The All-pay Auction with Non-monotonic Payoff

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Abstract

This article analyzes a two-bidder first-price all-pay auction under complete information where the winning payoff is non-monotonic in own bids. We derive the conditions for the existence of pure strategy Nash equilibria and fully characterize the unique mixed strategy Nash equilibrium when the pure strategy equilibria do not exist. Unlike the standard all-pay auction results as in Baye et al (1996) or Siegel (2009), under this non-monotonic payoff structure, the stronger bidder has two distinct mass points in his/her equilibrium mixed strategy and the equilibrium support of the weaker player is not continuous. When the bidders face common value, then in the equilibrium mixed strategy both bidders place mass points at the same point of support. The equilibrium payoff conditions stated in Siegel (2009) do not hold in case of pure strategy Nash equilibria. Possible real life applications are discussed.

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1. Introduction

All-pay auctions are a type of auction where the bidders simultaneously bid for prize(s) and pay their bids irrespective of the outcome. Because of its applicability in real life situations such as patent races, innovation tournaments, electoral contests, rent-seeking activities and legal disputes, the all-pay auction has become a popular area of research.

The basic first-price all-pay auction (where a single prize is awarded to only the highest bidder) equilibrium under complete information is fully characterized by Baye et al (1990, 1996).[1] They show that if there are unique bidders with highest and second-highest valuations for the prize, then a symmetric mixed strategy Nash equilibrium exists. For more than two bidders with the second-highest valuation, a continuum of asymmetric mixed strategy Nash equilibria exist. Also, the two highest valuation bidders randomize their bids from zero to the second-highest valuation and the other bidders bid zero. Only the highest valuation bidder earns a positive expected payoff. However, in their structure the size/ valuation of prize is not directly affected by the bid and hence the payoff is monotonically decreasing in own bids. Also, the highest bidder always wins the prize with certainty.

There are situations when the size of the prize in an all-pay auction is affected by the bid. Real life examples include the dependence of a patent's value on R&D expenditure, the relation of the amount of gain on lobbying expenses etc. Kaplan et al (2002) are the first to analyze this sort of problems. They construct an incomplete information model where the prize is separable in bidder-type. They derive conditions under which a decrease in prize value can increase bids. Kaplan et al (2003) construct a complete information model and consider 'innovation time' as the choice variable. Here a higher reward as well as a higher cost is incurred with a choice of lower time. The authors characterize equilibria under both symmetric and asymmetric valuation cases. Che and Gale (2006) model lobbying as an all-pay auction and take into account a possible cap on bidding. This structure can also be used for solving the problem of bid-dependent valuation where the choice variable positively influences the cost as well as the prize value.

Recently, Bos and Ranger (2008) and Sacco and Schmutzler (2008) analyze all-pay auctions under complete information with a specific emphasis on the bid-dependent prize valuation. These independent works are closely related to the present analysis. Bos and

[1] See also Hilman and Riley (1989)

Ranger (2008) construct an all-pay auction where the prize value is increasing in own bids in a constant-returns-to-scale fashion. They make a strong assumption that makes the winning payoff monotonically decreasing in own bids. They characterize the unique mixed strategy Nash equilibrium, which is very similar to the one obtained by Baye et al (1996). Sacco and Schumtzler (2008) construct an n-bidder model of an all-pay auction. They assume that the winning prize value is an increasing concave function of own bid minus the second highest bid and that the cost is convex. They find conditions under which a pure strategy Nash equilibrium can be obtained. However, to solve the mixed strategy equilibria, they make a strong assumption of monotonically decreasing payoff in bids. They also find equilibrium mixed strategies similar to those of Baye et al (1996).

Siegel (2009a) constructs a general family of games called 'all-pay contests'. This model provides a generic structure that incorporates the majority of the features of the previous analyses in the literature.[2] Specifically, it is an n-bidder model under complete information where the bidders posses a degree of asymmetry in terms of their prize valuations and cost functions. In addition, the bidders choose a costly 'score' (similar to a bid) that monotonically affects the prize value. Siegel (2009a) gives a generic formula for the equilibrium payoffs of this type of auctions. But, even in this generic structure, the highest bidder wins a prize with certainty and the winning payoff is assumed to be monotonically decreasing in own bids. Siegel (2009b) is an extension of Siegel (2009a) where the author characterizes the equilibrium strategies and participation rules under similar assumptions.

It is interesting to note two particular features of all the existing models: firstly, the highest bidder wins a prize with certainty- there is no possibility of no-win in any of the models; secondly, none of the models investigate the case when the winning payoff is not monotonically decreasing in own bids. Both the features of no-win and non-monotonicity of winning payoff are inconsistent with some of the real life phenomena that the all-pay auction framework is used to model. For example, in a patent race two firms can make costly investments in order to innovate a new product. But there is a chance that none of them is successful (Loury (1979)).[3] In another case, two firms can expend resources to create prototypes of a product and place the prototype for a procurement auction. There is always a chance that the demand side governing body does not like any of the prototypes

[2] Gonzalez-Diaz (2008) also constructs a general structure to unify different contests including all-pay auctions. But the payoff, again, is assumed to be monotonically decreasing in own bids.

[3] Nti (1997) incorporates the no-win possibility under a Tullock contest.

and rejects both. Interest groups may lobby a government agency to influence the details of regulations, yet regulations may be issued that do not favor any special interest. Che and Gale (2003) give examples of both kinds. Interestingly enough, under both cases the winning payoff may turn out to be non-monotonic in own bids. [4]

In the current article we construct a 2-bidder single prize all-pay auction model where there is a possibility that none of the bidders wins the prize and the prize value becomes increasing and concave in own bids. This results in non-monotonicity of the winning payoff in own bids. We find sufficient conditions for the existence of pure strategy Nash equilibria and fully characterize the unique mixed strategy Nash equilibrium when there are no pure strategy equilibria.

2. Model

2.1 Construction of the All-pay Auction with Non-monotonic Payoff

There are two bidders 1 and 2 with initial value for a prize V_1 and V_2 with $V_1 \geq V_2 > 0$. The bidders place costly bids to win the prize and lowest bidder never wins the prize. The bids are denoted as x_1 and x_2. There is a possibility that none of them wins the prize. We can explain this as a 'No success' case of innovation driven by nature or quality standard of the buying party in a procurement auction. We incorporate this by including a random threshold $\tilde{R}$ with known cumulative probability distribution $G(.)$ described by nature, where $G(0) = 0, G'(.)=g(.)>0$, and $G''(.)=g'(.)<0$. The winner is determined by the highest bid that is higher than the random threshold $\tilde{R}$. Irrespective of the result, the bidders bear cost according to the cost function $C(.)$. The cost function starts from origin, is increasing and weakly convex in own bids i.e., $C(0) = 0, C'(.)=c(.)>0$, and $C''(.)=c'(.)\geq 0$. Hence, the payoff function (neglecting a tie) is written as:

$$\pi_t(x_t, x_{-t}) = \begin{cases} V_t\text{-}C(x_t) & \text{if } \; x_t > Max(x_{-t}, \tilde{R}) \\ \text{-}C(x_t) & \text{otherwise} \end{cases} \qquad (2.1)$$

The expected prize value for the winner becomes $G(x_t)V_t\text{-}x_t$, if $x_t > x_{-t}$, where $-t$ is denoted as the bidder 'not t'. In case of a tie in asymmetric initial value ($V_1 > V_2$), if both

