Generalized projection dynamics in evolutionary game theory

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January 30, 2009

Abstract

We introduce the ray-projection dynamics in evolutionary game theory by employing a ray projection of the relative fitness (vector) function both locally and globally. By global (local) ray projection we mean a projection of the vector (close to the unit simplex) unto the unit simplex along a ray through the origin. For these dynamics, we prove that every interior evolutionarily stable strategy is an asymptotically stable fixed point, and that every strict equilibrium is an evolutionarily stable state and an evolutionarily stable equilibrium.

Then, we employ these projections on a set of functions related to the relative fitness function which yields a class containing e.g., best-response, logit, replicator, and Brown-Von-Neumann dynamics.

Key words: evolutionary games, ray-projection dynamics, dynamic and evolutionary stability.
JEL-Codes: A12; C62; C72; C73; D83

1 Introduction

We introduce a class of dynamics to model evolutionary changes in game theory. We draw inspiration from rather early literature on price-adjustment processes as introduced by Samuelson [1941, 1947] and subsequent results by Arrow & Hurwicz [1958, 1960a,b] and Arrow, Block & Hurwicz [1959]. Our second source of inspiration is recent work featuring projection dynamics, e.g., Lahkar & Sandholm [2008], Hofbauer & Sandholm [2008].

In the latter papers it is shown that if a so-called stable game possesses an interior evolutionarily stable state (*ESS*, Maynard Smith & Price [1973]), the so-called projection dynamics converge to it from any starting point. In fact, the proofs imply that for these dynamics every interior evolutionarily stable state is an evolutionarily stable equilibrium (*ESE*, Joosten [1996]),

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i.e., trajectories converge to the equilibrium and along any such trajectory the Euclidean distance to it decreases strictly in time.

In the literature on price-adjustment processes, a similar result[1] was established about half a century ago, see e.g., Uzawa [1961], Negishi [1962]. If the Weak Axiom of Revealed Preferences (*WARP,* Samuelson [1938]) holds, the price-adjustment process, or tâtonnement, of Samuelson [1947] given by

$$\dot{x} = \frac{dx}{dt} = f(x) \text{ for all } x \in \mathbb{P} = \mathbb{R}_{+}^{n+1} \backslash \{0^{n+1}\},$$

converges to an economic equilibrium. Here, x denotes a vector of prices for $n+1$ commodities in the price space $\mathbb{P} = \mathbb{R}_{+}^{n+1} \backslash \{0^{n+1}\}$, 0^{n+1} denotes the $n+1$-vector of zeros, and the (vector) function $f : \mathbb{P} \to \mathbb{R}^{n+1}$ is an excess demand function. An excess demand function gives for each commodity the difference between its demand and supply given a price for each commodity. An equilibrium is a price vector for which there exists no positive excess demand for any commodity, i.e., y is an equilibrium iff $f(y) \leq 0^{n+1}$.

Our basic idea is to project a(ny) trajectory of Samuelson's tâtonnement process in $\mathbb{P}$ on the n-dimensional unit simplex such that every point of the original is projected on the unit simplex along the ray through this point and the origin. By the convergence result of the unrestricted dynamics under *WARP* mentioned, it follows that the projected dynamics also converge to an equilibrium. Which means that for these dynamics applied to a game theoretical model, each interior *ESS* is an asymptotically stable fixed point. We show that the ray-projection dynamics of Samuelson tâtonnement process on the unit simplex are for every $y = \lambda x \in int\ \mathbb{R}_{+}^{n+1} \backslash \{0^{n+1}\}$ given by

$$\dot{x} = \frac{1}{\lambda} \left[f(x) - x \left(\sum_{i=1}^{n+1} f_i(x) \right) \right],$$

where $\lambda = \sum_{i=1}^{n+1} y_i$ and $x \in S^n = \{z \in \mathbb{R}^{n+1} | z_j \geq 0 \text{ for all } j \in \{1, 2, ..., n+1\}$ and $\sum_{j=1}^{n+1} z_j = 1\}$.

One might think that the dynamics obtained in that manner, are equivalent to the projection dynamics of Lahkar & Sandholm [2008] on the interior of the unit simplex, and if not globally then at least locally. By a global projection, we mean a projection of an arbitrary trajectory unto the unit simplex. By local projection, we mean that the trajectory is started on the unit simplex and then continuously be forced back on the unit simplex by projection, i.e., $\lambda = 1$ always. This intuition is false, as the orthogonal-projection dynamics of Lahkar & Sandholm [2008], as we will call them, are for $x \in int\ S^n$ given in our notations, where $i = (1, ..., 1) \in \mathbb{R}^{n+1}$, by

[1] For correspondences between models, concepts, results and dynamics in economics and biology, we refer to Joosten [1996, 2006]. For instance, an implication of *WARP* in economics is similar to an implication of *ESS* in mathematical biology.

$$\dot{x} = f(x) - \frac{1}{n+1}\left(\sum_{i=1}^{n+1} f_i(x)\right) i.$$

Here, $f : S^n \to \mathbb{R}^{n+1}$ is a relative fitness function (cf., Joosten [1996]).

We demonstrate that under the ray-projection dynamics every interior *ESS* is an asymptotically stable fixed point. We also show that the concept of a strict equilibrium unifies two notions of evolutionary stability, namely static evolutionary stability as embodied by the *ESS* and dynamic evolutionary stability as embodied by *ESE*.

A geometric interpretation of the former result is the following. Samuelson's process moves on a sphere with the origin as its center and with a fixed radius. Points having equal Euclidean distance to the equilibrium form a circle on this sphere.[2] Connecting this circle to the origin yields a cone. This cone is intersected by the unit simplex, a subset of a plane. Hence, the projection of the circle unto the unit simplex is an ellipse. Since the unrestricted process always moves inwards relative to the circle around the equilibrium on which the process happens to be, the process projected unto unit simplex moves inwards relative to the ellipse it happens to be on.

Then, we generalize the approach with projections by employing modifications of the relative fitness function. As it turns out, the best-response dynamics of Matsui [1992], the dynamics of Brown & Von Neumann [1950], the logit dynamics of Fudenberg & Levine [1998], but also the replicator dynamics of Taylor & Jonker [1978], can be represented as projection dynamics by choosing appropriate variants of the relative fitness function.

Next, we present our ideas leading to the ray-projection dynamics. In Section 3 we generalize both ray-projection and orthogonal-projection dynamics. Well-known dynamics appear as special cases of generalized projection dynamics. Section 4 deals with conditions guaranteeing that the dynamics do not cross the boundary of the unit simplex. Section 5 concludes, all proofs are to be found in the Appendix.

2 Comparing the old and the new

In Joosten [2006] connections were highlighted between models formalizing evolutionary dynamics and price-adjustment processes. For instance, a condition resulting from the Weak Axiom of Revealed Preferences (*WARP*) can be translated almost one-to-one to a condition resulting from the evolutionarily stable strategy (*ESS*). This section continues in a similar vein, space limitations require us to be extremely brief, the reader interested in correspondences between evolutionary dynamics and price adjustment dynamics beyond what is being presented, is referred to e.g., Joosten [1996, 2006].

[2] For all of these objects in $\mathbb{R}^3$ proper higher– and lower-dimensional parallels exist.

