'-1})} \, (x_{\beta_{T'}}, \beta_{T'})$. Let $t^* \equiv \arg\min\{0 \leq t \leq T : \beta_t \in C\}$. The index t^* is well defined because $\beta_{T'} \in C$. Moreover, $t^* > 0$ since $\beta_0 \in A \setminus C$. Since for each $\beta \in A \setminus C$, $\mu^{-1}(\beta) = (\mu')^{-1}(\beta)$, then $(x_{\beta_0}, \beta_0) \, I_{(\mu')^{-1}(\beta_0)} \ldots I_{(\mu')^{-1}(\beta_{t^*-1})} \, (x_{\beta_{t^*}}, \beta_{t^*})$. Thus, $\beta_0 \succeq(R_{-i}, R'_i, z') \, \beta_{t^*}$. Since $\beta_{t^*} \succeq (R_{-i}, R'_i, z') \, \alpha_0$, then by Lemma 4, $\beta_0 \succeq(R_{-i}, R'_i, z') \, \alpha_0$.

Let $R'_i \in \mathcal{R}$ be such that for each $\alpha \in A \setminus \{\alpha_0\}$, $z'_i \, R'_i \, (x'_\alpha, \alpha)$. We claim that $z' \in F(R_{-i}, R'_i)$. Since for each $\alpha \in A \setminus \{\alpha_0\}$, $z'_i \, R'_i \, (x'_\alpha, \alpha)$, then it remains to prove that for each $k \subseteq K \setminus \{i\}$ and each $l \in N$, $z'_k \, R_k \, z'_l$. There are two cases.

Case 1: $\mu(k) \in A \setminus C$. Since $z'_k = z_k$, $z \in F(R)$, and $x' = x$, then for each $l \in N$, $z'_k \, R_k \, z'_l$.

Case 2: There is $t \in \{0, \ldots, T-1\}$ such that $\mu(k) = \alpha_t$. Since $z'_k = (x_{\alpha_{t+1}}, \alpha_{t+1})$, $(x_{\alpha_t}, \alpha_t) \, I_k \, (x_{\alpha_{t+1}}, \alpha_{t+1})$, $z \in F(R)$, and $x' = x$, then for each $l \in N$, $z'_k \, R_k \, z_l$.

We claim that there exists $R'_i \in \mathcal{Q}$ such that

$$O\langle \mathcal{D}, \Re^f, r\rangle(R_{-i}, R'_i, z) \, P^0_i \, O\langle \mathcal{D}, \Re^f, r\rangle(R, z).$$

There are two cases.

Case 1: $\alpha_0 \in \arg\min_{\alpha \in A} f_\alpha(x_\alpha)$. Let $R'_i \in \mathcal{Q}$ be a quasi-linear preference such that for each $\alpha \in A \setminus \{\alpha_0\}$, $z'_i \, P'_i \, (x'_\alpha, \alpha)$. We claim that $z' \in \Re^f(R_{-i}, R'_i)$. Since $z' \in F(R_{-i}, R'_i)$, $\alpha_0 \in \arg\min_{\alpha \in A} f_\alpha(x'_\alpha)$, and for each $\alpha \in A$, $\alpha \succeq(R_{-i}, R'_i, z') \, \alpha_0$, then by Proposition 1, $z' \in \Re^f(R_{-i}, R'_i)$.

Now, we claim that for each $\widehat{z} \in \Re^f(R_{-i}, R'_i)$, $\widehat{z}_i = z'_i$. Let $\widehat{z} = (\widehat{x}, \widehat{\mu}) \in \Re^f(R_{-i}, R'_i)$. By Lemma 1, $\widehat{z}_i \, I'_i \, z'_i$. Since $\Re^f(R_{-i}, R'_i) \subseteq F(R_{-i}, R'_i)$ and for each $\alpha \in A \setminus \{\alpha_0\}$, $z'_i \, P'_i \, (x'_\alpha, \alpha)$, then $\widehat{\mu}(i) = \mu'(i)$. Thus, for each $\widehat{z} \in \Re^f(R_{-i}, R'_i)$, $\widehat{z} \, P^0_i \, z$. Consequently,

$$O\langle \mathcal{D}, \Re^f, r\rangle(R_{-i}, R'_i, z) \, P^0_i \, O\langle \mathcal{D}, \Re^f, r\rangle(R, z).$$

Thus, $(R,z) \notin \mathcal{N}\langle \mathcal{D}, \Re^f, r, R^0\rangle$. This is a contradiction.

