Deriving Nash Equilibria as the Supercore for a Relational System*

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Abstract

In this paper, under a binary relation that refines the standard relation which only accounts for single profitable deviations, we obtain that the set of NE strategy profiles of every finite non-cooperative game in normal form coincides with the supercore (Roth, 1976) of its associated abstract system. Further, under the standard relation we show when these two solution concepts coincide.

Keywords: Normal form games, Nash Equilibria, Supercore.

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1 Introduction

In game theory there has been some interest in the study of the equivalence between solution concepts for non-cooperative games in normal form and those for the abstract system (*i.e.*, an abstract set endowed with a binary relation) associated with them.[1] These analyses have focussed on defining a binary relation for which the solution concepts under study coincide.[2] This is important because finding such a binary relation contributes to improving our understanding of how solution concepts in game theory relate to each other.

For instance, Kalai and Schmeidler (1977) associate the mixed extension of a normal form game with an abstract system using a binary relation that only accounts for profitable single deviations. For this binary relation, they find no equivalence between the Nash equilibrium (NE) solution (Nash, 1951) and the admissible set (Kalai, Schmeidler and Pazner, 1976). The coincidence, however, is achieved under a somewhat different binary relation that incorporates the idea of rationalizability. Greenberg (1989), and Kahn and Mookherjee (1992) for the case of infinite games, show that the Coalition Proof Nash Equilibrium solution is equivalent to the von Neumann and Morgenstern stable sets solution under a binary relation that allows for coalitional deviations.

More recently, Inarra, Larrea and Saracho (2007) study the supercore (Roth, 1976) for the abstract system associated with a finite game in normal form by considering the binary relation used by Kalai and Schmeidler (1977). For the case of pure strategies, a sequence of games is given and it is shown that the supercore for the abstract system associated with the first game in the sequence coincides with the set of NE strategy profiles of the last game in that sequence. With regard to the mixed extension of the game, it is shown that the set of NE strategy profiles coincides with the supercore for games with a finite number of NE.

Along this line of research, the purpose of this paper is to define a suitable binary relation under which we may obtain that the set of NE strategy profiles

[1] Abstract systems are considered by von Neumann and Morgenstern in their book Theory of Games and Economic Behavior, 1947. A generalization of this notion, the general system, has been defined by Luo (2001).

[2] Greenberg (1980) considers different binary relations that associate games in normal form with abstract relational systems.

for the mixed extension of *every* finite non-cooperative game in normal form coincides with the supercore for the abstract system associated with it. The new relation that we propose simply refines the conventional one (Kalai and Schmeidler (1977)) by incorporating individual deviations that do not require a strictly positive gain.

The contributions of the paper may be summarized as follows. First, using the new binary relation we obtain that the set of NE strategy profiles coincides with the supercore for *every* finite game. Second, using numerical examples we show that under the standard relation this coincidence need not hold for games with *infinite* NE profiles.[3] Lastly, we establish that the two solution concepts under study coincide *if and only if* there is not a *non-NE strategy profile* in which a player's payoff is equal to her payoff in some NE strategy profile and that is *only* dominated by "some strategy profile which is dominated by some NE strategy profile."

The rest of the paper is organized as follows. Section 2 contains the preliminaries. In Section 3 we give some examples to provide some intuition for the upcoming results. Section 4 contains the definition of the new binary relation and the results.

2 Preliminaries

An *abstract system* is a pair (X, R), where X is a set of elements and R is an irreflexive binary relation, which may be partial, defined on X. The relation R reads "dominates." Hence, if for two elements x, x' in X we have xRx', then we say that x dominates x'.

For any $x \in X$, let $\mathcal{D}(x)$ denote the dominion of x, *i.e.*, $\mathcal{D}(x) = \{x' \in X : xRx'\}$. Given a non-empty subset A of X, we may define the following sets: $\mathcal{D}(A) = \bigcup_{x \in A} \mathcal{D}(x)$ is the set of elements dominated by some element of A and $\mathcal{U}(A) = X - \mathcal{D}(A)$ is the set of elements undominated by any element of A.

A set $A \subseteq X$ is the *core* for (X, R) if $A = \mathcal{U}(X)$.