We first give a very brief introduction of pure exchange economies and price-adjustment dynamics, then we show that the well-known price-adjustment dynamics of Samuelson [1947] can be projected on the unit simplex and we provide explicit formulas for these projected dynamics. Next, we give a very brief introduction on dynamics and equilibria in evolutionary game theory to continue with projection dynamics in an evolutionary framework; we discuss the dynamics of Lahkar & Sandholm [2008] and propose our own variant of projection dynamics as evolutionary dynamics. The final subsection is devoted to stability of interior equilibria.

2.1 On price-adjustment dynamics

The condition implied by *WARP*, cf., e.g., Uzawa [1961], is the following

$$(y-x)\cdot f(x) > 0,$$

for all $x, y \in \mathbb{P} = \mathbb{R}^{n+1}_{+} \backslash \{0^{n+1}\}$ such that $y \in E = \left\{z \in \mathbb{P}\}|\ f(z) \leq 0^{n+1}\right\}$, $x \notin E$. Here, $f : \mathbb{P} \to \mathbb{R}^{n+1}$ satisfies continuity, **homogeneity (of degree zero in prices)**, i.e., $f(\lambda x) = f(x)$ for all $\lambda > 0$, and **complementarity**, i.e., $x \cdot f(x) = 0$ for all $x \in \mathbb{P}$. Often, since the function f satisfies homogeneity of degree zero, analysis is restricted to the n-dimensional unit simplex S^n, i.e.,

$$S^n = \left\{ x \in \mathbb{P} \,\middle|\, \sum_{j \in I^{n+1}} x_j = 1 \right\},$$

where $I^{n+1} = \{1, ..., n+1\}$.

In economics, $x \in S^n$ represents a vector of relative prices adding up to unity; the function f represents a so called **generalized excess demand function**. A price vector $y \in S^n$ satisfying $f(y) \leq 0^{n+1}$ is called an **equilibrium** or a **Walrasian equilibrium**. At an equilibrium no commodity has positive excess demand. *Existence of an equilibrium (ray)* is readily shown by using homogeneity in order to restrict analysis to the unit simplex, constructing an adequate continuous function from this unit simplex unto itself, and then using Brouwer's fixed point theorem.

The work of Sonnenschein [1972, 1973], Mantel [1974] and Debreu [1974] shows that any function satisfying continuity, complementarity and desirability[3], can be approximated by an excess demand function on an arbitrarily large subset of the interior of the unit simplex resulting from a pure exchange economy with as many agents as commodities in which each of the agents has well-behaved preferences and positive initial endowments of all commodities. If the property of desirability is dropped one obtains a generalized excess demand function, and if one furthermore restricts attention to

[3]Desirability of all goods means that if the price of a commodity equals zero, then the supply of that good can not exceed its demand, i.e., $x_j = 0$ implies $f_j(x) \geq 0$.

the unit simplex, homogeneity of degree zero in prices becomes void. So, a generalized excess demand function on the unit simplex is characterized by continuity and complementarity.

A well-known result by Arrow & Hurwicz [1958,1960a,b], Arrow *et al.* [1959] is that the tâtonnement process of Samuelson [1947]:

$$\dot{x} = \frac{dx}{dt} = f(x), \tag{1}$$

converges to an equilibrium if $(y - x) \cdot f(x) > 0$ for all $y \in E$, and $x \notin E$ and if desirability holds. Here, $E = \left\{x \in \mathbb{R}^{n+1} | \; f(x) \leq 0^{n+1}\right\}$ denotes the set of (economic) equilibria, and if the condition mentioned holds, it can be shown that E is convex (cf., Arrow & Hurwicz [1960b]).

The sketch of the proof is straightforward. Complementarity of f implies

$$\frac{d||x||^2}{dt} = \sum_{i \in I^{n+1}} 2x_i \frac{dx_i}{dt} = 2 \sum_{i \in I^{n+1}} x_i f_i(x) = 2x \cdot f(x) = 0.$$

Hence, continuity and desirability of all commodities imply that if the process starts in the non-negative orthant it remains on the sphere in this orthant having the origin as its center and containing the starting point. Furthermore, let $y \in E$ and let $x \notin E$ satisfy $||x|| = ||y||$, $x \neq y$, then

$$||y - x||^2 > 0, \text{ moreover } \frac{d||y - x||^2}{dt} = -2(y - x) \cdot f(x) < 0.$$

So, under the dynamics the Euclidean distance to y decreases monotonically in time. The actual proof uses Lyapunov's second method, and the Euclidean distance can be interpreted as a so-called Lyapunov function. Recall that by homogeneity of degree zero of f, a ray $\{\lambda y\}_{\lambda > 0}$ exists satisfying $f(x) = 0^{n+1}$ for all $x \in \{\lambda y\}_{\lambda > 0}$.

2.2 Ray-projection of Samuelson's tâtonnement process

Now, we derive the dynamics being the projection of Samuelson's tâtonnement process on the unit simplex. Note that the trajectory $\{y_t\}_{t \geq 0}$ with $y_0 \in \mathbb{P}$ under (1) may be approximated at $y \in \{y_t\}_{t \geq 0}$ by $y + \Delta t f(y)$. The projection of $y + \Delta t f(y)$ unto the unit simplex is given by

$$\frac{y + \Delta t f(y)}{\sum_{i=1}^{n+1} y_i + \Delta t \sum_{i=1}^{n+1} f_i(y)}.$$

Here, Δt is the length of the time interval elapsed, $\sum_{i=1}^{n+1} y_i + \Delta t \sum_{i=1}^{n+1} f_i(y)$ is a number, whereas y and $f(y)$ are vectors. Then, this implies a move from $x = \frac{y}{\sum_{i=1}^{n+1} y_i} \in S^n$ to $\frac{y + \Delta t f(y)}{\sum_{i=1}^{n+1} y_i + \Delta t \sum_{i=1}^{n+1} f_i(y)} \in S^n$ and therefore

$$\Delta x = \frac{y + \Delta t f(y)}{\sum_{i=1}^{n+1} y_i + \Delta t \sum_{i=1}^{n+1} f_i(y)} - \frac{y}{\sum_{i=1}^{n+1} y_i}$$

$$
\begin{aligned}
&\overset{y=\lambda x}{=} \frac{\lambda x + \Delta t f(\lambda x)}{\sum_{i=1}^{n+1} \lambda x_i + \Delta t \sum_{i=1}^{n+1} f_i(\lambda x)} - \frac{\lambda x}{\sum_{i=1}^{n+1} \lambda x_i} \\
&\overset{\sum_{i=1}^{n+1} \lambda x_i = \lambda}{=} \frac{\lambda x + \Delta t f(\lambda x)}{\lambda + \Delta t \sum_{i=1}^{n+1} f_i(\lambda x)} - x \\
= &\frac{\lambda x + \Delta t f(\lambda x) - x\left(\lambda + \Delta t \sum_{i=1}^{n+1} f_i(\lambda x)\right)}{\lambda + \Delta t \sum_{i=1}^{n+1} f_i(\lambda x)} \\
&\overset{f(\lambda x) = f(x)}{=} \Delta t \frac{f(x) - x\left(\sum_{i=1}^{n+1} f_i(x)\right)}{\lambda + \Delta t \sum_{i=1}^{n+1} f_i(x)}.
\end{aligned}
$$

So, this means that

$$
\dot{x} = \lim_{\Delta t \downarrow 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \downarrow 0} \frac{\Delta t}{\Delta t} \frac{f(x) - x\left(\sum_{i=1}^{n+1} f_i(x)\right)}{\lambda + \Delta t \sum_{i=1}^{n+1} f_i(x)} = \frac{1}{\lambda}\left[f(x) - x\left(\sum_{i=1}^{n+1} f_i(x)\right)\right].
$$

The term $\frac{1}{\lambda}$ has no influence on the direction of the dynamics. This motivated the following definition, see Figure 1 for an illustration.