Case 2: $\alpha_0 \notin \arg\min_{\alpha\in A} f_\alpha(x_\alpha)$. Let $R'_i \in \mathcal{Q}$ be a quasi-linear preference such that for each $\alpha \in A\setminus\{\alpha_0\}$, $z'_i\, R'_i\, (x'_\alpha, \alpha)$. Let $C \subseteq A$ be the set of objects α such that there are $\{\beta_0,\ldots,\beta_T\} \subseteq A\setminus\{\alpha_0\}$ such that $\beta_0 = \alpha$, $\beta_T \in \arg\min_{\delta\in A} f_\delta(x_\delta)$, and

$$(x'_{\beta_0}, \beta_0)\, I_{(\mu')^{-1}(\beta_0)}\, \ldots I_{(\mu')^{-1}(\beta_{T-1})}\, (x'_{\beta_T}, \beta_T).$$

Since $\alpha_0 \notin \arg\min_{\alpha\in A} f_\alpha(x_\alpha)$, then $\arg\min_{\alpha\in A} f_\alpha(x_\alpha) \subseteq C$ and thus, $C \neq \emptyset$. Moreover, $\alpha_0 \in A\setminus C$. Let $N(C) \equiv (\mu')^{-1}(C)$ and $e|_C \equiv (C, R|_{N(C)}, \sum_{\alpha\in C} x_\alpha)$. Let $z'|_C \equiv (x'|_C, \mu'|_{N(C)}) \in Z(e|_C)$. Since $z' \in F(R_{-i}, R_i)$, then $z'|_C \in F(e|_C)$. Observe that for each $\alpha \in C$, there are $\{\beta_0,\ldots,\beta_{T'}\} \subseteq C$ such that $\beta_0 = \alpha$, $\beta_{T'} \in \arg\min_{\delta\in A} f_\alpha(x_\delta)$, and

$$(x'_{\beta_0}, \beta_0)\, I_{(\mu')^{-1}(\beta_0)}\, \ldots I_{(\mu')^{-1}(\beta_{T'-1})}\, (x'_{\beta_{T'}}, \beta_{T'}).$$

Thus, for each $\alpha \in C$ there is $\beta \in \arg\min_{\delta\in C} f_\delta(x_\delta)$ such that $\alpha \succeq (R|_C, z'|_C)\, \beta$. By Proposition 1, $z'|_C \in \Re^f(e|_C)$.

