The Primal Auction: a new design for multi-commodity double auctions

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Abstract

It this paper, we propose an auction design for a multi-commodity double auction where participants simultaneously submit their valuations (bids) for the commodities. We label this the Primal Auction (PA) mechanism. The auctioneer computes the prevailing market price as the average over the bids and allocates the goods over the bidders in accordance with the relative bid of each bidder compared to this market price. Under the assumption of money metric utility functions, we show convergence of this process to an efficient equilibrium, but only if truth telling by all participants can be enforced. Commitment of all players to pay the prevailing market price at each round of the auction for the commodities allocated to them provides a strong incentive for truthful revelation, since lying means that the bidder has to pay the market price for a non-optimal quantity. However, to address concerns on shill bidding and bid shielding, we implement a stronger test on truth telling by endowing the auctioneer with the power to inactivate bids that are inconsistent with Revealed Preference. If bids cannot be refuted under his rule, then this implies that the existence of a concave utility function cannot be ruled out, and this is a sufficient condition for convergence of the projected gradient path represented by the auction design. There is no need to actually estimate this utility function: it is sufficient that bids are rationalizable. The PA mechanism can be extended to include a learning phase after which automata can finish the auction, which makes it also a suitable design for Internet auctions such as eBay. Finally, we link the PA mechanism to general equilibrium theory by showing that it is the dual of Walrasian tatonnement procedures, with the important advantage that at each step, commodity balances are maintained.

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1. Introduction

Auction theory seeks to understand the mechanisms that govern actual trade. Theoretically, the framework for 'mainstream' analysis used is the game theoretical model of incomplete information, and research focuses on proving properties of equilibrium outcomes under different auction mechanisms and under relaxations of assumptions on players' behavior and knowledge (see e.g. Klemperer, 2004, Milgrom, 2004, Maasland and Onderstal, 2006 for overviews of auction theory). Despite the direct involvement of auction theorists in the design of auctions for the sale of state assets from the 1990s onwards (Cameron et al., 1997; Cramton, 1995; Cramton et al., 2007; Bulow et al., 2009), the proposed designs not always proved to be successful (e.g. see Malvey et al. 1996, Reinhart and Belzer, 1996, Van Damme, 2002). Furthermore, critique on the game theoretic approach has been increasing from the side of practitioners (e.g. see Rothkopf and Harstad, 1994; Bapna et al., 2003), where, in particular, the assumptions being made on the abilities of the participants of the auctions are considered unrealistic, and the sensitivity of outcomes to specific assumptions is perceived to be an important hindrance to application, as much of the information needed to test such assumptions is not available (including the distribution of valuations across the bidders).

At the same time, since the launch of eBay in 1995, new issues have sprung up that were not or not adequately addressed in the mainstream auction literature. Specifically, the anonymous environment of the Internet made shill bidding and bid shielding much easier than it is in

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traditional auction rooms, and this undermines the efficiency of traditional mechanism designs (e.g. see Lucking-Reiley, 2000; Pinker et al., 2003; Kauffman and Wood, 2003; Wang et al., 2000). In addition, the fact that physical presence in a room is not required allows the bidders to participate in multiple auctions while setting at their desks, which boosted the study of combinatorial, multi-unit, and double auctions. Finally, the large amounts of data that are available from Internet auctions also make it possible to learn from auction data and this has given rise to a new, statistical, approach to auctions which may provide new insights into the auction process itself.

In this paper, we first review the literature related to Internet auctions, both empirical and theoretical.

Then, we present a new generalization of the Vickrey auction that we label the "Primal Auction" (PA). In deriving the results on efficiency and robustness against shill bidding and bid shielding, we rely heavily on the theory of Revealed Preference and in particular, we use the Afriat (1967) inequalities that link observed consumer choices to a possible underlying concave utility function.

We show that application of the Revealed Preference (RP)-rule is sufficient to ensure convergence of a specific auctioning process, where players simultaneously place their bids, the auctioneer computes the new market price as the average of these bids, and assigns quantities of commodities to the players in accordance with the relative magnitude of their bids. For the process to converge, the auctioneer does not have to estimate the underlying utility functions themselves, essentially because bids that are rationalizable through RP cannot rule out the existence of concave utility functions, even if, as Matzkin and Richter (1991) already noticed, this condition does not preclude very weak types of rational behavior such as maximization of pseudo transitive or semi transitive preferences. If this behavior is observationally equivalent to maximization of continuous, strictly concave and strictly monotonic utility functions, then convergence of the auction is guaranteed. Secondly, we show that the RP-rule for auctions is empirically implementable. Thirdly, we show how, after some initial learning phase, the bidding process can be finished by automata that are required to issue bids consistent with the RP-rule, which would provide an alternative to the procedures used now, for example, on eBay. Finally, we also link the PA to general equilibrium theory in showing that the PA is the dual of the Walrasian tatonnement procedure, and hence represents an alternative algorithm to solve social welfare programs, which is interpretable at every step and, in contrast to the Walrasian procedure, does not suffer from out-of-equilibrium infeasibility.

1.1. Structure of the paper

The structure of the paper therefore is as follows. In section 2, we discuss the main concerns in designing auction mechanisms for Internet auctions and discuss the literature in this field. In section 3, we introduce the PA mechanism. For the most general case (multiple commodity double auctions), we show how the PA mechanism can be implemented and interpreted as a practical device and we prove the convergence of the process described by the algorithm to equilibrium. While section 3 assumes the existence of concave utility functions, section 4 concentrates on observed bids only and introduces RP as a test on the consistency of a sequence of bids made by a player. We show that convergence of the auction process is guaranteed under two mild assumptions: (1) if a player is offered an assignment of goods that is identical to an offer made earlier in the process, then his bids should also be the same, (2) not all participants in the auction are disqualified by the RP rule. We show how an initial learning phase in the auction process can be followed by a stage where automata bid for the participants in a

strictly consistent way. In section 5, we link the PA mechanism to general equilibrium theory by showing that the PA mechanism is the dual of Walrasian tatonnement. Finally, section 6 concludes and provides directions for further research.

2. Internet auctions: concerns and design issues

The rise of Internet auctions has given a new impulse to the design of optimal auctions, as the special features of the Internet (anonymity, easy access, lack of geographical constraints etc) required rethinking auction designs, especially for multi-unit and double auctions. Already in 2000, Lucking-Reilly observed that "Internet technology increases the feasibility of ascending-bid multi-unit auctions" (Lucking-Reilly, p. 23), and Bapna et al (2000) actually observed that "the majority of on-line auctions are multi-item auctions" (Bapna et al., p. 2), who but many other types of auctions, including double auctions, are held via the Internet.. The sharp rise in the number of these auctions brought the question of designing efficient and robust multi-unit and double auctions back to the forefront. In addition, the large amount of data that has become available also spurred contributions that analyze auction processes in a statistical way. Although a full overview of the literature clearly is beyond the scope of the paper, we briefly discuss contributions made in the areas of multi-unit auctions, combinatorial auctions, and double auctions, in the areas of mechanism design as well as statistics.

