Competition and Cooperation in Decentralized Distribution

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Abstract

Any decentralized retail or wholesale system of competing entities has a benefit sharing arrangement when collaborating with regards to demand realizations. We study a distribution system similar to the observed behavior of independent car dealerships. If a dealership does not have in stock the vehicle requested by a customer, it might consider acquiring it from a competing dealer. This raises questions about procurement strategies that achieve a system optimal (first-best) outcome. We examine such a decentralized distribution system with respect to: (a) Does a unique first-best solution imply unique Nash equilibrium procurement strategies? (b) If some of the participants do not select Nash procurement strategies, what are the implications on the benefit sharing? (c) When demand parameters are not of common knowledge the system might not encourage truth revelation. (d) How are the above results affected if we relax the assumption of satisfying local demand first? We show that the profit sharing rules like the ones found in the literature will result in a stable collaborative outcome that achieves first-best only if (i) individual demand parameters satisfy a number of restrictive conditions, (ii) complete information assumption holds, and (iii) all the parties select Nash equilibrium strategy.

1 Introduction

Anupindi et al. (2001) proposed a framework to study a decentralized inventory system. Decentral-

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ized inventory systems are common in a broad range of supply chain networks and impact many aspects of a daily commercial behavior. However, it is not obvious how and if one can 'engineer' an operational process so that the decentralized system performance matches an optimal centralized setting. In order to induce parties to behave in a manner that results in collectively best outcome one usually relies on some form of transfer payments. For example, consider a case of a car dealership acquiring a car for its customer from a neighboring dealership. Clearly all parties have to be appropriately compensated. To fully understand the stable first-best solution that allows competing entities to rationally collaborate, we have to carefully examine the solution's sensitivity to all relevant levers.

This paper studies a single-commodity multi-player inventory procurement and storage operations in a decentralized two-stage decision system similar to one described by Anupindi et al. (2001), referred from now on as ABZ. A precursor to this line of research is an extensive body of work that we refer to as we progress with our analysis. Our results reveal that the outcome of stable solution proposed in past work is sensitive to a number of crucial assumptions regarding Nash play by all participants, complete information, and more. However before turning to all the more technical details we first describe the basic problem and the commonly encountered assumptions.

The remainder of this paper is organized as follows. §2 describes our basic setup of decentralized distribution systems. §3 critiques the existence and uniqueness of first-best Nash equilibrium for the decentralized distribution systems that adopts the ABZ's transferred payment approach. We state a set of conditions on cost parameters and distributions that guarantee uniqueness of first-best Nash equilibrium. The implications of failure in satisfying the necessary conditions is examined next. §4 presents the effect of non-Nash strategy on the decentralized distribution system. §5 discusses the situation when the players' complete information assumption does not hold. That is, we examine the transferred payment approach with respect to incentive compatibility property. Both §4 and §5 provide insight into strategic limitation of implementing collaboration in a decentralized distribution system. Section §6 expands the scope of the previous model. It presents an alternate model for decentralized distribution systems that relaxes the assumption of satisfying local demand first. We assume that retailer is allowed to transship her inventory regardless of the local demand status if such a transshipment increases her profit. As a main contribution, this paper provides important insight and clarifications regarding collaboration in decentralized distribution systems. The more involved technical details and proofs are presented in the Appendix.

2 Model Description

Assume a decentralized setting (see ABZ) with competitive independent retailers who face random demands. In the first-stage, inventories are ordered based on anticipated demands and retailers may end up with excess demand or supply. In the second-stage, these retailers use pooling of residual, i.e., excess demand at one retailer's local inventory can be satisfied from surplus transshipped from other retailers' local inventories. ABZ assume that each retailer will choose to satisfy local demand from the local stock before sharing the residual demand with other retailers. This assumption is applicable in situation when the transshipment cost is high and the differences in costs or selling prices cannot make up for the transshipment cost. ABZ also introduced the notion of *claims* for units stored in centralized warehouse. Claims indicate ownership for each unit of inventory. The claim holder pays for inventory holding cost of the unit and can decide on where the unit will be transshipped to. For simplicity, we exclude the option of shared warehouses.

In contrast to the decentralized system, in the centralized inventory system, all retailers cooperate fully and both inventory decisions and transshipment decisions are made to maximize the expected profit of the overall system. The solution for a centralized inventory system is referred to as the *first-best solution* and the maximum expected profit of the overall distribution system as the *first-best profits*. This first-best solution does not consider how the profit will be shared among retailers. Clearly, the total profit of a decentralized inventory system cannot exceed the first-best profit. ABZ propose a set of conditions and claim that their conditions result in the decentralized inventory system achieving the first-best profit. We examine these conditions in some detail below.

2.1 Game theory terminology

Since the cooperation of retailers has the flavor of coalitional game with transferable utility, we introduce some basic cooperative game terminology. Let $N = \{1, \ldots, n\}$ be a set of retailers. In our inventory game, retailers are players. The set N is referred to as the *grand coalition*. A nonempty subset $S \subseteq N$ is a *coalition*. There are $2^N - 1$ different coalitions that can be formed. A *characteristic function* v is a set function such that $v(\emptyset) = 0$ and associates a real number

$v(S) \in \mathbb{R}$ with each subset $S \subseteq N$. We can think of the characteristic function as an amount of profit that retailers who are members of S generate as a result of forming a coalition S to transship products only among themselves. The pair (N, v) denotes a cooperative game. The decision on how the profits are shared is called an *allocation rule*. An allocation rule α ($\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$) determines an allocation of profit to each individual retailer. An allocation that enables stable cooperation (no subset of retailers has an incentive to withdraw from the grand coalition) is called a *core allocation*. The *core* is a set of core allocations. An allocation α is in the core of game (N, v) if $\sum_{i \in S} \alpha_i \geq v(S)$ for all $S \subseteq N$ and $\sum_{i \in N} \alpha_i = v(N)$.

In addition to cooperative game terminology, the decentralized inventory system requires an understanding of competitive game terminology. Let $\mathbb{S}_i$ be set of inventory strategies available for player i and a strategy $s_i \in \mathbb{S}_i$ denote a strategy carried out by a player i. In our inventory game, a set of strategies is a non-negative amount of inventory ordered by each retailer in the first stage. A *payoff function* u_i of player i associates a real number $u_i(s_1, \ldots, s_n) \in \mathbb{R}$ with the strategies $s_1, \ldots, s_n$ chosen by players individually. We can think of the payoff function as an amount of profit that retailer i expects to get over an infinite sequence of repeated decentralized inventory games as a function of the chosen strategies. We use a tuple $(\mathbb{S}_1, \ldots, \mathbb{S}_n; u_1, \ldots, u_n)$ to denote a competitive inventory game. The strategies $(s_1^*, \ldots, s_n^*)$ are a *Nash equilibrium* if, for each player i, s_i^* is player i's best response to the strategies $(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$ chosen by other players; that is, s_i^* solves $\max_{s_i \in S} u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*)$. Unique pure strategy Nash equilibrium (PSNE) provides arguably a rational prediction of what options players may pursue whenever unique PSNE exists in a game.

2.2 Past assumptions

Now we present the conditions as stated by ABZ that allow the decentralized inventory system to achieve the first-best profit. The conditions in ABZ are: 1) profit allocation must be in the *core* of *a snapshot allocation game* (defined below); 2) a *unique PSNE* must exist in the first stage; and 3) the first-stage inventory decisions must result in the same inventory levels as the centralized system. First, we discuss the snapshot allocation game (SAG). This game is the game that occurs in the second-stage of the decentralized inventory system. The SAG is defined as a transshipment game that occurs at a given retailers' inventory level $[Z]$ and a specific demand realization $\vec{D}$.

The characteristic function of SAG is defined as a maximum excess profit from transshipment (in addition to what could be achieved without pooling of residual and stock) available to be shared among retailers who join the transshipment coalition. Since the SAG game is superadditive, the largest possible excess profit is the excess profit achievable by the grand coalition of retailers that uses transshipping to maximize the total profit. A profit allocation is in the core of SAG game if no subset of retailers receives smaller amount of excess profit than they can earn on their own. At this point we note that the first-best profit (distribution system profit) is the maximal expected profit. However, the SAG is in turn played with respect to the demand realizations.

For the SAG, an *allocation rule based on dual prices* for the solution of transshipment profit maximization problem is in the core. This well-known result is based on the previous works of Shapley and Shubik (1975), and Samet and Zemel (1984). A transshipment game or a transportation game is an extension of an assignment game. It belongs to a class of games called linear programming games (LP-games) which has been extensively studies in the past (Samet and Zemel, 1984; Sánchez-Soriano et al., 2001). Each LP-game has a dual problem and an associated dual optimal solution. We can see such solution as a vector of shadow prices (dual prices) on the various resources in the original LP-game. This vector can be used to define an allocation rule so that each retailer is paid an amount which corresponds to the value of his resources. In our case, resources are excess demand and excess supply. The dual prices are determined for each unit of excess demand in each retailer's location (local inventory or warehouses) as well as for each unit of excess supply in each retailer's location. The allocation rule based on dual prices is in the core of SAG.

The second condition for the solution of the decentralized inventory system is an existence of a unique PSNE in the first stage. The first-stage game can be defined as a tuple $(\mathbb{S}_1, \ldots, \mathbb{S}_n; \tilde{u}_1, \ldots, \tilde{u}_n)$. A retailer i chooses a strategy $s_i = X_i$ from her set of available strategies $\mathbb{S}_i \subset \mathbb{R}_+$. Hence, the set of game strategies $\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2 \times \cdots \times \mathbb{S}_n \subset \mathbb{R}^n_+$ represents a set of possible non-negative amounts of inventory ordered by retailers in the first stage. The function $\tilde{u}_i$ is an individual payoff function. The value of $\tilde{u}_i$ ($\tilde{u}_i : \mathbb{S} \to \mathbb{R}$) is equal to the sum of expected profit earned by retailer i in the first stage and an expected profit to be allocated to retailer i in the second stage for the excess demand and surplus distribution. If all retailers predict that a unique PSNE will occur at a specific inventory level in the first stage, then no retailer has an incentive to deviate from such inventory level. ABZ claim that there exists a unique PSNE for the decentralized inventory game if the second-stage profit function of each retailer is simultaneously continuous in inventory levels at all retailers, unimodal in each retailer's own inventory level, and the demand distribution function belongs to the class of Polya Frequency Functions of order 2 (PF2), say, normal, exponential, or uniform distribution among others.

The last condition for the solution of the decentralized inventory system requires that the first-stage independent inventory decisions result in the same inventory levels as for the centralized system. The purpose of this condition is to cause the decentralized inventory system achieve the first-best profit. However, an allocation based on dual price, although in the core of SAG and attains a unique PSNE, does not necessarily imply that the retailers order the same inventory levels as in the centralized system. For that reason ABZ construct an allocation rule based on a scheme of ex-post side payments between the retailers, restated below and in §2.3, that is claimed to satisfy all three rules.

Say a group of retailers selects ex ante an allocation rule α for the SAG that attains a unique PSNE and first-best profit but ex post is not necessarily in the core of SAG. This allocation makes each retailer choose her inventory level in the first stage to maximize her expected profit, while simultaneously maximizes the total expected profit of the grand coalition. However, at some realizations of demand, such allocation α is not necessarily in the core of SAG. Thus, the solution to the second-stage transshipment game (the SAG) may not be enforceable (rational). To counter this fact, ABZ create a new allocation $\tilde{\alpha}$ by adding side payments to allocation α. The calculation rule of side payments is based on another allocation — $\alpha^x(\vec{D})$. Allocation $\alpha^x(\vec{D})$ is required to be in the core of SAG and is computed in turn for each demand realization, but may not necessarily be an allocation based on dual prices of the transshipment game. Now assume that there is a unique PSNE for an allocation α. The side payments are equal to the difference between the allocation α and the allocation $\alpha^x(\vec{D})$, evaluated at that unique PSNE. Hence, when the modified allocation $\tilde{\alpha}$ is used, we obtain an allocation in the core of SAG in the second stage by making side payments (adjusting α) based on α^x. The solution of the expected profit maximization in the first-stage remains unchanged.