[4] Another example of no-win is successful technological innovations that are not marketable. For example, jetpacks and teleportation were invented long back in 1961 and 1993, but because of non-marketability issues none of them gave any profit to the inventors. See Wilson (2007) for details.

bidders bid more than the random threshold ($\widetilde{R}$), then the highest initial value bidder, i.e., bidder 1 wins the prize. In the case of common initial value ($V_1 = V_2$), such a tie is resolved by a coin toss. Given the conditions, we can rewrite the payoff function as:

$$\pi_t(x_t, x_{-t}) = \begin{cases} G(x_t)V_t - C(x_t) & \text{if } (x_t > x_{-t}) \text{ or } (x_t = x_{-t}) \text{ and } (V_t > V_{-t}) \\ G(x_t)\frac{V_t}{2} - C(x_t) & \text{if } (x_t = x_{-t}) \text{ and } (V_t = V_{-t}) \\ -C(x_t) & \text{Otherwise} \end{cases} \quad (2.2)$$

Let us call the payoff at the winning state as the winning payoff and denote the same for bidder t as $W_t(x_t)$. The losing payoff is: $L_t(x_t)$. Hence $W_t(x_t) = G(x_t)V_t$-$C(x_t)$ and $L_t(x_t) = -C(x_t)$. Denote the game as $\Gamma(1,2)$. Also, call the graph of the winning (losing) payoff in the bid-payoff space as the winning (losing) curve.

2.2 Claims about the Shape of the Winning Curve

Claim 1. The winning curves for both players start from the origin and are strictly concave.
Proof: Given the assumptions $G(0) = C(0) = 0$ we get $W_t(0)=0$, for t = 1, 2 i.e., the winning curves start from the origin. Further, note that $\frac{d^2W_t(x_t)}{dx_t^2} = G''(x_t)V_t$- $C''(x_t) < 0$ as $G''(.)<0$ and $C''(.) \geq 0$. Hence, the winning curves are strictly concave. ■

Claim 2. If $\frac{C'(0)}{G'(0)} \geq V_t$ then any winning payoff is non-positive.

Proof: The slope of the winning curve is $\frac{dW_t(x_t)}{dx_t} = G'(x_t)V_t$- $C'(x_t)$. Recall from Claim 1 that the winning curves start from the origin. If $\frac{C'(0)}{G'(0)} \geq V_t$ then starting from the origin (Claim 1) the slope of the winning curve is non-positive throughout the bid range and consequently any winning payoff is also non-positive. ■

Claim 3. If $\frac{C'(0)}{G'(0)} < V_t$, then starting from the origin the winning curve is inverted U-shaped with unique maxima and as bid increases, eventually the winning curve cuts the X-axis at a unique point and winning payoff becomes negative.

Proof: Starting from the origin (Claim 1), as $\frac{C'(0)}{G'(0)} < V_t$, the winning curve has positive slope at the origin (Claim 2). But as winning curves are strictly concave (Claim 1) slope declines as bid increases; also as $G''(.)<0$ and $C''(.) \geq 0$, eventually at some unique point $\frac{C'(x_t)}{G'(x_t)} = V_t$ (follows from the uniqueness of a maximizer of a strictly concave function) and the winning curve reaches a unique maximum. After that point, $\frac{C\prime(x_t)}{G\prime(x_t)} > V_t$ and winning curve has a strictly negative slope. As a result, as x_t increases winning curve declines and cuts the X-axis at a unique point and as x_t increases further, W_t becomes negative. ■

Claim 4. Starting with no-difference, W_1 and W_2 diverge away from each other and the difference tends to the initial value difference $(V_1 - V_2)$ as bid increases to infinity.

Proof: From the properties of $G(x)$, $(W_1 - W_2)$=$G(x)(V_1 - V_2)$. Hence, $(W_1$-$W_2)|_0$=$(V_1 - V_2)G(x)|_0 = 0$. Also, $\frac{d(W_1-W_2)}{dx_t} = G'(x_t)(V_1 - V_2)$>0 and $\frac{d^2(W_1-W_2)}{d{x_t}^2} = G''(x_t)(V_1 - V_2)$<0 . Finally, $\lim_{x\to\infty}(W_1 - W_2) = (V_1 - V_2)\lim_{x\to\infty} G(x) = (V_1 - V_2)$. ■

Claim 5. If $\frac{C\prime(0)}{G\prime(0)} < V_2$, define $x_t^{Wmax} = argmax(W_t(x_t))$; then $x_1^{Wmax} \geq x_2^{Wmax}$.

Proof: From Claim 3, $argmax(W_t(x_t))$ is the solution to the first order condition $\frac{dW_t(x_t)}{dx_t} = G'(x_t)V_t$-$C'(x_t) = 0$ or $\frac{C\prime(x_t)}{G\prime(x_t)} = V_t$. Define $K(x_t)$=$\frac{C\prime(x_t)}{G\prime(x_t)}$. Note that $\frac{d\left(\frac{C\prime(x_t)}{G\prime(x_t)}\right)}{dx_t} > 0$, hence the inverse of $K(x_t)$ exists and is also monotonically increasing function. Define $K^{-1}(.)$=$H(.)$; thus, $argmax(W_t(x_t)) = H(V_t)$. By assumption $V_1 \geq V_2$ and by construction $H(.)$ is a monotonically increasing function, hence $x_1^{Wmax} \geq x_2^{Wmax}$. ■

Claim 6. $\max W_1 \geq \max W_2$.

Proof: From Claim 5, $\max W_t = G(H(V_t))V_t - C(H(V_t))$. Hence $\frac{dmaxW_t}{dV_t} = G'(.)H'(.)V_t + G(.) - C'(.)H'(.) = H'(.)[G'(.)V_t - C'(.)] + G(.) = G(.) > 0$ as $[G'(.)V_t - C'(.)] = 0$ for maximization and $V_t > 0$ for $t = 1, 2$. Given $V_1 \geq V_2$, we confirm $\max W_1 \geq \max W_2$. ■

Claim 7. Define $\bar{x}_t = \left\{x_t \neq 0\text{: } \frac{C\prime(0)}{G\prime(0)} < V_t \;\&\; W_t(x_t) = 0\right\}$ i.e., $\bar{x}_t$ is the unique positive bid by bidder t (Claim 3) for which his/her winning payoff is zero.[5] Then $\bar{x}_1 > \bar{x}_2$.

Proof: From Claim 4, $(W_1(x) - W_2(x)) > 0 \;\forall\, x > 0$ and by definition $W_t(\bar{x}_t) = 0$. Consequently, $W_1(\bar{x}_2) > W_2(\bar{x}_2) = 0 = W_1(\bar{x}_1)$. Hence, the inverted U-shape of winning curve $W_1(.)$ (Claim 3) confirms $\bar{x}_1 > \bar{x}_2$. ■

Claims 1 through 7 characterize the shape of the winning curves. It is trivial to check the shape of the losing curve. Given the shapes of the curves, below we characterize the

[5] $\bar{x}_t$ is defined as the 'reach of player t' in Siegel (2009 a, b)

equilibria of the game. Subsections 2.3 and 2.4 deal with the initial asymmetric value case ($V_1 > V_2$) whereas Subsection 2.5 deals with the initial common value case ($V_1 = V_2$).