2.1. Multi-unit auctions

For multi-unit auctions, the main issue is designing an auction that has a Pareto-efficient equilibrium, and that satisfies incentive compatibility and individual rationality. A major issue in the literature is to avoid 'demand reduction', where players are bidding lower than their true values in order to pay a lower equilibrium price. Especially uniform price auctions suffer from this problem (see e.g. Engelbrecht-Wiggans and Kahn, 1998 for a theoretical contribution and Engelbrecht-Wiggans et al., 2006 for an empirical analysis of the severity of the problem).

The (generalized) Vickrey Auction (GVA) (Vickrey, 1961; Varian, 1995) does not share this problem, as it is a dominant strategy for players to bid their true valuation. Yet, this auction is not seen much in practice, one of the main reasons being that the auction is too complex for bidders to understand. This led to the design of alternative auctions that share the desirable properties with the GVA, where the ascending bid auction (Ausubel and Cramton, 2002; Ausubel, 2004), that relies on the idea of "clinching" and the clock proxy auction (Ausubel et al., 2006) are the best-known examples. Yet, especially the clock-proxy auction is also criticized for being too complex. In addition, the desirable properties of the GVA and related auctions are no longer maintained if bidders can submit bids under false names (shill bidding), a practice that is very difficult to detect in Internet auctions (Sakurai et al, 1999). In response to this, protocols were developed for Internet auctions that should prevent shill bidding (e.g. see Yokoo et al. (2001a, b), but these protocols place high demands on the auctioneer, since they rely on bundling the units in an optimal way. Even if the auctioneer has some knowledge of the distribution of agents' evaluation values, the auctioneer must solve a very complicated optimization problem to find an appropriate division set. Summarizing, the main issue in multi-unit auctions seems to be to design a simple and implementable auction that is robust against shill bidding and demand reduction.

2.2. Double auctions

For double auctions, the concern of designing an auction that has a Pareto-efficient equilibrium, and that satisfies incentive compatibility and individual rationality, is complemented by the need to prove that a (non-trivial) equilibrium exists. Many of the existence results rely on specific assumptions on symmetry (Milgrom and Weber, 1982), monotonicity (Athey, 2001; Maskin and Riley, 2000) and the number of players (Fudenberg et al., 2004). In a more general setting, Jackson and Swinkels (2005) prove existence of equilibrium, but for the solution to be non-trivial, the introduction of a non-strategic agent is necessary.

With respect to efficiency, individual rationality and incentive compatibility, the most prominent concept is that of the *k-double auction* introduced by Chatterjee and Samuelson (1983), and many refinements and extensions were proposed since then (see e.g. Parsons et al., 2006, for a recent overview of the literature). In this auction, sellers and buyers submit offers and bids that are aggregated to construct supply and demand curves. The crossing of the graphs determines a price interval from which a market clearing price can be selected as a linear combination. Trade occurs among buyers who bid at least this price and sellers who offer at no more than this price. Wilson (1985) extended the concept to a multilateral k-double auction by means of a clearing house construction where traders submit sealed bids. The main result of this paper is that the design is incentive efficient, but this result hinges on there being enough players in the market. McAfeee (1992) designed a double auction protocol where truth telling is an equilibrium strategy independent of the number of players, by introducing a specific pricing rule that deviates from that in Wilson (1985), and can also be viewed as a refinement of the Buyer's Bid Double Auction (e.g. see Williams, 1991). However, using Internet auctions as the general setting, Yokoo et al (2005) show that Preston protocol is no longer dominant strategy incentive compatible when false bidding can occur. There "TPD" protocol relies on the auctioneer setting a threshold price, but it has the same weakness as the protocols developed for the multi-unit auction (Yokoo et al., 2001a,b), namely that very high demands are placed on the auctioneer, which makes actual implementation hard to imagine.

Summarizing, for double auctions, the main issues are to prove existence of equilibrium with as few assumptions on the players in the auction as possible and without introducing artificial constructs, and to design an implementable auction that is robust against shill bidding.

2.3. Statistical analysis of auction data

The rise of online auctions brought with it a very large amount of electronic data describing the auction process. Statistical analysis of these data has now become increasingly popular as a means to predict end-prices of auctions (e.g. Jank and Shmueli, 2003, Xuefeng et al, 2006, Wang et al., 2008, 2009) and to understand the influence of auction design on outcomes, including the condition for auctions to end (hard or soft end-time, e.g. Roth and Ockenfels, 2002; Ariely et al., 2005, Ockenfels and Roth, 2006), and mechanisms to detect and deter shill bidding and bid shielding (Kauffman and Wood, 2003). Although general conclusions with respect to bidder's behavior are drawn (e.g. see Bajari and Hortaçsu, 2003, 2004 for overviews), the spillover from this line of research to theoretical design of auctions seems to be limited yet.

3. Introducing the Primal Auction (PA) mechanism

In this section, we introduce our generalization of the single good, single-side Vickrey auction. Contrary to the usual definition (e.g. see Friedman, 1993) of a double auction, we will assume first that goods are divisible, and then show how the principle can be extended to the case of indivisible goods. We largely follow Varian (1995) in assuming the following:

1. There are I agents, indexed i and K+1 goods, indexed k, where the last commodity acts as a numeraire, with unit price.
2. Goods are divisible
3. Utility functions are quasi-linear, i.e. of the form $u_i(x_{i1},...,x_{iK}) + x_{i,K+1}$, with the first term a concave, strictly quasi-concave and continuously differentiable function with $\bar{p}$ as upper bound.
4. At the start of the auction, agent i owns quantity ω_{ik} of commodity k.

Given that even large multicommodity auctions will only cover a limited set of the total goods traded in an economy, the assumption of a quasilinear utility function in this context seems very well defendable, and following Varian (1995) a reasonable objective is to allocate the goods among the consumers so as to maximize the sum of utilities:

$$W = \max_{x_{ik} \geq 0; x_{i,K+1}} \sum_i u_i(x_{i1},...,x_{iK}) + x_{i,K+1}$$

subject to **(1)**

$$\sum_i x_{ik} = \sum_i \omega_{ik}, \ k = 1,...,K+1 \quad (p_k)$$

Where p_k denoted the Lagrange multiplies associated with the commodity balance.

