Unbeatable Imitation

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Abstract

We show that the simple decision rule "imitate-the-best" can not be beaten even by a dynamic relative payoff optimizer in many classes of symmetric games. These classes comprise of all symmetric 2x2 games, games for which the relative payoff function is quasiconcave or a valuation, aggregative quasiconcave quasisubmodular games, and symmetric quasiconcave zero sum games. Examples include Cournot oligopoly, rent seeking, public goods games, common pool resource games, minimum effort coordination games, arms race etc. It suggests that prior theoretical studies of imitation in those games are less ad hoc than previously thought.

Keywords: Imitate-the-best, quasisubmodularity, aggregative games, finite population ESS, relative payoff, symmetric zero-sum games.

JEL-Classifications: C72, C73, D43.

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"Whoever wants to set a good example must add a grain of foolishness to his virtue: then others can imitate and yet at the same time surpass the one they imitate - which human beings love to do." Friedrich Nietzsche

1 Introduction

Psychologists stress the role of simple heuristics or rules for human decision making under limited computational capabilities (see Gigerenzer and Selten, 2002). While such heuristics may lead to successful decisions in some particular tasks, they may be suboptimal in others. In particular, a heuristic may be easily exploited by rational opponents in strategic interactions. This paper is to challenge this view. We will show that the simple decision heuristic "imitate-the-best" is essentially unbeatable in large classes of games that are highly relevant for economics such as Prisoners' dilemma, Cournot oligopoly, rent seeking, public goods games, common pool resource games, minimum effort coordination games, arms races etc.

The idea for the paper emerged from a prior observation in an experiment. In Dürsch et al. (2009) we observed in an experimental repeated linear Cournot duopoly that subjects were on average worse off than their computer opponents when the computer was programmed to "imitate-the-best" (while they were on average better off than the computer if the computer was programmed to some other decision heuristics). A player is said to follow "imitate-the-best" if she mimics the action of the most successful player in the previous round. In Dürsch et al. (2009), we also proved for for a particular linear Cournot duopoly that a dynamic relative payoff maximizer can beat "imitate-the-best" at most by a small fixed amount. In the current paper we ask how general this finding is. We show that the result extends far beyond linear Cournot duopoly. In particular, we show that a dynamic relative payoff maximizer can beat imitate-the-best by at most a finite amount for a finite periods although the game may be repeated for an infinite number of times. The reason for pitting imitation against a dynamic relative payoff maximizer is that if imitation is essentially unbeatable by a dynamic relative payoff maximizer, then it is also essentially unbeatable by a dynamic absolute payoff maximizer or any other decision rule.

To gain some intuition for why imitation is hard to beat, consider the simple 2x2 game of "Hawk-Dove" presented in the following payoff matrix:

$$\begin{array}{c} \\ \text{Dove} \\ \text{Hawk} \end{array} \begin{array}{c} \begin{array}{cc} \text{Dove} & \text{Hawk} \end{array} \\ \left(\begin{array}{cc} 3,3 & 1,4 \\ 4,1 & 0,0 \end{array} \right) \end{array}$$

Suppose that initially the imitator starts out with playing "Dove". What should a dynamic relative payoff maximizer do? If she decides to play "Hawk", she will earn more than the imitator today but will be copied by the imitator tomorrow. From their own the imitator will stay with "Hawk" forever. If she decides to play "Dove" today, then she will earn not less than the imitator. In fact, the imitator will stay with "Dove" as long as the opponent stays with "Dove". So the dynamic relative payoff maximizer can beat the imitator at most by 1 for one period only. In Section 4.1 we show that this arguments extend to all symmetric 2x2 games.

Imitate-the-best has been previously studied theoretically and experimentally mostly in Cournot oligopoly. Vega-Redondo (1997) shows that in symmetric Cournot oligopoly with imitators, the long run outcome converges to the competitive output if small mistakes are allowed. Huck, Normann and Oechssler (1999) and Offerman, Potters and Sonnemans (2002) provide some experimental evidence. Vega-Redondo's result has been generalized to aggregative quasisubmodular games by Schipper (2003) and Alos-Ferrer and Ania (2005). In Cournot oligopoly with imitators and myopic best reply players, Schipper (2009) shows that the imitators' long run average per period payoff is strictly higher than the best reply players' long run average per period payoff.

The article is organized as follows: In the next section, we introduce the model. In Section 3 we develop some preliminary results on relative payoff games that are used in the later analysis. Our main results for various classes of games are presented in Section 4. We finish with a discussion in Section 5. All proofs not included in the main sections are collected in an appendix.

2 Model

We consider a symmetric two player game (X, π) with a (finite or infinite) set of pure strategies X and a symmetric and bounded payoff function $\pi : X^2 \longrightarrow \mathbb{R}$.[1] Since the game is symmetric, we can denote the *payoff differential* in period t between a player choosing x_t and a player choosing y_t by

$$\Delta(x_t, y_t) := \pi(x_t, y_t) - \pi(y_t, x_t).$$

We will frequently refer to the *relative payoff game* (X, Δ) derived from (X, π) by setting $\Delta(x, y) = \pi(x, y) - \pi(y, x)$. Note that by construction, $\Delta(x, y) = -\Delta(y, x)$ and hence (X, Δ) is a symmetric zero–sum game.

There are two types of players, imitators and relative payoff maximizers. In each period, the imitator follows the rule "imitate–the–best". To be precise, the imitator adopts the maximizer's action if and only if in the previous round the maximizer's payoff was strictly higher than that of the imitator.

The relative payoff maximizer, from now on call her the "maximizer", maximizes the sum of all future payoff differentials between her and the imitator,

$$D(T) := \sum_{t=0}^{T} \Delta(x_t, y_t). \tag{1}$$

[1] Our results can be generalized two n-player symmetric games with one optimizer and $n-1$ imitators if we assume that all imitators adjust at the same time (no inertia).

