Games on Manifolds

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Abstract

We consider games played on smooth manifolds with smooth utilities. All strategically relevant information can be expressed in terms of a smooth 1-form, which we call a game form. We characterize which 1-forms correspond to game forms and which of those correspond to exact potential games. We then show that this topological setup leads naturally to generalizations of the notion of Nash equilibrium defined in terms of local deviations. These in turn suggest definitions for generalized classes of "games" with non-transitive preferences. These preferences cannot be written in terms of utility functions but can be expressed naturally using 1-forms. We examine the existence and non-existence of equilibria in this more general setting.

1 Extended abstract

Our goal is to explore the game theoretic implications of the structure of pure strategy spaces, in particular the interaction between local and global topology. This extends the work of Ekeland [1]. Since our primary interest at this point is to discover what kinds of behavior can be expected, we make no attempt to strive for the greatest degree of generality. Rather, we will simplify the setup by requiring strategy spaces to belong to the most well-behaved category of topological spaces, the compact connected smooth manifolds (without boundary). For similar reasons, we will require the utilities to be smooth (C^∞) functions. For simplicity of exposition we will consider only the case when there are two players, but all the results below generalize in an obvious way to the multiplayer case. The players' strategy spaces will be M and N, their utilities $u, v \in C^\infty(M \times N)$, and strategies will typically be denoted $m \in M$ or $n \in N$.

We will show that all strategically relevant information from both utility functions can be naturally combined into a single smooth 1-form on $M \times N$ which we call a **game form**.

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Different utilities may lead to the same game form, but only if they are strategically equivalent in the sense that the difference of the two utilities for player 1 is a function of player 2's strategy only and vice versa. Exact potential games correspond to game forms which are exact as 1-forms. We show that a game form is exact if an only if it is closed, or in other words, the condition of being an exact potential game is purely local; there can be no global obstructions.

Concentrating on the local structure of the pure strategy spaces suggests a natural generalization of the concept of Nash equilibirum, which we call a **local equilibrium**. This is a strategy profile in which no player has an incentive to deviate within a small neighborhood of his strategy. If players only consider local deviations, it makes sense to enlarge the class of games to consider interactions in which the players' preferences look locally like they come from a utility function, but may not be transitive under global deviations. The resulting **local games** cannot necessarily be expressed in terms of utility functions but again have a natural representation by 1-forms of a certain type.

At any local equilibrium both players' utility functions will be constant to first order. This condition suggests a further natural relaxation of the concept of equilibrium, which we call a **first order equilibrium**. First order equilibria are the same as points where the (local) game form vanishes. We can further relax the notion of a game accordingly by dropping the locally consistency condition and obtain the class of **first order games**. These are represented by arbitrary 1-forms.

We will show that on all pure strategy manifolds M and N there exists a game which admits no local equilibria. On the other hand, there exist certain manifolds, such as those which are orientable and have nonzero Euler characteristic, on which all games (even local and first order) have a first order equilibrium. This generalizes a theorem of Ekeland [1].

A natural followup question is what topological conditions on the strategy manifolds and utilities can ensure the existence of a Nash, local, or first-order equilibrium. We leave this problem for future work.

References

[1] Ivar Ekeland. Topologie différentielle et théorie des jeux. *Topology*, 13:375 – 388, 1974.