2.3 Characterization of Equilibria under Initial Asymmetric Values: Pure Strategy Cases

Lemma 1. An equilibrium in pure strategies for the game $\Gamma(1,2)$ exists under condition (i) $\left\{\frac{C\prime(0)}{G\prime(0)} \geq V_1\right\}$ or (ii) $\left\{\frac{C\prime(0)}{G\prime(0)} \in [V_2, V_1)\right\}$ or (iii) $\left\{\frac{C\prime(0)}{G\prime(0)} < V_2 \text{ and } (x_1^{Wmax} \geq \bar{x}_2)\right\}$. Moreover, under condition (i) there exist unique equilibrium strategies $(x_1^*, x_2^*) = (0,0)$, whereas under condition (ii) or (iii) the unique equilibrium strategies are $(x_1^*, x_2^*) = (x_1^{Wmax}, 0)$.

Proof: (i) If $\frac{C\prime(0)}{G\prime(0)} \geq V_1$ then by Claim 2 the winning payoffs are always non-positive and bidding any positive amount with positive probability ensures loss. So, in equilibrium both the bidders bid zero, i.e., $x_1^* = x_2^* = 0$.

(ii) If $\frac{C\prime(0)}{G\prime(0)} \in [V_2, V_1)$ then bidder 2's winning payoff is always non-positive and following the same logic as in (i), $x_2^* = 0$. Bidder 1's winning curve is inverted U-shaped and given bidder 2 bids 0 with certainty, bidder 1 maximizes its payoff by always bidding $x_1^* = argmax(W_1(x_1)) = x_1^{Wmax} > 0$.

Figure 2.1 PSNE case (i)

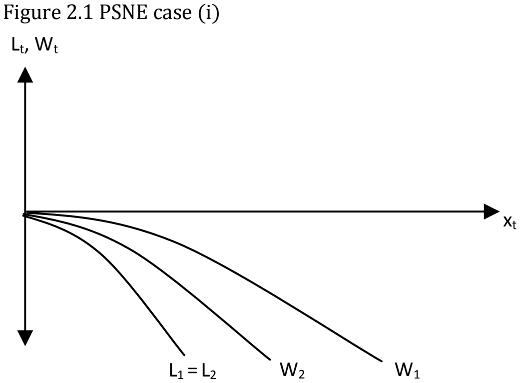


Figure 2.2 PSNE case (ii)

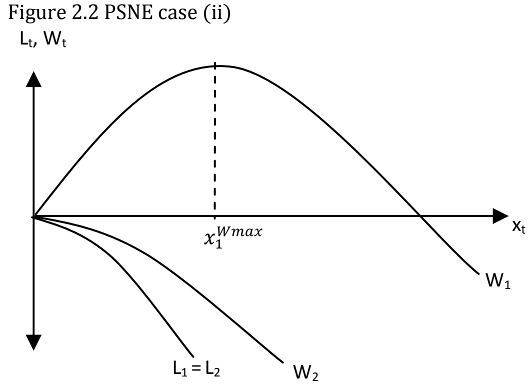
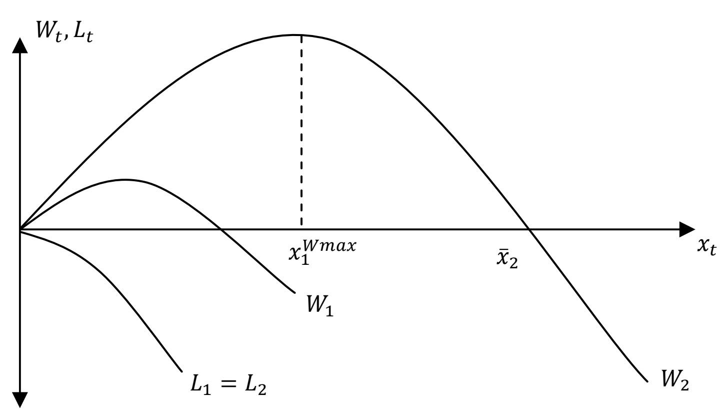


Figure 2.3 PSNE case (iii)



(iii) If $\frac{C\prime(0)}{G\prime(0)} < V_2$, then in some sufficiently small neighborhood of zero bid, the winning payoff is positive for both players. When $(x_1^{Wmax} \geq \bar{x}_2)$ then knowing bidder 2 never bids on or over $\bar{x}_2$ (as that will result in a negative payoff whereas a zero-bid ensures a zero payoff), any bid between $\bar{x}_2$ and $\bar{x}_1$ gives a sure positive payoff to bidder 1. The sure payoff reflected by the winning curve is maximized at x_1^{Wmax}. Hence, bidder 1 bids at $x_1^* = x_1^{Wmax} > \bar{x}_2$ with certainty. Knowing this, bidder 2 bids x_2^* =0 with certainty. ■

Lemma 2. If $\frac{C\prime(0)}{G\prime(0)} < V_2$ and $(x_1^{Wmax} < \bar{x}_2)$ then there exists no pure strategy Nash equilibrium for the game $\Gamma(1,2)$.[6]

Proof: A pure strategy Nash equilibrium in this game is a set of bids $\{x_1^*, x_2^*\}$ where bidder t cannot increase its payoff by deviating from x_t^* given rival bid x_{-t}^*. Suppose there exists PSNE for the game $\Gamma(1,2)$ under the stated condition in Lemma 2. Also let $\{\tilde{x}_t\}$ be the set of maximum bids among the PSNE bids. Therefore, either $\{\tilde{x}_t\}$ is a singleton set or $\tilde{x}_1 = \tilde{x}_2$.

If $\{\tilde{x}_t\}$ is singleton and $\pi_t(\tilde{x}_t) > 0$ then bidder 1 is the highest bidder as bidder 2 never bids more than $\bar{x}_2 < \bar{x}_1$ and bidding $\bar{x}_2$ gives bidder 1 a sure payoff of $W_1(\bar{x}_2)$. Because $x_1^{Wmax} < \bar{x}_2$, bidding more than $\bar{x}_2$ decreases payoff for bidder 1. But if bidder 1 bids $\bar{x}_2$ then the best response for bidder 2 would be to bid zero. Consequently, if bidder 2 bids zero, then the best response for bidder 1 is to bid at x_1^{Wmax}. As $x_1^{Wmax} < \bar{x}_2$ bidder 2 can overbid bidder 1 and make a positive payoff by bidding $x_1^{Wmax} < x_2 < \bar{x}_2$(by the continuity of the payoff functions). Hence there exists no pure strategy Nash equilibrium when $\{\tilde{x}_t\}$ is singleton and $\pi_t(\tilde{x}_t) > 0$.

If $\{\tilde{x}_t\}$ is singleton and $\pi_t(\tilde{x}_t) = 0$ then by construction the highest bidder, say bidder t, bids at $\bar{x}_t$. Bidder 1 never bids at $\bar{x}_1$ as placing a bid $x_1 \in (\bar{x}_2, \bar{x}_1)$ strictly increases payoff. Bidder 2 also never bids at $\bar{x}_2$ as bidder 1 can always place a bid $(\bar{x}_1 + \varepsilon)$ where $\varepsilon > 0$ and that will result in negative payoff for bidder 2. So, there exists no PSNE in this case.

If $\{\tilde{x}_t\}$ is singleton and $\pi_t(\tilde{x}_t) < 0$ then the highest bidder can always make a zero payoff by bidding zero; implying no PSNE. Therefore, there exists no PSNE with $\{\tilde{x}_t\}$ being singleton.