Since this sum may become infinite for $T \to \infty$, we assume that the maximizer evaluates her strategies according to the overtaking–criterion (see e.g. Osborne and Rubinstein, 1994, p. 139). Accordingly, a sequence of relative payoffs $\{\Delta(x_t, y_t)\}_{t=0}^{\infty}$ is strictly preferred to a sequence $\{\Delta(x'_t, y'_t)\}_{t=0}^{\infty}$ if $\lim_T \inf \sum_{t=0}^{T} (\Delta(x_t, y_t) - \Delta(x'_t, y'_t)) > 0$.

Definition 1 (Essentially Unbeatable) *We say that* imitation is essentially unbeatable *if there exists a bound $M \in \mathbb{R}_+$ such that,*

$$\lim_{T \to \infty} \sup \sum_{t=0}^{T} \Delta(x_t, y_t) \leq M. \quad (2)$$

That is, imitation is essentially unbeatable if it can be beaten only in a finite number of periods although the game between the imitator and the maximizer runs for an infinite number of periods. In some cases we can show that imitation can in fact not be beaten by more than the payoff differential from a single period.

3 Preliminary Results

Lemma 1 *Consider a game (X, π). In a dynamically optimal strategy, the maximizer will never choose an action x_t such that $\Delta(x_t, y_t) < 0$.*

Proof. Suppose to the contrary that the maximizer chooses an action x_t such that $\Delta(x_t, y_t) < 0$. Then in period $t + 1$, the imitator will not imitate her period t action. But then, she could improve her relative payoff in t by setting the same action in t as the imitator without influencing the actions of the imitator in period $t + 1$ or any other future period, contradicting the fact that her action in t was dynamically optimal. □

The relationship between a finite population ESS (Schaffer, 1988) and imitation processes is well understood (see e.g. Schipper, 2003; Alos–Ferrer and Ania, 2005). As it turns out, this concept is also relevant in the current setting where one imitator meets a relative payoff maximizer.

Definition 2 (fESS) *An action $x^* \in X$ is a* finite population evolutionary stable strategy (fESS) *of the game (X, π) if*

$$\Delta(x^*, y) \geq 0, \forall y \in X.$$

Lemma 2 (Schaffer, 1988, 1989) *Let (X, Δ) be the relative payoff game derived from the symmetric game (X, π) by setting $\Delta(x, y) = \pi(x, y) - \pi(y, x)$. Then the following statements are equivalent:*

(i) x^* is a fESS of (X, π)

(ii) x^* *is a symmetric pure strategy Nash equilibrium of* (X, Δ)

(iii) x^* *is a symmetric pure strategy saddle point* (X, Δ)

Proof. We show that (i) if and only if (ii). x^* is a Nash equilibrium of (X, Δ) if

$$\Delta(x^*, x^*) \geq \Delta(x, x^*) \text{ for all } x \in X.$$

By symmetry of payoffs,

$$\Delta(x, x^*) = -\Delta(x^*, x).$$

Hence, the inequality is equivalent to

$$\Delta(x^*, x^*) + \Delta(x^*, x) \geq 0.$$

Since $\Delta(x^*, x^*) = 0$ by definition, we have

$$\Delta(x^*, x) \geq 0$$

which is precisely the definition of fESS.

Since (X, Δ) is a 2-player zero sum game, a strategy is a Nash equilibrium if and only if it a saddle point. □

This observation will be used in the following results on the relative payoff equivalence of fESS and the existence and. We are not aware of such results in the literature.

First, the uniqueness of the fESS does not matter in the following sense that all fESS are relative payoff equivalent.

Lemma 3 *All fESS are relative payoff equivalent.*

Proof. This follows from interchangeability of Nash equilibrium actions in the two player zero sum game (X, Δ). I.e., if (x'', x'') and (x', x') are Nash equilibria, then so are (x'', x') and (x', x''). □

Next, we show the existence of fESS. More generally, consider a symmetric two-player zero sum game (X, u) with an at most countable action set X and payoff function u. Note that we can write u as matrix of payoffs of player 1 (row player).

Definition 3 (Quasiconcave) *We say that the at most countable symmetric two-player zero sum game* (X, u) *is* quasiconcave *if there exists a total order on* X *such that we can enumerate the actions by* 1 *to* m*, write the payoff matrix as* $u = (u_{r,c})_{r,c=1,\dots,m}$*, and for each column* c *there is a* k_c *with*

$$u_{1,c} \leq u_{2,c} \leq \dots \leq u_{k_c,c} \geq u_{k_c+1,c} \geq \dots \geq u_{m,c}. \tag{3}$$

Note that since (X, u) is a two player zero sum game, the payoff function u is quasiconcave in its first argument if and only if it is quasiconvex in its second argument.

Proposition 1 *Every quasiconcave at most countable symmetric two-player zero sum game has a symmetric pure strategy saddle point.*

The proof is contained in the appendix.

Note that if the the game is not symmetric, then it does not need to have a pure strategy saddle point. A counter example is presented in Radzik (1991, p. 26).

Note further that the result does not follow from standard results of existence of pure strategy equilibrium in continuous quasiconcave games with convex, compact and real-valued action (see Debreu, 1952) sets since the action space in our game is not convex. We are interested in Proposition 1 because in the later analysis we will work with finite quasiconcave games.

Note however that there is a natural connection between at most countable quasiconcave games and quasiconcave games. Consider a two player zero sum game (X, u) in which X is a convex subset of the real line. u is quasiconcave in the first argument if for all $x, x', y \in X$ and for all $\lambda \in [0, 1]$ we have

$$u(\lambda x' + (1 - \lambda)x, y) \geq \min\{u(x', y), u(x, y)\}.$$

Consider now any at most countable grid of X. Then the game in which X is replaced by the grid is at most countable quasiconcave. This is relevant in applications of our theory where often a game may be given with a continuous action space (e.g. Cournot duopoly) but we need to consider a version with a finite grid.