Definition 1 *Let $f : \mathbb{P} \to \mathbb{R}^{n+1}$ satisfy continuity, complementarity, and (positive) homogeneity of degree zero. Let for all $y \in \mathbb{P}$, $\dot{y} = \frac{dy}{dt} = f(y)$ and $\lambda_y = \sum_{i=1}^{n+1} y_i$. Then, the* ***ray-projection dynamics*** *on the unit simplex are for every $x = \frac{1}{\lambda_y} y \in int\ S^n$ given by*

$$
\dot{x} = \frac{1}{\lambda_y}\left[f(x) - x\left(\sum_{i=1}^{n+1} f_i(x)\right)\right].
$$

Remark 1 *If $\lambda_y = 1$, i.e., $x = y \in S^n$, we call the ray-projection dynamics local, and global otherwise. Local and global ray-projection dynamics can be transformed one into the other by a transformation of time.*

Here, we are not concerned for the behavior of these dynamics on the boundary of the unit simplex, as price-adjustment processes tend to stay away from the boundary of $\mathbb{P}$ (boundary behavior is treated in Section 4).

2.3 On dynamics and equilibria in evolutionary game theory

In evolutionary game theory, for a population having $n+1$ distinguishable subgroups, $x \in S^n$ is a vector of population shares for each subgroup. Let $F : S^n \to \mathbb{R}^{n+1}$ be a fitness function, i.e., a function attributing to each subgroup in the population its fitness. The fitness of a subgroup may be interpreted as its potential to reproduce depending on the composition of the population, i.e., $x \in S^n$.

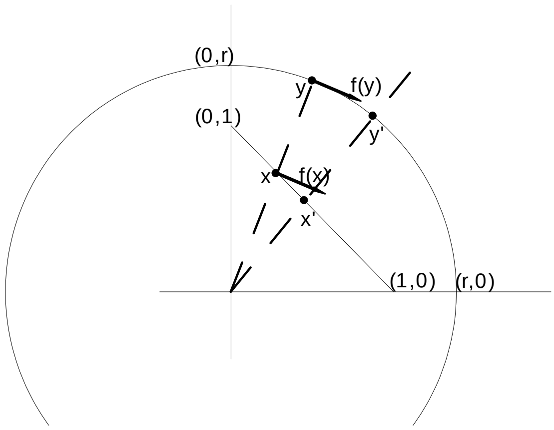

Figure 1: Samuelson's tâtonnement inducing a trajectory from y to y', is projected unto S^1. The projection moves from x towards x'. We have depicted vectors $f(x) = f(y)$.

The **relative fitness function** $f : S^n \to \mathbb{R}^{n+1}$ is given by

$$f_i(x) = F_i(x) - x \cdot F(x) \text{ for all } x \in S^n \text{ and all } i \in I^{n+1}.$$

So, a relative fitness function (cf., Joosten [1996]) attributes to each subgroup the difference of its fitness and the population share weighted average fitness of the population. If the fitness function F is continuous, the same property follows immediately for the relative fitness function f. Observe furthermore that for all $x \in S^n$, it holds that $x \cdot f(x) = 0$.

The evolution of the composition of the population is usually represented by a system of $n+1$ autonomous differential equations:

$$\dot{x} = \frac{dx}{dt} = h(x).$$

Here, the function $h : S^n \to \mathbb{R}^{n+1}$ is connected to the relative fitness function f in one of the ways proposed, cf., e.g., Nachbar [1990], Friedman [1991], Swinkels [1993], Joosten [1996], Ritzberger & Weibull [1995]. (Lipschitz) continuity of h implies existence (and uniqueness) of a solution to the differential equation for every starting point $x_0 \in S^n$; differentiability of h implies both existence and uniqueness (cf., e.g., Perko [1991]). We are reluctant to impose conditions on the function h at this point since many interesting evolutionary dynamics are neither differentiable, nor continuous.

For **sign-compatible dynamics**, we have

$$sign\ h_i(x) = sign\ f_i(x) \text{ whenever } x_i > 0.$$

i.e., the change in population share of each subgroup with positive population share corresponds in sign with its relative fitness; for **weakly sign-compatible dynamics**, at least one subgroup with positive relative fitness grows in population share. A more general alternative than sign compatibility is provided by Friedman [1991], evolutionary dynamics are **weakly compatible** if $f(x) \cdot h(x) \geq 0$ for all $x \in S^n$.

The state $y \in S^n$ is a **saturated equilibrium** if $f(y) \leq \mathbf{0}^{n+1}$, a **fixed point** if $h(y) = \mathbf{0}^{n+1}$; a fixed point y is **(asymptotically) stable** if, for any neighborhood $U \subset S^n$ of y, there exists an open neighborhood $V \subset U$ of y such that any trajectory starting in V remains in U (and converges to y). A **limit point** is a point $y \in S^n$ satisfying $\lim_{t\to\infty} x_t = y$ for at least one solution $\{x_t\}_{t\geq 0}$ to $x_0 \in S^n$ and the differential equation above.

At a saturated equilibrium all subgroups with below average fitness have population share equal to zero. So, rather than 'survival of the fittest', we have 'extinction of the less fit'. If the fitness function is given by $F(x) = Ax$ for some square matrix A, every saturated equilibrium coincides to a Nash equilibrium of the evolutionary game at hand. The term is due to Hofbauer & Sigmund [1988], in the sequel we may omit the term 'saturated'.

The fixed point $y \in S^n$ is a **generalized evolutionarily stable state** (*GESS*, Joosten [1996]) if and only if there exists an open neighborhood $U \subset S^n$ of y satisfying

$$(y - x) \cdot f(x) > 0 \text{ for all } x \in U\backslash\{y\}. \tag{2}$$

A geometric interpretation of (2) is that the angle between the vector pointing from x towards the equilibrium, i.e., $(y - x)$, and the vector $f(x)$ is always acute. The *GESS* generalizes the concept of an *ESS* of Maynard Smith & Price [1973] in order to deal with *arbitrary* (relative) fitness functions. For the more standard fitness functions, the two notions coincide.

Taylor & Jonker [1978] introduced the replicator dynamics into mathematical biology and gave conditions guaranteeing that an *ESS* is an asymptotically stable fixed point of these dynamics. Zeeman [1981] extended this result and pointed out that the conditions formulated by Taylor and Jonker [1978] are almost always satisfied. The most general result on asymptotic stability regarding the replicator dynamics for the *ESS* is probably Hofbauer *et al.* [1979] as it stipulates an equivalence of the *ESS* and existence of a Lyapunov function of which the time derivative is similar to Eq. (2).

Friedman [1991] has an elegant way of coping with evolutionary stability as he defines any asymptotically stable fixed point of given evolutionary dynamics as an evolutionary equilibrium. Most approaches however, deal with conditions on the underlying system in order to come up with a viable evolutionary equilibrium concept, or deal with refinements of the asymptotically stable fixed point concept (e.g., Weissing [1990]).

In Joosten [1996] we defined an evolutionary equilibrium concept on the dynamic system, wishing to rule out some asymptotically stable fixed points.