If $\{\tilde{x}_t\}$ is not singleton and $\tilde{x}_1 = \tilde{x}_2 = 0$ then bidder t can improve payoff by placing a bid of x_t^{Wmax}. If $\tilde{x}_1 = \tilde{x}_2 \neq 0$ then $\tilde{x}_t < \bar{x}_2$ as placing a bid more than or equal to $\bar{x}_2$ ensures loss

[6] We prove the non-existence of PSNE by following the same procedure as in Kaplan et al. (2003), however, in their structure the payoff is monotonically decreasing in own bids.

for bidder 2 (recall the tie breaking rule). Finally, when $\tilde{x}_t \in (0, \bar{x}_2)$ then from the tie breaking rule $\pi_1(\tilde{x}_1) = (G(\tilde{x}_1)V_1 - C(\tilde{x}_1))>0$ and $\pi_2(\tilde{x}_2) = -C(\tilde{x}_2) < 0$. But bidder 2 can always bid $(\tilde{x}_1 + \varepsilon)$ (where $\varepsilon > 0$) and earn $\pi_2(\tilde{x}_1 + \varepsilon) = W_2(\tilde{x}_1 + \varepsilon)>0>= -C(\tilde{x}_2)$. Hence, again, there exists no PSNE when $\{\tilde{x}_t\}$ is not singleton. ■

Proposition 1. A pure strategy equilibrium for the game $\Gamma(1,2)$ exists *if and only if* any of the conditions (i) $\left\{\frac{C\prime(0)}{G\prime(0)} \geq V_1\right\}$ or (ii) $\left\{\frac{C\prime(0)}{G\prime(0)} \in [V_2, V_1)\right\}$ or (iii) $\left\{\frac{C\prime(0)}{G\prime(0)} < V_2 \text{ and } (x_1^{Wmax} \geq \bar{x}_2)\right\}$ holds. Moreover, under condition (i) there exist unique equilibrium strategies $(x_1^*, x_2^*) = (0,0)$ whereas under condition (ii) or (iii) the unique equilibrium strategies are $(x_1^*, x_2^*) = (x_1^{Wmax}, 0)$.

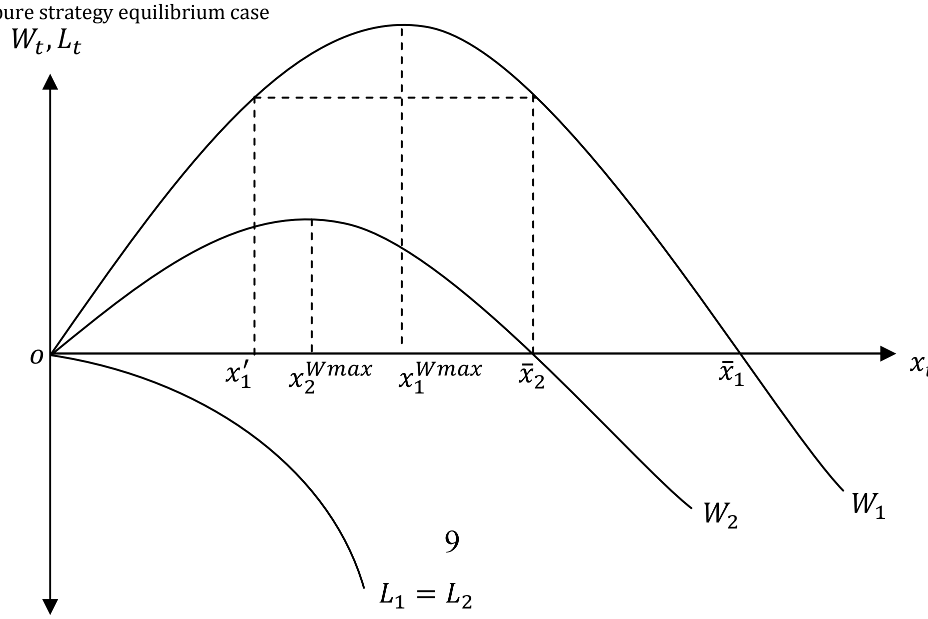
Proof : Combination of Lemma 1 and Lemma 2 proves Proposition 1. ■

It is important to note that unlike the standard all-pay auction results as in Baye et al (1996) or Siegel (2009a, b), under the non-monotonic payoff case we might end up attaining pure strategy Nash equilibria. More interestingly, here the payoff characterization results of Siegel (2009a) and strategy characterization results of Siegel (2009b) do not hold.

2.4 Characterization of Equilibria under Initial Asymmetric Values: Mixed Strategy Case

Under this section we discuss only the case of $\left\{\frac{C\prime(0)}{G\prime(0)} < V_2 \text{ and } (x_1^{Wmax} < \bar{x}_2)\right\}$, i.e., the case with no PSNE. We fully characterize the mixed strategy Nash equilibrium for the game $\Gamma(1,2)$ under this condition. This, in turn, proves the existence of equilibrium in mixed strategies that also comes directly from theorem 5 of Dasgupta and Maskin (1986). We define the No-arbitrage Bid Function of bidder t to keep bidder $-t$ indifferent as $F_t(x)$.

Figure 3. No pure strategy equilibrium case



Lemma 3. Denote $\bar{s}_t = inf\{x: F_t(x) = 1\}$ and $\underline{s}_t = sup\{x: F_t(x) = 0\}$, then $0 \leq \underline{s}_t \leq \bar{s}_t \leq \bar{x}_t$.

Proof: $0 \leq \underline{s}_t$, as by construction bid cannot be negative. $\underline{s}_t \leq \bar{s}_t$ comes from the definitions of $\underline{s}_t$ and $\bar{s}_t$ and strict inequality holds if there is no pure strategy for any of the bidders. By definition $\bar{s}_t = inf\{x: F_t(x) = 1\}$. If bidder t places a mass on any bid more than $\bar{x}_t$ then it will make a sure negative payoff for that mass and as a result the expected payoff will fall. He can always increase the expected payoff by placing that mass at 0. $\therefore \bar{s}_t \leq \bar{x}_{t.}$ ■

Lemma 4. $\bar{s}_t \leq \bar{x}_2$.

Proof: From Lemma 3, $\bar{s}_2 \leq \bar{x}_2$. At $\bar{x}_2$ bidder 1 earns a sure payoff of $W_1(\bar{x}_2)$.[7] No PSNE case implies $x_1^{Wmax} < \bar{x}_2$, hence $W_1(.)$ is falling at $\bar{x}_2$ and placing any bid above $\bar{x}_2$ with positive probability strictly reduces expected payoff for bidder 1. So, bidder 1 never places a bid above $\bar{x}_2$, i. e., $\bar{s}_1 \leq \bar{x}_2$ as well. ■

Lemma 5. Define $x_1' = \{x \neq \bar{x}_2 : W_1(x) = W_1(\bar{x}_2)\}$, then $x_1' < x_1^{Wmax} (< \bar{x}_2)$.

Proof: From Claim 3, $W_1(x)$ curve is inverted U shaped and from the definition of x_1', $W_1(x_1') = W_1(\bar{x}_2)$. Given the stated condition of no PSNE $x_1^{Wmax} < \bar{x}_2$, we must have $x_1' < x_1^{Wmax}$. Note that the strict concavity property of $G(.)$ function ensures a unique x_1'. ■

Lemma 6. $\underline{s}_1 \geq x_1'$.