The specific contribution of this paper is to show that there exists an algorithm that can be interpreted as an auction, is sufficiently strategy-proof and can allow for an approximate solution of program 1 when terminated early. Auctioning is seen here as a procedure where agents express bids rather than desired net trade, and the auctioneer's task is then to assign the available goods to the bidders in accordance with their bids. Formally, in discrete time, the procedure is as follows: given a positive step size factor μ small enough and exclusively under the control of the auctioneer, we consider the projected gradient process:

$$x_{ik}^{t+1} = \max(x_{ik}^t + \mu(\pi_{ik}^t - p_k^t), 0), \quad t = 0,1,... \quad \textbf{(2)}$$

Starting from given $x_i^0 = \omega_i$, for bids

$$\pi_{ik}^t = u_{ik}^{'}(x_i^t) \quad \textbf{(3)}$$

And reference prices p_k^t set by the auctioneer so that, throughout the iterations, the following feasibility condition holds:

$$\sum_i x_{ik}^t = \sum_i \max(x_{ik}^t + \mu(\pi_{ik}^t - p_k^t), 0) \quad \textbf{(4)}$$

Through this restriction, the adjustment rule projects the gradient vectors π_i^t of individual bids onto the compact and convex set $X = \left\{ (x_1,...,x_I) \geq 0 \middle| \sum_i x_i = \sum_i \omega_i \right\}$. Hence, the procedure defines a specific step in a projected gradient algorithm. From this and the concavity of utility functions it follows, first, that convergence is assured for a step size μ small enough and, secondly, that all commodity balances are respected throughout the process that starts from autarky. Furthermore, given the bids of all the participants in a specific round of the auction, the auctioneer can compute the new reference price as the mean bid of the active participants and distribute the goods over the participants in accordance with their relative bids. Proposition 1 formalizes this:

Proposition 1 (computing the reference price): the constraint set in (4) defines, for each commodity separately, reference prices p_k^t as continuous functions of bids $(\pi_{1k}^t,...,\pi_{Ik}^t)$ and current assignments $(x_{1k}^t,...,x_{Ik}^t)$ that can be computed in a finite number of iterations.

Proof. If all entries in $(\pi_{1k}^t,...,\pi_{Ik}^t)$ are identical, p_k^t is also equal to this value and hence uniquely defined. Otherwise, at given $(\pi_{1k}^t,...,\pi_{Ik}^t)$ and $(x_{1k}^t,...,x_{Ik}^t)$ the right-hand side of (4) is a continuous function of p_k^t, that is larger that the fixed left-hand side at $p_k^t = \underline{p}_k^t = \min_i \pi_{ik}^t$, smaller at $p_k^t = \overline{p}_k^t = \max_i \pi_{ik}^t$ and strictly increasing in between. Hence, p_k^t solves uniquely and is a continuous function of $(\pi_{1k}^t,...,\pi_{Ik}^t)$ and $(x_{1k}^t,...,x_{Ik}^t)$. To compute its value a Newton-Raphson procedure, or a bisection procedure, or a combination of both could be used. It would converge but not necessarily in a finite number of iterations. To obtain finite termination, we compute the target value as the mean bid of the active participants:

$$p_k^{*t} = \frac{\sum_{i \in I_k^t} \pi_{ik}^t}{\sum_{i \in I_k^i} 1}, \text{ where } I_k^t = \left\{ i = 1,...,I : x_k^t > 0 \right\} \tag{5}$$

If $I_k^t = I_k^{t-1}$ and $p_k^{*t} = p_k^{t-1}$, this exactly calculates the final price, and the algorithm terminates. If $I_k^t \neq I_k^{t-1}$, we evaluate excess demand and can apply standard bisection on p_k^t, proving convergence in a finite number of iterations.

Since (5) solves in a finite number of iterations, it is implementable and interpretable as a practical device. The resulting value is equal to the average bids over all the agents, except those with zero consumption and a private valuation below p_k^t. The procedure allows for variable participation to the auctions and the non-participants who do not have any claim over goods and are not interested in getting one do not affect the price, as required.

Auctioning proceeds on a commodity-by-commodity basis but interdependence across commodities is accounted for via the full vector of goods possessed (demanded) at every iteration (x_k^i). The key difference from the regular auction is that, at each iteration, the algorithm asks every consumer for a bid reflecting what the bidder would be willing to pay *to keep* the given quantity, rather than *to acquire* it. Next, the mean bid p_k^t is calculated and each bid is separately compared to it. Finally, those willing to pay more than the mean receive an increased allocation,

the others a reduced one, and everyone has the assurance that the process will not end until one either leaves it without demand, or receives the amount one is willing to pay for.

Payments eventually take place at the last price p_k^T of the auction, although goods are actually being distributed at every iteration. The obligation on the part of the participants is to accept the quantity, without payment, but with the expectation that they will have to pay p_k^t.

To highlight the basic operation of the adjustments, we will first describe them in a continuous time version: starting from $x_i(0) = \omega_i$, the continuous time procedure reads:

$$\dot{x}_{ik} = \kappa_{ik}(x(t))(\pi_{ik}(x_i(t)) - p_k(x(t))) \tag{6a}$$

For

$$\pi_{ik}(x) = u'_{ik}(x_i) \tag{6b}$$

$$p_k(x) = \frac{\sum_i \kappa_{ik}(x)\pi_{ik}(x_i)}{\sum_i \kappa_{ik}(x)} \tag{6c}$$

Computed in a finite number of iterations (less than the number of agents), so that

$$\kappa_{ik}(x) = \left\{0 \text{ if } x_{ik} = 0 \text{ and } p_k(x) > \pi_{ik}(x_i), \text{ and } 1 \text{ otherwise}\right\} \tag{6d}$$

We now show convergence of our auctioning process:

Proposition 2 (Primal auction) Assume that utility functions $u_i : R_+^K \to R$ are continuously differentiable, concave and that u_1 is increasing. Then, continuous time Primal auction mapping (6a-d) converges to the market equilibrium and there exists a positive μ for which this also holds for discrete time auction (1-3).

Proof. As long as $\dot{x}_{ik} \neq 0$ for some i,k (and hence $\kappa_{ik}(x) = 1$ for some i,k, which is assured since u_1 is increasing, monotonicity of the criterion follows since:

$$\dot{w} = \sum_i \dot{u}_i = \sum_i \dot{u}_i - \sum_k p_k \sum_i \kappa_{ik}(x)(u'_{ik} - p_k)\dot{x}_{ik} = \sum_i \sum_k (\dot{x}_{ik})^2 > 0,$$

Where the second equality holds because $\sum_i \kappa_{ik}(x)\dot{x}_{ik} = 0$, i.e. commodity balances hold throughout, while by continuity of the derivative switches in $\kappa_{ik}(x)$ can occur only finite number of times on any compact interval. Hence, social welfare rises monotonically along the path and since commodity balances hold and supply is bounded, it is bounded from above and must, therefore, converge to a stationary point. By concavity of the utility, this is a welfare optimum and hence a market equilibrium. This proves convergence for the continuous time process. Regarding convergence for the discrete time version, by continuity of the bids as function of x_i^t, and continuity of the reference price (Proposition 1), (1-3) defines a projected gradient process for x_i^{t+1} as function of $(x_1^t,...,x_I^t)$,which for step size μ small enough, by concavity of the underlying utility functions converges to a global optimum that solves the welfare program.

A "corollary" of Proposition 2 is that it puts one of the main findings of empirical research on the performance of double auctions, the "surprising competitive properties" of the outcome (even for small groups of traders) in a clear theoretical perspective[2]: the Primal auction solution is a

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[2] Smith (1962), generally regarded as the start of this line of research, called this finding a "scientific mystery"

competitive market equilibrium, while the incentive for truth telling under the Primal auction design does not depend on the number of participants in the auction room (recall that in the Walrasian auction, strategic behavior is ruled out only if the number of participants is large).