[3] Inarra et al. (2007) proves the coincidence result for *finite* games with a finite number of NE profiles. Wilson (1971) shows that in "almost all" finite games the number of NE is finite and odd. See also Harsanyi (1973).

A set $A \subset X$ is a *vN&M stable set* of (X, R) if $A = \mathcal{U}(A)$. Hence $A \subset \mathcal{U}(A)$, which is known as the internal stability condition, and $\mathcal{U}(A) \subset A$, known as the external stability condition.

A *subsolution* of (X, R) is a subset A of X such that *(i)* $A \subset \mathcal{U}(A)$ and *(ii)* $A = \mathcal{U}^2(A)$, where $\mathcal{U}^2(A) = \mathcal{U}(\mathcal{U}(A))$.

Let $\mathcal{P}(A) = \mathcal{U}(A) - A$ be the set of elements undominated by any element of A excluding the elements of A. Given a subsolution A, the set X may be partitioned into three sets: A, $\mathcal{D}(A)$ and $\mathcal{P}(A)$.[4] Moreover, if $A \subset \mathcal{U}(A)$, given that $\mathcal{U}(\mathcal{U}(A)) = \mathcal{U}(A \cup \mathcal{P}(A)) = X - \mathcal{D}(A \cup \mathcal{P}(A)) = A \cup \mathcal{P}(A) - \mathcal{D}(\mathcal{P}(A))$, then we have $\mathcal{U}^2(A) = A \Longleftrightarrow \mathcal{P}(A) \subset \mathcal{D}(\mathcal{P}(A))$ and $A \cap \mathcal{D}(\mathcal{P}(A)) = \emptyset$.

The most significant subsolution of (X, R) is the intersection of all subsolutions which is called the *supercore*.[5]

This solution concept is a generalization of the vN&M stable set. Note that if A is a vN&M stable set then A is a subsolution with $\mathcal{P}(A) = \emptyset$.

A finite *normal form game* Γ^N is a triple $< N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} >$ where $N = \{1, ..., n\}$ is the finite set of players, S_i is the finite set of strategies for player i and $u_i : S = \times_{i \in N} S_i \longrightarrow \mathbb{R}$ is player i's payoff function.

A *mixed extension of the game* Γ^N is a triple $< N, \{\Delta S_i\}_{i \in N}, \{U_i\}_{i \in N} >$ where ΔS_i is the simplex of the mixed strategies for player i, and $U_i : \Delta(S) = \times_{i \in N} \Delta(S_i) \longrightarrow \mathbb{R}$, assigns to $\sigma \in \Delta(S)$, where σ denotes a mixed strategy profile, the expected value under u_i of the lottery over S that is induced by σ, so that $U_i(\sigma) = \sum_{s \in S} (\prod_{j \in N} \sigma_j(s_j) u_i(s))$.

The strategy profile $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ is a *Nash equilibrium* in the mixed extension of the game Γ^N if σ_i^* is a best response to $\sigma_{-i}^* = (\sigma_1^*, ..., \sigma_{i-1}^*, \sigma_{i+1}^*, ..., \sigma_n^*)$ for all $i \in N$. The set of NE strategy profiles of a *mixed extension of the game* Γ^N is denoted by Σ^*.

In order to associate an abstract system (X, R) to the mixed extension of a finite normal form game we follow the approach developed by Greenberg (1990). He proposes a negotiation procedure among players that can be described as follows.[6] Suppose that the strategy profile σ is proposed to players. Then each

[4] These sets are the good, the ugly and the bad in terms of Kahn and Mookherjee (1992).
[5] If $\mathcal{U}(X) = \emptyset$, then the supercore of (X, R) is the empty set (Roth, 1976).
[6] In Greenberg (1980) this procedure is called an individual contingent threat situation.

individual player can object to the prevailing profile and can threat the others by saying that she will choose another strategy. If she does this, player i induces σ' from σ. The set of profiles that player i can induce from σ is defined as:

$$\gamma_i(\sigma) = \{\sigma' \in \Delta(S) : \sigma'_j = \sigma_j \text{ for all } j \neq i, j \in N\}.$$

Following this negotiation procedure among players and considering the strategy profiles of $\Delta(S)$, the following abstract system can be defined.