For instance, consider a fractional allocation that is calculated by, first, combining retailers overall profits (both local and transshipment profit), then re-distributing those profits using previously agreed fractions, and finally, paying each retailer that fractional amount minus her local profit. The agreed fractions could be of any value but overall add up to one. In practice, each such fraction might depend on the retailer's bargaining power. At some realizations of demand, the fractional allocation may not be in the core of SAG. But, the fractional allocation attains unique PSNE and first-best profit because it encourages retailers to order the same inventory level as a centralized system. ABZ constructed a new allocation by modifying the fractional allocation using side payments. The side payments are fixed to the difference between the allocation based on dual prices of transshipment game and the fractional allocation, evaluated at the unique PSNE of the game that uses the modified allocation. ABZ claim that this can be done because the allocation based on dual prices of the corresponding transshipment game is always in the core of SAG regardless of the first-stage inventory decision. They claim that these side payments retain the modified allocation in the core of SAG while preserving unique PSNE and the first-best profit when unique PSNE inventory level is ordered. For clarity and completeness the details of ABZ claim are restated in the next section.

2.3 Details of the ABZ Decentralized Distribution Model

Consider a case of two retailers who make independent inventory stocking decision but agree to cooperate on second-stage transshipment decision. Assume that all inventory is stored locally. Let r_i, c_i, and v_i, where $i = 1, 2$, represent unit revenue, unit cost, and unit salvage value of a retailer i, respectively. Let $t_{1,2}$ and $t_{2,1}$ represent the transshipping cost from retailer 1 to retailer 2, and the transshipping cost from retailer 2 to retailer 1, respectively.

In the first stage, each retailer makes decision on her inventory level. Let the vector $\vec{X} = (X_1, X_2)$ denote the levels of inventory ordered in the first stage. Then, the demand represented by the vector $\vec{D} = (D_1, D_2)$ is realized at both retailers. Retailer i sells $B_i = \min\{X_i, D_i\}$ units and may have $H_i = \max\{X_i - D_i, 0\}$ unit surplus or $E_i = \max\{D_i - X_i, 0\}$ unit shortage.[1] The profit expected at each retailer is $J_i(\vec{X}) = E_{\vec{D}}(P_i(\vec{X}, \vec{D}))$ where $P_i(\vec{X}, \vec{D}) = [r_iB_i + v_iH_i - c_iX_i] + \alpha_i(\vec{X}, \vec{D})$.

The function $\alpha_i(\vec{X}, \vec{D})$ is the profit allocated to retailer i as a result of transshipment game such that $\alpha_1(\vec{X}, \vec{D}) + \alpha_2(\vec{X}, \vec{D}) = (r_1 - v_2 - t_{2,1}) \min\{E_1, H_2\} + (r_2 - v_1 - t_{1,2}) \min\{E_2, H_1\}$.

Assume that these two retailers agree to allocate profit using the allocation rule proposed by

[1] In this section as in ABZ, we assume that retailers must satisfy their local demand first. An alternate model which relaxes this assumption is discussed in §6.

ABZ in their Corollary 5.1. Before restating this corollary, we state three definitions:

Definition 1 *In a case of two retailers, for a given inventory level $\vec{X}$ and demand realization $\vec{D}$, the* combined profit *is represented by:*

$$\begin{aligned} P_N^c(\vec{X}, \vec{D}) \;\; &= \;\; [r_1 B_1 + v_1 H_1 - c_1 X_1] + [r_2 B_2 + v_2 H_2 - c_2 X_2] \\ &\quad + (r_1 - v_2 - t_{2,1}) \min\{E_1, H_2\} \\ &\quad + (r_2 - v_1 - t_{1,2}) \min\{E_2, H_1\}. \end{aligned}$$

The expected combined profit is $J_N^c(\vec{X}) = E_{\vec{D}}(P_N^c(\vec{X}, \vec{D}))$. The first-best solution *$\vec{X}^{c^*}$ is the solution that maximizes the expected combined profit assuming that retailers make centralized decisions in both stages. The* first-best profit *$J_N^c(\vec{X}^{c^*})$ is the expected combined profit when the first-best solution $\vec{X}^{c^*}$ is played. We assume $J_N^c(\vec{X}^{c^*}) \geq 0$. This definition is generalizable in a straight forward fashion to $n > 2$ retailers.*

Definition 2 *Let the* fractional allocation *$\alpha_i^f(\vec{X}, \vec{D})$ be defined as $\alpha_i^f(\vec{X}, \vec{D}) = \gamma_i P_N^c(\vec{X}, \vec{D}) - [r_i B_i + v_i H_i - c_i X_i]$, where γ_i is a fraction agreed by all retailers such that $\sum_{i \in N} \gamma_i = 1$ and for all i, $\gamma_i \in (0, 1)$. Note that $\alpha_i^f(\vec{X}, \vec{D})$ can be negative.*

Definition 3 *Let the* dual allocation *$\alpha_i^d(\vec{X}, \vec{D})$ be defined as the allocation based on dual price of transshipment game. That is $\alpha_i^d(\vec{X}, \vec{D}) = \lambda_i H_i + \delta_i E_i$ and $\sum_{i \in N} \alpha_i^d(\vec{X}, \vec{D}) = W_N(\vec{X}, \vec{D})$ where $W_N(\vec{X}, \vec{D})$ is the transshipment problem represented by:*

$$\begin{aligned} W_N(\vec{X}, \vec{D}) = \;\; & \max_{\vec{y}} \;\; \sum_{i \in N} \sum_{j \in N, j \neq i} (r_j - v_i - t_{i,j}) y_{i,j} \\ & \text{s.t.} \;\; \sum_{j \in N, j \neq i} y_{i,j} \leq H_i \textit{ for all } i \in N \\ & \qquad \sum_{i \in N, i \neq j} y_{i,j} \leq E_j \textit{ for all } j \in N \\ & \qquad \textit{for all } y_{i,j} \geq 0. \end{aligned} \tag{1}$$

The dual prices λ_i and δ_j are obtained from the solution of the above transshipment problem. The quantity $y_{i,j}$ represents a number of units of product transshipped from retailer i to retailer j in the second stage.

Claim 1 *(Corollary 5.1 in ABZ). Consider a modified fractional allocation rule that allocates the residual profits to player $i \in N$ as follows:*

$$
\begin{aligned}
\alpha_i^m(\vec{X},\vec{D}) &= \alpha_i^f(\vec{X},\vec{D}) + \alpha_i^d(\vec{X}^{c^*},\vec{D}) - \alpha_i^f(\vec{X}^{c^*},\vec{D}) \\
&= \gamma_i P_N^c(\vec{X},\vec{D}) - [r_iB_i + v_iH_i - c_iX_i] \\
&\quad + \lambda_i^{c^*} H_i^{c^*} + \delta_i^{c^*} E_i^{c^*} \\
&\quad - \gamma_i P_N^c(\vec{X}^{c^*},\vec{D}) + [r_iB_i^{c^*} + v_iH_i^{c^*} - c_iX_i^{c^*}],
\end{aligned}
$$

where $\vec{X}^{c^*}$ is the first-best solution. Then the PSNE using $\alpha_i^m(\vec{X},\vec{D})$ is first-best and the $\alpha_i^m(\vec{X}^{c^*},\vec{D})$ allocation values are in the core of the transshipment game.

Proof. Consider when $\vec{X} = \vec{X}^{c^*}$.

$$
\begin{aligned}
\alpha_i^m(\vec{X}^{c^*},\vec{D}) &= \gamma_i P_N^c(\vec{X}^{c^*},\vec{D}) - [r_iB_i^{c^*} + v_iH_i^{c^*} - c_iX_i^{c^*}] \\
&\quad + \lambda_i^{c^*} H_i^{c^*} + \delta_i^{c^*} E_i^{c^*} \\
&\quad - \gamma_i P_N^c(\vec{X}^{c^*},\vec{D}) + [r_iB_i^{c^*} + v_iH_i^{c^*} - c_iX_i^{c^*}] \\
&= \lambda_i^{c^*} H_i^{c^*} + \delta_i^{c^*} E_i^{c^*} \\
&= \alpha_i^d(\vec{X}^{c^*},\vec{D})
\end{aligned}
$$

The allocation $\alpha_i^m(\vec{X}^{c^*},\vec{D})$ is equal to the allocation $\alpha_i^d(\vec{X}^{c^*},\vec{D})$. Recall that the allocation $\alpha_i^d(\vec{X},\vec{D})$ is always in the core of the transshipment game (Samet and Zemel, 1984). Hence, the allocation $\alpha_i^m(\vec{X}^{c^*},\vec{D})$ will also be in the core of the transshipment game when the inventory levels $\vec{X}^{c^*}$ are ordered by all retailers.

Now, we check if the first-best solution $\vec{X}^{c^*}$ is the Nash equilibrium solution. We reduce the function $P_i(\vec{X},\vec{D})$ as follows:

$$
\begin{aligned}
P_i(\vec{X},\vec{D}) &= [r_iB_i + v_iH_i - c_iX_i] + \alpha_i^m(\vec{X},\vec{D}) \\
&= [r_iB_i + v_iH_i - c_iX_i] + \gamma_i P_N^c(\vec{X},\vec{D}) - [r_iB_i + v_iH_i - c_iX_i] \\
&\quad + \alpha_i^d(\vec{X}^{c^*},\vec{D}) - \alpha_i^f(\vec{X}^{c^*},\vec{D}) \\
&= \gamma_i P_N^c(\vec{X},\vec{D}) + \alpha_i^d(\vec{X}^{c^*},\vec{D}) - \alpha_i^f(\vec{X}^{c^*},\vec{D}).
\end{aligned}
$$

The expected payoff of retailer i is then:

$$J_i(\vec{X}) \quad = \quad \gamma_i E_{\vec{D}}(P_N^c(\vec{X},\vec{D})) + E_{\vec{D}}(\alpha_i^d(\vec{X}^{c^*},\vec{D})) - E_{\vec{D}}(\alpha_i^f(\vec{X}^{c^*},\vec{D})).$$

The value $E_{\vec{D}}(\alpha_i^d(\vec{X}^{c^*},\vec{D}))$ and $E_{\vec{D}}(\alpha_i^f(\vec{X}^{c^*},\vec{D}))$ are essentially constants as $\vec{X}^{c^*}$ only depends on $\vec{D}$. Thus, the vector $\vec{X}$ that maximizes $J_i(\vec{X})$ is the same as $\vec{X}$ that maximizes $E_{\vec{D}}(P_N^c(\vec{X},\vec{D}))$. Therefore, the first-best solution $\vec{X}^{c^*}$ is the Nash equilibrium solution and the first-best profit can be achieved using this allocation $\alpha_i^m(\vec{X},\vec{D})$. $\square$

3 Existence and Uniqueness of First-Best Nash Equilibrium

Corollary 5.1 of ABZ assumes that there exists a unique PSNE. In this section, we show that there always exists a PSNE for a game that uses ABZ allocation rule. Note that such proof was omitted in ABZ. We also show that under certain conditions uniqueness of the expected centralized profit function implies uniqueness of PSNE. Such conditions were not discussed in ABZ. Finally, we discuss a situations when there are multiple PSNE.

3.1 Conditions for the Existence of PSNE

In this section, we discuss the condition for the existence of PSNE by examining retailer i's expected profit function $J_i(\vec{X})$ when ABZ allocation rule is used. In ABZ's work, the conditions for existence of PSNE include: 1) the expected profit function $J_i(\vec{X})$ for retailer i is simultaneously continuous in $\vec{X}$, and 2) $J_i(\vec{X})$ is unimodal in X_i for every $\vec{X}_{N\backslash i}$.