So far, we have concentrated the discussion on the case of divisible goods. We now move to the more usual case of indivisible goods. Starting from a discrete choice model, we consider agent i faced with H indivisible objects, indexed h, with corresponding money metric utilities $\bar{u}_{ih}$, all different across objects and suppose that individual agents have additive utilities:

$$u_i(d_{i1},...,d_{iH}) = \sum_h \bar{u}_{ih} d_{ih} \tag{7}$$

For binary valued d_{ih}. Free disposal can be represented via zero evaluation. The associated social planning problem reads:

$$\max_{d_{ih} \in \{0,1\}} \left\{ \sum_i \sum_h \bar{u}_{ih} d_{ih} \mid \sum_i d_{ih} = 1 \right\}$$

We now drop the indivisibility of d_{ih} replacing it by the real-valued non-negative δ_{ih}. We maintain the assumption that all $\bar{u}_{ih}$ are different across objects and agents, and hence, the solution of:

$$\max_{\delta_{ih} \geq 0} \sum_i \sum_h \bar{u}_{ih} \delta_{ih}$$
$$subject\ to$$
$$\sum_i \delta_{ih} = 1 \qquad (r_h)$$

will be unaffected and hence binary-valued as required. The Primal auction updates the "ownership shares" δ_{ih}^t in each round of the auction as[3]:

$$\delta_{ih}^{t+1} = \max\left(\delta_{ih}^t + \mu\left(\rho_{ih}^t - r_i^t\right), 0\right), \quad t=0,1,...$$

Starting from given δ_{ih}^0, for bids

$$\rho_{ih}^t = \bar{u}_{ih}$$

And reference prices r_h^t set by the auctioneer so as to fulfill feasibility condition

$$\sum_i \delta_{ih}^t = \sum_i \max\left(\delta_{ih}^t + \mu\left(\rho_{ih}^t - r_i^t\right), 0\right)$$

Then, the allocation would for μ small enough, converge to a binary valued solution. In case several agents have the same valuation for a commodity, convergence of the process is not affected, but the allocation will be non-binary in general, in some tie-breaking rule has to be imposed. The resulting binary allocation will still be Pareto-efficient. In the converse case, where one agent has the same valuation of different commodities with positive δ_{ih}-values in the optimum, implies that some other agent(s) must also have a non-specialized outcome. This case can in general not be solved by a tie-breaking rule, because the price impacts will be different:

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[3] Note that this differs from the share auctions inspired by Wilson (1979) since in the PA, the players do not indicate the share themselves, but only issue a price bid.

only if the winners of the auction are willing to share the property over the object will an efficient outcome result.

4. Revealed Preference as test on admissibility of bids

Section 3 assumed the existence of a concave utility function from which the bids made by the players are derived as the marginal utilities. However, as was stated in the introduction, one of the main concerns in auction theory is to design procedures that will provide the participants with the correct incentives to behave in accordance with their true preferences.

If participants have no way to affect the price, they have no incentive to lie about the quantity they want and have to pay for as this would only leave them with the wrong quantity. In this sense, the Primal auction is truth telling device, since it makes it difficult, in the absence of collusion, to impact in a predictable manner on price, and at the same time, offers a means to signal at each iteration possible discontent about the prevailing allocation. However, as was explained in section 2, the incentive may be too weak, as in practice some individuals may be large enough to have an impact, or think they are, on in any other way destabilize the process through their bids or impair further on the (necessarily reduced) efficiency of the auction. Since bids are the only observed actions of the participants of the auction, a rule has to be devised that allows the auctioneer to judge the bids. Ausubel et al. (2006) propose an activity rule that is based on revealed preference, but in contrast to our approach, it evaluates quantity bids made by players, and is stated in a combinatorial auction setting, not in the context of double auction. Wilson (2006) introduces revealed preference as a guiding principle for activity rules in a double auction setting, also for quantity bids, but there is no formal development of the argument made, and no explicit link with convergence of the auction process.

Here, we provide a formal argument for an activity rule by which the auctioneer accepts the bid (price) of a participant if this bid is consistent with a rationalizing preference and to inactivate participants otherwise. For ease of exposition, we return to the case of divisible goods here. Operationally, this rule implies that the auctioneer tests whether the bids placed by a player obey RP. Hence, as is also mentioned by Wilson (2006) as time passes, the test on RP becomes stricter as the bidding progresses. If a player's bids violate RP, there are essentially three possibilities. The first is to impose a fine and restart the auction, the second to continue without the perpetrators, while excluding them from further bidding and change in demand, but having them pay at the end of the auction for the quantity they possessed upon their disqualification. Hence, disqualified agents keep $x_{ik}^{t+1} = x_{ik}^t$ indefinitely from their disqualification onwards but they have to settle the eventual payment $\sum_k p_k^*(x_{ik}^t - \omega_{ik})$ at the end of the auction. The third possibility is to neglect their bid and leave them with their current assignment while allowing them to bid again in the next iteration.[4] We focus on the third possibility, since the other two can be shown to be special cases. First, we follow Ausubel et al. (2006) in defining RP for bids – note that the definition is different from that usually used in consumer theory. In fact, taking the Afriat conditions as reference, it adds a fourth condition for concavity of a function,

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[4] We note that in Ausubel et al. (2006), it is not explicitly stated what sanctions follow if a player violates the activity rule; it seems to be tacitly assumed that the player will, after receiving a warning, revise his bid in accordance with the rule. In Wilson (2006), the portion of the submitted tender that violates the rules is discarded by the auctioneer.

Definition 1 (RP for bids) The bids of consumer i satisfy Revealed Preference if $(\pi_i(x_i) - \pi_i(x_i'))^T (x_i - x_i') \leq 0$ for any pair $x_i, x_i' \in D_0$.

RP for bids stand in direct relation to the concavity of the associated utility function, as the following interpretation of Afriat (1967) for money metric utility functions shows

Proposition 3 (concavity of utility functions) consider a convex set $D \subset R_+^K$ and a differentiable money metric utility function $u_i : D \to R, u_i(x_i)$ and let $\pi_i(x_i) = \dfrac{\partial u_i(x_i)}{\partial x_i}$. Then,

(1) u_i is concave on $D_0 \subset D$ if and only if it meets RP or $u_i(x_i') \leq u_i(x_i) + \pi_i(x_i)^T (x_i' - x_i)^T$.

(2) u_i is strictly concave on $D_0 \subset D$ if RP is met with strict inequality or $u_i(x_i') < u_i(x_i) + \pi_i(x_i)^T (x_i' - x_i)^T$, whenever $x_i \neq x_i'$.