Corollary 1 *If the relative payoff game (X, Δ) is at most countable and quasiconcave, then a fESS exists.*

Since every symmetric 2x2 game is quasiconcave, we obtain the following corollary:

Corollary 2 *Every symmetric 2x2 game has a fESS.*

4 Unbeatable imitation

In this section we collect four propositions that demonstrate for different classes of games that imitation is essentially unbeatable.

4.1 Symmetric 2x2 Games

In this section, we extend the "Hawk-Dove" example of the introduction to all symmetric 2x2 games.

Proposition 2 *In any symmetric 2x2 game, imitation is essentially unbeatable. In fact, it can be beaten by at most the maximal one-period payoff* $\hat{\pi} := \max_{x,y} \pi(x, y)$.

Proof. Let $X = \{x, x'\}$. Consider a period t in which the maximizer achieves a strictly positive relative payoff, $\Delta(x, x') > 0$. By definition, $\Delta(x, x') \leq \hat{\pi}$. Since $\Delta(x, x') > 0$, the imitator imitates x in period $t+1$. For there to be another period in which the maximizer achieves a strictly positive relative payoff, it must hold that $\Delta(x', x) > 0$, which however yields a contradiction as $\Delta(x', x) = -\Delta(x, x')$. Thus there can only be one period in which the maximizer achieves a strictly positive relative payoff. □

Note that "Matching pennies" is not a counter example since it is not symmetric. The following example illustrates the result for symmetric 2x2 zero sum games:

Example 1 *Consider the symmetric 2x2 zero sum game given in the following payoff matrix:*

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \left(\begin{array}{cc} 0,0 & 1,-1 \\ -1,1 & 0,0 \end{array} \right) \end{array}$$

Initially, let the imitator start with B. *Then the optimizer can gain* 1 *from playing* A, *after which the imitator would play* A *as well. No further gains by the optimizer are possible.* (A, A) *is the unique saddle point and fESS of this game.*

The following example shows that inefficient non-equilibrium outcomes may emerge.

Example 2 *Consider the symmetric 2x2 game given in the following payoff matrix:*

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \left(\begin{array}{cc} 4,4 & 1,2 \\ 2,1 & 0,0 \end{array} \right) \end{array}$$

This game has a unique Nash equilibrium, (A, A), *that is efficient and in strict dominant actions. Yet, the unique fESS is* (B, B). *Initially, let the imitator start with* A. *Then the dynamic relative payoff maximizer can gain at most* 2 *for one period from playing* B. *No gains are possible if the imitator starts with* B.

4.2 Quasiconcave Relative Payoff Games

In Section 2 we showed that fESS exists in games for which relative payoffs are quasiconcave. Here we take up this property again and show that imitation is essentially unbeatable if the relative payoff game is finite and quasiconcave.

Lemma 4 *Let* (X, π) *be a symmetric game and suppose* $\Delta(x, y)$ *is quasiconcave in* x.

1. *If x is between some y and some fESS x^*, then*

$$\Delta(x, y) = -\Delta(y, x) \geq 0.$$

2. *If x^* and x^{**} are fESS, then so are all x between x^* and x^{**}.*

Proof. (1) Let x be between y and x^*. By definition of the fESS, $\Delta(x^*, y) \geq 0$. By symmetry of payoffs, $\Delta(y, y) = 0$. Part (1) of the lemma follows then by quasiconcavity.

(2) Let x^* and x^{**} both be fESS. Thus, $\Delta(x^*, x^{**}) \geq 0$ and $\Delta(x^{**}, x^*) \geq 0$. Since $\Delta(x^*, x^{**}) = -\Delta(x^{**}, x^*)$, we must have $\Delta(x^*, x^{**}) = 0$. By quasiconcavity, $\Delta(x', x^{**}) = \Delta(x', x^*) = 0$ for all x' between x^* and x^{**}. By part (1) of the Lemma, $\Delta(x', x) \geq 0$ for all $x \in X$. Hence, all x' between x^* and x^{**} are fESS as well. □

Proposition 3 *Let (X, π) be a finite game with a fESS. If $\Delta(x, y)$ is quasiconcave in x, then imitation is essentially unbeatable.*

Proof. We will show that the imitator's play must reach the set of fESS in finitely many steps, which implies that imitation is essentially unbeatable.

We need the following notation. Let E denote the (finite) set of fESS of (X, π) and $x^* := \min E$, $x^{**} := \max E$ the smallest and largest fESS, respectively. For any value $y \in X$, $y > x^{**}$, we define the following lower bound (which need not always exist)

$$l(y) := \max \{x \in X \ \text{ s.t. } \ \Delta(x, y) < 0, x < x^*\}.$$

See Figure **??** for an illustration. Likewise, for $y \in X$, $y < x^*$, we define the following upper bound

$$u(y) := \min \{x \in X \ \text{ s.t. } \ \Delta(x, y) < 0, x > x^{**}\}.$$

Let $y_0 > x^{**}$ be the arbitrary starting value of the imitator (the case of $y_0 < x^*$ follows analogously). Let us consider all possible choices of the maximizer. By Lemma 1, the maximizer will not choose any strategy x such that $\Delta(x, y_0) < 0$. If the maximizer chooses any x such that $\Delta(x, y_0) = 0$, then the maximizer will not be imitated and the situation in $t = 1$ will be identical to $t = 0$. Thus, from now on we can restrict attention to $x \in X$ such that $\Delta(x, y_0) > 0$.

We claim that $\Delta(x, y_0) > 0$ can occur only if $x < y_0$ and $l(y_0) < x$, where the second requirement is empty should $l(y_0)$ not exist. To see that we can exclude $x < l(y_0)$ note that $\Delta(l(y_0), y_0) < 0$ by definition. By Lemma 4 this also holds for all x with $x < l(y_0)$. To see that we can exclude $x \geq y_0$ note that $\Delta(y_0, y_0) = 0$. By Lemma 4, $\Delta(x, y_0) \leq 0$, for all $x > y_0$. This proves the claim.