Proposition 1 *There exists a PSNE for a decentralized distribution system that adopts ABZ allocation rule.*

Proof. We know that if a vector $\vec{X}$ is a first-best solution, then the vector $\vec{X}$ is also a member of a set of PSNE. This is because the best-response to other retailers' playing first-best strategies is to play first-best strategy. So, if there exists a first-best solution, then there must exist a PSNE. According to well-known Weierstrass's theorem, if a function $f : C \to \mathbb{R}$ is continuous and its domain is a compact subset C of $\mathbb{R}^n$, then there are vectors in C that maximize the function f. In our case, if the inventory domain constitutes a compact subset of $\mathbb{R}^n$ and the expected centralized profit $J_N^c(\vec{X})$ is continuous in $\vec{X}$, then there exists a first-best solution and it follows that there exists a PSNE.

In our setting, each retailer's inventory level falls in a closed and bounded interval of $\mathbb{R}$, hence the domain is a compact subset of $\mathbb{R}^n$. We proceed to check whether the expected centralized profit $J_N^c(\vec{X})$ is continuous in $\vec{X}$

Recall that

$$P_N^c(\vec{X}, \vec{D}) = \sum_{i \in N} r_i B_i + v_i H_i - c_i X_i + W_N(\vec{X}, \vec{D})$$

where $W_N(\vec{X}, \vec{D})$ is the profit from transshipment as defined in (1). For any given $\vec{D}$, $W_N(\vec{X}, \vec{D})$ is continuous in $\vec{X}$ because there is no fixed cost related to transshipment profits. In addition, there is no fixed cost related to local profits at any retailers. As a result, the centralized profit $P_N^c(\vec{X}, \vec{D})$ is continuous in $\vec{X}$.

According to Kolmogorov and Fomin (1970, p.109), a real function continuous on a compact metric space $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$. In our case, for all i, X_i are defined on nonempty compact convex subsets of $\mathbb{R}$ and $P_N^c(\vec{X}, \vec{D})$ is continuous in $\vec{X}$. Hence, $P_N^c(\vec{X}, \vec{D})$ is uniformly continuous in $\vec{X}$, and it follows that $J_N^c(\vec{X})$ is continuous in $\vec{X}$. $\square$

3.2 Uniqueness of PSNE

When using ABZ allocation rule, it is important to ascertain that the PSNE/first-best inventory level of the decentralized distribution system is unique because the side payment calculation is based on the value of the unique PSNE/first-best inventory level as shown in ABZ's Corollary 5.1. ABZ did not provide a direct proof of uniqueness of PSNE, but state (in Theorem 5.3) that if the distribution system exhibits a unique first-best solution, then the PSNE under their allocation rule is unique. This, however is not true. In this section, we introduce conditions regarding the unique first-best solution that imply a unique PSNE.

Lemma 1 *Suppose that $J_N^c(\vec{X})$ is strictly quasi-concave in X_i for each $i \in N$. If there is a unique point $\vec{X}^*$ where $J_N^c(\vec{X})$ is* strictly *increasing in X_i for $X_i < X_i^*$, and $J_N^c(\vec{X})$ is* strictly *decreasing in X_i for $X_i > X_i^*$ for all $i \in N$, then there is a unique PSNE that corresponds to the first-best* *solution.*

Proof. Recall that the expected profit function for player i is:

$$J_i(\vec{X}) \quad = \quad \gamma_i J_N^c(\vec{X}) + E_{\vec{D}}(\alpha_i^d(\vec{X}^{c^*}, \vec{D})) - E_{\vec{D}}(\alpha_i^f(\vec{X}^{c^*}, \vec{D})).$$

Both $E_{\vec{D}}(\alpha_i^d(\vec{X}^{c^*}, \vec{D}))$ and $E_{\vec{D}}(\alpha_i^f(\vec{X}^{c^*}, \vec{D}))$ do not depend on $\vec{X}$. Hence, a player i's strategy X_i that maximizes her expected profit function, also maximizes the expected centralized profit $J_N^c(\vec{X})$.

Because there is a unique point $\vec{X}^*$ where $J_N^c(\vec{X})$ is strictly increasing in X_i for $X_i < X_i^*$, and $J_N^c(\vec{X})$ is strictly decreasing in X_i for $X_i > X_i^*$ for all $i \in N$, no other points are local maxima. Therefore, the point $\vec{X}^*$ is a global maximum, i.e., a unique first-best solution.

The point $\vec{X}^*$ is also a PSNE because (i) no player has an incentive to deviate from it, and (ii) for any other point, says at $\vec{X}^\circ \neq \vec{X}^*$, each player i would be better off not playing X_i°, given that other players play $\vec{X}^\circ_{N\setminus i}$.

In other words, we can state that uniqueness of first-best solution implies uniqueness of Nash equilibrium if the conditions in Lemma 1 are satisfied. $\square$

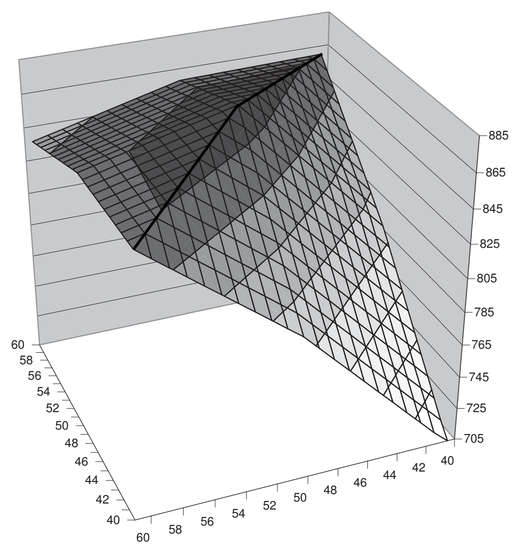
A pertinent question is what are the demand distributions and cost parameters that would satisfy the conditions in Lemma 1, that is, assure a unique PSNE. Strict quasi-concavity in X_i for each $i \in N$ does not necessarily imply such unique PSNE point. However, given strict quasi-concavity in X_i but not necessarily in $\vec{X}$, there are only two cases that allow for multiple PSNE points.

The first case is the existence of multiple strict local maxima. At each such strict local maximum $\vec{X}^*$, there is a neighborhood of $\vec{X}^*$ so that $J_N^c(\vec{X})$ is strictly increasing for an $\vec{X}$ in this neighborhood, $\vec{X} < \vec{X}^*$ in each component, and strictly decreasing for an $\vec{X}$ in this neighborhood $\vec{X} > \vec{X}^*$ in each component. Each of these local maxima is strictly quasi-concave in X_i's separately but not necessarily in $\vec{X}$. Thus, each such strict local maximum corresponds to PSNE. To ensure that there are no multiple strict local maxima, it is sufficient to require $J_N^c(\vec{X})$ to be quasi-concave in $\vec{X}$.

The other case of $J_N^c(\vec{X})$ with multiple PSNE is when $J_N^c(\vec{X})$ has what we call a "ridge". To demonstrate the role of the ridge, consider the following example.

Example 1 *Assume that the cost parameters is symmetric with $r_i = 10$, $c_i = 1.2$, $v_i = -1$ for $i = 1, 2$, and $t_{1,2} = t_{2,1} = 2$. Retailers agree on using ABZ allocation rule with $\gamma_i = 0.5$. Let the demand be known and fixed at 50 for both retailers. This system has a unique first-best solution at $(X_1^{c^*}, X_2^{c^*}) = (50, 50)$ with the first-best profit of \$880. The unique first-best solution is one of the PSNE. This system has infinite number of PSNE, e.g. (49,51) with expected profit of (\$439,\$439), (48,52) with expected profit of (\$438 ,\$438),(47,53) with expected profit of (\$437,\$437), and so on. This centralized profit $J_N^c(\vec{X})$ is strictly quasi-concave in X_i for each $i \in N$ and quasi-concave in $\vec{X}$. However, there is a ridge along the line $X_1 + X_2 = 100$. $J_N^c(\vec{X})$ is strictly increasing on the left hand side, and strictly decreasing on the right hand side for each retailer. In this case, every point X_i^* on the ridge line (see Figure 1) correspond to PSNE.*

Figure 1: Centralized profit $J_N^c(\vec{X})$ with Ridge



Note that weak local maxima (flat plateau) are just another instance of a ridge. However, not all points on a ridge are a local maximum. When $J_N^c(\vec{X})$ is differentiable everywhere, we can also describe a ridge as the case when best response functions of two retailers describe the same graph. A ridge is defined more formally below.

Definition 4 *Let $f : \vec{X} \to \mathbb{R}$ be a continuous function. There exists a ridge for function f if, for an $|\vec{\epsilon}| > 0$ and a point $\vec{X}^* = (X_1^*, \ldots, X_n^*)$ with $X_i^* = \arg\max_{X_i} f(X_i, \vec{X}^*_{N\setminus i})$ for all $i \in N$, there is a point $\vec{X}' = \vec{X}^* + \vec{\epsilon}$ such that $X_i' = \arg\max_{X_i} f(X_i, \vec{X}'_{N\setminus i})$ for all $i \in N$.*

The existence of a ridge violates the conditions of Lemma 1, more specifically, the uniqueness of $\vec{X}^*$. In the case of strict quasi-concave X_i's and single maximum, the following conditions (see Lemma 2 and the subsequent discussion) will also eliminate the possible existence of a ridge, thus guaranteeing the uniqueness of PSNE.

Lemma 2 *Given $J_N^c(\vec{X})$ strictly quasi-concave in X_i, let $\overline{D}_i$ and $\underline{D}_i$ represent a lowest and highest possible demand for retailer i, and $\overline{X}_i$ and $\underline{X}_i$ represent upper and lower bound of inventory level for retailer i. There exists a ridge if*

(a) there are at least two retailers, i and j, such that $\overline{X}_i > \overline{D}_i$ and $\underline{X}_j < \underline{D}_j$, and

(b) there exists a PSNE at $\vec{X}^* = (X_1^*, \ldots, X_n^*)$ where $X_i^* > \overline{D}_i$ and $X_j^* < \underline{D}_j$.

The proof is given in the Appendix.

In view of Lemma 2, we first restrict the strategy space to be within the demand distribution range. That is $\underline{D}_i < \underline{X}_i < \overline{X}_i < \overline{D}_i$ for all $i \in N$. Second, we restrict the implicit best-response function of each retailer. For instance, consider $J_N^c(\vec{X})$ that is differentiable everywhere. An implicit best-response function of retailer i is:

$$\frac{\partial J_N^c(\vec{X})}{\partial X_i} = 0.$$

Say we write explicitly a best-response function of retailer i as $Br_i(\vec{X}_{N\backslash i})$. That is, $Br_i(\vec{X}_{N\backslash i}) : (\mathbb{X}_1 \times \cdots \times \mathbb{X}_{i-1} \times \mathbb{X}_{i+1} \times \cdots \times \mathbb{X}_n) \to \mathbb{X}_i$, where $\mathbb{X}_i = \left[\underline{X}_i, \overline{X}_i\right]$.

For any two retailers i and j and any fixed $\vec{X}_{N\backslash\{i,j\}}$, define $\hat{Br}_i(X_j) = Br_i(X_j, \vec{X}_{N\backslash\{i,j\}})$ and $\hat{Br}_j(X_i) = Br_j(X_i, \vec{X}_{N\backslash\{i,j\}})$. We can plot $\hat{Br}_i(X_j)$ on a two dimensional plane (X_i, X_j) as a graph where $X_i = \hat{Br}_i(X_j)$. On the same plane, we can also plot an inverse function of $\hat{Br}_j(X_i)$ as a graph where $X_i = \hat{Br}_j^{-1}(X_j)$. If the two functions describe the same graph, then there is a ridge. Thus, to assure a unique PSNE we require that for any pair (i, j) the two graphs cross only once within the strategy space.