We claim that for each $\beta \in A\setminus C$, there are $\{\beta_t\}_{t=0}^T \subseteq A\setminus C$ such that $\beta_0 = \beta$, $\beta_T = \alpha_0$, and $(x'_{\beta_0}, \beta_0)\, I_{(\mu')^{-1}(\beta_0)}\, \ldots I_{(\mu')^{-1}(\beta_{T-1})}\, (x'_{\beta_T}, \beta_T)$. Since for each $\beta \in A$, and in particular for each $\beta \in A\setminus C$, $\beta \succeq (R_{-i}, R'_i, z')\, \alpha_0$, then there are $\{\delta_t\}_{t=0}^T \subseteq A$ such that $\delta_0 = \beta$, $\delta_T = \alpha_0$, and $(x'_{\delta_0}, \delta_0)\, I_{(\mu')^{-1}(\delta_0)}\, \ldots I_{(\mu')^{-1}(\delta_{T-1})}\, (x'_{\delta_T}, \delta_T)$. We claim that $\{\delta_t\}_{t=0}^T \subseteq A\setminus C$. Suppose by contradiction that there is $0 < t^* \leq T$ such that $\delta_{t^*} \in C$. Since $(x'_{\delta_{t^*-1}}, \delta_{t^*-1})\, I_{(\mu')^{-1}(\delta_{t^*-1})}\, (x'_{\delta_{t^*}}, \delta_{t^*})$, then $\delta_{t^*-1} \in C$. The recursive argument shows that $\beta = \delta_0 \in C$. This is a contradiction. Let $R' \equiv (R_{-i}, R'_i)$, $N(A\setminus C) \equiv (\mu')^{-1}(A\setminus C)$ and $e|_{A\setminus C} \equiv (A\setminus C, R'|_{N(A\setminus C)}, \sum_{\alpha\in A\setminus C} x_\alpha)$. Let $z'|_{A\setminus C} \equiv (x'|_{A\setminus C}, \mu'|_{N(A\setminus C)}) \in Z(e|_{A\setminus C})$. Since $z' \in F(R')$, then $z'|_{A\setminus C} \in F(e|_{A\setminus C})$. Now, since for each $\beta \in A\setminus C$ there are $\{\beta_t\}_{t=0}^T \subseteq A\setminus C$ such that $\beta_0 = \beta$, $\beta_T = \alpha_0$, and $(x'_{\beta_0}, \beta_0)\, I_{(\mu')^{-1}(\beta_0)}\, \ldots I_{(\mu')^{-1}(\beta_{T-1})}\, (x'_{\beta_T}, \beta_T)$, then for each $\beta \in A\setminus C$, $\beta \succeq (R'|_{N(A\setminus C)}, z'|_{N(A\setminus C)})\, \alpha_0$. Now, we claim that for each $k \in N(A\setminus C)\setminus\{i\}$ and each $j \in N(C)$, $z'_k\, P_k\, z'_j$. Since $z' \in F(R')$, then if there are $k \in N(A\setminus C)\setminus\{i\}$ and $j \in N(C)$ such that $\neg(z'_k\, P_k\, z'_j)$, then $z'_k\, I_k\, z'_j$. Thus, $\mu'(k) \in C$ and therefore $k \in N(C)$. This is a contradiction. Thus, there is $\varepsilon \in \mathbb{R}_{++}$ such that for each for each $k \in N(A\setminus C)\setminus\{i\}$ and each $j \in N(C)$, $(x_{\mu'(k)} - \varepsilon, \mu'(k))\, P_k\, (x_{\mu'(j)} + \varepsilon, \mu'(j))$. Since $\arg\min_{\alpha\in A} f_\delta(x'_\alpha) \subset C$, then ε can be chosen small enough that $\min_{\alpha\in C} f_\alpha(x'_\alpha + \varepsilon) < \min_{\alpha\in A\setminus C} f_\alpha(x_\alpha - \varepsilon)$. Moreover, since $(x_{\alpha_0}, \alpha_0)\, P^0_i\, (x_{\mu(i)}, \mu(i))$, then ε can be chosen small enough that $(x_{\alpha_0} - \varepsilon, \alpha_0)\, P^0_i\, (x_{\mu(i)} + \varepsilon, \mu(i))$.

Let $y \equiv (u,\sigma) \in \Re^f(C, R|_{N(C)}, \sum_{\alpha\in C} x_\alpha + \varepsilon)$. By Corollary 1, $y \gg x'|_C$. Thus, for each $\alpha \in C$, $u_\alpha < x'_\alpha + \varepsilon$.

Since $z'|_{A\setminus C} \in F(e|_{A\setminus C})$ and for each $\beta \in A\setminus C$, $\beta \succeq (R'|_{N(A\setminus C)}, z'|_{N(A\setminus C)})\, \alpha_0$, then by Lemma 8, there is $v \equiv (w,\lambda) \in (A\setminus C, R'|_{N(A\setminus C)}, \sum_{\alpha\in A\setminus C} x_\alpha - \varepsilon)$ such that: (1) $x'|_{N(A\setminus C)} \gg w$, (2) for each $k \in N(A\setminus C)\setminus\{i\}$, and each $l \in N(A\setminus C)\setminus\{i\}$, $v_k\, R_k\, v_l$, (3) $\lambda(i) = \mu'(i)$, and (4) for each $\beta \in A\setminus C$, $\beta \succeq (R'|_{N(A\setminus C)}, v)\alpha_0$. Thus, for each $\beta \in A\setminus C$, $w_\beta > x'_\beta - \varepsilon$.