Proof: Bidder 1 can always bid $\bar{x}_2$ to earn a sure payoff of $W_1(\bar{x}_2)$. $x_1' < x_1^{Wmax}$ (Lemma 5); i.e., at x_1', $W_1(.)$ is increasing. Hence, if bidder 1 bids $x_1 < x_1'$, then $W_1(x_1) < W_1(x_1') = W_1(\bar{x}_2)$. i.e., even winning the bid provides less payoff to bidder 1 than the sure payoff. Thus, bidder 1 never places a positive probability to bid less than x_1' i.e., $\underline{s}_1 \geq x_1'$. ■

Lemma 7. Support for $2 \in \{0, [x_1', \bar{x}_2]\}$.

Proof: From Lemma 6, $\underline{s}_1 \geq x_1'$, i.e., bidder 1 never places a bid less than x_1' with positive probability. Knowing this, bidder 2 also never places a positive probability of bidding in $(0, x_1')$ as that ensures a negative payoff. So, bidder 2 places positive probability of bidding in either 0 or between $[x_1', \bar{x}_2]$, i.e., bidder 2's support is in the set $\{0, [x_1', \bar{x}_2]\}$. ■

Lemma 8. The possible equilibrium payoff of bidder 1, $\pi_1^* \geq W_1(\bar{x}_2) > 0$ and the possible equilibrium payoff of bidder 2, $\pi_2^* \geq 0$.

[7] $W_1(\bar{x}_2)$ is the 'power' of bidder 1 as in Siegel (2009 a, b).

Proof: The sure payoff of bidder 1 is $W_1(\bar{x}_2) > 0$. Therefore, if the expected payoff of bidder 1 is not at least as high as $W_1(\bar{x}_2)$, then it is not an equilibrium. So, $\pi_1^* \geq W_1(\bar{x}_2) > 0$. Similarly, bidder 2 can always earn a zero payoff by not submitting any bid, hence $\pi_2^* \geq 0$. ■

Lemma 9. $\exists t \in \{1,2\}$ such that $\underline{s}_t \leq \underline{s}_{-t}$ and $F_{-t}(\underline{s}_t) = 0$.

Proof: Suppose not, i.e., $\nexists t \in \{1,2\}$: $\underline{s}_t \leq \underline{s}_{-t}$ & $F_{-t}(\underline{s}_t) = 0$. Then $\forall t \in \{1,2\}$ either (i) $\underline{s}_t \leq \underline{s}_{-t}$ & $F_{-t}(\underline{s}_t) > 0$ or (ii) $\underline{s}_t > \underline{s}_{-t}$ & $F_{-t}(\underline{s}_t) = 0$ or (iii) $\underline{s}_t > \underline{s}_{-t}$ & $F_{-t}(\underline{s}_t) > 0$. Cases (i) and (ii) cannot be true from the definition of $\underline{s}_t$ and case (iii) cannot be simultaneously true for bidder 1 and 2. Hence we arrive at a contradiction. In consequence, $\exists t \in \{1,2\}$ such that $\underline{s}_t \leq \underline{s}_{-t}$ and $F_{-t}(\underline{s}_t) = 0$. ■

Lemma 10. $\exists k \in \{1,2\}\ s.t. \pi_k^*$=0.

Proof: Suppose not. Then at $\underline{s}_k$, $\pi_k^*|_{\underline{s}_k} > 0$ i.e., $\underline{s}_k > 0$ (as $\pi_1^* > 0$ and $\pi_2^* \geq 0$ from Lemma 8). But from Lemma 9, if $\underline{s}_t \leq \underline{s}_{-t}$ then $F_t(\underline{s}_{-t}) = 0$ i.e., $\pi_k^*|_{\underline{s}_k} < 0$ for some $k \in \{1,2\}$: a contradiction. Hence, we must have some $k \in \{1,2\}$ such that π_k^*=0. ■

Lemma 11. π_2^*=0 i.e., k=2 and $\underline{s}_2$=0.

Proof: Combining Lemma 8: $\pi_1^* > 0$ and Lemma 10: $\exists k \in \{1,2\}\ s.t. \pi_k^*$=0 we conclude π_2^*=0. Combining Lemma 9 with π_2^*=0 and the fact that bidder 1 must win with positive probability over the whole support to attain $\pi_1^* > 0$ we must have $\underline{s}_2$=0. ■

Lemma 12. $\bar{s}_1 = \bar{x}_2$ and $\pi_1^* = W_1(\bar{x}_2)$.

Proof: From Lemma 4, $\bar{s}_1 \leq \bar{x}_2$. Suppose $\bar{s}_1 < \bar{x}_2$ then bidding any $x_2 \in (\bar{s}_1, \bar{x}_2)$ ensures bidder 2 a strictly positive payoff; which is a contradiction with Lemma 11. Also, $\bar{s}_1 = \bar{x}_2$ implies $\pi_1(\bar{s}_1) = W_1(\bar{x}_2)$. Hence, in equilibrium $\pi_1^* = W_1(\bar{x}_2)$ throughout the support. ■

Lemma 13. $\bar{s}_2 = \bar{x}_2$.

Proof: From Lemma 4, $\bar{s}_2 \leq \bar{x}_2$ and from Lemma 12, $\bar{s}_1 = \bar{x}_2$. Suppose $\bar{s}_2 < \bar{x}_2$ then because $W_1(.)$ is decreasing at $\bar{x}_2$, placing any bid $x_1 \in [\bar{s}_2, \bar{x}_2)$ ensures bidder 1 a sure payoff of $W_1(x_1) > W_1(\bar{x}_2)$: a contradiction with Lemma 12. Hence $\bar{s}_2 = \bar{x}_2$. ■

Lemma 14. $F_2(\underline{s}_1) = F_2(0)$.

Proof: Suppose not. Then bidder 2 places a positive probability of bidding in the semi-open interval $(0, \underline{s}_1]$. But that ensures a negative payoff which is contradictory to Lemma 11. ■

Lemma 15. If $\alpha_t(s)$ is the amount of mass bidder t places at point s, then $\alpha_2(0) \in (0,1)$.

Proof: From Lemma 14, $\text{Prob}(x_2 < \underline{s}_1) = \alpha_2(0)$. $\therefore \pi_1(\underline{s}_1) = \alpha_2(0)G(\underline{s}_1)V_1 - C(\underline{s}_1)$. Hence, if $\alpha_2(0) = 0$ then $\pi_1(\underline{s}_1) = -C(\underline{s}_1) < W_1(\bar{x}_2)$ and if $\alpha_2(0) = 1$ then bidding any $x_1 \in (x_1', \bar{x}_2)$ ensures bidder 1 a payoff $W_1(x_1) > W_1(x_1') = W_1(\bar{x}_2)$ both of which are contradictory with Lemma 12. $\therefore \alpha_2(0) \in (0,1)$. ■

Lemma 16. $\alpha_1(\underline{s}_1) \in (0,1)$.