When the maximizer chooses any x such that $l(y_0) < x < y_0$, the imitator imitates x and chooses $y_1 = x$ in the next period. Suppose $y_1 < x^*$. We claim that $u(y_1) \leq y_0$. This follows because $\Delta(y_1, y_0) > 0$ and hence $\Delta(y_0, y_1) < 0$. By quasi–concavity and the definition of fESS we have $\Delta(x', y_1) < 0$ for all $x' \geq y_0$. Hence, $u(y_1) \leq y_0$. Next, suppose that $y_1 \geq x^{**}$. In that case simply restart the procedure with the new starting value y_1.

Thus, in this first step we have strictly narrowed down the range of possible choices of the imitator in period $t = 1$ to $l(y_0) < y_1 < y_0$. Since X is finite, when we repeat this step, the imitator must reach the the set of fESS in a finite number of steps. Once the imitator has reached a fESS, he has reached a stationary state since then $\Delta(x, x^*) \leq 0$ for all x. The imitator will never leave x^* and the maximizer will never again obtain a positive relative payoff. Since there are only finitely many rounds in which $\Delta(x_t, y_t) > 0$, imitation is essentially unbeatable. □

[an example for what goes wrong w/o the tie-braking rule of the imitator is a homogenous Bertrand model. Start with p=MC for the imitator. If the maximizer chooses a $p' > MC$ and can be imitated, the maximizer can start the moeny pump by undercutting the imitator until they reach again p=MC and the start again.]

add a nice application

Example 3 (Peter's Game) *Consider a symmetric two-player game with the payoff function given by $\pi(x, y) = \frac{x}{y}$ with $x, y \in X \subseteq [1, 2]$. This game's relative payoff function is quasiconcave. Thus our result implies that imitation is essentially unbeatable.*

4.3 Quasiconcave Zero-Sum Games

Since imitation is a rigid decision heuristics, one may conjecture that it could be exploited by a sophisticated player in strictly competitive games. Yet, our result on symmetric 2x2 games suggests already that this may not be true. Example **??** illustrates this in a symmetric 2x2 zero sum game. Does this result extend beyond 2x2 zero sum games?

Example 4 (Rock-Scissors-Paper) *Consider the the symmetric 3x3 zero sum game "(R)ock-(S)cissors-(P)aper" given by the following matrix:*

$$\begin{array}{c} \\ R \\ S \\ P \end{array} \begin{array}{c} \begin{array}{ccc} R & S & P \end{array} \\ \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}$$

Clearly, if the imitator starts for instance with R, then the dynamically optimal strategy of the optimizer is the cycle S-P-R... In this way, the optimizer wins in every round and the imitator looses in every round. Over time, the payoff difference will grow without bound in favor of the optimizer.

How general is this result really? If (X, π) is a symmetric two-player zero sum game, then $\Delta = 2\pi$ since $\Delta(x, y) = \pi(x, y) - \pi(y, x) = \pi(x, y) + \pi(x, y)$.

Hence, we obtain the following corollary to Proposition 3:

Corollary 3 *If (X, π) is finite quasiconcave symmetric two player zero sum game with a pure strategy saddle point, then imitation is essentially unbeatable and the imitator's play converges to an symmetric pure strategy saddle point.*

Note that the "Rock-Scissors-Paper" game is not quasiconcave and does not posses a pure strategy saddle point. Zero-sum games with pure strategy saddle points are games that do not require any player to make the opponent uncertain through mixing. So the result may be loosely interpreted as imitation is essentially unbeatable in zero-sum games that do not require mixing.

`Is the existence of a pure strategy saddle point also necessary for essential unbeatability in zero sum games?`

4.4 Relative Payoff Valuations

In the previous sections, quasiconcavity - a "second order" property - played a crucial role for the results. Other "second order" properties such as increasing or decreasing differences are often useful for convergence of learning processes. Is imitation essentially unbeatable if Δ satisfies increasing or decreasing differences?

Definition 4 *Let X be a totally ordered set. Δ has decreasing (resp. increasing) differences on $X \times X$ if for all $x'', x', y'', y' \in X$ with $x'' > x'$ and $y'' > y'$,*

$$\Delta(x'', y'') - \Delta(x', y'') \leq (\geq) \Delta(x'', y') - \Delta(x', y'). \tag{4}$$

Δ is a valuation (or has constant differences in $X \times X$) if it has both decreasing and increasing differences.

The following observation does not only apply to any relative payoff game derived from some symmetric game but to any two-player zero sum games. We did not find it in the literature.

Lemma 5 *Let (X, Δ) be any two-player zero sum game for which X is a totally ordered set. Then the following statements are equivalent:*

(i) Δ has decreasing differences on $X \times X$,

(ii) Δ has increasing differences on $X \times X$,

(iii) Δ is a valuation.

Proof. Δ has decreasing differences on $X \times X$ if for all $x''', x'', x', x \in X$ with $x''' > x'$ and $x'' > x$,

$$\Delta(x''', x'') - \Delta(x', x'') \leq \Delta(x''', x) - \Delta(x', x).$$

Since (X, Δ) is two-player zero sum game, $\Delta(x', x) = -\Delta(x, x')$ for all $x, x' \in X$. Hence, we can rewrite this inequality

$$-\Delta(x'', x''') + \Delta(x'', x') \leq -\Delta(x, x''') + \Delta(x, x'). \tag{5}$$

Rearranging inequality (5) yields the definition of increasing differences,

$$\Delta(x'', x') - \Delta(x, x') \leq \Delta(x'', x''') - \Delta(x, x''').$$

Hence (i) if and only if (ii). (i) and (ii) are equivalent to (iii). □

Remark 1 *If Δ is a valuation and u.s.c. in the first argument, then an fESS exists.*

Proof. If Δ is a valuation and u.s.c. then the arg max correspondence is nonempty and constant. Hence a Nash equilibrium exists. □

Note that if X is finite, then Δ is trivially u.s.c.. For the following result we do not necessarily impose finiteness.