Proof. See e.g. Ortega and Rheinboldt (1970, p. 84-86)

The far-reaching implications of RP for theory as well as for empirical work are well-known and it also proves effective here. In practical terms, the problem for the auctioneer is that he has no way of knowing a priori whether the bids made are expressions of true preference, let alone whether they are gradients of some well-behaved utility function. The key point that was not observed by Ausubel et al. (2006) or Wilson (2006) is that the gradient algorithm does not have to know this either to converge. As long as the bids could be gradients, convergence is guaranteed. In other words, as long as the bids are rationalizable as being the derivatives of a utility function, the algorithm will converge to a welfare-maximizing solution. Formally:

Proposition 4 (rationalizable bids) Any series π_i^s, $s = 0,...,t$ of consumer bids that satisfies RP is rationalizable as marginal valuation with the concave utility function,

$$u_i^t(x_i) = \min_{s \in \{0,t\}} (\hat{u}_i^s + \pi_i^{sT}(x_i - x_i^s)),$$

For constants $\hat{u}_i^t$, estimated from the linear program:

$$C_i^t = \min_{\varepsilon_i^s \geq \hat{u}_i^s} \left\{ \sum_{s=0}^{t-1} \varepsilon_i^s \,\middle|\, \hat{u}_i^t \leq \hat{u}_i^s + \pi_i^{sT}(x_i^t - x_i^s) + \varepsilon_i^s \right\}, \text{ with outcome } C_i^t = 0. \tag{8}$$

Proof ***To be inserted***.

We also remark that the utility functions estimated via this definition are not necessarily strictly concave or differentiable. Strict concavity is important for the uniqueness of consumer choice, and Proposition 3 already indicated that strict inequality is sufficient to ensure strict concavity. On an empirical data set, this inequality will in general hold and strict concavity cannot be rejected.

In addition to concavity, differentiability is essential for the uniqueness of bids and the continuity of the underlying differential equations. Along an auction path, it is (for divisible goods) unlikely that exactly the same x_i^t-vector will be repeated. As long as these differ by more

than a constant, differentiability of the underlying utility function cannot be ruled out. For now, the next proposition established in a constructive way that the differentiability requirement may not be too demanding, while showing at the same time that uncertainty be accounted for that may prevail in the individual's mind about the utility level attainable from given *x*. Specifically, additive perturbation of utility according to a differentiable density generates an expected utility function that is concave as well as differentiable (Keyzer and Van Wesenbeeck, 2005):

Proposition 5 (Mollifier mapping) Consider the concave function $u(x)$, the differentiable density function $\ell : R^K \to R_+, \ell(\varepsilon)$, and define the mollifier mapping $\ddot{u}(x) = Eu(x - \varepsilon) = \int u(x - \varepsilon)\ell(\varepsilon)d\varepsilon$. Then the expected utility function $\ddot{u}(x)$ is concave and differentiable.

Proof Concavity follows because the integral is a linear operation, and the weighted sum of concave functions is concave. Differentiability is verified via the Sobolev (1963) averaging principle, i.e by substituting $x - \varepsilon = q$ which gives: $\ddot{u}(x) = \int u(q)\ell(x - q)dq$, showing that $\ddot{u}(x)$ inherits the differentiability properties of the likelihood density

Since we are free in choosing a density, the perturbation it imposes can have arbitrary width as well. In applications, this is usually represented via a scalar window size θ:

$$\ddot{u}(x) = \int u(\frac{x - \varepsilon}{\theta})\ell(\varepsilon)d\varepsilon$$

that can be shown to converge to the underlying function u for $\theta \downarrow 0$, confirming that RP compliant choices can for auctions that do not replicate the same x_i^t more than once, be represented by an underlying concave and differentiable utility function, which if strict inequality holds, is also strictly concave.

While we have seen in Proposition 5 that the differentiability requirement may not be that demanding theoretically, in a real auction, discontinuity cannot be ruled out, It could be addressed by requiring the participant to stick to an earlier bid whenever the assignment is the same (or arbitrarily close) to an earlier one but since in numerical practice full replication will rarely occur, the continuity test defined here will rarely be activated as a safeguard.

Definition 2 Continuity rule. Participants in the auction are entitled to express modified bids at iteration *t* as long as $\left\| x_i^t - x_i^s \right\| \ge \varepsilon_0$ with respect to all observations $s = 0,1,...,t-1$ collected so far. For $\left\| x_i^t - x_i^s \right\| < \varepsilon_0$, the auctioneer assigns the earlier recorded bids $\breve{\pi}_i^t = \pi_i^s$ and quantities $\breve{x}_i^t = x_i^s$ prior to all further allocations, for use in later RP tests.

We take ε_0 to be chosen small enough not to disturb convergence via the shift from x_{ik}^t to $\breve{x}_{ik}^t$. We accordingly denote the set of agents with distinct bids by $I^t = \left\{ i = 1,...,I \middle| \left\| x_i^t - x_i^s \right\| \ge \varepsilon_0, s = 0,1,...,t-1 \right\}$. This modifies the process for $t = 0,1,...$ as:

$$x_{ik}^{t+1} = \max(x_{ik}^t + \mu(\pi_i^t - p_k^t),0) \text{ if } i \in I^t \text{ and } \breve{x}_{ik}^t \text{ otherwise} \tag{9}$$

And reference prices p_k^t set by the auctioneer to meet feasibility condition:

$$\sum_i x_{ik}^t = \sum_{i \in I^t} \max(x_{ik}^t + \mu(\pi_i^t - p_k^t), 0) + \sum_{i \notin I^t} \breve{x}_{ik}^t \qquad (10)$$

Now for the case that all bids satisfy RP, while the auctioneer applies the continuation rule, we can formulate the following proposition:

Proposition 6 (bid functions satisfying RP under continuation) For bids π_i^t satisfying RP across all observations $t = 0, 1, ...$ gathered throughout an auction conducted under the continuation rule (Definition 2), there exist positive constants μ and ε_0 small enough to ensure that the projected gradient algorithm (10,11) converges toward an equilibrium and ends after a finite number of steps at an approximate equilibrium, modulo ε_0.

Proof. ***to be inserted***

The proposition confirms that only bids made in the course of the process matter. In other words, convergence holds as long as participants behave 'as if" for the feasible allocations emerging in the course of the action. Moreover, operational estimates of utility functions used in the process can be obtained via Proposition 4.

4.1. RP rule in practice: exclusion of participants and the possibility of learning

Finally, we now represent the case in which RP is not met by all participants at every iteration. As before, we specify an auctioning process that converges to an approximate equilibrium, but now, perpetrators may end up with an allocation they will not consider optimal. In the Primal auction, the new bids are the free choice variables of individual agents, signaling their marginal valuation of the goods currently assigned to them. Hence, surplus maximizing consumers would, under truthful revelation have to meet the RP rule:

Definition 3 (RP rule) participants in the auction are entitled to receive modified assignments at iteration t only if their bid satisfies RP with respect to observations $s = 1, 2, ...t - 1$ collected so far that passed the RP test as well as the continuity test.