Proof: From Lemma 6, $\underline{s}_1 > 0$ and from Lemma 15, $\alpha_2(0) > 0$. If $\alpha_1(\underline{s}_1) = 0$, then at $\underline{s}_1$ bidder 2 loses with certainty and payoff of bidder 2 becomes $-C(\underline{s}_1) < 0$; and if $\alpha_1(\underline{s}_1) = 1$ then for any small $\varepsilon > 0$, bidding $(\underline{s}_1 + \varepsilon)$ gives bidder 2 a sure payoff of $W_2(\underline{s}_1 + \varepsilon) > 0$: both of which are contradictory with Lemma 11. $\therefore \alpha_1(\underline{s}_1) \in (0,1)$. ■

Lemma 17. A *No-arbitrage Bid Function* (NBF) of bidder 1 to keep bidder 2 indifferent is: $F_1(s) = \frac{C(s)}{V_2 G(s)}$, and a No-arbitrage Bid Function of bidder 2 to keep bidder 1 indifferent is: $F_2(s) = \frac{C(s)+W_1(\bar{x}_2)}{V_1 G(s)}$.

Proof: To keep bidder 1 indifferent, bidder 2 places the bid function $F_2(.)$ in a way such that $F_2(s)V_1G(s) - C(s) = \pi_1^* = W_1(\bar{x}_2)$. Solving for $F_2(.)$ yields the NBF of bidder 2: $F_2(s) = \frac{C(s)+W_1(\bar{x}_2)}{V_1 G(s)}$. Similarly, bidder 1 places the bid function $F_1(.)$ in a way such that $F_1(s)V_2G(s) - C(s) = \pi_2^* = 0$. Solving for $F_1(.)$ yields NBF of bidder 2 $F_1(s) = \frac{C(s)}{V_2 G(s)}$. ■

Lemma 18. $\lim_{s\to 0} F_1(s), F_1(x_1'), F_1(\bar{x}_2) < 1$.

Proof: Using L'Hospital rule: $\lim_{s\to 0} F_1(s) = \frac{\lim_{s\to 0}\frac{dC(s)}{ds}}{\lim_{s\to 0}\frac{d(V_2G(s))}{ds}} = \frac{C'(0)}{V_2 G'(0)} < 1$ as $\frac{C'(0)}{G'(0)} < V_2$.

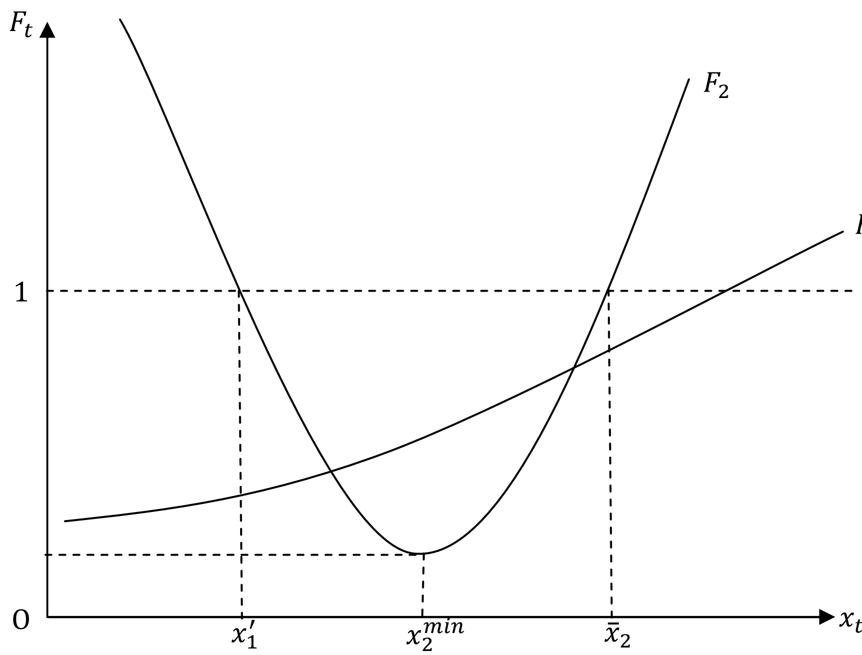
Also, $F_1(x_1') = \frac{C(x_1')}{V_2 G(x_1')} = \frac{C(x_1')}{\{V_2G(x_1') - C(x_1')\} + C(x_1')} = \frac{x_1'}{W_1(x_1') + C(x_1')} < 1$ as $W_1(x_1') = W_1(\bar{x}_2) > 0$.

And $F_1(\bar{x}_2) = \frac{C(\bar{x}_2)}{V_2 G(\bar{x}_2)} = \frac{C(\bar{x}_2)}{\{V_2G(\bar{x}_2) - C(\bar{x}_2)\} + C(\bar{x}_2)} = \frac{C(\bar{x}_2)}{W_1(\bar{x}_2) + C(\bar{x}_2)}$ <1 as $W_1(\bar{x}_2) > 0$. ■

Lemma 19. $F_1(s)$ is monotonically increasing in the closed interval $[x_1', \bar{x}_2]$.

Proof: From Lemma 18, $\frac{1}{F_1(x_1')} = \frac{W_1(x_1') + C(x_1')}{C(x_1')} = 1 + \frac{W_1(\bar{x}_2)}{C(x_1')}$, and similarly $\frac{1}{F_1(\bar{x}_2)} = 1 + \frac{W_1(\bar{x}_2)}{C(\bar{x}_2)}$. We know $W_1(x_1') = W_1(\bar{x}_2)$. And $\bar{x}_2 > x_1'$ implies $C(\bar{x}_2) > C(x_1')$, hence we obtain $\frac{1}{F_1(x_1')} > \frac{1}{F_1(\bar{x}_2)}$, i.e., $F_1(\bar{x}_2) > F_1(x_1')$. If there exists no extreme point of $F_1(.)$ within the open interval $(x_1', \bar{x}_2)$ then it means that $F_1(s)$ is monotonically increasing in $(x_1', \bar{x}_2)$. In any extreme point of $F_1(s)$, $\frac{\mathrm{dF_1(s)}}{\mathrm{ds}} = \frac{\mathrm{V_2G(s)C'(s) - C(s)V_2G'(s)}}{\mathrm{[V_2G(s)]^2}} = 0$, i.e., $[\mathrm{G(s)C'(s) - C(s)G'(s)}] = 0$. But, $[\mathrm{G(s)C'(s) - C(s)G'(s)}]$ is a strictly upward rising curve from origin.[8] Hence there is no solution except origin for $[\mathrm{G(s)C'(s) - C(s)G'(s)}] = 0$, i.e., there exists no extreme point for $F_1(s)$ within the interval$(x_1', \bar{x}_2)$. Thus, $F_1(s)$ is monotonically increasing in $[x_1', \bar{x}_2]$. ■

Figure 4. No-arbitrage Bid Function of the bidders



Lemma 20. $F_2(s)$ starts from infinity, monotonically decreases to 1 at $s = x_1'$, reaches unique minimum within the open interval $(x_1', \bar{x}_2)$ and then monotonically increases to 1 at $s = \bar{x}_2$.