In summary, if (i) $J_N^c(\vec{X})$ is strictly quasi-concave in each X_i, (ii) weakly quasi-concave in $\vec{X}$, and (iii) there is no ridge present for $J_N^c(\vec{X})$, then there is a unique PSNE. Now, we are ready to discuss demand distributions and cost parameters that satisfy (i) and (ii). First, we characterize $P_N^c(\vec{X}, \vec{D})$ and demand distributions. Then, we characterize the cost parameters.

Proposition 2 *If the following statements are satisfied then $J_N^c(\vec{X})$ is strictly quasi-concave in each X_i and weakly quasi-concave in $\vec{X}$.*

(a) The demand density function $f(\vec{D})$ is strictly *log-concave in $\vec{D}$.*

(b) $P_N^c(\vec{X}, \vec{D})$ is weakly *log-concave in $(\vec{X}, \vec{D})$.*

(c) $P_N^c(\vec{X}, \vec{D})$ is strictly *log-concave in X_i for all $i \in N$.*

Proof. Since (strictly) log-concave implies (strictly) quasi-concave, we require that $J_N^c(\vec{X})$ be strictly log-concave in each X_i and weakly log-concave in $\vec{X}$.

According to Prékopa (1973, Theorem 6), if the integrand is log-concave in its argument (in our case a vector $(\vec{X}, \vec{D})$) and the domain of integration is a convex subset of $\mathbb{R}^N$, then the integral is log-concave.

Recall that

$$J_N^c(\vec{X}) \quad = \quad \int_\Omega P_N^c(\vec{X}, \vec{D}) f(\vec{D}) d\vec{D}$$

where Ω is the support of $f(\vec{D})$ – the probability density function of demand. First, the expected centralized profit $J_N^c(\vec{X})$ is weakly log-concave in $\vec{X}$ if $P_N^c(\vec{X}, \vec{D}) f(\vec{D})$ is log-concave in $(\vec{X}, \vec{D})$. We can achieve that by requiring $f(\vec{D})$ to be log-concave in $\vec{D}$ and $P_N^c(\vec{X}, \vec{D})$ to be log-concave in $(\vec{X}, \vec{D})$ (as stated in assumption (b)) because log-concavity is preserved under multiplication. Notice that 'strict' is not require at this point.

Secondly, the expected centralized profit $J_N^c(\vec{X})$ is strictly log-concave in X_i for all $i \in N$ if $P_N^c(\vec{X}, \vec{D}) f(\vec{D})$ is strictly log-concave in $(X_i, \vec{D})$.

Let $f(\vec{D})$ be strictly log-concave in $\vec{D}$ (as stated assumption (a)) and $P_N^c(\vec{X}, \vec{D})$ be (weakly) log-concave in $(\vec{X}, \vec{D})$ (as stated in assumption (b)). Assumption (b) implies that $P_N^c(\vec{X}, \vec{D})$ is (weakly) log-concave in $(X_i, \vec{D})$. Define a function $G(X_i, \vec{D}) = P_N^c(\vec{X}_{N\setminus i}, X_i, \vec{D}) f(\vec{D})$. Note that $\vec{X}_{N\setminus i}$ is fixed.

We separate our analysis to 2 cases. The first case is for any two points $A = (X_i^A, \vec{D}^A)$ and $B = (X_i^B, \vec{D}^B)$ such that $\vec{D}^A \neq \vec{D}^B$. Clearly, $G(\lambda A + (1-\lambda)B) > G(A)^\lambda G(B)^{(1-\lambda)}$ because of strict log-concavity of $f(\vec{D})$. The second case is for any two points $A = (X_i^A, \vec{D}^A)$ and $B = (X_i^B, \vec{D}^B)$ such that $X_i^A \neq X_i^B$ and $\vec{D}^A = \vec{D}^B$. In this case, if $P_N^c(\vec{X}, \vec{D})$ is strictly concave in X_i (as stated in assumption (c)), then $G(\lambda A + (1-\lambda)B) > G(A)^\lambda G(B)^{(1-\lambda)}$ by strict concavity of $P_N^c(\vec{X}, \vec{D})$ in

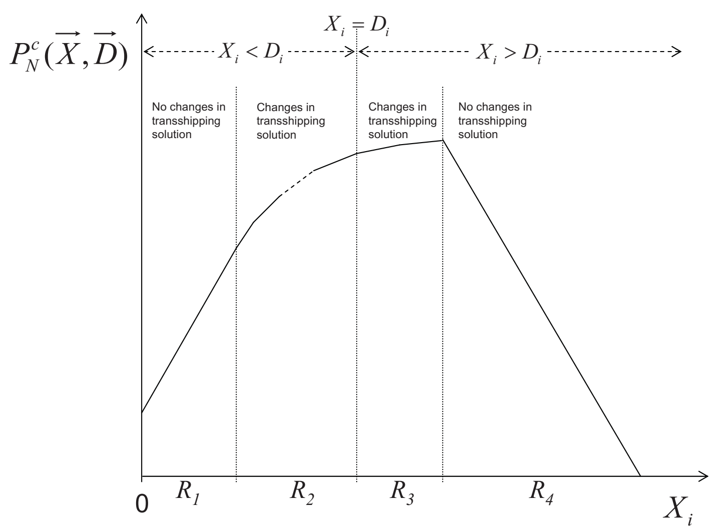
X_i. $\square$

There are many probability density functions that are strictly log-concave, such as normal distribution and exponential distribution. (See Bagnoli and Bergstrom, 2005.)

At this point, we examine the conditions on cost parameters so that $P_N^c(\vec{X}, \vec{D})$ be weakly log-concave in $(\vec{X}, \vec{D})$ and strictly log-concave in X_i for all $i \in N$.

Consider $P_N^c(\vec{X}, \vec{D})$ for a given $\vec{D}$ as X_i changes as depicted in Figure 2. To avoid the possibility of a ridge we assume through out this analysis that salvage value v_i is less than unit cost c_i for all $i \in N$. In addition, to avoid degeneracy for any two retailers i and j, $r_i - t_{j,i} \neq c_j$, $r_j - t_{i,j} \neq c_i$, and $v_i > v_j - t_{i,j}$ (it does not pay to transship in terms of the salvage value).

Figure 2: Centralized profit function



The graph in Figure 2 essentially has four regions. Figure 2 illustrates the case when the peak is between the third and the fourth region. However, the graph does not necessarily have all four regions present for all possible demand realization. For instance, when the system has a large amount of surplus, the first and third regions may vanish and the peak would be between the second and fourth regions.

Consider the first region $R_1 = \left\{X_i : X_i + \sum_{j \in N, j \neq i} y^*_{j,i} \leq D_i\right\}$. The demand at retailer i is large and as X_i increases there is no change in an optimal transshipping solution y^*. The slope of the profit function (marginal profit) in this region is constant and equal to $\rho^\circ := r_i - c_i$.

The second region $R_2 = \left\{ X_i : X_i \leq D_i \text{ and } X_i + \sum_{j \in N, j \neq i} y^*_{j,i} > D_i \right\}$ follows the first region. In the second region, as X_i increases, the optimal transshipping solution changes. Assume that we increase X_i by a small $\epsilon > 0$. The ϵ at retailer i will generate centralized profit at the rate of ρ°. Moreover, the ϵ amount of inventory from some other retailers is freed and is reallocated to other retailers with shortage or be disposed at salvage rate of $v_j, j \neq i$ of the j^{th} original owners of the ϵ amount of inventory. The marginal profit in this region cannot be greater than ρ° because of the following:

Consider a point in this second region, say at $\vec{X}'$ where $X'_i = X_i + \epsilon', \epsilon' > 0$. The marginal profit at this point is $\rho' = \rho^\circ - \frac{\partial W_N(\vec{X}', \vec{D})}{\partial E_i}$. Recall that surplus $H_i = \max\{X_i - D_i, 0\}$ and shortage $E_i = \max\{D_i - X_i, 0\}$ for all $i \in N$. An increase of ϵ amount of inventory to retailer i makes E_i smaller. The question is whether $\frac{\partial W_N(\vec{X}', \vec{D})}{\partial E_i}$ can be negative. If it is negative, then the marginal profit in the second region will be greater than ρ°. We know that an increase in E_i induces a change in transshipping solution only if it is better. So, the lower bound of $\frac{\partial W_N(\vec{X}', \vec{D})}{\partial E_i}$ is 0. Hence $\frac{\partial W_N(\vec{X}, \vec{D})}{\partial E_i}$ is non-negative.

Using the similar approach, let the next point, say at $\vec{X}''$ where $X''_i = X'_i + \epsilon'', \epsilon'' > 0$, have the marginal profit $\rho'' = \rho^\circ - \frac{\partial W_N(\vec{X}'', \vec{D})}{\partial E_i}$. Obviously, $\frac{\partial W_N(\vec{X}'', \vec{D})}{\partial E_i} \geq \frac{\partial W_N(\vec{X}', \vec{D})}{\partial E_i}$ because of the nature of the transshipment profit maximization. Therefore, $\rho'' \leq \rho'$. So, we can be certain that the graph in this region is strictly log-concave.

The third region is $R_3 = \left\{ X_i : X_i > D_i \text{ and } X_i \leq D_i + \sum_{j \in N, j \neq i} y^*_{i,j} \right\}$. The retailer i is now a transshipment source in the transportation problem. This region, if exists, will also be strictly log-concave because of non-degeneracy requirement $r_j - t_{i,j} \neq c_i$ and because the transshipment solution only changes when it is profitable, i.e., $r_j - v_i - t_{i,j} > 0$ for $i \neq j$. The best transshipment solution should fulfill the retailers that generate the more transshipment profit before fulfill the retailers that generate less transshipment profit. However, note that the slope in this region $\frac{\partial W_N(\vec{X}, \vec{D})}{\partial H_i}$ might be negative near the fourth region such that $r_j - c_i - t_{i,j} < 0$, while the transshipment profit $r_j - v_i - t_{i,j} > 0$.

The last region is $R_4 = \left\{ X_i : X_i > D_i \text{ and } X_i > D_i + \sum_{j \in N, j \neq i} y^*_{i,j} \right\}$. We enter the last region when an increasing in inventory at retailer i does not change an optimal transshipment solution and overall profit declines. The centralized profit decreases at the rate of $v_i - c_i \leq 0$ assuming that local excess inventory is disposed only at local retailer because the salvage value minus transshipment cost at any other retailer is below the local salvage value. Note that the slope of the fourth region is always lower than the slope of the third region because $r_j - c_i - t_{i,j} > v_i - c_i$ for any retailer j involved the third region.

Another part that we have not discussed is the transition between the second region and the third region. The reader can view this discussion in the Appendix where the following proposition is proven.

Proposition 3 *If the marginal profits from own selling at each one of the retailers are more than or equal to the marginal profits from units sold through transshipment, then the profit function is strictly log-concave in each X_i.*

We also know that if Proposition 3 is satisfied $P_N^c(\vec{X}, \vec{D})$ is also concave (and therefore log-concave) in D_i for all $i \in N$. This is because the profit increases as D_i increases (up to the sum of all X_i). At most the profit increases at the rate of $r_i - c_i$ from sales at local retailer i. Then rate of increase in profit would decline as the transshipment solution changes. When the demand is more than the inventory in the system, the profit $P_N^c(\vec{X}, \vec{D})$ is stable.

Up to this point, we can restrict the cost parameter such that $P_N^c(\vec{X}, \vec{D})$ is strictly log-concave in each X_i and log-concave in each D_i. Next, we show the requirement for log-concavity of $P_N^c(\vec{X}, \vec{D})$.