Namely, the ones which induce trajectories starting nearby, but going far away from the equilibrium before converging to it in the end. The fixed point $y \in S^n$ is an **evolutionarily stable equilibrium** if and only if there exists an open neighborhood $U \subset S^n$ of y satisfying

$$(y - x) \cdot h(x) > 0 \text{ for all } x \in U \backslash \{y\}. \tag{3}$$

A geometric interpretation of (3) is that sufficiently close to the equilibrium the angle between $(y - x)$ and the vector representing the direction of the dynamics is always acute. The concept was inspired by the Euclidean distance approach of early contributions in economics as mentioned, since (3) implies that the Euclidean distance is a (strict) Lyapunov function for U.

2.4 Projection dynamics in evolutionary games

Lahkar & Sandholm [2008] introduce the following dynamics into evolutionary game theory quoting Nagurney & Zhang [1996] as a source of inspiration.

Definition 2 *Let $f : S^n \to \mathbb{R}^{n+1}$ be a relative fitness function, $C_f(x) = \sum_{i=1}^{n+1} f_i(x)$ and $\overline{x} = \frac{1}{n+1}(1, ..., 1)$. Then, the **orthogonal-projection dynamics** are for every $x \in$ int S^n given by: $\dot{x} = f(x) - \overline{x}C_f(x)$.*

Here, $\overline{x}$ is the barycenter of S^n. For the time being, we are only interested in the behavior of the dynamics of Lahkar & Sandholm [2008] for the interior of the unit simplex. The authors actually define their dynamics on the fitness function but for the interior of the unit simplex their definition and the one given above concur. Below, we present the ray-projection dynamics, corresponding to the local variant of the definition given in the economic framework.

Definition 3 *Let $f : S^n \to \mathbb{R}^{n+1}$ be a relative fitness function and $C_f(x) = \sum_{i=1}^{n+1} f_i(x)$. Then, the **ray-projection dynamics** are for every $x \in$ int S^n given by: $\dot{x} = f(x) - xC_f(x)$.*

Informally stated, both processes move from $x \in S^n$ into the direction $f(x)$, hence outside the unit simplex in general. Lahkar and Sandholm's dynamics return to the unit simplex by continuously changing all components with identical amounts, whereas our dynamics are brought back to the unit simplex by continuously changing all components proportional to x. For the framework presented, we have the following result.

Lemma 4 *Every interior equilibrium is a fixed point of the both types of projection dynamics and every interior fixed point of both types of projection dynamics is an equilibrium.*

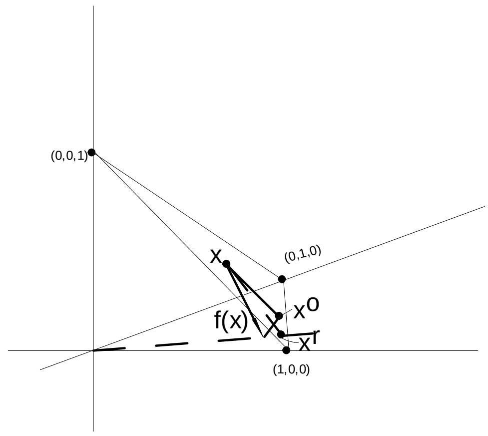

Figure 2: The point x^o is the orthogonal projection of $x + f(x)$ on the S^2; x^r is the ray-projection of $x + f(x)$ on S^n.

2.5 On stability of interior equilibria

Hofbauer & Sandholm [2008] introduce the class of stable games. A stable game is a game in which the following property holds:

$$(y - x) \cdot (F(y) - F(x)) \leq 0 \text{ for all } x, y \in S^n.$$

Here, F is a *fitness function*, but it follows easily that in our notations using the *relative fitness function* f we obtain

$$(y - x) \cdot (f(y) - f(x)) \leq 0 \text{ for all } x, y \in S^n.$$

The property which defines a stable game is called monotonicity (MON) elsewhere and is connected to a multitude of important results guaranteeing uniqueness and dynamic stability of equilibria and fixed points (see Joosten [2006], Harker & Pang [1990]). MON is a weaker version of strict monotonicity ($SMON$) which can be written as

$$(y - x) \cdot (f(y) - f(x)) < 0 \text{ for all } x, y \in S^n, x \neq y.$$

A game in which $SMON$ holds for all states $x, y \in S^n, x \neq y$, is called a strictly stable game by Hofbauer & Sandholm [2008]. It can be shown that $SMON$ implies that there is a unique saturated equilibrium, and that MON implies that the set of equilibria is compact and convex.

Joosten [2006] showed that if the relative fitness function is given by $f(x) = Ax - (xAx)\, i$ for all $x \in S^n$, then strict monotonicity is equivalent to Haigh's criterion (Haigh [1975]) which can be written as

$$\xi A \xi < 0 \text{ whenever } i \cdot \xi = 0.$$

The version where $\xi A \xi \leq 0$ replaces $\xi A \xi < 0$, is equivalent to MON.

For an interior equilibrium $y \in S^n$, $(S)MON$ implies

$$(y - x) \cdot f(x) \geq (>) 0 \text{ for all } x \in S^n \backslash \{y\}.$$

So, every interior equilibrium of a strictly stable game is a *GESS* (cf., Joosten [1996]) for which the neighborhood U in Eq. (2) can be expanded to include the entire unit simplex. For every stable game, every interior equilibrium is a *neutrally stable state* following Joosten [2006] and Maynard Smith [1982]. Under the replicator dynamics every (generalized) evolutionarily stable state is an asymptotically stable fixed point and every neutrally stable state is stable (cf., e.g., Hofbauer & Sigmund [1998]).

For the orthogonal-projection dynamics it can be seen that every *interior evolutionarily stable equilibrium* is a *generalized evolutionarily stable state* and every *interior generalized evolutionarily stable state* is an *evolutionarily stable equilibrium*, as for $y \in int\ S^n$ we have

$$\begin{gathered} (y - x) \cdot h(x) > 0 \Longleftrightarrow \\ (y - x) \cdot f(x) - C_f(x)(y - x)\overline{x} > 0 \Longleftrightarrow \\ (y - x) \cdot f(x) > 0. \end{gathered}$$

So, we have shown the validity of the following generalization, albeit for the interior of the unit simplex, of a result in Hofbauer & Sandholm [2008].

Proposition 5 *For the interior of the unit simplex, every generalized evolutionarily stable state is an evolutionarily stable equilibrium under the orthogonal-projection dynamics and vice versa.*

We now present a corresponding result for ray-projection dynamics. Our strategy of proof is the following. From a given relative fitness function we construct a function on the relevant positive orthant, connect dynamics to that function and construct a trajectory under the dynamics converging to an equilibrium corresponding to a full-dimensional expansion of the interior evolutionarily stable state. Then we project this trajectory unto the unit simplex using the ray-projection. This projected trajectory converges then to the projected equilibrium point. The corresponding dynamics on the unit simplex are the ray-projection dynamics.

Theorem 6 *Under the ray-projection dynamics, every interior generalized evolutionarily stable state is an asymptotically stable equilibrium.*

3 Generalizations of projection dynamics

Here, we pursue the idea of generalizing both projection dynamics presented. For this purpose we define some $g : S^n \to \mathbb{R}^{n+1}$. We intend to examine dynamics induced by g in two variants:

$$\overset{.}{x}^{r}_{g} = g(x) - xC_g(x),$$
$$\overset{.}{x}^{o}_{g} = g(x) - \overline{x}C_g(x).$$

Superscript r (o) refers to the ray-projection (orthogonal-projection) dynamics and subscript g refers to the function g, $\overline{x}$ is the barycenter of S^n and $C_g(x) = \sum_{i=1}^{n+1} g_i(x)$.