A dominance system of the mixed extension of a game Γ^N is a pair $(\Delta(S), \succ)$ where $\succ$ is the binary relation defined on $\Delta(S)$ such that:

$$\sigma' \succ \sigma \text{ if there exists } i \in N \text{ such that } \sigma' \in \gamma_i(\sigma) \text{ and } U_i(\sigma') > U_i(\sigma).$$

To conclude these preliminaries, note that Σ^* is included in the supercore for the abstract system $(\Delta(S), \succ)$ since the strategy profiles in Σ^* are the undominated profiles in $\Delta(S)$. That is, $Core(\Delta(S), \succ) = \Sigma^*$.

The two examples in the following section show that in order to derive the coincidence between Σ^* and the supercore for the abstract system associated with the mixed extension of the game Γ^N it is necessary to define a binary relation that is different from $\succ$. In addition, the examples provide some intuition as to the kind of binary relation that may be required to obtain that coincidence.

3 Some Examples

Inarra et al. (2007) show that if Σ^* is finite then Σ^* is the supercore for the abstract system $(\Delta(S), \succ)$. Example 1 shows that it is not necessary that Σ^* is finite to obtain Σ^* as the supercore for $(\Delta(S), \succ)$. In Example 2, however, there is no coincidence between Σ^* and the supercore. As we shall see, the strategy profiles that belong to the supercore are the profiles of Σ^* and some undominated profiles by the elements of Σ^* which have the following property: In all of them there is a player with a payoff equal to her payoff in some NE profile. We next analyze these examples in some detail.

Let $\sigma^* \in \Sigma^*$ be a NE strategy profile. The dominion of Σ^* is $\mathcal{D}(\Sigma^*) = \bigcup_{\sigma^* \in \Sigma^*} \mathcal{D}(\sigma^*)$. Let $\mathcal{U}(\Sigma^*) = \Delta(S) - \mathcal{D}(\Sigma^*)$ and $\mathcal{P}(\Sigma^*) = \mathcal{U}(\Sigma^*) - \Sigma^*$.

Example 1. Consider the mixed extension of the following game:

	b_1	b_2
a_1	1,1	0,1
a_2	0,0	0,1

Let p be the probability that player 1 chooses a_1 and let q be the probability that player 2 chooses b_1. In this example $U_1(\sigma) = pq$, $U_2(\sigma) = 1 - q + pq$, and the players' best response functions are:

$$BR_1(q,\ 1-q) = \begin{cases} (p,\ 1-p) & \text{if } q = 0 \\ (1, 0) & \text{otherwise} \end{cases}$$

and

$$BR_2(p,\ 1-p) = \begin{cases} (q,\ 1-q) & \text{if } p = 1 \\ (0, 1) & \text{otherwise.} \end{cases}$$

It is easy to check that $\Sigma^* = \{(p,\, 1-p,\, 0,\, 1) : 0 \le p \le 1\} \cup \{(1,\, 0,\, q,\, (1-q)) : 0 \le q \le 1\}$ and $\mathcal{D}(\Sigma^*) = \{(p, 1-p, q, 1-q) : 0 \le p < 1,\ 0 < q \le 1\}$. Since $\Sigma^* \cup \mathcal{D}(\Sigma^*) = \Delta(S)$ then $\mathcal{P}(\Sigma^*) = \emptyset$. Hence, Σ^* is the supercore for $(\Delta(S), \succ)$. Since $\mathcal{P}(\Sigma^*) = \emptyset$, Σ^* is the vN&M stable set for $(\Delta(S),\ \succ)$.