Let x^ε be the vector obtained by concatenating u and w, and let μ^ε be the bijection that coincides with σ on $N(C)$ and with λ on $N(C)$. Let $z^\varepsilon \equiv (x^\varepsilon, \mu^\varepsilon)$. Since $\sum_{\alpha\in C} u_\alpha + \sum_{\beta\in A\setminus C} w_\beta = \sum_{\delta\in A} x'_\delta$, then $z^\varepsilon \in Z(A, R', M)$. Let $\rho \in \arg\min_{\alpha\in C} f_\delta(u_\alpha)$. Since $\min_{\alpha\in C} f_\alpha(x'_\alpha + \varepsilon) < \min_{\alpha\in A\setminus C} f_\alpha(x_\alpha - \varepsilon)$, then $\rho \in \arg\min_{\alpha\in C} f_\delta(x^\varepsilon_\alpha)$. Let us complete the definition of R'_i. Assume that $(x^\varepsilon_\rho, \rho)\, I'_i\, (x^\varepsilon_{\alpha_0}, \alpha_0)$ and for each $\delta \in A \setminus \{\alpha_0, \rho\}$, $(w_{\alpha_0}, \alpha_0)\, P'_i\, (x_\delta, \delta)$.[14] A similar argument to the one in Lemma 7, shows that $z^\varepsilon \in \Re^f(R_{-i}, R'_i)$ and for each $\widehat{z} \in \Re^f(R_{-i}, R'_i)$, $\widehat{z}_i = z^\varepsilon_i$. Thus, for each $\widehat{z} \in \Re^f(R_{-i}, R'_i)$, $\widehat{z}\, P^0_i\, z$. Consequently,

$$O\langle \mathcal{D}, \Re^f, r\rangle(R_{-i}, R'_i, z)\, P^0_i\, O\langle \mathcal{D}, \Re^f, r\rangle(R, z).$$

Thus, $(R, z) \notin \mathcal{N}\langle \mathcal{D}, \Re^f, r, R^0\rangle$. This is a contradiction. □

The proof of Theorem 2 is a straightforward implication of Lemma 9.

Proof of Theorem 2. Since $\mathcal{R} \supset \mathcal{Q}$, then by Lemma 9, for each $r \in \Re^f$, and each $R^0 \in \mathcal{R}^N$, $\mathcal{O}\langle \mathcal{R}, \Re^f, r, R^0\rangle = F(R^0)$. Thus, for each $R^0 \in \mathcal{R}^N$, $\mathcal{O}\langle \mathcal{R}, \Re^f, R^0\rangle = F(R^0)$. □

5 Implementation of the fair correspondence

In this section we investigate the implications of our results for the so called implementation of the fair correspondence. First, each GMRF correspondence naturally implements F on $\mathcal{R}$.

Corollary 4. For each $f \in \mathcal{I}^A$, $\Re^f$ naturally implements F on $\mathcal{R}$.

Corollary 4 follows from Theorem 2. We omit the proof.

Each GMRF correspondence not only naturally implements F on $\mathcal{R}$, but also implements it in strong Nash equilibria on $\mathcal{R}$:

Proposition 2. For each $f \in \mathcal{I}^A$, $\Re^f$ naturally implements F in strong Nash equilibria on $\mathcal{R}$.

Proof. Let $f \in \mathcal{I}^A$ and $r \in \Re^f$. We will prove that for each $R^0 \in \mathcal{R}^N$, $\mathcal{O}^*\langle \mathcal{R}, \Re^f, r, R^0\rangle = F(R^0)$. Since $\mathcal{O}^*\langle \mathcal{R}, \Re^f, r, R^0\rangle \subseteq \mathcal{O}\langle \mathcal{R}, \Re^f, r, R^0\rangle$, then by Theorem 2, it is enough to prove that $F(R^0) \subseteq \mathcal{O}^*\langle \mathcal{R}, \Re^f, r, R^0\rangle$. Let $z \equiv (x, \mu) \in F(R^0)$. Let $R \in \mathcal{Q}^N$ be such that for each $\{i, j\} \subseteq N$, $z_i\, I_i\, z_j$. We claim that $(R, z) \in \mathcal{N}^*\langle \mathcal{D}, \Re^f, r, R^0\rangle$. Suppose that there is $N' \subseteq N$, $i \in N'$ and $(R'_{N'}, z'_{N'}) \in \mathcal{D} \times (\mathbb{R} \times A)$ such that

$$O\langle \mathcal{D}, \Re^f, r\rangle(R_{-N'}, R'_{N'}, z_{-N'}, , z'_{N'})\, P^0_i\, O\langle \mathcal{D}, \Re^f, r\rangle(R, z).$$