The following result is straightforward, its proof is left to the reader.

Lemma 7 *Let* $g : S^n \to \mathbb{R}^{n+1}$.

- *If g satisfies $C_g(x) = 0$ for al $x \in S^n$, then the local and global ray-projection dynamics, and the orthogonal-projection dynamics concur.*
- *If g is weakly compatible f, i.e., $g(x) \cdot f(x) \geq 0$ for all $x \in int\ S^n$, then the associated ray-projection dynamics are weakly compatible, too.*
- *If g is non-negative, i.e., $g : S^n \to \mathbb{R}^{n+1}_{+}$, then the ray-projection dynamics remain on the unit simplex.*

Note that (trivially) all evolutionary dynamics on the unit simplex are projected 'unto themselves', hence in that case by the first statement of the lemma, ray-projection and orthogonal projection dynamics concur. The second statement of the lemma gives a criterion to determine the status of the ensuing ray-projection dynamics. Recall that evolutionary dynamics should be connected with the relative fitness function and weak compatibility of Friedman [1991] is one of the ways to accomplish this. The final statement deals with a criterion to guarantee that ray-projection dynamics do not cross the boundary of the unit simplex.

In order to be relevant in an evolutionary framework it is of utmost importance to link the function g to the relative fitness function. It is not the purpose of this section to give a classification of functions suitable for evolutionary modeling purposes. Instead we show that several well-known dynamics can be represented as ray- or orthogonal-projection dynamics for appropriately chosen functions.

Example 8 ***(Replicator dynamics)*** *We can have the function driving both projection dynamics depend on the fitness function $F : S^n \to \mathbb{R}^{n+1}$. Let $\widetilde{g} : S^n \to \mathbb{R}^{n+1}$ be given by $\widetilde{g}_i(x) = x_i F_i(x)$ for all $x \in int\ S^n$, $i \in I^{n+1}$. Then for all $i \in I^{n+1}$:*

$$\left(\overset{.}{x}^{r}_{\widetilde{g}}\right)_i = x_i F_i(x) - x_i \sum_{j=1}^{n+1} x_j F_j(x) = x_i \left[F_i(x) - x \cdot F(x)\right] = x_i f_i(x),$$

$$\left(\overset{.}{x}^{o}_{\widetilde{g}}\right)_i = x_i F_i(x) - \frac{1}{n+1} \sum_{j=1}^{n+1} x_j F_j(x).$$

So, the generalized ray-projection dynamics connected to the function $\widetilde{g}$ *as defined yield the replicator dynamics.*
Another way of obtaining similar dynamics is particularly interesting in case the fitness function is given by $F(x) = Ax$ *for a symmetric matrix* A*. Let* $\underline{a} \leq \min(0, \min_{ij} a_{ij})$ *. Then, let* $\widehat{g} : S^n \to \mathbb{R}^{n+1}$ *be given by* $\widehat{g}_i(x) = x_i f_i(x) - \underline{a}$ *for all* $x \in int\ S^n$, $i \in I^{n+1}$*. Then,*

$$\begin{aligned}\left(\overset{\cdot o}{x}_{\widehat{g}}\right)_i &= x_i f_i(x) - \underline{a} - \frac{1}{n+1}\left(\sum_{j=1}^{n+1}[x_j f_j(x) - \underline{a}]\right)\\ &= x_i f_i(x) \text{ for all } i \in I^{n+1}.\end{aligned}$$

The ray-projection dynamics are given by

$$\left(\overset{\cdot r}{x}_{\widehat{g}}\right)_i = x_i f_i(x) - \underline{a}(1 - x_i(n+1)) \text{ for all } i \in I^{n+1}.$$

An advantage of this function is that $\widehat{g}_i(x) = x_i f_i(x) - a \geq 0$ *for all* $x \in int\ S^n$, $i \in I^{n+1}$*. So, the dynamics can not cross on the boundary of* S^n*. Here, orthogonal-projection dynamics yield the replicator dynamics.* ■

Example 9 ***(Best-response dynamics)*** *Let* $e_k \in \mathbb{R}^{n+1}$ *denote the k-th unit vector, and for given* $x \in S^n$, $j^* = \min\{h \in I^{n+1} |\ f_h(x) = \max_{k \in I^{n+1}} f_k(x) > 0\}$*. Let* $g : S^n \to \mathbb{R}^{n+1}$ *for all* $x \in S^n$ *and* $i \in I^{n+1}$*, be given by*

$$g_i(x) = \begin{cases} 1 & \text{if } i = j^*, \\ 0 & \text{otherwise.}\end{cases}$$

Then, we obtain

$$\begin{aligned}\left(\overset{\cdot r}{x}_g\right)_i &= \begin{cases} 0 & \text{if } x \in E, \\ (e_{j^*})_i - x_i & \text{otherwise.}\end{cases} \quad \text{and}\\ \left(\overset{\cdot o}{x}_g\right)_i &= \begin{cases} 0 & \text{if } x \in E, \\ (e_{j^*})_i - \frac{1}{n+1} & \text{otherwise.}\end{cases}\end{aligned}$$

The ray-projection dynamics form a special case of the best-response dynamics of Matsui [1992]. We introduced two slight changes to the original, one implying that $f(y) \leq \mathbf{0}^{n+1}$ *implies* $\dot{y} = \mathbf{0}^{n+1}$*, and a tie-breaker for the case that multiple best-responses exist.* ■

BR-dynamics have a predecessor in the continuous fictitious-play dynamics of Rosenmüller [1971], a continuous-time version of fictitious play (Brown [1951]). Brown formulated this process in order to compute a solution (i.e., a Nash equilibrium) of a zero-sum game. Brown has conceived several other ideas on dynamics to compute equilibria. The following example deals with one of them and variations thereof.

Example 10 ***(Generalized "Brownian motions")*** *The term including the quotation marks is due to Hofbauer [2000] after G.W. Brown (not botanist Robert Brown, the (re)discoverer of Brownian motion). As a tâtonnement process Nikaidô [1959] used $g_i(x) = \max\{0, f_i(x)\}$ for all $x \in \mathbb{P}$, $i \in I^{n+1}$ which yields*

$$\begin{aligned}
\left(\overset{\cdot r}{x}_g\right)_i &= \max\{0, f_i(x)\} - x_i \sum_{j \in I^{n+1}} \max\{0, f_j(x)\}, \\
\left(\overset{\cdot o}{x}_g\right)_i &= \max\{0, f_i(x)\} - \frac{1}{n+1} \sum_{j \in I^{n+1}} \max\{0, f_j(x)\}.
\end{aligned}$$

The ray-projection dynamics coincide with those of Brown & Von Neumann [1950] on the interior of the unit simplex; the orthogonal-projection dynamics have not been studied as far as we know. For both types of dynamics, each equilibrium is a fixed point, and each limit point is an equilibrium.
More generally, let $z : \mathbb{R}_+ \to \mathbb{R}_+$ be given by $z(0) = 0$ and $z(x) > 0$ for all $x > 0$. Then, defining $g^z : S^n \to \mathbb{R}^{n+1}$ by $g_i^z(x) = z(\max\{0, f_i(x)\})$ for all $i \in I^{n+1}$, we obtain

$$\begin{aligned}
\left(\overset{\cdot r}{x}_{g^z}\right)_i &= z(\max\{0, f_i(x)\}) - x_i C_{g^z}(x), \\
\left(\overset{\cdot o}{x}_{g^z}\right)_i &= z(\max\{0, f_i(x)\}) - \frac{1}{n+1} C_{g^z}(x).
\end{aligned}$$