Under the RP rule, the new observation is, *at every iteration*, checked against all previous ones on a grid of sufficiently distinct points. This is a fully individualized condition that can be evaluated irrespective of the fact that assignments x_i^s will until equilibrium not be optimal at prices p^s. The condition expresses that, like proximate bids that fail the continuity test, non-RP bids are not being tested against later on. The RP rule also has the advantage that it prevents agents from relying on seemingly independent intermediaries, say, to continue participation when one component has been disqualified, as this will only make the RP requirement on the collective more severe than under the consolidated requirement. At the same time, the condition also indicates that the auctioneer cannot make any decision on a commodity-by-commodity base. This is a disadvantage because it means that the need for coordination is important, and new assignments can only be made once all the bids have been collected. Yet, in practice it may be possible to achieve further decentralization when interdependencies in demand prove to be

limited. Specifically, it the utility function is additively separable into non-overlapping commodity groups, each group can under the prevailing quasilinearity assumption be treated as a separate consumer with respect to the RP rule. Incorporation of the RP rule can be effectuated by including a parameter ξ_i^s in the process, which equals 1 when bids satisfy the RP rule and 0 otherwise. Then, process (9,10) is reformulated as:

$$x_{ik}^{t+1} = \max(x_{ik}^t + \mu \xi_i^t (\pi_{ik}^t - p_k^t), 0) \text{ if } i \in I^t \text{ and } \breve{x}_{ik}^t \text{ otherwise}$$

With reference prices p_k^t set by the auctioneer to maintain commodity balances:

$$\sum_i x_{ik}^t = \sum_{i \in I^t} \max(x_{ik}^t + \mu \xi_i^t (\pi_{ik}^t - p_k^t), 0) + \sum_{i \notin I^t} \breve{x}_{ik}^t \ .$$

Finally, we recall that the earlier process required a step size μ to be given and small enough. The RP rule offers as major advantage that it applies globally, irrespective of step size. This makes it possible for the auctioneer to infer from possible non-convergence, i.e. non-increasingness of the estimated aggregated utility across from one iteration to the next, both estimated with the utility function of the latest iteration, that the step size is too large and has to be reduced, as oppose to suspecting participants of any false revelation. Clearly, discarding bids may lead to a situation where the auction ends prematurely due to lack of active participants. This possibility is ruled out by the following assumption:

Definition 4 (nonstationarity) If $\sum_i \sum_k |\pi_{ik}^t - p_k^t| > 0$, *then* $\sum_{i \in I^t} \xi_i^t \sum_k |\pi_{ik}^t - p_k^t| > 0$.

Under this nonstationary assumption, the process will roll on until the social optimum is reached:

Proposition 7 Exchange with inactivation of bids If the auctioneer applies the continuity rule and the RP rule on the sequence of bids $\pi_i^t, t = 0,1,..$ there exist positive constants μ and ε_0 small enough to ensure that projected gradient algorithm (10,11) converges toward an equilibrium and ends after a finite number of iterations. If the nonstationarity assumption holds, throughout the iterations, this equilibrium is approximately (modulo ε_0) Pareto optimal.

***Proof** **to be inserted**.*

In general, the Primal auction design can be modified to allow for a "learning" phase in the bidding process (that could include a feed back from the market price to the own valuation of the goods at auction), where failure of the RP test is accepted without concluding that bids are not rationalizable, analogous to tests on "goodness of fit" applied in the literature on testing for RP on given, finite datasets of consumer choice.[5]. Such a learning phase could precede the automated

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[5] Afriat's 1967 "partial efficiency" measures how well a given set of data satisfies utility maximization. Different measures of the "goodness of fit" of the data are developed, including the number of violations (Whitney and Swofford, 1987), the fraction of violations (Famulari, 1995), and statistically testing the magnitude of the adjustment of the data needed to satisfy the Afriat inequalities (Varian, 1982; see e.g. De Peretti, 2005 for a recent contribution along this line). The central idea is that consumers should be allowed to make some "mistakes" without having to conclude that their observed behaviour is inconsistent with the existence of a concave utility function. Except for Forges and Minelli (2008), testing of consistency of decisions takes place after the data is collected and there is no explicit time dimension in the application of the RP test.

bidding common in Internet auctions, which would then be robust against shill bidding and bid shielding, as we have shown that the RP-rule induces truthful revelation by players. In our framework, the possibility of errors in the early stages of the auctions implies that Proposition 4 is reformulated as follows:

Proposition 8 (weak rationalization of bids) Any series of bids $\pi_i^s, s = 0,...,t$ associated to assignments x_i^s in the linear program:

$$\tilde{C}_i^t = \min_{\varepsilon_i^s, C_i^t \geq 0, \hat{u}_i^s, \hat{\eta}_i^{s+}, \hat{\eta}_i^{s-} \geq 0} C_i^t + \gamma \sum_{s=\underline{t}}^{t-1} (\beta)^{t-s} (\hat{\eta}_i^{s+} + \hat{\eta}_i^{s-})$$

subject to

$$\hat{u}_i^t \leq u_i^s + (\pi_i^s + \hat{\eta}_i^{s+} - \hat{\eta}_i^{s-})^T (x_i^t - x_i^s) + \varepsilon_i^s \tag{11}$$

$$\pi_i^s + \hat{\eta}_i^{s+} - \hat{\eta}_i^{s-} \geq 0$$

$$C_i^t = \sum_{s=\underline{t}}^{t-1} (\beta)^{t-s} \varepsilon_i^s$$

For which $C_i^t = 0$ holds throughout, is weakly rationalizable as marginal valuation of the concave utility function:

$$u_i^t(x_i) = \min_{s \in \{0,t\}} (\hat{u}_i^s + (\pi_i^s + \hat{\eta}_i^{s+} - \hat{\eta}_i^{s-})(x_i - x_i^s)) \tag{12}$$

And satisfies RP on the series $(\pi_i^s + \hat{\eta}_i^{s+} - \hat{\eta}_i^{s-}, x_i^s), s = 0,...,t$.

Proof *The proof is along the same lines as for Proposition 4.**to be completed***

We remark that because of the error, the subdifferential of $u_i^t(x_i)$ now becomes time dependent. This is the reason to impose discounting, so as to ensure that old bids gradually become discarded in (12) and higher precision is demanded of new bids. Therefore, as the auction progresses, eventually players' bids will have to become fully compatible with RP in order to be accepted, and it also implies that from that point onwards, automata could take over the auction process.

5. PA and Walrasian tatonnement

In the optimum, the program decentralizes to the competitive equilibrium where each consumer maximizes surplus according to:

$$\max_{x_{ik} \geq 0} u_i(x_{i1},...,x_{iK}) - \sum_k p_k x_{ik} ,$$

which is equivalent to utility maximization subject to a budget constraint:

$$\max_{x_{ik} \geq 0, x_{i,K+1}} \left\{ u_i(x_{i1},...,x_{iK}) + x_{i,K+1} \Big| \sum_k p_k x_{ik} + x_{i,K+1} = \sum_k p_k \omega_{ik} + \omega_{i,K+1} \right\}.$$

Prices are such that markets clear:

$$p_k \geq 0, \ k = 1,...,K : \sum_i x_{ik} \leq \sum_i \omega_{ik}$$

Because of quasilinearity of the underlying utility functions, the welfare program decentralizes as a competitive equilibrium and Walrasian auctioneering (with price adjustments done by a fictive auctioneer in the direction of excess demand) can be invoked as a process to reach the equilibrium price and associated quantities. This because clear when we consider the Lagrangean of the first K commodities:

$$V(p) = \max_{x_{ik} \geq 0} \sum_i u_i(x_{i1},...,x_{iK}) - \sum_k p_k \sum_i (x_{ik} - \omega_{ik}) \qquad \textbf{(13)}$$

Equilibrium prices emerge from the associated dual problem: $\min_{p \geq 0} V(p)$, and the dual gradient algorithm operates on V's derivative $V'(p)$, which is unique because of the assumed strict quasiconcavity of utility and, by the Envelope Theorem, satisfies:

$$V'(p) = \sum_i (x_{ik}(p) - \omega_{ik}),$$

From which a Walrasian tatonnement rule for the stepwise adjustment of prices by the auctioneer in response to expressed trades by the market participants can be formulated as:

$$p_k^{t+1} = \min(\max(p_k^t + \lambda \sum_i x_{ik}(p^t) - \omega_{ik}),0), \overline{p}_k), t = 0,1,...$$, given p^0. Under this tatonnement, the commodities are auctioned in parallel and interdependencies between the goods are taken into account via the full price vector available and net demands are publicly known, as sealing of demands would take away relevant information from the other participants.