Proposition 4 *$P_N^c(\vec{X}, \vec{D})$ is (weakly) log-concave in $(\vec{X}, \vec{D})$ if it is more profitable to satisfy local demand first.*

Proof. Because (weak) concavity implies (weak) log-concavity, we prove that $P_N^c(\vec{X}, \vec{D})$ is log-concave in $(\vec{X}, \vec{D})$ by showing that $P_N^c(\vec{X}, \vec{D})$ is concave in $(\vec{X}, \vec{D})$ when the condition in Proposition 3 is satisfied.

From Theorem 3.4.1 of Topkis (1998), the transportation problem is submodular in the vector of its sources and sinks. Submodularity implies concavity. Hence, the transportation problem is concave in its vector of supply and demand. (See also Lemma 2 of Karaesmen and van Ryzin, 2004).

Recall that

$$P_N^c(\vec{X},\vec{D}) = \sum_{i\in N} r_i B_i + v_i H_i - c_i X_i + W_N(\vec{X},\vec{D})$$

where the transshipment problem is defined as:

$$\begin{aligned} W_N(\vec{X},\vec{D}) = \quad & \max_{\vec{y}} \sum_{i\in N}\sum_{j\in N, j\neq i} (r_j - v_i - t_{i,j}) y_{i,j} \\ \text{s.t.} \quad & \sum_{j\in N, j\neq i} y_{i,j} \le \max\{X_i - D_i, 0\} \text{ for all } i \in N \\ & \sum_{i\in N, i\neq j} y_{i,j} \le \max\{D_j - X_j, 0\} \text{ for all } j \in N \\ & \text{for all } y_{i,j} \ge 0. \end{aligned} \tag{2}$$

The only difference between $P_N^c(\vec{X},\vec{D})$ and the transportation problem is that our setting requires retailers to satisfy local demand first. Hence, $P_N^c(\vec{X},\vec{D})$ is not necessarily concave if local demand does not generate profit as much as transshipping to other retailers. A restriction on cost parameters is required to make $P_N^c(\vec{X},\vec{D})$ coincide with the transportation problem.

Consider when the condition to satisfy local demand first is relaxed. The right hand side of constraints becomes X_i and D_j instead of $\max\{X_i - D_i, 0\}$ and $\max\{D_j - X_j, 0\}$. The profit function $P_N^c(\vec{X},\vec{D})$ is changed such that $\sum_{i\in N} r_i B_i + v_i H_i - c_i X_i$ is removed since it would be included in the transportation problem. In this case, $P_N^c(\vec{X},\vec{D})$ is concave in $(\vec{X},\vec{D})$.

To make the solution of this new setting coincides with our original problem, the solution of the new setting should be a transshipment pattern such that the maximum overall profit can be achieved when retailers satisfy their local demand first. This is possible when the profit from satisfying local demand is greater than profit from any transshipment pairs. □

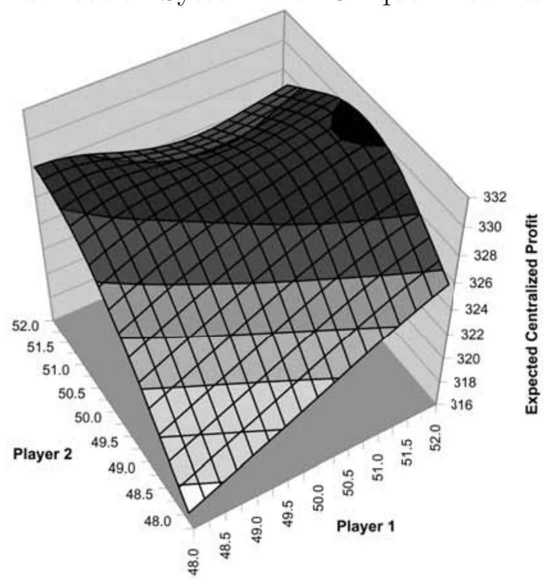
In summary, there exists a unique PSNE for decentralized distribution system that use ABZ's allocation rule if Proposition 3 and Proposition 4 are satisfied, the strategy space is limited to domain of demand distributions, and density function of demand is strictly log-concave.

When the conditions of Proposition 3 or Proposition 4 are omitted, multiple PSNE may exist as shown in the following example.

Example 2 *When $r_1 = 5.4$, $r_2 = 5.6$, $c_1 = 3.2$, $c_2 = 1.2$, $v_1 = 4$, $v_2 = -1$, $t_{1,2} = 0$, $t_{2,1} = 2$, and*

$\gamma_i = 0.5$. *The demands are independent and uniformly distributed in the range of [49,51]. Assume that the strategy space for inventory levels is bounded in [48,52]. This system has a unique first-best solution at (52,50) with expected profit of \$330.35. However, it has an infinite number of PSNE, e.g. (48,52), (48.5,51.5), etc., with the combined expected profit of \$328.53 as shown in Figure 3.*

Figure 3: Decentralized Distribution System with Unique First-Best and Non-Unique PSNE



Why is uniqueness of PSNE important for ABZ allocation rule? It is important because the side payment calculations are based on the value of the unique PSNE/first-best inventory level. For instance, consider an n-retailers decentralized distribution system that has two different PSNE/first-best inventory levels: $\vec{X}^{*A}$ and $\vec{X}^{*B}$ where

$$\begin{aligned}\vec{X}^{*A} &= \left\{X_1^{*A}, X_2^{*A}, \ldots, X_n^{*A}\right\} \\ \vec{X}^{*B} &= \left\{X_1^{*B}, X_2^{*B}, \ldots, X_n^{*B}\right\}.\end{aligned}$$

Which PSNE/first-best inventory levels should be used when calculating the side payment? Retailer 1 may hope to calculate the side payment from PSNE at A because it benefit her more. Therefore, she will choose inventory level X_1^{*A}. On the other hand, retailer 2 may be better off with PSNE at B and choose inventory level X_2^{*B}. In this case, the resulting inventory levels are not a member of PSNE, hence, the first-best expected profit is not achieved.

4 Effect of Non-Nash Strategy

The setup of ABZ's allocation rule $\alpha_i^m(\vec{X},\vec{D})$ also calls for additional discussion. It is reasonable to assume that the allocation of profit from transshipment should only be shared by retailers who participate in the transshipment. That is, if a retailer is not involved in transshipment, she should not receive any of its profit. This, however, does not apply to the fractional allocation $\alpha_i^f(\vec{X},\vec{D})$ because it is allowed to be a non-core allocation of the transshipment game as intended by ABZ. Recall that $\alpha_i^f(\vec{X},\vec{D})$ may be negative. The retailer with negative $\alpha_i^f(\vec{X},\vec{D})$ would be better off not joining the coalition for transshipment. With above understanding, we redefine the allocation $\alpha_i^m(\vec{X},\vec{D})$ as follows:

$$\alpha_i^{\tilde{m}}(\vec{X},\vec{D}) = \begin{cases} \alpha_i^f(\vec{X},\vec{D}) + \alpha_i^d(\vec{X}^{c^*},\vec{D}) - \alpha_i^f(\vec{X}^{c^*},\vec{D}) & \text{if } (i \in \Phi_{\vec{X},\vec{D}}) \\ +\frac{1}{\phi}\sum_{j\in N\backslash\Phi_{\vec{X},\vec{D}}}\left[\alpha_j^f(\vec{X},\vec{D}) + \alpha_j^d(\vec{X}^{c^*},\vec{D}) - \alpha_j^f(\vec{X}^{c^*},\vec{D})\right] & \\ & \\ 0 & \text{otherwise} \end{cases}$$

where $\Phi_{\vec{X},\vec{D}}$ is a set of retailers who are involved in the transshipment solution after realization, and ϕ is a number of retailers in $\Phi_{\vec{X},\vec{D}}$. Essentially, the amount that would have been paid to retailers not involved in the transshipment solution is distributed equally among retailers who are involved in the transshipment solution. Note that this allocation $\alpha_i^{\tilde{m}}(\vec{X},\vec{D})$ is always in the core of the transshipment game if the first-best solution $\vec{X}^{c^*}$ is played because the value of $\alpha_j^d(\vec{X}^{c^*},\vec{D})$ is zero for all retailer $j \in N\backslash\Phi_{\vec{X},\vec{D}}$.

Proposition 5 *The Nash equilibrium solution using the allocation rule $\alpha_i^{\tilde{m}}(\vec{X},\vec{D})$ is not necessarily the first-best solution.*

Proof. We derive the payoff function $P_i(\vec{X},\vec{D})$ as:

$$P_i(\vec{X},\vec{D}) = \begin{cases} \gamma_i P_N^c(\vec{X},\vec{D}) + \alpha_i^d(\vec{X}^{c^*},\vec{D}) - \alpha_i^f(\vec{X}^{c^*},\vec{D}) & \text{if } (i \in \Phi_{\vec{X},\vec{D}}) \\ +\frac{1}{\phi}\sum_{j\in N\backslash\Phi_{\vec{X},\vec{D}}}\left[\alpha_j^f(\vec{X},\vec{D}) + \alpha_j^d(\vec{X}^{c^*},\vec{D}) - \alpha_j^f(\vec{X}^{c^*},\vec{D})\right] & \\ & \\ [r_iB_i + v_iH_i - c_iX_i] & \text{otherwise} \end{cases}$$

We further reduce it to $P_i(\vec{X},\vec{D}) = \gamma_i P_N^c(\vec{X},\vec{D}) + \alpha_i^d(\vec{X}^{c^*},\vec{D}) - \alpha_i^f(\vec{X}^{c^*},\vec{D}) + Q_i(\vec{X},\vec{D})$ where

$$Q_i(\vec{X},\vec{D}) = \begin{cases} \frac{1}{\phi}\sum_{j\in N\setminus\Phi_{\vec{X},\vec{D}}}\left[\alpha_j^f(\vec{X},\vec{D}) + \alpha_j^d(\vec{X}^{c^*},\vec{D}) - \alpha_j^f(\vec{X}^{c^*},\vec{D})\right] & \text{if } (i \in \Phi_{\vec{X},\vec{D}}) \\ \\ -\alpha_i^f(\vec{X},\vec{D}) - \alpha_i^d(\vec{X}^{c^*},\vec{D}) + \alpha_i^f(\vec{X}^{c^*},\vec{D}) & \text{otherwise} \end{cases}$$

The best response function (the inventory ordering strategy) of each player is a strategy that maximizes the expected payoff $J_i(\vec{X}) = E_{\vec{D}}(P_i(\vec{X},\vec{D}))$. We see that it is not the same as maximizing the centralized profit $P_N^c(\vec{X},\vec{D})$ because $E_{\vec{D}}(Q_i(\vec{X},\vec{D}))$ is not constant as in (9). Hence, with the allocation $\alpha_n^{\tilde{m}}(\vec{X},\vec{D})$, the first-best solution will not necessarily be the one selected. $\square$

Proposition 5 indirectly points out a potential misuse or misinterpretation with regards to the allocation $\alpha_i^m(\vec{X},\vec{D})$. Consider the case that one (or more) retailer chooses to order inventory that is not the first-best strategy. In that case, for the allocation $\alpha_i^m(\vec{X},\vec{D})$, the corresponding SAG game is likely to have an empty core. This is due to the fact that the additional SAG profit may be shared with retailers who do not participate in the corresponding transshipment solution. For instance, consider retailer A who would have participated in the transshipment solution for some demand realization $\vec{D}$ if a first-best strategy have been played by all. Retailer A might be excluded for the transshipment solution if another retailer selected a strategy that is different from first-best. As a result, even though retailer A does not participate in the transshipment solution, she still receives some profit from transshipment because of the side payments that are part of the $\alpha_i^m(\vec{X},\vec{D})$ allocation.