Note that if $z(x) = x^\alpha$ for $\alpha > 0, x \geq 0$, then clearly $\alpha = 1$ yields the BN-dynamics. An interesting case is then to let $\alpha \to \infty$, where the dynamics are very similar to the best-response dynamics.
Another 'Brownian motion' is due to Nikaidô & Uzawa [1960] in the framework of ***price-adjustment****. These dynamics are driven by the following function defined component-wise and for strictly positive ρ by*

$$\widetilde{g}_i^\rho(x) = \max\{0, \rho f_i(x) + x_i\} - x_i.$$

Now, using our projections we obtain

$$\begin{aligned}
\left(\overset{\cdot r}{x}_{\widetilde{g}^\rho}\right)_i &= \max\{0, \rho f_i(x) + x_i\} - x_i C_{g^\rho}(x), \\
\left(\overset{\cdot o}{x}_{\widetilde{g}^\rho}\right)_i &= \max\{0, \rho f_i(x) + x_i\} - x_i + \frac{1}{n+1} C_{g^\rho}(x).
\end{aligned}$$

It is easy to check that the ray-projection dynamics are weakly compatible. ■

BN-dynamics converge to a Nash equilibrium, if the relative fitness function $f(x) = Ax - (x \cdot Ax)\,i$ is such that for matrix $A : a_{ij} = -a_{ji}$ for all $i, j \in I^{n+1}$. Moreover, BN-dynamics are globally stable under strict monotonicity (*SMON*) of the generalized excess demand function (or relative fitness function) (cf., Nikaidô [1959]). Hofbauer [2000] treats families of dynamics including (smoothed) BN-dynamics, BR-dynamics and replicator dynamics. His convergence results on the *ESS* complement Nikaidô's. The majority of results in Hofbauer [2000] rely on the weak version of Haigh's criterion, for the stronger one Hofbauer [1995] already has parallels.

Nikaidô & Uzawa [1960] show that any interior Walrasian equilibrium is an asymptotically stable fixed point of their dynamics under *WARP.* For $\rho \to +\infty$ the ray-projection of the process of Nikaidô & Uzawa 'approximates' the BN-dynamics 'almost everywhere'; for $\rho \downarrow 0$ the ray-projection dynamics are equivalent to the ray-projection of Samuelson's process 'almost everywhere'. Clearly, for any interior equilibrium, there exists a neighborhood such that the processes of Nikaidô & Uzawa and Samuelson concur. So, any interior *ESS* is an asymptotically stable fixed point of the ray-projection dynamics, and an *ESE* for the orthogonal-projection dynamics.

Example 11 ***(Logit type dynamics)*** *Now, let* $\beta > 0$, $g^{\beta} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *be given by* $g_i^{\beta}(x) = e^{\beta f_i(x)}$. *Then, we obtain projection dynamics given by*

$$\begin{aligned}\left(\overset{\cdot}{x}{}^{r}_{g^{\beta}}\right)_i &= e^{\beta f_i(x)} - x_i \sum_{j=1}^{n+1} e^{\beta f_j(x)} \\ \left(\overset{\cdot}{x}{}^{o}_{g^{\beta}}\right)_i &= e^{\beta f_i(x)} - \frac{1}{n+1} \sum_{j=1}^{n+1} e^{\beta f_j(x)}.\end{aligned}$$

Clearly, the ray-projection dynamics do not cross the boundary of S^n, *as* $x_i = 0$ *implies* $\dot{x}_i = e^{\beta f_i(x)} \geq 0$. *Furthermore, for very large values of* β *only best-responses increase in population share under both variants. The former dynamics are known as the logit dynamics (Fudenberg & Levine [1998]), where* $\frac{1}{\beta}$ *is interpreted as an error term. For error terms going to zero, i.e.,* β*'s going to infinity, the dynamics become more and more similar to the best response dynamics, but remain continuous. Note that Fudenberg & Levine [1998] actually write*

$$\dot{x}_i = \frac{e^{\beta F_i(x)}}{\sum_{j=1}^{n+1} e^{\beta F_j(x)}} - x_i \text{ for all } x \in S^n,\ i \in I^{n+1}.$$

However, observe that

$$\left(\overset{\cdot}{x}{}^{r}_{g^{\beta}}\right)_i = e^{\beta f_i(x)} - x_i \sum_{j=1}^{n+1} e^{\beta f_j(x)} = \xi(x) \left[\frac{e^{\beta F_i(x)}}{\sum_{j=1}^{n+1} e^{\beta F_j(x)}} - x_i\right].$$

Since, $\xi(x) = \frac{\sum_{j=1}^{n+1} e^{\beta F_j(x)}}{e^{\beta x \cdot F(x)}}$ *does not depend on the subgroup at hand, it follows that both dynamics have the same direction, but may differ in speed.*

A glaring shortcoming of the logit dynamics is that an interior equilibrium need not be a fixed point of the dynamics. In this sense, the orthogonal-projection dynamics are perhaps more interesting than the ray-projection variant, as $f(y) = 0^{n+1}$ *implies* $\overset{.o}{x}_{g^\beta} = 0^{n+1}$.
Logit-type dynamics which possess the property that an interior equilibrium is a fixed point of the dynamics are generated by

$$g_i^\beta(x) = \frac{x_i e^{\beta f_i(x)}}{\sum_{j=1}^{n+1} x_j e^{\beta f_j(x)}} \text{ for all } i \in I^{n+1},$$

which yields

$$\begin{aligned} \left(\overset{.r}{x}_{g^\beta}\right)_i &= x_i \left(\frac{e^{\beta f_i(x)}}{\sum_{j=1}^{n+1} x_j e^{\beta f_j(x)}} - 1 \right), \\ \left(\overset{.o}{x}_{g^\beta}\right)_i &= \frac{x_i e^{\beta f_i(x)}}{\sum_{j=1}^{n+1} x_j e^{\beta f_j(x)}} - \frac{1}{n+1}. \end{aligned}$$

The ray-projection dynamics feature in e.g., Björnerstedt & Weibull [1996], and in Cabrales & Sobel [1992] in a discrete-time version . ■

We refer to Hopkins [1999] and Hofbauer [2000] for stability results of the *ESS* for the ray-projection variant of the logit dynamics. Sandholm [2007] provides a microfoundation for these dynamics (see also Fudenberg & Levine [1998], Hopkins [2002]).

4 Boundary conditions

The standard way of dealing with Samuelson's dynamics on the boundary of $\mathbb{P}$ is to define them as being zero for every zero component of the state variable, see e.g., Arrow & Hurwicz [1958, 1960a,b], Arrow *et al.* 1959]. In our notations the extension to include the boundary of $\mathbb{P}$ would be given by

$$\dot{x}_i = \begin{cases} 0 & \text{if } x_i = 0, \\ f_i(x) & \text{otherwise.} \end{cases}$$

So, the dynamics extended to the boundary may be **discontinuous**. For the ray-projection dynamics this extension to the boundary does not pose great problems as we may (re)define

$$\dot{x}_i = \begin{cases} 0 & \text{if } x_i = 0, \\ f_i(x) - \left(\sum_{j:x_j>0} f_j(x)\right) & \text{otherwise.} \end{cases} \qquad \text{(a)}$$

Under (a), a trajectory might in finite time reach the boundary of the unit simplex, and then remain on it while the relative fitness of a subgroup with population share zero becomes positive again.