6. Concluding remarks

This paper presented an auction design for the efficient allocation of commodities in multi-commodity auctions with both sellers and buyers acting. It is clear that the one-sided auction is a special case of our set-up, and that for the buyers' auction, it is very different from existing auction designs for multiple non-identical objects, in particular the Vickrey-Clarke-Groves (VCG) mechanism, which is the most well-known in this field. Under the VCG mechanism, each player announced his "type", which is a vector of parameters of his (money-metric) utility function. Based on these announcements, an allocation of the commodities is made over the players, and the price to be paid by the participants is calculated as the externality the player imposes on the others. Although the VCG mechanism has an efficient equilibrium, bidding is complex as bidders have to specify a bid for all packages of objects they desire to win (have to specify their utility function for all possible combinations); more competition leads to lower prices; and the mechanism is not robust against "shill bidding", the hiring by a real bidder of an additional player to affect prices in his advantage (Milgrom, 2004). In the Primal auction, increasing the number of participants in a buyers' auction would lead to higher prices, as demand increases while supply is fixed, and shill bidding only leads to a tightening of the RP test for the player attempting it. Finally, as we have shown, the RP test only requires players to act *as if* their

bids are based on a concave utility function, thereby allowing players to have their own - simple or very complex - decision models to arrive at a bidding strategy.

In this paper, we have assumed throughout that the marginal utilities of the players (their private valuations of the goods) are independent of the observed bids of the other players, since we focused on the design of an efficient auction mechanism in a multi-commodity auction. Of course, a significant part of the auction literature assumes that such dependence exists, and hence views auctions as non-cooperative games, where subjective and objective probability assessments are made and optimal reactions to expected actions by the other players are defined. As this was not the approach taken in this paper, we only note that, for the single commodity auction, our results also hold in the situation where the private valuation of each player is an increasing (possibly discontinuous) function of the bids π^t and non-decreasing in t. However, this does not imply that the process will converge to a socially efficient price.

Another assumption made throughout the paper was that the different commodities were not complementary to each other (e.g. see Milgrom, 2007 for a recent contribution in this field). Allowing for complementarity between markets implies that the assumption of fixed endowments ω_i is replaced by an explicit description of the production process with associated input demand.

Within Walrasian auctions, the inclusion of production and input demand is relatively straightforward, in particular when firm technologies can be represented by strictly convex production sets. Within the setting of the Primal auction, the explicit representation of production creates a difficulty in principle because the projection on the feasible space of commodity allocations involves the technology itself. A next step will therefore by to show that the Primal auction mechanism can be extended to allow for production, and convergence results even in the case of multiple production processes with constant returns to scale, where the Walrasian auction fails due to the set-valuedness of net supply, and in the presence of increasing returns to scale.

References

S.N. Afriat (1967) The construction of utility functions from expenditure data. International Economic Review, 8(1): 67-77.

D. Ariely, A. Ockenfels, and A.E. Roth (2005) An experimental analysis of ending rules in Internet auctions. The Rand Journal of Economics, 36(4): 890-907.

Athey, S. (2001) Single crossing properties and the existence of pure strategy equilibrium n games of incomplete information, Econometrica 69: 861-890.

L.M. Ausubel and P. Cramton (2002) Demand reduction and inefficiency in multi-unit auctions. Universty of Maryland Working Paper, available at http://www.ausubel.com/auction-papers/demand-reduction-r.pdf

L.M. Ausubel (2004) An efficient ascending-bid auction for multiple objects. American Economic Review, 94: 1452-1475.

L.M. Ausubel, P. Cramton, and P. Milgrom (2006) The Clock-Proxy Auction: a practical combinatorial auction design. In: P. Cramton, Y. Shoham and R. Steinberg (eds.) *Combinatorial Auctions*, Cambridge (Mass.), MIT Press, pp. 115-138.

P. Bajari and A. Hortaçsu (2003) The winner's curse, reserve prices, and endogenous entry: empirical insights from eBay auctions, The RAND journal of economics, 34(2): 329-355.

P. Bajari and A. Hortaçsu (2004) Economic insights from Internet auctions, Journal of Economic Literature, 42: 457-486.

R. Bapna, P. Goes and A. Gupta (2000) A theoretical and empirical investigation of multi-item on-line auctions. Information technology and management, 1: 1-23.

R. Bapna, P. Goes and A. Gupta (2003) Analysis and design of business-to-consumer online auctions. Management Science, 49(1): 85-101.

J. Bulow, J. Levin and P. Milgrom (2009) Winning play in spectrum auctions, Working paper downloadable at: http://www.milgrom.net/downloads/Winning%20Play.pdf

L. Cameron, P. Cramton and R. Wilson (1997) Using auctions to divest generating assets. Electricity Journal, 10: 22-31.

K. Chatterjee and W. Samuelson (1983) Bargaining under incomplete information. Operations Research, 31(5): 835-851.

P.C. Cramton (1995) Money out of thin air: the nationwide narrowband PCS auction, Journal of Economic and Management Strategy 4: 267-343.

P.C. Cramton, A. Skrzypacz and R. Wilson (2007) Economic comments on the design of the 700 MHz Spectrum auction, submitted with testimony of H.L. Barksdale to the U.S. Senate Committee on Commerce, Science, and Transportation, 14 June 2007.

P. De Peretti (2005) Testing the significance of the departures from utility maximization. Macroeconomic dynamics, 9: 372-397.

R.Engelbrecht-Wiggans and C.M. Kahn (1998) Multi-unit auctions with uniform prices. Economic Theory 12(2): 227-258.

R.Engelbrecht-Wiggans, J.A List and D. H. Reiley (2006) Demand reduction in multi-unit auctions with varying numbers of bidders: theory and evidence from a field experiment. International Economic Review, 47(): 203-231.

M. Famulari (1995) A household-based, nonparametric test of demand theory. Review of economics and statistics, 77: 372-383.

F. Forges, and E. Minelli (2008) Afriat's Theorem for general budget sets. Journal of Economic Theory, accepted manuscript

D. Friedman (1993) The double auction institution: a survey. In: D. Friedman and J. Rust (eds.) The double auction market: institutions, theories and evidence. Santa Fe Institute Studies in the Sciences of Complexity, chapter 1, pp. 3-25. Perseus Publishing, Cambridge, Mass.