If we assume that in this decentralized distribution system, every player has complete information and all players are rational, then we would expect retailers/players to order the first-best inventory level in the first-stage and expect that the modified allocation $\alpha_i^m(\vec{X},\vec{D})$ would be in the core of the transshipment game. However, in reality, we may not be able to assume complete information and/or rationality and guarantee that every player will order the first-best/Nash equilibrium inventory level. For instance, Aumann (1997) states that "polls and laboratory experiments indicate that people often fail to conform to some of the basic assumptions of rational decision theory." Aumann and Maschler (1995) also claims that "unlike the situations treated in classical game theory, a participant in a real-life conflict situation usually lacks information on the strategies that are available to him and to his opponent, on the actual outcomes and their utility to each of the participants, and on the amount of information that the other participants possess." In this section, we examine the allocation $\alpha_i^m(\vec{X}, \vec{D})$ when non-Nash strategy is played by some of the players.

To analyze the effect of a strategy that is not PSNE strategy, we need to examine an issue of implementation and demand realizations. In centralized inventory game literature, it is proven that some cost games only have a nonempty core for the expected cost game (Hartman and Dror, 2005). For any specific demand realization, the core of the game is likely to be empty. Fortunately, our decentralized distribution systems do not have that pitfall because the core of the allocation based on dual prices of our second-stage transshipment game is always non-empty regardless of demand realization. But, there is another issue to consider. When retailers agreed on the profit allocation ex-ante, the profit allocation in the core of SAG is calculated in an expectation. Once the demand is realized at every retailer, the retailers would re-evaluate the allocation (ex-post) given the actual demand and inventory levels chosen in the first stage. If the chosen inventory levels are at unique PSNE, ABZ allocation is guaranteed to be in the core. Otherwise, the allocation is not guaranteed to be in the core. We examine what might happen in the case when a retailer chooses to order non-PSNE inventory level in the first-stage. Specifically, we examine if allocation $\alpha_i^m(\vec{X}, \vec{D})$ is always in the core of the realized transshipment game.

Example 3 *Consider the numerical example 3 provided by ABZ. Recall that cost structure are as following: for* $i = 1, 2,\ r_i = 10, c_i = 1.2, v_i = -1, t_{1,2} = 1,$ *and* $t_{2,1} = 2$. *The demands are assumed to be independent and uniformly distributed between [0,100] at each separate retailer. The inventory position based on the allocation* $\alpha_i^m(\vec{X}, \vec{D})$ *with* $\gamma_i = 0.5$ *is (76.81,62.35), i.e., the (first-best) Nash equilibrium inventory levels* $X_1^{c^*} = 76.81$ *and* $X_2^{c^*} = 62.35$. *Let us assume that an extreme scenario happens such that retailer 1 does not order any units of inventory. The reason could be that she did not have enough funding or she has faulty information about distribution of demand. Let us also assume that retailer 2 orders 62.35 units of inventory. Pick arbitrary 75 and 70 for the demand realized at retailer 1 and 2, respectively. In this case, there is no transshipment since both retailers face shortages. Note that the resulting profit $548.68 is generated by retailer 2 alone. Given the allocation* $\alpha_i^m(\vec{X}, \vec{D})$ *with* $\gamma_i = 0.5$ *as proposed by ABZ, retailer 1 would receive $337.06 from* *retailer 2. As a result, retailer 2 would want to break from cooperation as she can do better on her own. Thus, the allocation $\alpha_i^m(\vec{X},\vec{D})$ is not always in the core of the corresponding transshipment game. Consequently, the following proposition is stated without proof.*

Proposition 6 *At non-PSNE inventory position, the allocation $\alpha_i^m(\vec{X},\vec{D})$ is not always in the core of the transshipment game when $\vec{X} \neq \vec{X}^{c^*}$. That is, there exists $S \subseteq N$ such that*

$$\sum_{i \in S} \alpha_i^m(\vec{X},\vec{D}) < W_S(\vec{X},\vec{D})$$

and

$$\sum_{i \in N} \alpha_i^m(\vec{X},\vec{D}) = W_N(\vec{X},\vec{D})$$

where $W_S(\vec{X},\vec{D})$ is the maximum amount of profit that could be generated during the transshipment game by coalition S given inventory position $\vec{X}$ and demand $\vec{D}$, and $W_N(\vec{X},\vec{D})$ is the maximum amount of profit that could be generated during the transshipment game by the grand coalition N.

Notice that the discussion in this section has a flavor of *open-loop* strategies. In non-cooperative setting, open-loop strategies are functions of calendar time alone, as oppose to closed-loop strategies which are functions of calendar time as well as the history of play until that date. According to Fudenberg and Tirole (1991), "If the players can condition their strategies on other variables in addition to calendar time, they may prefer not to use open-loop strategies in order to react ... to possible deviations by their rivals from the equilibrium strategies." In our setting, we assume that the second-stage transshipment profit allocation rule is decided ex-ante among retailers. So, we can consider the open-loop non-cooperative strategy of retailer i as to cooperate with the grand coalition and receive the payoff of α_i^m in the second-stage transshipment game. This strategy might not be optimal if some retailers do not choose PSNE inventory position and retailer i might be better off breaking from the grand coalition if she has an option to do so in response to other retailers' deviations. Moreover, we can say that the two-stage strategy tuple (Ordering PSNE inventory level in the first stage, Cooperating with grand coalition in the second stage) is not a subgame-perfect equilibrium because retailers do not respond optimally to unanticipated strategy deviations.

On the other hand, consider a decentralized distribution system that uses dual allocation rule. The two-stage open-loop strategy tuple of this game (Ordering PSNE inventory level in the first stage, Cooperating with grand coalition in the second stage) is a subgame-perfect equilibrium because retailers still respond optimally since the dual allocation is always in the core of the transshipment game.

5 Allocation Rules and Incentive Compatibility

The above decentralized distribution system is modeled based on a number of assumptions. The assumption discussed in this section is that of complete information. We assume that retailers share their information of unit revenue r_i, unit cost c_i, unit salvage value v_i, transshipping cost $t_{i,j}$, and distribution of demand D_i in the first stage. This information is considered common knowledge and the retailers have a right to order any inventory levels they prefer. In this two-stage game, such complete information assumption might be difficult to verify, especially information related to distribution of demand. It might be difficult to check whether a retailer lies about her distribution of demand. However, it is important to know whether a retailer has an incentive to lie.

Consider retailer A who shares her information with a number of other retailers with an agreement to cooperate on transshipments. Assume that at the last minute before ordering the inventory in the first stage based on demand parameters μ_A, σ_A, she learns that her demand has changed such that $\tilde{\mu}_A > \mu_A$ and $\tilde{\sigma}_A < \sigma_A$. She has to decide whether to inform the other retailers about the change and then choose an optimization solution that best fits her problem parameters. She faces a few options. We consider only the three options below.

- (a) She chooses not to inform others about the change in demand distribution parameters and chooses to maximize her expected profit using a single newsvendor model assuming that she will not join any coalition in the second stage.

- (b) She chooses not to inform others about the change. She, then, assumes that the other retailers will choose their inventory levels based on the first-best/PSNE solution $\vec{X}^{c^*}$ calculated using the original demand distribution. Hence, she will choose an inventory level X_i that maximizes her expected profit, including profit from transshipment, based on her new demand distribution.

- (c) She chooses to inform others about the change and chooses an optimization model that maximizes her expected profit assuming that the other retailers will choose their inventory levels based on the first-best/PSNE solution $\vec{X}^{c^*}$ calculated using the new demand distribution given by her.

We assume for cases (b) and (c) that all retailers will share all shortage/surplus for transshipment and omit the possibility that some retailers may hold back shortage/surplus. This option was discussed in Granot and Sošić (2003). In their paper, Granot and Sošić (2003) consider the decentralized distribution system as a three-stage model. Retailers choose inventory levels in the first stage. After demand is realized, each retailer fulfills her local demand and at the second stage decides how much of her shortage/surplus she should share with the other retailers. At the third stage, the collaborated transshipment decisions take place. The main result of Granot and Sošić (2003) is that dual allocation rules may induce retailers not to share their shortage/surplus. Granot and Sošić (2003) analyze allocation rules on completely sharing property, value preserving property, and efficient property. An allocation rule is called completely sharing if it induces all the retailers to share their total residual supply/demand with other retailers, it is called value preserving if it induces all the retailers to share their residual supply/demand in amounts that do not result in a decrease in the total transshipment profit, and it is called efficient if the full amount of transshipment profit is allocated to retailers. Based on their model, Granot and Sošić (2003) proposed a fractional allocation rule that is efficient value preserving, is a Nash equilibrium profile, and also induces a first-best solution, but may not always be in the core of the third stage transshipment game. (See Granot and Sošić, 2003, Theorem 13.) We assume that all retailers are binded by contract to share all of their shortage/surplus for transshipment and pursue a different line of analysis than Granot and Sošić, 2003.

For the cases (b) and (c), we ask whether retailer A has an incentive to hold back her new distribution information. If she has incentive to do so, then the proposed allocation is unlikely to result in optimal (and first-best) profit. To check whether retailer A has an incentive to hold back her new distribution information is essentially to check whether a proposed allocation satisfies *incentive compatibility* property. In mechanism design, when a competitive game has incentive compatibility property, incentive for every player of telling the truth is of higher utility than telling a lie. Note that the first stage of our game is competitive in nature, none of the retailers are obligated to disclose their information. In our decentralized distribution system, the allocation with incentive compatibility property should encourage retailers to choose option (c) described above. That is, retailer A would see the highest expected profit when she truthfully reveals her true distribution of demand. On the other hand, if options (a) or (b) are better than option (c), then retailer A has an incentive not to share the information about her true distribution of demand.

Unfortunately, we cannot conclude that the allocation proposed by ABZ has the incentive compatibility property. Notice that the three options mentioned above are calculated based on different probability distributions and that retailers' action may cause the agreed allocation not be a member of the core of the corresponding transshipment game. For instance, option (a) is calculated using only retailer A's distribution of demand, regardless of other retailers' distributions of demand and might result in a higher expected profit than option (b) and (c). Another instance, with some arbitrary realization, option (b) causes the allocation proposed by ABZ not be a member of the non-empty core of the transshipment game. The analysis of the subsequent actions of retailers when this case happens is out of the scope of this paper. In general, retailers may decide to leave the grand coalition because of suboptimal payoff, or may stay with the grand coalition because of long-term (repeated game) incentive. We refer the reader to Hartman and Dror (2005) (see also Dror et al., 2008) for a discussion of inventory centralization games for which allocations may converge to a selected core solution in the long run.

Proposition 7 *Assume that all retailers will share all shortage/surplus for transshipment. If the demand distribution is not of common knowledge, the decentralized distribution system that adopts ABZ allocation rules is not necessarily incentive compatible.*

Example 4 *We apply the idea to the numerical example 3 provided by ABZ that we mention earlier in §4. Recall that cost structure are as following: for* $i = 1, 2$, $r_i = 10, c_i = 1.2, v_i = -1, t_{1,2} = 1$, *and* $t_{2,1} = 2$. *Retailers agree on using ABZ allocation rule with* $\gamma_i = 0.5$. *Let the distribution of demand assumed by retailer 2 be independent and uniform on [0,100] for both players, while retailer 1 knows that the true distribution of demand is uniform on [0,110] for herself and [0,100] for retailer 2. If retailer 1 communicates the true distribution of demand to retailer 2, then retailers*

1 and 2 will order 84.20 and 62.00 units, and receive \$406.99 and \$375.16, respectively. If retailer 1 does not communicate the true distribution of demand to retailer 2, then retailers 1 and 2 will order 79.31 and 62.35 units, and receive \$407.23 and \$374.15, respectively. (Note that retailer 2 would not know that her expected profit is \$374.15) We can see that retailer 1 would choose to not communicate the true demand distribution to retailer 2.