An alternative is to define the dynamics extended as

$$\dot{x}_i = \begin{cases} 0 & \text{if } x_i = 0 \text{ and } f_i(x) < 0, \\ f_i(x) - x_i \left(\sum_{\substack{j: x_j > 0 \text{ or} \\ f_j(x) \geq 0}} f_j(x) \right) & \text{otherwise.} \end{cases} \tag{b}$$

This way, the dynamics escape the boundary of S^n as soon as $f_i(x) > 0$. So, at a limit point $y \in bd\ S^n$, we can never have $y_i = 0$ and $f_i(y) > 0$.

The following small result has interesting implications. Let, $ZP = \{x \in S^n |\ f(x) = 0^{n+1}\}$ and $FP = \{x \in S^n |\ \dot{x} = 0^{n+1}\}$.

Lemma 12 *Let $\{x_t\}_{t \geq 0}$ be a trajectory under the ray-projection dynamics and let $y = \lim_{t \to \infty} x_t$. If t^* exists such that $\{x_t\}_{t \geq t^*} \subset int\ S^n$, then $y \in ZP$; otherwise, $y \in bd\ S^n$ and under (a) $y \in FP$, under (b) $y \in E$.*

So, if a trajectory converges from the interior of the unit simplex to a boundary state, then under (a) the latter is a fixed point, whereas under (b) it is an equilibrium. Boundary conditions are of high relevance for boundary equilibria, fixed points and limit points. A refinement of the saturated equilibrium concept is the **strict saturated equilibrium** (cf., Joosten [1996]) which is a saturated equilibrium satisfying $f_j(y) = 0$ for precisely one $j \in I^{n+1}$. For this type of equilibrium we have the following result.

Theorem 13 *Every strict saturated equilibrium is an evolutionarily stable equilibrium of the ray-projection dynamics.*

Let $SSAT$, $ASFP$, and LP denote the sets of strict saturated equilibria, asymptotically stable fixed points, and limit points respectively; let LP_{int*} denote the set of limit points satisfying there is at least one $\{x_t\}_{t \geq 0}$ with $y = \lim_{t \to \infty} x_t$ satisfying that some t^* exists such that $\{x_t\}_{t \geq t^*} \subset int\ S^n$. Note that in Joosten [1996] it was shown that $SSAT \subseteq GESS \subseteq E$, then the following summarizes results.

Corollary 14 *For arbitrary dynamics, $SSAT \subseteq GESS \subseteq E$. For the ray-projection dynamics: $LP_{int*} \subseteq ZP \subseteq E \subseteq FP$; (a) implies $SSAT \subseteq ESE \subseteq ASFP \subseteq LP \subseteq FP$; (b) implies $SSAT \subseteq ESE \subseteq ASFP \subseteq LP \subseteq E \subseteq FP$.*

5 Conclusions

We introduced new evolutionary dynamics in game theory, the ray-projection dynamics. We have shown that every interior (generalized) evolutionarily stable strategy is an asymptotically stable fixed point of the ray-projection dynamics. We showed that each strict saturated (Hofbauer & Sigmund

[1988]) equilibrium is both a (generalized) evolutionarily stable strategy (*ESS*, Maynard Smith & Price [1973], *GESS*, Joosten [1996]) and an evolutionarily stable equilibrium (*ESE*, Joosten [1996]) for ray-projection dynamics.

We applied both projections to dynamics driven by functions connected to the relative fitness function. It turns out that well-known dynamics in evolutionary game theory can be represented as projection dynamics for appropriately chosen functions. Even if well-known dynamics can not be recovered in full, attractive elements may be used for new ray- or orthogonal-projection dynamics. For instance, the generalized replicator dynamics of Sethi [1998] introduced in a learning framework in which strategies are not equally easily adopted, can not be recovered by either type of projection. Yet, the 'inflows' incorporating the possible differences in which strategies can be adopted, can be taken to motivate new evolutionary dynamics.

The strategy of proof for our first major result contains some promise for future research. We transformed a dynamic process on the unit simplex into a dynamic process in the positive orthant, then projected the latter unto the unit simplex. We took a know result on price-adjustment dynamics in the positive orthant to show stability of the unrestriced dynamics, i.e., convergence to an equilibrium ray, implying the same properties for the connected ray-projection dynamics on the unit simplex. It should be noted that there is an abundance of stability results on both restricted and unrestricted tâtonnements (cf., e.g., Uzawa [1961], Negishi [1962], Harker & Pang [1990]) which may be used to derive stability results for evolutionary dynamics using a similar strategy of proof. In this context, an important topic for further research is to find a classification for the functions admissible for projection unto the unit simplex.

Microfoundations were not a theme of this paper, but connections between the ones given by e.g., Lahkar & Sandholm [2008] seem immediate. Tsakas & Voorneveld [2008] show that target-projection dynamics (Sandholm [2005]) can be associated to rational choice behavior if control costs (as in e.g., Van Damme [1991]) can be assumed (see also Mattson & Weibull [2002], Voorneveld [2006]). Further research must reveal which dynamics can be motivated with such microeconomic foundations.

6 Appendix

Proof of Lemma 4. The part 'interior equilibrium implies fixed point' is evident. Conversely, let $y \in int\ S^n$ be a fixed point of the ray-projection dynamics. Then, $f_i(y) - y_i \left(\sum_{j=1}^{n+1} f_j(y)\right) = 0$ for all $i \in I^{n+1}$. This in turn implies $y_i f_i(y) = y_i^2 \left(\sum_{j=1}^{n+1} f_j(y)\right)$ for all $i \in I^{n+1}$. Then, summing over all $i \in I^{n+1}$ and complementarity of f lead to $0 = y \cdot f(y) = \sum_{i=1}^{n+1} y_i^2 \left(\sum_{j=1}^{n+1} f_j(y)\right)$.

This can only hold if $\sum_{j=1}^{n+1} f_j(y) = 0$, hence $f(y) = 0^{n+1}$. For orthogonal-projection dynamics, the reasoning is similar. ■

Proof of Theorem 6. Let $f : S^n \to \mathbb{R}^{n+1}$ be a continuous relative fitness function. Define $\widetilde{f} : \mathbb{P} \to \mathbb{R}^{n+1}$ by $\widetilde{f}(\lambda x) = f(x)$ for all $\lambda > 0$. Then, $\widetilde{f}$ is continuous, homogeneous of degree zero, and satisfies complementarity. Define for all $x \in \mathbb{P}$:

$$\dot{x} = \widetilde{f}(x). \tag{4}$$

Clearly, this implies that $\frac{d||x||^2}{dt} = 2\sum_{j=1}^{n+1} x_j \dot{x}_j = 2\sum_{j=1}^{n+1} x_j \widetilde{f}_j(x) = 0$. Let $\{x_t\}_{t\geq 0}$ denote a solution to $x_0 \in \mathbb{P}$ and Eq. (4). Then, $\{x_t\}_{t\geq 0}$ remains on the sphere with the origin as center and with radius $r = ||x_0||$.