Proposition 4 *If (X, Δ) is a relative payoff game for which Δ is a valuation, then imitation is essentially unbeatable. In particular, imitation can be beaten by at most the maximal one–period profit $\hat{\pi}$ and the imitator's play converges to a fESS.*

Proof. Since Δ is a valuation, we have that for all $x'', x', x \in X$,

$$\Delta(x'', x) - \Delta(x', x) = \Delta(x'', x') - \Delta(x', x'),$$

which is equivalent to

$$\Delta(x'', x) = \Delta(x'', x') + \Delta(x', x). \tag{6}$$

By induction, (6) implies that one large step is just as profitable as *any* number of steps. Suppose three steps were optimal for the maximizer. By (6) the maximizer is no worse off by merging two of the three steps to one larger step. Applying (6) again yields the claim.

Thus, the maximizer cannot to better than jumping directly to a fESS x^* since for all $x, y \in X$,

$$\Delta(x^*, y) = \Delta(x^*, x) + \Delta(x, y) \geq \Delta(x, y),$$

where the equality follows from (6) and the inequality from the definition of fESS. Once the maximizer has chosen x^*, the imitator will follow and remain there for ever. □

Corollary 4 *Consider a relative payoff game (X, Δ) for which the relative payoff function is additive separable, i.e. $\Delta(x, y) = f(x) + g(y)$ for some functions $f, g : X \longrightarrow \mathbb{R}$. Then imitation is essentially unbeatable. In particular, imitation can be beaten by at most the maximal one–period profit $\hat{\pi} := \max_{x,y} \pi(x, y)$.*

Proof. By Topkis (1998, Theorem 2.6.4.), $\Delta(x, y)$ is additive separable on $X \times X$ if and only if $\Delta(x, y)$ has constant differences. Hence the result follows from Proposition 4. $\square$

While the assumption that the absolute payoff function has constant differences is very restrictive, it is much less restrictive to assume that the relative payoff function has constant difference.

Corollary 5 *Consider a game* (X, π) *for which the payoff function can be written as* $\pi(x, y) = f(x) + g(y) + a(x, y)$ *for some functions* $f, g : X \longrightarrow \mathbb{R}$ *and a symmetric real-valued function* $a : X \times X \longrightarrow \mathbb{R}$ *(i.e.,* $a(x, y) = a(y, x)$ *for all* $x, y \in X$*). Then imitation is essentially unbeatable. In particular, imitation can be beaten by at most the maximal one–period profit* $\hat{\pi} := \max_{x,y} \pi(x, y)$*.*

The following examples illustrate the applicability of the result:

Example 5 (Cournot Duopoly with Linear Demand) *Consider a Cournot duopoly given by symmetric payoff function by* $\pi(x, y) = x(a - x - y) - c(x)$ *with* $a > 0$*. Such games are special cases of games whose relative payoff functions are valuations. To see this, note that* $\Delta(x, y) = \pi(x, y) - \pi(y, x) = (ax - x^2 - c(x)) - (ay - y^2 - c(y))$*. Our results imply that imitation can not be beaten in any such Cournot duopoly except by at most the maximal one–period profit.*

Example 6 (Public Goods) `add references` *Consider the class of symmetric public good games defined by* $\pi(x, y) = g(x, y) - c(x)$ *where* $g(x, y)$ *is some symmetric monotone increasing benefit function and* $c(x)$ *is an increasing cost function. Usually, it is assumed that* g *is an increasing function of the the sum of provisions, that is the sum* $x+y$*. Various assumptions on* g *have been studied in the literature such as increasing or decreasing returns. In any case, this class of games is special case of games whose relative payoff functions are valuations. Our results imply that imitation can not be beaten in any such public good game except by at most the maximal one–period profit.*

Example 7 (Common Pool Resources) *Consider a common pool resource game with two appropriators. Each appropriator has an endowment* $e > 0$ *that she can invest in an outside activity with marginal payoff* $c > 0$ *or into the common pool resource.* $x \in X \subseteq [0, e]$ *denotes the optimizer's investment into the common pool resource (likewise* y *denotes the imitator's investment). The return from investment into the common pool resource is* $\frac{x}{x+y}(a(x+y) - b(x+y)^2)$*, with* $a, b > 0$*. So the symmetric payoff function is given by* $\pi(x, y) = c(e - x) + \frac{x}{x+y}(a(x + y) - b(x + y)^2)$ *if* $x, y > 0$ *and* ce *otherwise. (See Walker, Gardner and Ostrom, 1990.) This is a game whose relative payoff function is a valuation. To see this note that* $\Delta(x, y) = (c(e-x)+ax-bx^2)-(c(e-y)+ay-by^2)$*. Thus our results imply that imitation can not be beaten in any such common pool resource game except by at most the maximal one–period profit.*

Example 8 (Minimum Effort Coordination) *Consider the class of minimum effort games given by the symmetric payoff function* $\pi(x, y) = \min\{x, y\} - c(x)$ *for some cost function c (see Bryant, 1983 and Van Huyck, Battalio and Beil, 1990). Clearly, this class of games constitutes an example of games whose relative payoff functions are valuations. Thus, imitation is essentially unbeatable.*

Example 9 (Arms Race) *Consider two countries engaged in an arms race. Each player chooses a level of arms in a compact totally ordered set* X*. The symmetric payoff function is given by* $\pi(x, y) = b(x-y) - c(x)$ *where b is concave function of the difference between both players' level of arms,* $x - y$*, satisfying* $b(x - y) = -b(y - x)$*. (See Milgrom and Roberts (1990, p. 1272).) This game is an example of games whose relative function is a valuation. Imitation can not be beaten in such an arms race game except by at most the maximal one–period payoff.*

4.5 Aggregative Games

We say that (X, Π) is an *aggregative game* if it satisfies the following properties.

(i) X is a totally ordered set of actions and Z is a totally ordered set.

(ii) There exists an aggregator $a : X \times X \longrightarrow Z$ that is

- monotone increasing in its arguments, i.e. if $(x'', y'') > (x', y')$, then $a(x'', y'') > a(x', y')$, and
- symmetric, i.e., $a(x, y) = a(y, x)$ for all $x, y \in X$.