6 Relaxing the Assumption on Satisfying Local Demand First

Prior to this section, we assumed that retailer must satisfy local demand first and dispose (salvage) excess inventory only locally. However, in the real market, an independent retailer might choose to transship products to other retailers if doing so is more profitable to her. For example, retailer A makes \$5 profit per units when selling locally, but earns \$7 when she transships her inventory to sell by retailer B. (The overall profit made by this transshipment could be greater or equal to \$7 but let's assume that retailer A gets exactly \$7 per unit transshipped.) Then, retailer A would transship her inventory to retailer B before satisfying her local demand. Similarly, if the salvage value at retailer A is lower than the salvage value at retailer B minus transshipment cost between them, then the retailer A would transship her excess inventory to dispose at retailer B. In this section, we model the decentralized distribution systems by relaxing assumptions on satisfying local demand first and also on disposing excess inventory only at local retailer. Then, we examine whether ABZ's allocation can still achieve the first-best profit. Thus, we extend the 'range' of our analysis.

As before, competitive independent retailers face random demands. In the first stage, inventories are independently ordered based on anticipated demands. Assume that all inventory is stored locally. In the second-stage, these retailers use pooling of stocks, i.e., any demand at one retailer can be satisfied from inventory transshipped from other retailers. We assume that retailers will cooperate and make profit maximizing centralized decision to transship inventory to satisfy all demands in the system.

Consider a case of N retailers as before. Let r_i, c_i, and v_i, where $i = 1, \ldots, N$, represent unit revenue, unit cost, and unit salvage value of a retailer i, respectively. Let $t_{i,j}$ represent the transshipping cost from retailer i to retailer j for all $i, j \in N$ and $t_{i,i} = 0$ all i. In the first stage, each retailer makes decision on her inventory level. Let the vector $\vec{X} = (X_1, \ldots, X_N)$ denote the levels of inventory ordered in the first stage by all retailers. Then, the demand represented by the vector $\vec{D} = (D_1, \ldots, D_N)$ is realized at all retailers. Note that this decentralized distribution system is equivalent to the decentralized distribution system explained in the earlier sections of this paper if $r_i - c_i > r_j - c_i - t_{i,j}$ and $v_i - c_i > v_j - c_i - t_{i,j}$ for all $i, j \in N$.

Given the above setting for a two-retailer case, if retailer 2 can order inventory at a cost c_2 that is higher than cost $c_1 + t_{1,2}$ of obtaining transshipment from retailer 1, one might misinterpret that retailer 2 would not order at all and let retailer 1 order for her. This is incorrect because the second-stage game is a cooperative game. The profit made from sales at retailer 2 must be shared with retailer 1 according to an agreed allocation rule. Retailer 2's share of profit per unit from transshipment might be lower than the profit per unit when retailer 2 sells from her own local inventory. Thus, the behavior of retailers depends on the allocation rules that they agreed on.

Assume that these retailers agree to allocate profit that is a result of transshipment game using allocation rule α^x. The profit expected at each retailer i is

$$J_i(\vec{X}) = E_{\vec{D}}(\alpha_i^x(\vec{X}, \vec{D}))$$

such that $\sum_{i \in N} \alpha_i^x(\vec{X}, \vec{D}) = \hat{W}_N(\vec{X}, \vec{D})$. The transshipment problem $\hat{W}_N(\vec{X}, \vec{D})$ is represented by:

$$\begin{aligned} \hat{W}_N(\vec{X}, \vec{D}) = \quad & \max_{\vec{y}} \; \sum_{i \in N} \sum_{j \in N} (r_j - c_i - t_{i,j}) y_{i,j} + \sum_{i \in N} \sum_{j \in N} (v_j - c_i - t_{i,j}) z_{i,j} \\ & \text{s. t.} \quad \sum_{j \in N} (y_{i,j} + z_{i,j}) = X_i \text{ for all } i \in N \\ & \qquad\quad \sum_{i \in N} y_{i,j} \leq D_j \text{ for all } j \in N \\ & \qquad\quad \text{for all } y_{i,j}, z_{i,j} \geq 0. \end{aligned}$$

The quantity $y_{i,j}$ represents a number of units of product transshipped from retailer i to sell by retailer j in the second stage. The quantity $z_{i,j}$ represents a number of units of product transshipped from retailer i to dispose at retailer j in the second stage. Note that $\hat{W}_N(\vec{X}, \vec{D})$ is continuous in $\vec{X}$ because there is no fixed cost related to local profits or transshipment profits at any retailers.

At this point, we examine the conditions that result in the decentralized distribution system achieving the first-best profit. Similar to the previous model, we need 1) profit allocation to be in the core of the transshipment game, 2) a unique PSNE in the first stage, and 3) the first-stage inventory decisions to result in the same inventory levels as the centralized system. Consider ABZ's allocation which is a combination of fractional allocation and allocation based on dual prices of transshipment game. The local profit term in fractional allocation is zero because the expected profit only relies on the profit allocated from transshipment. The allocation based on dual prices of transshipment game $\alpha_i^d(\vec{X}, \vec{D})$ is:

$$\alpha_i^d(\vec{X}, \vec{D}) = \lambda_i X_i + \delta_i D_i.$$

Using ABZ's allocation rule, the side payment moves the profit allocation based on fractional rule to the allocation according to the dual price of the transshipment game. Therefore, ABZ's allocation will be in the core of the transshipment game if Nash equilibrium inventory level is chosen by each individual retailer as previously discussed in §4. The characteristic of the transshipment problem $\hat{W}_N(\vec{X}, \vec{D})$ is still the same as before. The dual price for inventory disposed at the salvage value will be assigned to the original owner of the inventory, not the retailer where the inventory is disposed because there are no limits on the disposal capacity, a dual price of zero will be assigned to the the retailer where the inventory is disposed.

We examine the individual expected profit $J_i(\vec{X})$ assuming that an ABZ allocation rule is applied. We know that if there exists a unique PSNE, then the first-stage inventory decisions result in the same inventory levels as the centralized system.

First, we check the existence of a PSNE. Recall that if there exists a first-best solution, then there exists a PSNE. In this setting, the strategy space for inventory level is nonempty compact convex subsets of a Euclidean space. Recall there exists a first-best solution if $J_N^c(\vec{x})$ is continuous in $\vec{X}$.

In this case, the centralized profit $P_N^c(\vec{X}, \vec{D})$ is equivalent to the transshipment profit $\hat{W}_N(\vec{X}, \vec{D})$. The transshipment profit function is uniformly continuous because there is no fixed cost related to transshipment profits. It follows that $J_N^c(\vec{x})$ is continuous in $\vec{X}$.

In terms of the uniqueness of PSNE, the relaxed model has fewer requirements for uniqueness.

There exists a unique PSNE if (i) there is no ridge present for $J_N^c(\vec{X})$ (see §3.2 for restriction on cost parameters), (ii) each demand density function is *strictly* log-concave, and (iii) non-degeneracy cost parameters are assumed, i.e., salvage value is less than unit cost, and for any two retailers i and j, $r_i - t_{j,i} \neq c_j$ and $r_j - t_{i,j} \neq c_i$.

We no longer have to require conditions for strict log-concavity of $P_N^c(\vec{X}, \vec{D})$ on each X_i and (weakly) log-concave in $(\vec{X}, \vec{D})$ as previously described in Proposition 3 and Proposition 4. This is because the solution of transportation problem already has those properties.

7 Discussion

This section describes assumptions of decentralized distribution systems and discusses some modeling issues. Potential applications of this paper are also described.

Decentralized distribution systems with cooperative transshipment are undoubtedly an important research area. Retailers, such as a car dealership, find this type of practice attractive because it improves customer satisfaction, reduces excess inventory, and may potentially generate higher profit than traditional decentralized distribution systems without transshipment.

Past literature on decentralized distribution systems with cooperative transshipment adopted two key assumptions when analyzing decentralized distribution systems with cooperative transshipment. First, they assumed complete information because the demand distributions and accurate cost parameters of all retailers are crucial for calculation of optimal inventory. The complete information assumption is technically feasible in today's advanced information systems and supply chain management software. However, in practice voluntary complete information sharing arrangement among competing players is somewhat questionable. In §5, we emphasized that noncooperative incentives may result if the assumption of complete information does not hold. For instance, a player may be induced to lie about her true demand distribution parameters. In future research it would be interesting to examine different profit allocation rules with regards to properties such as incentive compatibility. Furthermore, if sharing of demand and pricing information is not completely free of charge, it might be valuable to consider mechanism design that ensures truth telling.

Another key assumption is that all players are individually rational in making their competitive decision in the first stage but are confined by an agreed ex-post transshipment profit allocation rule in the second stage. This is assumed even in the case that such profit allocation rule may not necessarily be the best course of action they could take in the second stage. For some allocation rules that are always in the core of the transshipment game, e.g. dual price allocation rule, this assumption generally holds and the accuracy of expected profit calculation is not affected. But for other allocation rules such as the ABZ rule, we show in §4 that if some players do not play PSNE strategy, the allocation may not necessarily be in the core of the transshipment game. Therefore, a number of players might be better-off without cooperation in the transshipment stage. When the cooperative outcome does not hold, the expected profit calculation is no longer straight forward. Further analysis is needed in this case.

With respect to the ABZ model, we have noted that a number of assumptions may influence its validity. For instance, the uniqueness of PSNE/first-best profit will be achieved only if demand distribution and cost parameters are restricted as we discussed in §3. It would be of interest to see an empirical study and compare real-life performance of this model to a traditional newsboy model with ad-hoc transshipment arrangement.

Decentralized inventory system is applicable to many industries and supply chain settings, not just limited to car dealerships. For instance, the operation of independent lumber companies in Scandinavia (that motivates the work of Sandsmark, 2009) and the pooling of spare parts inventories at air carrier companies in Brussels (Wong et al., 2006) are all of similar flavor. Since most realistic situations involve random variability in demand, even with the best forecasting technique, the likelihood of correctly matching demand with supply is not very promising. Shared resources and capabilities help companies cope effectively with unexpected or unusual demands for products and services.

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Appendix

Proof of Lemma 2. The general idea is to show that there is a point in the neighborhood that has equal or higher expected centralized profit at $\vec{X}^*$.

If only one of retailer i or j changes her inventory by a small $\epsilon, |\epsilon| > 0$, then the expected centralized profit $J_N^c(\vec{X})$ decreases because $\vec{X}^*$ was her best response given all other retailers choose $\vec{X}^*_{N\setminus\{i,j\}}$. In addition, ϵ would not change the situation of a retailer having shortage or overage.