Proof: From Lemma 17, $F_2(s) = \frac{C(s)+W_1(\bar{x}_2)}{V_1G(s)}$. $\therefore F_2(0) = \infty$. And $F_2(x_1') = \frac{C(x_1')+W_1(\bar{x}_2)}{V_1G(x_1')} = \frac{C(x_1')+W_1(x_1')}{V_1G(x_1')} = \frac{C(x_1')+\{V_1G(x_1')-C(x_1')\}}{V_1G(x_1')} = 1$. Also, $F_2(\bar{x}_2) = \frac{C(\bar{x}_2)+W_1(\bar{x}_2)}{V_1G(\bar{x}_2)} = \frac{C(\bar{x}_2)+\{V_1G(\bar{x}_2)-C(\bar{x}_2)\}}{V_1G(\bar{x}_2)} = 1$. If we prove that $F_2(.)$ is decreasing at x_1' then there will be at least one minimum point of $F_2(.)$ in the open interval $(x_1', \bar{x}_2)$. Note that $\frac{\mathrm{dF_2(s)}}{\mathrm{ds}} = \frac{V_1G(s)C'(s)-\{C(s)+W_1(\bar{x}_2)\}V_1G'(s)}{[V_1G(s)]^2}$. Hence $Sign\left(\frac{\mathrm{dF_2(s)}}{\mathrm{ds}}\right) = Sign\big(G(s)C'(s) - \{C(s) + W_1(\bar{x}_2)\}G'(s)\big)$. At point x_1', it can be shown that $Sign\left(\frac{\mathrm{dF_2(s)}}{\mathrm{ds}}|_{x_1'}\right) = Sign\big(C'(x_1') - V_1G'(x_1')\big) = Sign\left(-\frac{\mathrm{dW_1(s)}}{\mathrm{ds}}|_{x_1'}\right) < 0$ as $\mathrm{W_1(.)}$ is upward rising at point x_1' (Claim 3 and Lemma 5). Thus $F_2(.)$ is decreasing at x_1' and consequently, there exists at least one minimum point of $F_2(.)$ in the open interval $(x_1', \bar{x}_2)$.

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[8] $[\mathrm{G(s)C'(s) - C(s)G'(s)}]|_0 = 0$, and $\frac{\mathrm{d[G(s)C'(s) - C(s)G'(s)]}}{\mathrm{ds}} = \mathrm{G(s)C''(s) - C(s)G''(s)} > 0 \;\; \forall \mathrm{s} > 0$.

Now, if we show that the minimum is unique then we will prove that (i) $F_2(.)$ decreases from infinity to 1 at x_1' and (ii) $F_2(.)$ the minimum in the interval $(x_1', \bar{x}_2)$. At a minimum, $\frac{dF_2(s)}{ds} = 0$, which implies $\frac{\left(G(s)C'(s) - C(s)G'(s)\right)}{G'(s)} = W_1(\bar{x}_2)$. Here RHS is a positive constant whereas LHS is an upward rising curve from origin.[9] Thus there exists unique solution for $\frac{\left(G(s)C'(s) - C(s)G'(s)\right)}{G'(s)} = W_1(\bar{x}_2)$, i.e., there exists unique minimum for $F_2(.)$. Because $F_2(x_1') = F_2(\bar{x}_2) = 1$ and $F_2(s)$ is decreasing at x_1', $\therefore \operatorname{argmin}\left(F_2(s)\right) = x_2^{min} \in (x_1', \bar{x}_2)$. ■

It is clear that the No-arbitrage Bid Functions are not strategies. In particular, $F_2(s)$ is not nondecreasing. However, following Osborne and Pitchick (1986) and Deneckere and Kovenock (1996), the NBFs remain the basis for the construction of equilibrium. Let $IF_2(s) = \underset{x \geq s}{\operatorname{Inf}}\left(F_2(x)\right)$ be the nondecreasing floor of $F_2(s)$. $IF_2(s)$ equals $F_2(s)$ except the interval $[0, x_2^{min})$. Then the strategy $Q_2(s) = \begin{cases} IF_2(s) & \text{for } (s < \bar{x}_2) \\ 1 & \text{for } (s \geq \bar{x}_2) \end{cases}$ is an equilibrium strategy for bidder 2. Note that $Q_2(s)$ is nondecreasing, non-negative, right continuous and is less than or equal to 1 for all s; hence, $Q_2(s)$ is a strategy. When bidder 2 does not bid according to $Q_2(s)$, it earns a strictly negative payoff . Given Q_2, if bidder 1 were indifferent between all bids in the interval $(0, \bar{x}_2]$, then F_1would be an equilibrium strategy, since it makes bidder 2 indifferent between all prices in the interval, and earns a strictly lower payoff otherwise. However, since Q_2 is strictly less than F_2 over the interval $[0, x_2^{min})$, bidder 1 will attach zero probability to those set of bids. Since bidder 1 must set the strategy that keeps bidder 2 indifferent in the points of support, it will place a mass point at x_2^{min}, the size of which equals $F_2\left(x_2^{min}\right)$. Finally, bidder 1 will place zero probability to bid more than $\bar{x}_2$, hence it must place another mass point at $\bar{x}_2$ with a size$\left(1 - F_2(\bar{x}_2)\right)$. Hence the strategy of bidder 1 is: $Q_1(s) = \begin{cases} 0 & \text{for } (s < \bar{x}_2) \text{ and} \left(F_2(s) > Q_2(s)\right) \\ F_1(s) & \text{for } (s < \bar{x}_2) \text{ and} \left(F_2(s) = Q_2(s)\right) \\ 1 & \text{for } (s \geq \bar{x}_2) \end{cases}$. It is also easy to check that $Q_1(s)$ is nondecreasing, non-negative, right continuous, less than or equal to 1 for all s; and hence, is a strategy. In the following proposition we show that $Q_1(x)$ and $Q_2(x)$constitute the unique mixed strategy Nash equilibrium of this game.

[9] Note that $\frac{\left(G(s)C'(s) - C(s)G'(s)\right)}{G'(s)}|_0 = 0$ and $\frac{d\left[\left(G(s)C'(s) - C(s)G'(s)\right)/G'(s)\right]}{ds} = \frac{G(s)\left[G'(s)C''(s) - C'(s)G''(s)\right]}{[G'(s)]^2} > 0$

Proposition 3. The unique mixed strategy Nash equilibrium of the game $\Gamma(1,2)$ is characterized by the CDF pair $Q_1^*(s)$ and $Q_2^*(s)$: For bidder 1

$$Q_1^*(s) = \begin{cases} 0 & for\ s < x_2^{min} \\ \frac{C(s)}{V_2 G(s)} & for\ s \in [x_2^{min}, \bar{x}_2) \\ 1 & for\ s \geq \bar{x}_2 \end{cases}$$

i.e., there are two atoms: at x_2^{min} with a mass of size $\alpha_1(x_2^{min}) = \frac{C(x_2^{min})}{V_2 G(x_2^{min})}$ and at $\bar{x}_2$ with a mass of size $\alpha_1(\bar{x}_2) = \left(\frac{W_2(\bar{x}_2)}{V_2 G(\bar{x}_2)}\right)$. And for bidder 2