Let $y \in S^n$ be an interior generalized evolutionarily stable state, i.e., an open neighborhood $U \subseteq int\ S^n$ containing y exists such that

$$(y - x) \cdot f(x) > 0 \text{ for all } x \in U \backslash \{y\}.$$

Let $E = \{x \in \mathbb{P} |\ x = \lambda y,\ \lambda > 0\}$. Define for $z \in \mathbb{P}$, $\lambda_z = \sum_{k=1}^{n+1} z_k$. Then, let $x^* \in \mathbb{P}$ satisfy $\frac{1}{\lambda_{x^*}} x^* \in U\backslash\{y\}$ and let $y^* \in E$ such that $||x^*|| = ||y^*||$. Then, obviously $d(x^*, y^*)^2 > 0$, $d(y^*, y^*)^2 = 0$ and under the dynamics we have

$$\frac{1}{2}\dot{d}(x, y^*)^2 = -\sum_{j=1}^{n+1}(y_j^* - x_j^*)\widetilde{f}_j(x^*) = -\sum_{j=1}^{n+1}(\lambda_{y^*} y_j - \lambda_{x^*} x_j)\widetilde{f}_j(\lambda_{x^*} x)$$

$$= -\sum_{j=1}^{n+1}(\lambda_{y^*} y_j - \lambda_{y^*} x_j + (\lambda_{y^*} - \lambda_{x^*}) x_j) f_j(x) = -\lambda_{y^*}(y - x) \cdot f(x) < 0.$$

This means that the squared (Euclidean) distance is a strict Lyapunov function for $U' = \left\{x \in \mathbb{P} |\ \frac{1}{\lambda_x} x \in U\right\}$. Hence, an open neighborhood U'' of y^* exists such that every trajectory $\{x_t\}_{t\geq 0}$ with $x_0 \in U''\backslash\{y^*\}$ such that $||x_0|| = ||y^*||$, converges to y^*, i.e., $\lim_{t\to\infty} x_t = y^*$.

The ray-projection $\{x_t'\}_{t\geq 0}$ of such a trajectory $\{x_t\}_{t\geq 0}$ with $x_0 \in U''\backslash\{y^*\}$ such that $||x_0|| = ||y^*||$, and $\lim_{t\to\infty} x_t = y^*$ is given by $x' = \frac{x_0}{\sum_{j=1}^{n+1}(x_0)_j}$ and

$$\dot{x}' = \frac{1}{\lambda_x}\left[f(x) - x\sum_{i=1}^{n+1} f_i(x)\right] \text{ for every } x \in \{x_t\}_{t\geq 0}.$$

Clearly, $\lim_{t\to\infty} x_t' = y$. As the factor $\frac{1}{\lambda_x}$ only influences the speed of the dynamics but not the direction, it follows that any trajectory $\{x_t\}_{t\geq 0}$with $x_0 \in U'''$ converges to y under the local ray-projection dynamics given by

$$\dot{x} = f(x) - x\sum_{i=1}^{n+1} f_i(x). \tag{5}$$

So, y is an asymptotically stable fixed point for (5). ■

Proof of Lemma 12. Let $h : S^n \to \mathbb{R}^{n+1}$ be given by $h(x) = f(x) - x\sum_{j=1}^{n+1} f_j(x)$ for all $x \in S^n$. Clearly, h is continuous because f is continuous on the unit simplex. Let $\{x_t\}_{t\geq 0}$ satisfy that some t^* exists such that $\{x_t\}_{t\geq t^*} \subset int\ S^n$ and $\lim_{t\to\infty} x_t = y$. If $y \in int\ S^n$, then by continuity of h it follows that $h(y) = 0^{n+1}$. So, y is an interior fixed point of the dynamics and our earlier result applies, i.e., $y \in E$.
If $y \in bd\ S^n$, then assume $y_j = 0$ and $f_j(y) > 0$. By continuity of h we have $h_j(y) > 0$, and an open neighborhood $U \ni y$ exists such that $h_j(x) > 0$ for all $x \in U$. However, since $y_j = 0$ and $x_j > 0$ for all $x \in \{x_t\}_{t\geq t^*}$ a subsequence $\{x_{t_k}\}_{k\in\mathbb{N}} \subseteq \{x_t\}_{t\geq t^*}$ must exist such that $\dot{(x_{t_k})}_j = h_j(x_{t_k}) < 0$ for all $k \in \mathbb{N}$. Since $\lim_{k\to\infty} x_{t_k} = y$, $\{x_{t_k}\}_{k\in\mathbb{N}} \cap U \neq \varnothing$. This yields a contradiction. Hence, $y_j = 0$ implies $f_j(y) \leq 0$. Furthermore, for $y_j > 0$ we have $h_j(y) = 0 = f_j(y) - y_j\left(\sum_{k=1}^{n+1} f_k(y)\right)$ by continuity which implies $f_j(y) = y_j\left(\sum_{k=1}^{n+1} f_k(x)\right)$. However, then $0 = \sum_{j:y_j>0} y_j f_j(y) = \sum_{j:y_j>0} y_j^2\left(\sum_{k=1}^{n+1} f_k(x)\right)$ and therefore $\sum_{k=1}^{n+1} f_k(x) = 0$ which in turn implies $f_j(y) = 0$ whenever $y_j > 0$, hence $f(y) = 0^{n+1}$.
Suppose $\{x_t\}_{t\geq 0} \overset{t\to\infty}{\to} y$ and it does not hold that t^* exists such that $\{x_t\}_{t\geq t^*} \subset int\ S^n$. Let $T = \{k \in I^{n+1} |\ y_k > 0$ or $[y_k = 0$ and $(x_t)_k > 0$ for all $t > t'$ for some $t' \geq 0]\}$. If follows from the above that for $k \in T$ it must hold that $f_k(y) = 0$. Now, let $h \in I^{n+1}\backslash T$ then $y_h = (x_t)_h = 0$. If (a) holds, then $\dot{x}_h = 0$ regardless whether $f_h(x) > 0$ or $f_h(x) \leq 0$, hence $y \in FP$. Under (b), $\dot{x}_h > 0$ whenever $f_h(x) > 0$ and therefore $f_h(y) \leq 0$ and $y \in E$. ■

Proof of Theorem 13. Let y be a strict saturated equilibrium, then $m = \max_{h\neq j} f_h(y) < 0$ and continuity implies that a neighborhood $U \ni y$ exists such that $\max_{h\neq j} f_h(x) \leq \frac{m}{2}$ for all $x \in U$. Complementarity implies $y = e_j$. Let $C_S(x) = \sum_{h\in S\cup\{j\}} f_h(x)$ for $\varnothing \neq S \subseteq I^{n+1}\backslash\{j\}$. Then, clearly $C_S(y) \leq m < 0$ for all nonempty $S \subseteq I^{n+1}\backslash\{j\}$ and a neighborhood $U' \ni y$ exists such that $\max_{S\subseteq I^{n+1}\backslash\{j\}} C_S(x) \leq \frac{m}{2}$ for all $x \in U'$. Next, let $x \in U \cap U'$, then $(y - x)\cdot\dot{x} = (e_j - x)\cdot f(x) - C_{S'}(x)(e_j - x)\cdot x \geq -\frac{\left(\sum_{h\neq j} x_h f_h(x)\right)}{x_j} - (x_j - x\cdot x)\frac{m}{2} \geq -\frac{1-x_j}{x_j}\frac{m}{2} - (1 - x_j)\frac{m}{2}(x_j - \max_{h\neq j} x_h) = -(1 - x_j)\frac{m}{2}\left(\frac{1}{x_j} + (x_j - \max_{h\neq j} x_h)\right) \geq -(1 - x_j)\frac{m}{2} \geq 0$. Here, we have a strict inequality whenever $x_j \neq 1$. So, $y \in ESE$. ■

7 References

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