[14] This assumption is compatible with our previous assumption that $R'_i \in \mathcal{Q}$ is such that for each $\alpha \in A \setminus \{\alpha_0\}$, $z'_i\, R'_i\, (x'_\alpha, \alpha)$.

Let $y \equiv (u, \sigma) \equiv O\langle \mathcal{D}, \Re^f, r\rangle(R_{-i}, R'_i, z_{-i}, z'_i)$. Since $z \in F(R^0)$, then $u_{\sigma(i)} > x_{\sigma(i)}$. Since $y \in \Re^f(R_{-N'}, R'_{N'}) \subseteq F(R_{-N'}, R'_{N'})$, then for each $k \in N \setminus N'$, $u_{\sigma(k)} > x_{\sigma(k)}$. Since $\sum_{\alpha \in A} x_\alpha = \sum_{\alpha \in A} u_\alpha$, then there is $j \in N'$ such that $u_{\mu(k)} < x_{\mu(k)}$. Since $z \in F(R^0)$, then

$$O\langle \mathcal{D}, \Re^f, r\rangle(R, z)\, P^0_j\, O\langle \mathcal{D}, \Re^f, r\rangle(R_{-N'}, R'_{N'}, z_{-N'}, , z'_{N'}).$$

Thus, $(R, z) \in \mathcal{N}^*\langle \mathcal{R}, \Re^f, r, R^0\rangle$. Since $z = O\langle \mathcal{D}, \Re^f, r\rangle(R, z)$, then $z \in \mathcal{O}^*\langle \mathcal{R}, \Re^f, r, R^0\rangle$. □

The following corollary states that for the purpose of implementing F, one can reduce the agents' strategy space to the sub-domain of quasi-linear preferences and the desired consumption bundle.

Corollary 5. Let $f \in \mathcal{I}^A$. For each $\mathcal{D} \subseteq \mathcal{R}$ such that $\mathcal{D} \supseteq \mathcal{Q}$, $\langle \mathcal{Q}, \Re^f\rangle$ implements F on $\mathcal{D}$ and $\langle \mathcal{Q}, \Re^f\rangle$ implements F in strong Nash equilibria on $\mathcal{D}$.

Proof. By Lemma 9, $\langle \mathcal{Q}, \Re\rangle$ implements F on $\mathcal{D}$. A similar argument to the one used to prove Proposition 4, shows that $\langle \mathcal{Q}, \Re^f\rangle$ implements F in strong Nash equilibria on $\mathcal{D}$. □

One can think that a game form with simple strategy spaces is more "realistic." For instance, it is more realistic to imagine agents reporting elements of a finite dimensional space as is the case in game form $\langle \mathcal{Q}, \Re, r\rangle$ for some $r \in \Re$. Another goal that one achieves with the reduction of strategy spaces is that the "complexity" of the mechanism may be reduced. In contrast to the unrestricted domain of preferences, given reports $(R, z) \in (\mathcal{Q} \times (\mathbb{R} \times A))^N$, there is a polynomially bounded algorithm to calculate $O\langle \mathcal{Q}, \Re, r\rangle(R, z)$ (Aragones, 1995).

The reduction of strategy space in Corollary 5 could be seen also as a consequence of a more general result. Let us first introduce some definitions.

Let $\mathcal{D}' \subseteq \mathcal{D} \subseteq \mathcal{R}$. Then, $\mathcal{D}'$ is a **rich sub-domain** of $\mathcal{D}$ if for each $R_0 \in \mathcal{D}$ and each $z_0 \in \mathbb{R} \times A$, there is $R'_0 \in \mathcal{D}'$, such that the indifference sets through z_i of R_i and R'_i coincide. One can see easily that $\mathcal{Q}$ is rich on $\mathcal{R}$.

