# Repeated Congestion Games with Local Information

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## 1 Introduction

In congestion games considerable attention has been devoted to the inefficiency of Nash equilibria and to the relation between equilibrium costs and efficient costs (see, e.g., Roughgarden, 2005; Roughgarden and Tardos, 2007; Anshelevich et al., 2008). In particular two measures of inefficiency have been studied: the *price of anarchy*, i.e., the ratio of the worst Nash equilibrium cost to the optimal cost, and the *price of stability*, i.e., the ratio of the best Nash equilibrium cost to the optimal cost.

One motivation for this work is to study inefficiency of equilibria in *repeated* congestion game. While it is impossible to reduce the price of anarchy, since the one-shot equilibrium is also an equilibrium of the repeated game, a folk-theorem argument shows that the price of stability can often be reduced to 1 in the repeated game.

The difficulty in proving a folk theorem for congestion games resides in the fact that the monitoring is only local, i.e., players observe only the routes that they go through. Our model does not allow to use any of the various versions of the folk theorem under imperfect monitoring. Hence we explicitly define a punishment strategy that implements an efficient equilibrium.

The critical model of Braess (1968, 2005) shows a network where the addition of a new edge worsens the equilibrium cost for every player, and is used as a paradigmatic example of inefficient equilibrium in congestion networks. Holzman and Law-yone (2003) show that basically every network that has an inefficient equilibrium must contain a subnetwork of the Braess type. When the network is of Braess type the efficient equilibrium of the repeated game is subgame perfect.

Note that we try to implement *Pareto*-optimal strategy profiles, which are not necessarily *socially* optimal. The social optimum need not be individually rational, in which case, it is impossible to implement it even in the repeated game.

## 2 Model

We consider a congestion game with one source and one destination. This is described by a network, where the cost of traveling along one edge is an increasing function of the proportion of players that use that edge. The game is repeated over time and costs are discounted.

### 2.1 The one-shot game

To describe the stage game we need the following elements:

- A finite set of players $N = \{1, \ldots, n\}$.

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- A directed graph $G = (V, E)$, where $V$ is the set of vertices and $E$ is the set of edges.
- Two vertices $s, d \in V$, where $s$ is the source and $d$ is the destination.
- For each edge $e \in E$, a cost function $c_e : [0, 1] \to \mathbb{R}$.

In the one-shot congestion game each player chooses a path from $s$ to $d$, and, on each edge $e$, she pays a cost $c_e(f_e)$, where $c_e$ is a nondecreasing function and $f_e$ is the proportion of players who travel on $e$. Each player wants to minimize the total traveling cost from $s$ to $d$.

Call $\mathcal{P}$ be the non-empty set of paths from $s$ to $d$. This is the set of pure strategies in the one-shot congestion game. Given a profile of strategies $S = (S_1, \ldots, S_n) \in \mathcal{P}^N$, we define *the flow* induced by $S$ as:

- $\forall p \in \mathcal{P},\ f_S(p) = \frac{1}{n} |\{i \in N : S_i = p\}|$.
- $\forall e \in \mathcal{E},\ f_S(e) = \frac{1}{n} |\{i \in N : e \in S_i\}| = \sum_{p\colon e \in p} f_S(p)$.

The one-shot congestion game has the following strategic form:

- The set of players is $N$.
- For each player $i \in N$, the set of strategies is $\mathcal{P}$.
- The cost for player $i$ given the strategies $S = (S_1, \ldots, S_n)$ is

$$C_i(S) = \sum_{e \in S_i} c_e(f_S(e)).$$

Notice that the game is modeled in terms of costs, not payoffs, and the goal of a player is to minimize her cost. We use the standard notation $S_{-i}$ for $(S_j)_{j \neq i}$. We say that a path is *minimal* if no vertex of this path is visited more than once. Clearly a choice of a non-minimal path is a dominated strategy. Therefore we will always implicitly assume that $\mathcal{P}$ contains only minimal paths.

We recall the following usual concepts:

- A *Nash equilibrium* is a strategy profile $S^*$ such that

$$C_i(S^*) \leq C_i(S_i, S^*_{-i})\ \forall i \in N, \forall S_i \in \mathcal{P}.$$

- A *social optimum* is a strategy profile that minimizes $\sum_{i \in N} C_i(S)$.
- A *Pareto optimum* is a strategy profile $S^o$ such that there does not exist $S$ with $C_i(S) \leq C_i(S^o)$ for each $i \in N$, and $C_i(S) < C_i(S^o)$ for at least one $i \in N$.

A social optimum is a Pareto optimum. A Nash equilibrium need not be a Pareto optimum, see Braess (1968, 2005).

## 2.2 The repeated game

We consider a situation where players play the congestion game each day and observe the traffic on the road they choose. The aim of this paper is to study the equilibria of the game when players are sufficiently patient.

The repeated game is described as follows: at each stage $t$, each player $i$ chooses a path $S_i^t$. Let $S^t$ be the associated strategy profile played at stage $t$ and let $f^t = f_{S^t}$ be the associated flow. The observation of player $i$ before stage $t+1$ is $(f^t(e))_{e\in S_i^t}$, i.e. the amount of traffic on each edge that she uses. The history of player $i$ at stage $t$ is

$$h_i^t = (S_i^1, (f_e^1)_{e\in S_i^1}, \ldots, S_i^t, (f_e^t)_{e\in S_i^t}).$$

A strategy for player $i$ in the repeated game is a mapping $\sigma_i$ from the set of histories to $\mathcal{P}$. A stategy profile $(\sigma^1, \ldots, \sigma^n)$ induces a sequence of profiles $(S_\sigma^t)_{t\geq 1}$. The average discounted cost of player $i$ is

$$\mathcal{C}_i^\delta(\sigma) = \sum_{t\geq 1}(1-\delta)\delta^{t-1}C_i(S_\sigma^t),$$

where $\delta < 1$ is a discount factor.

# 3 Main results

**Theorem 3.1.** *Let $\bar{S}$ be a Pareto-optimal strategy of the stage congestion game and $S^*$ be a Nash equilibrium such that $C_i(\bar{S}) < C_i(S^*)$ for all $i \in N$. The vector $(C_i(\bar{S}))_{i\in N}$ is an equilibrium cost of the repeated game provided that the discount factor is close enough to* 1.

**Remark 3.2.** In the tradition of the folk theorem, this result shows that while it is possible that all equilibria of the one-shot game are not optimal, one may find an optimal equilibrium in the repeated game.

*Sketch of proof.* We construct a strategy profile $\sigma$ in the repeated game which induces the cost $(C_i(\bar{S}))$ and which is an equilibrium of the repeated game for high discount factor. The idea is as follows. Each player should play $\bar{S}_i$ at each stage. If the monitoring were perfect, then unilateral deviations would be deterred by the threat of playing the Nash equilibrium $S^*$ as soon as a deviation is noticed. However, since only local information is available to players, a unilateral deviation is only noticed locally. Therefore we need to devise a punishment strategy that first spreads the news about the deviation. This is achieved by asking all players who noticed a deviation to travel along several roads according to a pre-specified calendar. At some point everybody will be informed and then the Nash equilibrium $S^*$ will be played forever. □

**Theorem 3.3.** *Let the conditions of Theorem 3.1 hold. If the network is of Braess type then the vector $(C_i(\bar{S}))_{i\in N}$ is a subgame perfect equilibrium cost of the repeated game provided that the discount factor is close enough to* 1.

*Sketch of proof.* In a Braess network every deviation is immediately detected by all players, so players can play the inefficient Nash equilibrium as soon as a deviation is observed. □

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