(iii) π is extendable to $\Pi : X \times Z \longrightarrow \mathbb{R}$ with $\Pi(x, a(x, y)) = \pi(x, y)$ for all $x, y \in X$.

We say that an aggregative game (X, Π) is *quasisubmodular (resp. quasisupermodular)* if Π *quasisubmodular (resp. quasisupermodular)* in (x, y) on $X \times Z$, i.e., for all $z'' > z'$, $x'' > x'$,

$$\Pi(x'', z'') - \Pi(x', z'') \geq 0 \quad \Rightarrow (\Leftarrow) \quad \Pi(x'', z') - \Pi(x', z') \geq 0 \tag{7}$$
$$\Pi(x'', z'') - \Pi(x', z'') > 0 \quad \Rightarrow (\Leftarrow) \quad \Pi(x'', z') - \Pi(x', z') > 0 \tag{8}$$

Quasisubmodularity (resp. quasisupermodularity) is implied by submodularity (resp. supermodularity). We say that an aggregative game (X, Π) is *submodular (resp. supermodular)* if Π has decreasing (resp. increasing) differences in (x, z) on $X \times Z$. I.e., for all $z'' > z'$, $x'' > x'$,

$$\Pi(x'', z'') - \Pi(x', z'') \leq (\geq) \Pi(x'', z') - \Pi(x', z'). \tag{9}$$

It is known that if an aggregative game (X, Π) is submodular (resp. supermodular), then it is quasisubmodular (resp. quasisupermodular). The converse is false.

We say that an aggregative game (X, Π) with $X \subseteq \mathbb{R}$ is *quasiconcave* if Π is quasiconcave in the first argument. I.e., for all $\lambda \in [0,1]$ and $x, x' \in X$ such that $\lambda x' + (1-\lambda)x \in X$ and all $z \in Z$ we have

$$\Pi(\lambda x' + (1-\lambda)x), z) \geq \min\{\Pi(x', z), \Pi(x, z)\}.$$

Note that Definition 2 can be adapted to aggregated games as follows. An action $x^* \in X$ is a *finite population evolutionary stable strategy (fESS)* of the aggregative game (X, Π) if

$$\Pi(x^*, a(x^*, x)) \geq \Pi(x, a(x^*, x)), \ \forall x \in X.$$

The key insight for our main result in this section is the following.

Lemma 6 *Suppose* (X, Π) *is a quasiconcave quasisubmodular aggregative game. If* x *is between some* x' *and the fESS* x^**, then*

$$\Pi(x, a(x, x')) \geq \Pi(x', a(x, x'))$$

Proof. Suppose that $x' \leq x \leq x^*$. The case $x' \geq x \geq x^*$ can be dealt with analogously. By the definition of a fESS

$$\Pi(x^*, a(x^*, x')) - \Pi(x', a(x^*, x')) \geq 0.$$

By quasiconcavity,

$$\Pi(x, a(x^*, x')) - \Pi(x', a(x^*, x')) \geq 0.$$

The result follows then by quasisubmodularity,

$$\Pi(x, a(x, x')) - \Pi(x', a(x, x')) \geq 0,$$

since $a(x^*, x') \leq a(x, x')$. □

Proposition 5 *If* (X, Π) *is a finite quasiconcave quasisubmodular aggregative game for which a unique* ***[momentan gehen wir noch von einem fESS aus]*** *fESS exists, then imitation is essentially unbeatable.*

Figure **??** illustrates three strategies of the relative profit maximizer leading to fEES that are not ruled out by Proposition 5.

Proof. We will show that from any initial action, the imitator reaches the fESS in a finite number of steps. Once reached, there are no further improvement possibilities for the maximizer by definition of the fESS. Thus, $\lim_{T \to \infty} \sup \sum_{t=0}^{T} \Delta(x_t, y_t)$ must be finite.

Step 1: Let $y_0 \in X$ be the starting action profile of the imitator. Assume that $y_0 < x^*$ (the proof for $y_0 > x^*$ works analogously). We claim that when the imitator switches to a new action $y_1 \neq y_0$, we must have that $y_1 > y_0$. Suppose by contradiction that $y_1 < y_0$. By Lemma 1, the imitators would only choose y_1 if in the previous period the maximizer chose $x = y_1$ and received a higher payoff,

$$\Delta(y_1, y_0) = \Pi\left(y_1, a(y_1, y_0)\right) - \Pi\left(y_0, a(y_1, y_0)\right) > 0 \tag{10}$$

But this contradicts Lemma 6 as $y_1 < y_0 < x^*$.

- If $y_1 = x^*$, we are done.
- If $y_0 < y_1 < x^*$, then take y_1 as the new starting action and repeat Step 1.
- Else, go to Step 2.

Step 2: We have that $y_1 > x^*$. We claim that when the imitators switches to a new action $y_2 \neq y_1$, we must have that $y_2 < y_1$. Suppose by contradiction that $y_2 > y_1$. By Lemma 1, the imitators would only choose y_2 if in the previous period the maximizer chose $x = y_2$ and received a higher payoff, $\Delta(y_2, y_1) > 0$. But this contradicts Lemma 6 as $y_2 > y_1 > x^*$. Thus $y_2 < y_1$.

- If $y_2 = x^*$, we are done.
- If $y_0 < y_2 < x^*$, then take y_2 as the new starting action and repeat Step 1.
- If $x^* < y_2 < y_1$, then take y_2 as the new starting action and repeat Step 2.

We claim that $y_2 \leq y_0$ can be ruled out. Since X is finite, the algorithm then stops after finite periods. Thus the proof of the proposition is complete once we verify this last claim.