An SCC S is **level-set-only** if for each $R \in \mathcal{R}^N$ and each $z \in S(R)$, if $R' \in \mathcal{R}^N$ is such that for each $i \in N$, the indifference sets through z_i of R'_i and R_i coincide, then $z \in S(R')$. One can see easily that F is *level-set-only*.[15]

The following proposition states conditions under which strategy spaces can be reduced.

Proposition 3. Let $\mathcal{D}'$ be a rich sub-domain of $\mathcal{D}$ and let S and S' be two correspondences. If S is *level-set-only* and S' naturally implements S on D', then $\langle \mathcal{D}', S'\rangle$ implements S on $\mathcal{D}$.

Proof. Let $\mathcal{D}'$, $\mathcal{D}$, S, and S' be as in the statement of the lemma. Let $s' \in S'$. We will prove that $\langle \mathcal{D}', S', s'\rangle$ implements S on $\mathcal{D}$. Let $R^0 \in \mathcal{D}^N$. We claim that $S(R^0) \subseteq \mathcal{O}\langle \mathcal{D}', S', s', R^0\rangle$. Let $z \in S(R^0)$. We claim that $z \in \mathcal{O}\langle \mathcal{D}', S', s', R^0\rangle$. Let $R' \in \mathcal{D}'$ be such that for each

[15] In classical economies, the Walrasian and Constrained Walrasian correspondences are *level-set-only*.

$i \in N$, the indifference sets through z_i of R'_i and R_i coincide. Since S is *level-set-only*, then $z \in S(R')$. Thus, there is $(R^*, z^*) \in \mathcal{N}\langle \mathcal{D}', S', s', R' \rangle$ such that $O\langle \mathcal{D}', S', s' \rangle(R^*, z^*) = z$. Consequently, for each $i \in N$ and each $(\widetilde{R}_i, \widetilde{z}_i) \in \mathcal{D}' \times (\mathbb{R} \times A)$,

$$O\langle \mathcal{D}', \Re, r \rangle(R^*, z^*)\, R'_i\, O\langle \mathcal{D}', S', s' \rangle(R^*_{-i}, \widetilde{R}_i, z_{-i}, \widetilde{z}_i).$$

Now, since for each for each $i \in N$, the indifference sets through z_i of R'_i and R_i coincide and preferences are money-monotone, then for each $i \in N$ and each $(\widetilde{R}_i, \widetilde{z}_i) \in \mathcal{D}' \times (\mathbb{R} \times A)$,

$$O\langle \mathcal{D}', \Re, r \rangle(R^*, z^*)\, R^0_i\, O\langle \mathcal{D}', S', s' \rangle(R^*_{-i}, \widetilde{R}_i, z_{-i}, \widetilde{z}_i).$$

Thus, $(R^*, z^*) \in \mathcal{N}\langle \mathcal{D}', S', s', R^0 \rangle$ and $z \in \mathcal{O}\langle \mathcal{D}', S', s', R^0 \rangle$.

Now, we claim that $\mathcal{O}\langle \mathcal{D}', S', s', R^0 \rangle \subseteq S(R^0)$. Let $z \in \mathcal{O}\langle \mathcal{D}', S', s', R^0 \rangle$. Then, there is $(R^*, z^*) \in \mathcal{N}\langle \mathcal{D}', S', s', R^0 \rangle$ such that $O\langle \mathcal{D}', S', s' \rangle(R^*, z^*) = z$. Let $R' \in \mathcal{D}'$ be such that for each $i \in N$, the indifference sets through z_i of R'_i and R^0_i coincide. A similar argument to the one above shows that $(R^*, z^*) \in \mathcal{N}\langle \mathcal{D}', S', s', R' \rangle$ and $z \in \mathcal{O}\langle \mathcal{D}', S', s', R' \rangle$. Thus, since S' naturally implements S on $\mathcal{D}'$, then $z \in S(R')$. Now, since S is *level-set-only* and for each $i \in N$, the indifference sets through z_i of R'_i and R_i coincide, then $z \in S(R^0)$. □

A parallel statement to Proposition 3 also holds for implementation in strong Nash equilibria.