D. Fudenberg, M.M. Mobius, and A. Szeidl (2004) Existence of equilibrium in large double auctions. Harvard Institute of Economic Research Working Paper 2033, Harvard University, Harvard (Mass.). Available at http://www.economics.harvard.edu/pub/hier/2004/HIER2033.pdf

M.O. Jackson and J.M. Swinkels (2005) Existence of equilibrium in single and double private value auctions. Econometrica, 73(1): 93-139

W. Jank and G. Shmueli (2003) Dynamic profiling of online auctions using curve clustering, available at: http://www.rhsmith.umd.edu/ceme/pdfs_docs/research/AuctionProfiling.pdf

R. L. Kauffman and C.A. Wood (2003) Running up the bid: detecting, predicting and preventing reserve price shilling in online auctions, Proceedings of the 5th international conference on Electronic commerce, Pittsburgh, Pennsylvania, 259 – 265.

M.A. Keyzer and C.F.A. van Wesenbeeck (2005) Equilibrium selection in games: the mollifier method. Journal of Mathematical Economics, 41(3): 285-301.

P. Klemperer (2004) A survey of auction theory. In: P. Klemperer: Auctions: Theory and Practice, Princeton: Princeton University Press.

D. Lucking-Reilly (2000) Auctions on the Internet: What's being Auctioned, and How? Journal of Industrial Economics, 48(3): 227-252.

E. Maasland and S. Onderstal (2006) Going, going, gone! A swift tour of auction theory and its applications. De Economist, 154: 197-249.

P.F. Malvey, C.M. Archibald and S.T. Flynn (1996) Uniform-price auctions: evaluation of the treasury experience. Working Paper, US Treasury, available at http://faculty-gsb.stanford.edu/wilson/archive/E542/classfiles/USTreasuryUniformPrice.pdf

E. Maskin and J. Riley (2000) Equilibrium in sealed high bid auctions. Review of Economic Studies 67: 439-452.

R.L. Matzkin and M.K. Richter (1991) Testing strictly concave rationality. Journal of Economic Theory 53: 287-303.

R. P. McAfee (1992) A dominant strategy double auction. Journal of economic theory 56: 434-450.

P. Milgrom and R.J. Weber (1982) A theory of auctions and competitive bidding. Econometrica, 50: 1089-1122.

P. Milgrom (2004) Putting auction theory to work. Cambridge (UK). Cambridge University Press

P. Milgrom (2007) Package auctions and exchanges. Econometrica 75(4): 935-965.

A. Ockenfels and A.E. Roth (2006) Late and Multiple Bidding in Second Price Internet Auctions: Theory and Evidence Concerning Different Rules for Ending an Auction. Games and Economic Behavior, 55(2): 297-320.

J.M. Ortega and W.C. Rheinboldt (1970) Iterative solutions of nonlinear equations in several variables. New York: Academic Press.

S. Parsons, M. Marcinkiewicz, K. Niu, and S. Phelps (2006) Everything you wanted to know about double auctions, but were afraid to (bid or) ask. Technical Report, Department of Computer & Information Science, Brooklyn College, City University of New York, 2006. Available at: http://www.sci.brooklyn.cuny.edu/~parsons/projects/mech-design/

E.J. Pinker, A. Seidman, and Y, Vakrat (2003) Managing online auctions: current business and research issues. Management Science, 49(11): 1457-1484.

V. Reinhart and G. Belzer (1996) Some recent evidence on bid shading and the use of information in the U.S. Treasury's auction experiment. Working Paper, Board of Governors of the Federal Reserve System.

A.E. Roth and A. Ockenfels (2002) Last-minute bidding and the rules for ending second-price auctions: evidence from eBay and Amazon auctions on the Internet, American Economic Review, 92(4): 1093-1103.

M.H. Rothkopf and R.M. Harstad (1994) Modelling competitive bidding: a critical essay. Management Science, 40(3): 364-384.

Y. Sakurai, M. Yokoo and S. Matsubara (1999) A limitation of the Generalized Vickrey Auction in electronic commerce: robustness against false-name bids. In: Proceedings of the 16th national conference on artificial intelligence, P. 82-92.

V.L. Smith (1962) An experimental study of competitive market behavior. The Journal of Political Economy, 70(2): 111-137.

S.L. Sobolev (1963) Applications of Functional Analysis in Mathematical Physics. American Mathematical Society, Translations of Mathematical Monographs, Volume 7, Providence, Rhode Island.

E.E.C. Van Damme (2002) The European UMTS auctions. European Economic Review 46(4-5), 846-858.

H.R. Varian (1982) The nonparametric approach to demand analysis. Econometrica, 50: 945-973.

H.R. Varian (1995) Economic mechanism design for computerized agents. Proceedings of the first Usenix workshop on electronic commerce. Available at http://people.ischool.berkeley.edu/~hal/Papers/mechanism-design.pdf

W. Vickrey (1961) Counterspeculation, auctions and competitive sealed tenders. Journal of Finance, 16(1): 8-37.

W. Wang, Z. Hidvégi and A.B. Whinston (2000) Economic mechanism design for securing online auctions. Proceedings of the 21st International Conference on Information Systems, p. 676-680.

S. Wang, W. Jank and G. Shmueli (2008) Explaining and Forecasting Online Auction Prices and their Dynamics using Functional Data Analysis. Journal of Business and Economic Statistics, 26(2): 144–160.

S. Wang, W. Jank, G. Shmueli and P. Smith (2009) Modeling price dynamics in eBay auctions using principal differential analysis, Journal of the American Statistical Association (forthcoming)

G.A. Whitney and J.L. Swofford (1987) Nonparametric tests of utility maximization and weak separability for consumption, leisure and money. The Review of Economic Statistics, 69(3): 458-464.

S.R. Williams (1991) Existence and convergence of equilibria in the buyer's bid double auction.eview of Economic Studies, 58, 351-374

R. Wilson (1977) A bidding model of perfect competition, Review of Economic Studies

R. Wilson (1979) Auction of shares, Quarterly Journal of Economics, 93: 675-689.

R. Wilson (1985) Incentive efficiency of double auctions. Econometrica, 53(5): 1101-1116.

R. Wilson (2006) Activity rules for an iterative double auction. K Chatterjee and W. F.Samuelson (eds.) Game Theory and Business Applications, International series in operations research and management science, 35 (Chapter 12).

L. Xuefeng, L. Lu, W. Luhua and Z. Zhao (2006) Predicting the price of online auction items. Expert systems with applications 31: 542-550.

M. Yokoo, Y. Sakurai and S. Matsubara (2001a) Rubust combinatorial auction protocol against false-name bids. In: Proceedings of the 17th International Joint Conference on Artificial Intelligence, p. 110-115.

M. Yokoo, Y. Sakurai and S. Matsubara (2001b) Robust multi-unit protocol against false-name bids. In: Proceedings of the 17th International Joint Conference on Artificial Intelligence, p. 1089-1094.

M. Yokoo, Y. Sakurai and S. Matsubara (2005) Robust double auction protocol against false-name bids. Decision support systems, 13(2): 241-252.