Suppose to the contrary that $y_2 \leq y_0$. By Lemma 1, the imitators would only choose y_2 if in the previous period the maximizer chose $x = y_2$ and received a higher payoff,

$$\Delta(y_2, y_1) = \Pi\left(y_2, a(y_2, y_1)\right) - \Pi\left(y_1, a(y_2, y_1)\right) > 0.$$

By quasiconcavity, we have

$$\Pi\left(y_0, a(y_2, y_1)\right) - \Pi\left(y_1, a(y_0, y_1)\right) > 0.$$

By quasisubmodularity, we have since $a(y_0, y_1) \geq a(y_0, y_1)$,

$$\Pi\left(y_0, a(y_0, y_1)\right) - \Pi\left(y_1, a(y_0, y_1)\right) > 0.$$

But this contradicts (10) and proves the claim. □

The following examples present applications of the previous result:

Example 10 (Cournot Duopoly) *Let the symmetric payoff function be* $\pi(x, y) = xp(x, y) - c(x)$*. Schipper (2009, Lemma 1) implies that any symmetric Cournot duopoly with an arbitrary decreasing price function* p *and arbitrary cost function* c *is an aggregative quasisubmodular game. Thus Proposition 5 implies that imitation is essentially unbeatable in Cournot duopoly.*

Example 11 (Rent Seeking) *Two contestants compete for a rent* $v > 0$ *by bidding* $x, y \in X \subseteq \mathbb{R}_+$*. A player's probability of winning is proportional to her bid,* $\frac{x}{x+y}$ *and zero if both players bid zero. The cost of bidding equals the bid. The symmetric payoff function is given by* $\pi(x, y) = \frac{x}{x+y}v - x$ *(see Tullock, 1980, and Hehenkamp, Leininger and Possajennikov, 2004). This game is an aggregative quasisubmodular game (see Schipper, 2003, Example 6, and Alos-Ferrer and Ania, 2005, Example 2). Thus Proposition 5 implies that imitation is essentially unbeatable in rent seeking.*

5 Discussion

5.1 Relationships between the Results

Note that the relative payoff function of rent seeking game in Example 11 is not a valuation. Thus Proposition 5 covers games that are not covered by Proposition 4. The next example shows that also Proposition 4 covers games that are not covered by Proposition 5. Hence neither result implies the other.

Example 12 *Consider a type of public good game with the symmetric payoff function* $\pi(x, y) = x(x + y) - x$*. The relative payoff function,* Δ*, is a valuation. Rewrite* $\Pi(x, z) = xz - x$*. We have aggregative* strict super*modularity* $\frac{\partial^2 \Pi(x,z)}{\partial x \partial z} = 1$ *(assume a differentiable version for simplicity of the argument.) Hence,* Δ *being a valuation does not imply aggregate quasisubmodularity of* Π*.*

Similarly, Example 3 shows that Proposition 3 is not implied by Propositions 4 or 5. It's relative payoff function is not a valuation nor is it aggregative quasisubmodular.

Proposition 3 implies our result for symmetric 2x2 games, Proposition 2, since every symmetric 2x2 is quasiconcave.

5.2 Aggregative games

Aggregative games have been studied under various assumptions by Acemoglu and Jensen (2009), Alos-Ferrer and Ania (2005), Corchon (1994), Cornes and Harley (2005), Jensen (2009), Schipper (2003), Selten (1970) and others. This class includes many games relevant in economics such as Cournot oligopoly, Bertrand oligopoly, Public goods games, Common pool resource games, Rent seeking games, Patent races, Search games etc. Acemoglu and Jensen (2009, Section 9) provide results on the comparative statics of optimal aggregate taking strategy for a class of aggregative games consider in this paper. Symmetric optimal aggregate taking strategy was originally introduced in Possajennikov (2003) as a generalization of the competitive solution in Cournot oligopoly. He also showed connections to finite population evolutionary stability originally introduced by Schaffer (1988, 1989). Schipper (2003) generalized Vega-Redondo's (1997) result on imitation in Cournot oligopoly to quasisubmodular aggregative games. Alos Ferrer and Ania (2005) showed that finite population evolutionary stability implies the optimal aggregative taking strategy in quasisupermodular aggregative games. Leininger (2006) shows that in quasisubmodular aggregative games, finite population evolutionary stability implies stability against many mutants.

Note that quasisubmodularity in (x, z) where z is the aggregate of all players' actions is different from quasisubmodularity in (x, y) where y is the aggregate of all opponents' actions. For instance, Schipper (2009, Lemma 1) showed quasisubmodularity in (x, y) where y is the aggregate of all players' actions is satisfied in a Cournot oligopoly if the inverse demand function is decreasing. No assumptions on costs are required. It is known from Amir (1996, Theorem 2.1) that further assumptions on the costs are required if the Cournot oligopoly should be quasisubmodular in (x, y) where y is the aggregate of all opponents' actions. So quasisubmodularity in (x, y) with y being the aggregate of all players' actions appears to weaker than quasisubmodularity in (x, y) where y is the aggregate of all opponents' actions. `Proof?`

A Proofs

A.1 Proof of Proposition 1

Let (X, u) be a quasiconcave at most countable symmetric two-player zero sum game. Since (X, u) is quasiconcave and at most countable, there exists an enumeration of actions from 1 to m. For the proof, we proceed by induction on the totally ordered set of actions.

Recall that a combination of actions (k, ℓ) is a *pure strategy saddle point* of (X, u) if $u_{k,\ell}$ is the largest element in column ℓ and the smallest element in row k.

For $m = 1$ the claim is trivial.

Now let $m > 1$ and assume that there exists a pure strategy saddle point of the symmetric upper block payoff matrix $(u_{r,c})_{r,c \leq n < m}$. We will show that there exists also a pure strategy saddle point of the symmetric upper block payoff matrix $(u_{r,c})_{r,c \leq n+1 \leq m}$.

Since $(u_{r,c})_{r,c \leq n < m}$ has a pure strategy saddle point, it has a symmetric pure strategy saddle point. This follows from the interchangeable property of saddle points. If (k, ℓ) is a pure strategy saddle point of $(u_{r,c})_{r,c \leq n < m}$, then so is (ℓ, k) by symmetry. By the interchangeablity of saddle points, also (k, k) and (ℓ, ℓ) are pure strategy saddle points.