Proposition 4. Let $\mathcal{D}'$ be a rich sub-domain of $\mathcal{D}$ and let S and S' be two correspondences. If S is *level-set-only* and S' naturally implements S in strong Nash equilibria on D', then $\langle \mathcal{D}', S' \rangle$ implements S in strong Nash equilibria on $\mathcal{D}$.

Proposition 4 follows from a similar argument to the one in the proof of Proposition 3. We omit the proof.[16]

Propositions 3 and 4 not only allow us to identify situations in which strategy space can be reduced, but also situations in which implementation results for small domains can be extended. For instance, Āzacis (2008) defines a selection, $S_{\bar{\text{A}}} \in F$, which naturally implements F on $\mathcal{Q}$ and also naturally implements F in strong Nash equilibria on $\mathcal{Q}$. Proposition 3 allows us to conclude that $\langle \mathcal{Q}, S_{\bar{\text{A}}} \rangle$ implements F on $\mathcal{R}$. Proposition 4 allows us to conclude that $\langle \mathcal{Q}, S_{\bar{\text{A}}} \rangle$ implements F in strong Nash equilibria on $\mathcal{R}$. The following corollary formalizes this result.

Corollary 6. $\langle \mathcal{Q}, S_{\bar{\text{A}}} \rangle$ implements F on $\mathcal{R}$. Moreover, $\langle \mathcal{Q}, S_{\bar{\text{A}}} \rangle$ implements F in strong Nash equilibria on $\mathcal{R}$.

We omit the proof.

[16]See, Section 8, Additional material for referees.

6 Discussion

In this section we discuss the extension of our results when individual consumptions of money are bounded. We concentrate on an upper bound. A symmetric argument applies to a lower bound.

If we interpret our model as the allocation of rooms and rent among housemates, one may want to consider the requirement that each agent contribute to the rent, i.e., that individual consumptions of money be non-positive.

We first observe that the set of fair allocations in which individual consumptions of money are non-positive may be empty on the unrestricted domain of preferences. What may happen is that preferences are such that the agents "coincide" in that some rooms are considerably "inferior" to the other. That is, it is necessary that the agents who receive such rooms be compensated in order for fairness to be satisfied.

Suppose that the rooms in a house do not differ from each other in such a stark contrast. Then, this restriction should be formalized and imposed in the domain of preferences in which agents' preferences are supposed to belong to. For instance let $M < 0$ be the rent to collect in a particular application. Suppose that preferences belong to the domain in which receiving each room for free is preferred to receiving any other room and paying an equal share of the rent:

$$\mathcal{D}^* \equiv \left\{ R_0 \in \mathcal{R} : \text{for each } \{\alpha, \beta\} \subseteq A,\ (0, \alpha)\, P_0 \left(\frac{1}{n} M, \beta \right) \right\}.$$

One can easily see that if preferences belong to $\mathcal{D}^*$, then at each fair allocation each agent contributes to the rent, i.e., has a negative consumption of money. Thus, all fair allocations are "interior" and all of our proofs are easily modified to deliver parallel results on this domain.

7 Conclusion

In this paper we studied incentives for the truthful revelation of preferences in the problem of fairly allocating a set of objects among a set of agents agents when monetary compensation is possible. Previous literature shows that incentives go against the truthful revelation of preferences for each selection of the fair correspondence (Tadenuma and Thomson, 1995). Nevertheless, we show that for the GMRF correspondences, the extent to which each agent can manipulate the outcome is restricted by the other agents' behavior. In equilibrium, and regardless that all agents may lie about their preferences, the outcomes that ensue from the manipulation of these correspondences are fair and efficient with respect to true preferences. We conclude that incentives are against the truthful revelation of preferences, but not against fairness and efficiency.

Our results also imply that the direct revelation game associated with each GMRF correspondence implements in Nash and Strong Nash equilibria the Fair correspondence. Moreover, this implementation result is maintained if agents' reports are restricted to quasi-linear preferences.

It is an open question to study incentives issues for the variation of the model presented in this paper in which agents experience consumption externalities as in Velez (2008).

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