For a positive ϵ, consider a neighborhood point $\vec{X}^{1\circ} = (X_i^* + \epsilon, X_j^* - \epsilon, \vec{X}^*_{N\setminus\{i,j\}})$ that represents the point after both retailers i and j change their inventories. Without finding the optimal transshipment pattern, one can calculate the lower bound $\underline{J}_N^c(\vec{X}^{1\circ})$ of expected centralized profit given that an increase in retailer i is transshipped to retailer j. Hence, almost surely (with probability 1)

$$\underline{J}_N^c(\vec{X}^{1\circ}) - J_N^c(\vec{X}^*) = (r_2 - c_1 - t_{1,2})\epsilon - (r_2 - c_2)\epsilon = (c_2 - c_1 - t_{1,2})\epsilon$$

For a negative ϵ, consider another neighborhood point $\vec{X}^{2\circ} = (X_i^* + \epsilon, X_j^* - \epsilon, \vec{X}^*_{N\setminus\{i,j\}})$. The lower bound $\underline{J}_N^c(\vec{X}^{2\circ})$ of expected centralized profit given that a decrease in retailer i reduces transshipment to retailer j. In addition, retailer j increase in inventory is used towards her local demand. Hence, almost surely

$$\underline{J}_N^c(\vec{X}^{2\circ}) - J_N^c(\vec{X}^*) = -(r_2 - c_1 - t_{1,2})(-\epsilon) + (r_2 - c_2)(-\epsilon) = (c_2 - c_1 - t_{1,2})\epsilon$$

Notice that if $(c_2 - c_1 - t_{1,2}) > 0$, a neighborhood point from adding a positive epsilon can improve upon $J_N^c(\vec{X}^*)$. On the other hand if $(c_2 - c_1 - t_{1,2}) < 0$, a neighborhood point from adding a negative epsilon can improve upon $J_N^c(\vec{X}^*)$. Therefore regardless of cost parameters, there always exists a point in the neighborhood that has equal or higher expected centralized profit at $\vec{X}^*$. Hence, the ridge always exists. $\square$

Proof of Proposition 3. Let set of X_i in the second region be

$$R_2 = \left\{ X_i : X_i \leq D_i \text{ and } X_i > D_i - \sum_{j\in N, j\neq i} y^*_{j,i} \right\}$$

and set of X_i in the third region be

$$R_3 = \left\{ X_i : X_i > D_i \text{ and } X_i \leq D_i + \sum_{j \in N, j \neq i} y^*_{i,j} \right\}.$$

To ensure that the profit function is concave, we need the minimum slope in the second region to be greater than the maximum slope in the third region. That is,

$$\rho^\circ - \max_{X_i \in R_2} \frac{\partial W_N(\vec{X}, \vec{D})}{\partial E_i} > \max_{X_i \in R_3} \frac{\partial W_N(\vec{X}, \vec{D})}{\partial H_i}. \tag{3}$$

We know that $\rho^\circ = r_i - c_i$. The upper bound of $\max_{X_i \in R_2} \frac{\partial W_N(\vec{X},\vec{D})}{\partial E_i}$ is $\max_k(r_i - v_k - t_{k,i}) + (v_k - c_k)$; that is the most profit per unit made from transshipping to retailer i. So, the lower bound of the left-hand-side term is $r_i - c_i - \max_k (r_i - c_k - t_{k,i})$.

For the right-hand-side, the upper bound of $\max_{X_i \in R_3} \frac{\partial W_N(\vec{X},\vec{D})}{\partial H_i}$ is $\max_j(r_j - v_i - t_{i,j}) + (v_i - c_i)$, i.e., the most profit made from transshipping from retailer i. So, the upper bound of the right-hand-side term is $\max_j (r_j - c_i - t_{i,j})$.

Rearrange (3), to get

$$\begin{aligned} r_i - c_i - \max_k (r_i - c_k - t_{k,i}) &> \max_j (r_j - c_i - t_{i,j}) \\ r_i - c_i &> \max_k (r_i - c_k - t_{k,i}) + \max_j (r_j - c_i - t_{i,j}). \end{aligned} \tag{4}$$

Hence, to ensure that the profit function is strictly log-concave we require the cost parameters to imply that, for any retailer, it is more profitable to satisfy local demand from its own inventory, rather than to ship from other retailer and transshipping its own inventory to sell at yet another retailer.

If there exist retailers j and k such that $r_i - c_i < (r_j - c_i - t_{i,j})$ and $r_i - c_i < (r_i - c_k - t_{k,i})$, then the lower bound of the left-hand-side term of (4) is a negative value and the slope in the third region is strictly positive and greater than $r_i - c_i$. So, we might have a downward slope in the second region and upward slope in the third region as shown in Figure 4. In such case, the profit function is neither concave nor quasiconcave.

However, if there exist retailers j and k such that $r_i - c_i < (r_j - c_i - t_{i,j})$ but $r_i - c_i \geq (r_i - c_k - t_{k,i})$,

Figure 4: Non-concave centralized profit function

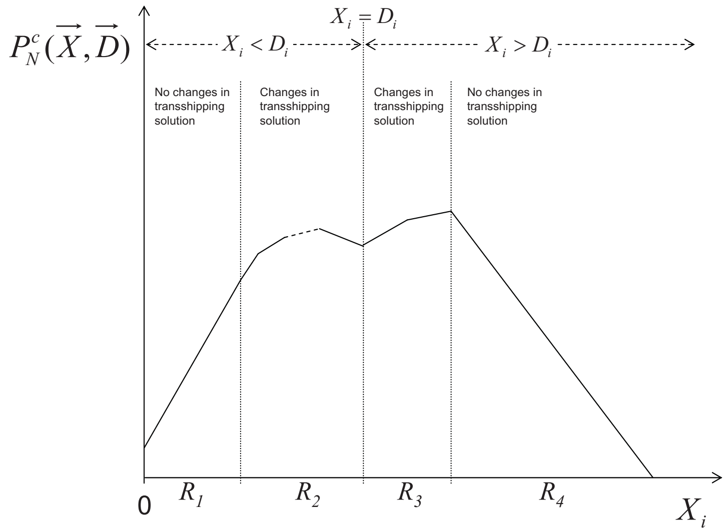
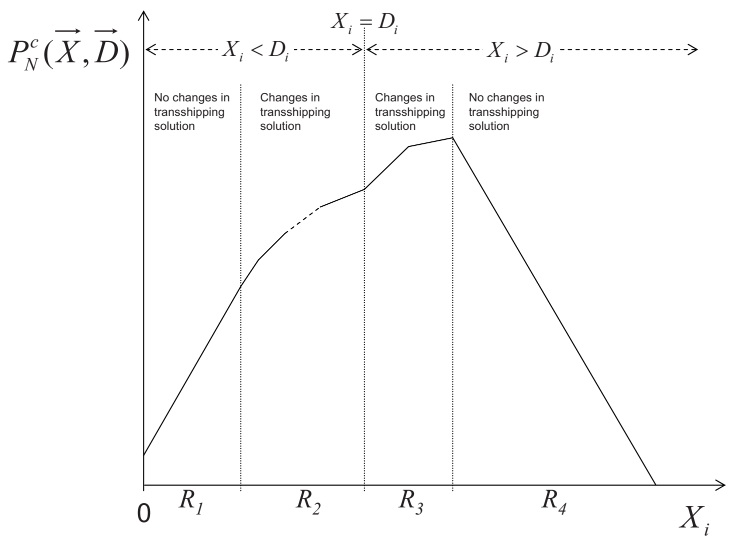


Figure 5: Quasi-concave centralized profit function



then we have a upward slope in the second region but the slope in the third region might be higher. The profit function is quasiconcave as shown in Figure 5. $\square$

Proof of Proposition 7. The incentive compatible allocation required that, for individual player, telling the truth is more profitable to her than telling a lie. Consider a decentralized distribution system that adopts ABZ allocation rules. Let the true cumulative demand distribution be $F^{true}(\vec{D})$. Assume this $F^{true}(\vec{D})$ is not a common knowledge and is only known to retailer i. Thus, the distribution of demand assumed by all the players but i is $F^{false}(\vec{D})$. We prove that the profit for

i is greater or equal under i's knowing $F^{true}(\vec{D})$ and the rest of players assuming $F^{false}(\vec{D})$ than when all players know $F^{true}(\vec{D})$.

Let the first-best Nash equilibrium inventory level for the decentralized distribution system with cumulative demand distribution $F^{true}(\vec{D})$ and $F^{false}(\vec{D})$ be $\vec{X}^{true} = (\vec{X}^{true}_{N\backslash i}, X^{true}_i)$ and $\vec{X}^{false} = (\vec{X}^{false}_{N\backslash i}, X^{false}_i)$, respectively. Let $\vec{X}^{response} = (\vec{X}^{false}_{N\backslash i}, X^{response}_i)$ be an inventory level when only retailer i has the knowledge of true demand distribution $F^{true}(\vec{D})$ while all other retailers assume the demand distribution is $F^{false}(\vec{D})$. Essentially, X^{true}_i and $X^{response}_i$ are two point on the best response function of retailer i. We prove that, there exists $F^{false}(\vec{D})$ such that:

$$J^{response}_i - J^{true}_i > 0,$$

where

$$\begin{aligned}
J^{true}_i &= \max_{X_i} \int_{\mathbb{R}^N_+} P_i((\vec{X}^{true}_{N\backslash i}, X_i), \vec{D}) dF^{true}(\vec{D}) \\
&= \int_{\mathbb{R}^N_+} P_i(\vec{X}^{true}, \vec{D}) dF^{true}(\vec{D}) \\
&= \int_{\mathbb{R}^N_+} \left[r_i B^{true}_i + v_i H^{true}_i - c_i X^{true}_i\right] + \alpha^m_i(\vec{X}^{true}, \vec{D}) dF^{true}(\vec{D}) \\
&= \int_{\mathbb{R}^N_+} \left[r_i B^{true}_i + v_i H^{true}_i - c_i X^{true}_i\right] + \alpha^d_i(\vec{X}^{true}, \vec{D}) dF^{true}(\vec{D})
\end{aligned}$$

and

$$\begin{aligned}
J^{response}_i &= \max_{X_i} \int_{\mathbb{R}^N_+} P_i((\vec{X}^{false}_{N\backslash i}, X_i), \vec{D}) dF^{true}(\vec{D}) \\
&= \int_{\mathbb{R}^N_+} P_i(\vec{X}^{response}, \vec{D}) dF^{true}(\vec{D}) \\
&= \int_{\mathbb{R}^N_+} [r_i B^{response}_i + v_i H^{response}_i - c_i X^{response}_i] + \alpha^m_i(\vec{X}^{response}, \vec{D}) dF^{true}(\vec{D}) \\
&= \int_{\mathbb{R}^N_+} \gamma_i P^c_N(\vec{X}^{response}, \vec{D}) + \alpha^d_i(\vec{X}^{false}, \vec{D}) - \alpha^f_i(\vec{X}^{false}, \vec{D}) dF^{true}(\vec{D}) \\
&= \int_{\mathbb{R}^N_+} \gamma_i P^c_N(\vec{X}^{response}, \vec{D}) + \alpha^d_i(\vec{X}^{false}, \vec{D}) - \gamma_i P^c_N(\vec{X}^{false}, \vec{D}) \\
&\quad + \left[r_i B^{false}_i + v_i H^{false}_i - c_i X^{false}_i\right] dF^{true}(\vec{D}).
\end{aligned}$$

So, we have

$$\begin{aligned} J_i^{response} - J_i^{true} &= \int_{\mathbb{R}_+^N} \Big[\gamma_i P_N^c(\vec{X}^{response}, \vec{D}) - \gamma_i P_N^c(\vec{X}^{false}, \vec{D})\Big] \\ &+ \Big[\Big[r_i B_i^{false} + v_i H_i^{false} - c_i X_i^{false}\Big] + \alpha_i^d(\vec{X}^{false}, \vec{D})\Big] \\ &- \Big[\big[r_i B_i^{true} + v_i H_i^{true} - c_i X_i^{true}\big] + \alpha_i^d(\vec{X}^{true}, \vec{D})\Big]\, dF^{true}(\vec{D}). \end{aligned} \tag{5}$$

Consider the case with sufficiently small γ_i, so we can disregards the contribution of the term $\Big[\gamma_i P_N^c(\vec{X}^{response}, \vec{D}) - \gamma_i P_N^c(\vec{X}^{false}, \vec{D})\Big]$. Recall that X_i^{true} maximizes centralized profit and, at the same time, maximizes retailer i profit given ABZ allocation, but does not necessarily maximize retailer i profit given allocation based on dual price as shown on the third line of (5). Hence, if there is an inventory level $X_i^{false} = X_i^{\#}$ that maximizes retailer i profit given allocation based on dual price as shown on the second line of (5), then $J_i^{response} - J_i^{true} > 0$. With the knowledge of X_i^{false}, retailer i may create $F^{false}(\vec{D})$, share this false information about the distribution to other retailers, and enjoy higher profit than sharing truthful information. Hence, it is not guaranteed that telling the truthful cumulative distribution function will always give highest profit to retailer i. $\square$

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