Let (k, k) be the largest symmetric pure strategy saddle point (with respect to the total order on X) of $(u_{r,c})_{r,c \leq n < m}$.

Case A: We claim that if $u_{n+1,k} \leq u_{k,k}$ then (k, k) is a symmetric pure strategy saddle point of $(u_{r,c})_{r,c \leq n+1 \leq m}$. To see this, note that since (X, u) is a symmetric two-player zero sum game, we must have $u_{k,k} = 0$, and therefore $u_{k,k} \leq u_{k,n+1}$. Hence, $u_{k,k}$ remains a largest element in column k and a smallest element in row k after adjoining row $n+1$ and column $n+1$ to $(u_{r,c})_{r,c \leq n < m}$. Thus (k, k) is a symmetric pure strategy saddle point of $(u_{r,c})_{r,c \leq n+1 \leq m}$.

Case B: Next we claim that if $n = k$ (i.e., (n, n) is the largest symmetric pure strategy saddle point of $(u_{r,c})_{r,c \leq n < m}$) and $u_{n+1,n} > u_{n,n}$, then $(n+1, n+1)$ is a pure strategy saddle point of $(u_{r,c})_{r,c \leq n+1 \leq m}$. To see this consider the payoff matrix given in Equation (11):

$$u = \begin{pmatrix} & & \vdots & & \vdots \\ \ddots & & \vdots & & \wedge \mid (3.) \\ & & u_{n,n} & \overset{(1.)}{>} & u_{n,n+1} \\ & & \wedge & & \wedge (2.) \\ \cdots & \overset{(5.)}{\geq} & u_{n+1,n} & \overset{(4.)}{>} & u_{n+1,n+1} \end{pmatrix} \tag{11}$$

We prove all inequalities numbered in the matrix in sequel:

1. Since $u_{n+1,n} > u_{n,n}$ and the fact that (X, u) is a symmetric two-player zero sum game, it follows that $u_{n,n} > u_{n,n+1}$.
2. Since $u_{n,n} = u_{n+1,n+1} = 0$ and $u_{n+1,n} > u_{n,n}$, we must have $u_{n,n+1} < u_{n+1,n+1}$.
3. From (2.) and quasiconcavity follows that $u_{n,n+1} \geq u_{r,n+1}$ for $r \leq n$.
4. From (2.) and the fact that (X, u) is a symmetric two-player zero sum game, it follows that $u_{n+1,n} > u_{n+1,n+1}$.
5. From (3.) and the fact that (X, u) is a symmetric two-player zero sum game, it follows that $u_{n+1,c} \geq u_{n+1,n}$ for $c \leq n$.

Hence $u_{n+1,n+1}$ is the largest element in column $n+1$ and the smallest element in row $n+1$. Thus $(n+1, n+1)$ is a symmetric pure strategy saddle point.

Case C: Finally, consider the case $k < n$ and $u_{n+1,k} > u_{k,k}$. Consider the payoff matrix given in Equation (12):

$$
u = \begin{pmatrix}
 & & \vdots & & \vdots & & & & \vdots \\
\ddots & & \vdots & & \vdots & & & & \wedge \shortmid (4.) \\
 & & u_{k,k} & \overset{(3.)}{=} & u_{k,k+1} & \overset{(3.)}{=} & \cdots & \overset{(1.)}{>} & u_{k,n+1} \\
 & & \shortparallel\ (2.) & & \shortparallel & & & & \\
 & & u_{k+1,k} & = & u_{k+1,k+1} & & \cdots & & \vdots \\
 & & \shortparallel\ (2.) & & & & & & \\
 & & \vdots & & & & & & \vdots \\
 & & \wedge & & & & & & \\
\cdots & \overset{(5.)}{\geq} & u_{n+1,k} & & & & \cdots & & u_{n+1,n+1}
\end{pmatrix} \tag{12}
$$

1. $u_{k,n} > u_{k,n+1}$ follows from the assumption $u_{n+1,k} > u_{k,k}$ and the fact that (X, u) is a symmetric two-player zero sum game.

2. We must have $u_{r,k} = u_{k,k}$ for $n + 1 > r \geq k$. To see this, note that if $u_{r,k} < u_{k,k}$, then we have a contradiction to quasiconcavity. If $u_{r,k} > u_{k,k}$, then we have a contradiction to (k, k) being a pure strategy saddle point.

3. From (2.) and the fact that (X, u) is a symmetric two-player zero sum game follow that $u_{k,k} = u_{k,c}$ for $n + 1 > c \geq k$.

4. Note that by (1.) $u_{k,n+1} < 0$. Since $u_{n+1,n+1} = 0$, it follows from quasiconcavity that $u_{k,n+1} \geq u_{r,n+1}$ for $k \geq r \geq 1$.

5. From (4.) and the fact that (X, u) is a symmetric two-player zero sum game follow that $u_{n+1,c} \geq u_{n+1,k}$ for $c \leq k$.

Since $(k+1, k+1)$ is not a saddle point (because (k, k) is the largest symmetric saddle point by assumption), there must exist a column c such that $u_{k+1,c} < u_{k+1,k+1}$ with either

(i) $c < k + 1$, or

(ii) $n + 1 > c > k + 1$.

Consider first case (i). Note that by (5.) in Matrix (12) we must have that $u_{n+1,c} > u_{k+1,c}$. Yet, we also have $u_{c,c} = 0 > u_{k+1,c}$. These two inequalities contradict quasiconcavity.

Consider now case (ii). Note that by (3.) in Matrix (12) we must have that $u_{k,c} = 0 > u_{k+1,c}$. Yet, we also have $u_{c,c} = 0 > u_{k+1,c}$. These two inequalities contradict quasiconcavity.

Thus, Case C leads to a contradiction.

Since Cases A, B and C exhaust all cases, we proved the induction step. This completes the proof of the theorem. □

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