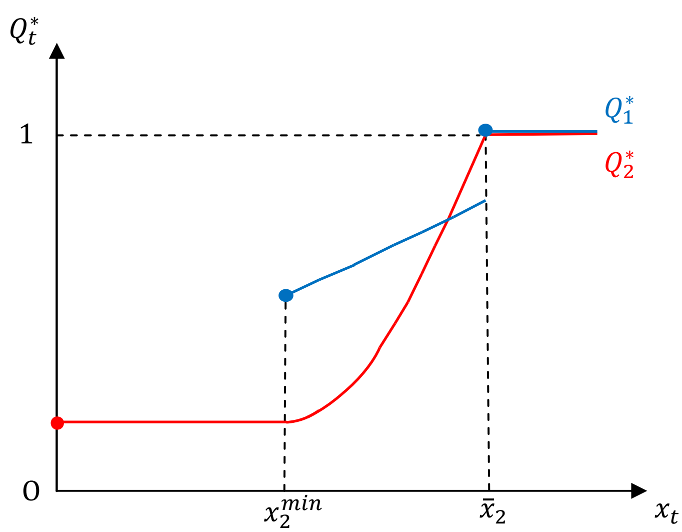
$$Q_2^*(s) = \begin{cases} \dfrac{C(x_2^{min}) + W_1(\bar{x}_2)}{V_1 G(x_2^{min})} & for\ s \leq x_2^{min} \\ \dfrac{C(s) + W_1(\bar{x}_2)}{V_1 G(s)} & for\ s \in [x_2^{min}, \bar{x}_2] \\ 1 & for\ s \geq \bar{x}_2 \end{cases}$$

i.e., there is an atom at 0 with the size of the mass: $\alpha_2(0) = \frac{C(x_2^{min}) + W_2(\bar{x}_2)}{V_1 G(x_2^{min})}$.

Proof: We prove this proposition into two parts. First, we conclude that the pair $\{Q_1(s), Q_2(s)\}$ indeed characterizes an equilibrium. Then we show that the equilibrium is unique.

It is easy to show that $Q_1^*(s) = \mathrm{Q}_1(\mathrm{s})$ and $Q_2^*(s) = \mathrm{Q}_1(\mathrm{s})$. Therefore, from the previous discussion, $Q_1^*(s)$and $Q_2^*(s)$ are strategies and are also best response to each other. Hence, the strategy pair $\{Q_1^*(s) Q_2^*(s)\}$ characterize a mixed strategy Nash equilibrium for the game $\Gamma(1,2)$. The diagrammatic representation of the equilibrium is described in Figure 5.

Figure 5. Equilibrium distribution functions



Now, if we show that the equilibrium support is unique, then the uniqueness of the equilibrium will also be proved. It is clear that the supports of equilibrium distribution coincide, and are equal to the interval $[x_2^{min}, \bar{x}_2]$. In addition, bidder 2 has a masspoint at 0. If $s \in \text{Support}(Q_t^*)$ then for any $F_t \neq Q_t^*$, $\pi_t(F_t(s, \pi_{-t}^*)) < \pi_t^*$. Also, if $\pi_t(F_t(s, \pi_{-t}^*)) > \pi_t^*$, then $s \notin \text{Support}(Q_t^*)$. Hence the support is unique and so is the equilibrium mixed strategies. ■

There are two important features of the equilibrium. First, unlike the standard all-pay auction equilibrium where the high value bidder places no atom, the bidder places two atoms at the two extreme points in its support. Also, the low value bidder's support has a discontinuous point at zero. Although the equilibrium distributions are different from the standard all-pay auction, the equilibrium payoffs of the bidders are similar to the standard case and resemble the payoff characterization results of Siegel (2009a). However, because of the possibility of no-reward, the expected payoff is lower than that of the standard case.

2.5. Characterization of Equilibria under Initial Common Value ($V_1 = V_2 = V$) Case

In the case of initial common value all-pay auction with non-monotonic payoff, define $\bar{x} = \left\{x \neq 0: \frac{C\prime(0)}{G'(0)} < V \ \& \ W(x) = 0\right\}$. It can easily be shown that for $\frac{C\prime(0)}{G'(0)} \geq V$ there exists unique PSNE (0,0). Following similar analyses as in section 2.2 to 2.4, under the case $\frac{C\prime(0)}{G'(0)} < V$ we derive the following proposition. The proof is obvious and is omitted.

Proposition 4. CDFs of the unique mixed strategy Nash equilibrium strategies $Q(s)$ for the initial common value case of the game $\Gamma(1,2)$ is $Q^*(s) = \begin{cases} \frac{C'(0)/G'(0)}{V} & for\ s = 0 \\ \frac{C(s)}{VG(s)} & for\ s \in (0, \bar{x}] \\ 1 & for\ s \geq \bar{x} \end{cases}$

Figure 6.1 Common Value Payoff functions

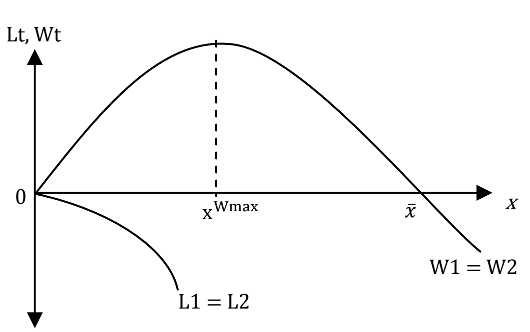
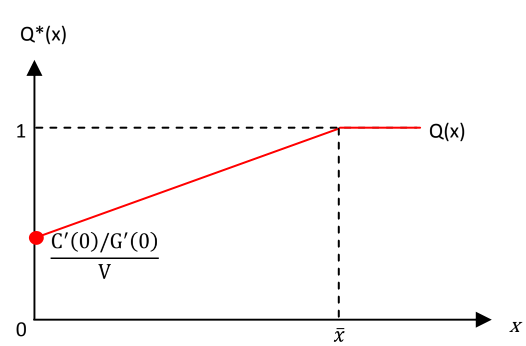


Figure 6.2 Common Value Equilibrium CDFs



i.e., the common support is $[0, \bar{x}]$ and unlike the standard all pay auction results, in the equilibrium both the bidders place the same amount of mass $\left(\frac{C\prime(0)}{VG\prime(0)}\right)$ at 0. Both bidders earn zero equilibrium payoffs.

In the standard 2-bidder all-pay auctions, because the winning payoff is positive at a zero-bid and is monotonically decreasing, bidders do not place atoms at the same point in equilibrium. For example, under Baye et al (1996) structure, if both bidders place mass points at zero, then shifting mass to a positive bid is strictly dominant strategy for both the bidders. In the current case, the winning curve starts from the origin and is continuous. Hence, if bidder *-t* shifts a mass of $\varepsilon > 0$ above zero, then its marginal payoff is zero.

2.6. Overall Characterization of Equilibria

Theorem. Propositions 1 to 4 fully characterize the equilibria for the all-pay auction with non-monotonic payoff described by the game $\Gamma(1,2)$.

3. Discussions

This study is one of the first attempts to analyze the all-pay auctions with bid-dependent prize schemes, where the winning payoff is not monotonic. We fully characterize the equilibrium and show that the equilibrium strategies are strikingly different from the standard all-pay auction results. The most useful results are the conditions for the existence of pure strategy equilibria, the existence of multiple mass points in the initial high value bidder's equilibrium strategy, and the common mass point in the initial common value case. The results indicate that under pure strategy equilibria, the payoff characterization results of Siegel (2009a) do not hold. It also indicates that the monotonocity of the payoff is not necessary for the existence of equilibrium. If the winning payoff eventually becomes negative for ever, then it is sufficient to ensure equilibrium in this structure.

This area is of high interest as this resembles real life situations such as patent race, procurement auction etc. The obvious ideas for further research would be to extend the model to n-bidders, to analyze the effects of change in initial prize value, using a more generic G(.) function, to show the effects of caps on bidding and to design experiments.

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