Example 2. Consider the mixed extension of the following game:

	b_1	b_2
a_1	1,0	0,1
a_2	0,1	0,0

In this example $U_1(\sigma) = pq$, $U_2(\sigma) = p(1-q) + (1-p)q$ and the players' best response functions are:

$$BR_1(q,\ 1-q) = \begin{cases} (p,\ 1-p) & \text{if } q = 0 \\ (1, 0) & \text{otherwise} \end{cases}$$

and

$$BR_2(p,\ 1-p) = \begin{cases} (q,\ 1-q) & \text{if } p = 1/2 \\ (1, 0) & \text{if } p < 1/2 \\ (0, 1) & \text{if } p > 1/2. \end{cases}$$

[Insert Figures 1 and 2 about here]

It is easy to check that $\Sigma^* = \{(p,\ 1-p,\ 0,\ 1) : \frac{1}{2} \leq p \leq 1\}$, which is represented by the thick line in Figure 1. The set $\mathcal{D}(\Sigma^*) = \{(p,\ 1-p,\ q,\ 1-q) : \frac{1}{2} < p \leq 1,\ 0 < q \leq 1\}$ is represented by the shaded area in Figure 1 and the set $\mathcal{P}(\Sigma^*) = \Delta(S) - \mathcal{D}(\Sigma^*) - \Sigma^* = \{(p, 1-p, q, 1-q) : 0 \leq p \leq \frac{1}{2}, 0 \leq q \leq 1\} - \{\frac{1}{2}, \frac{1}{2}, 0, 1\}$ is the remaining area[7].

Every profile of $\mathcal{P}(\Sigma^*)$ is dominated by some profile of this set except the profiles in $\{(\frac{1}{2}, \frac{1}{2}, q, 1-q),\ 0 < q \leq 1\}$, the thick dotted line in Figure 1. These profiles are *only* dominated by some profile of $\mathcal{D}(\Sigma^*)$. Hence $\mathcal{P}(\Sigma^*) \not\subset \mathcal{D}(\mathcal{P}(\Sigma^*))$ and the supercore does not coincide with Σ^*. Further, it is important to note that the profiles of $\{(\frac{1}{2}, \frac{1}{2}, q, 1-q),\ 0 < q \leq 1\}$ share a common feature: The payoff for player 2 $U_2(\frac{1}{2}, \frac{1}{2}, q, 1-q) = \frac{1}{2}$, coincides with her payoff in the NE strategy profile $(\frac{1}{2}, \frac{1}{2}, 0, 1)$. The union of the set of these "tied-profiles" and Σ^* gives a new set Σ^1 which is represented by the thick line in Figure 2. Some profiles of $\mathcal{P}(\Sigma^*)$ are in $\mathcal{D}(\Sigma^1)$. Specifically, $\mathcal{D}(\Sigma^1) = \{(p,\ 1-p,\ q,\ 1-q) : 0 \leq p \leq 1, 0 < q \leq 1\} - \Sigma^1$, the shaded area in Figure 2. Therefore $\mathcal{P}(\Sigma^1) = \{(p,\ 1-p,\ q,\ 1-q) : 0 \leq p < \frac{1}{2},\ q = 0\}$, the thick dotted line in Figure 2. The profiles of $\mathcal{P}(\Sigma^1)$ do not dominate each other, and hence the supercore does not coincide with Σ^1. These profiles share a common feature: The payoff for player 1 $U_1((p,\ 1-p,\ 0,\ 1) : 0 \leq p < \frac{1}{2})$ is equal to $U_1(\frac{1}{2}, \frac{1}{2}, 0, 1) = 0$ where $(\frac{1}{2}, \frac{1}{2}, 0, 1) \in \Sigma^*$. The union of the set Σ^1 and the set of these additional "tied-profiles" forms the set Σ^2 and $\mathcal{P}(\Sigma^2) = \emptyset$. Hence Σ^2 is the supercore for the system associated to the game of this example, represented by the thick lines and the thick dotted line in Figure 2. Moreover, since $\mathcal{P}(\Sigma^2) = \emptyset$ we have that Σ^2 is the vN&M stable set for $(\Delta(S), \succ)$.

Finally, let us show why Σ^2 is the supercore for $(\Delta(S), \succ)$. Inarra et al. (2007) observe that A is the supercore for (X, R) if and only if A is the supercore for (Y, R) where $Y = X - \mathcal{D}(Core(X, R))$ in (X, R).

In our context $X = \Delta(S)$ and $R = \succ$. For $(\Delta(S) \succ)$, the $Core(\Delta(S), \succ) = \Sigma^*$ and $Y = \Delta(S) - \mathcal{D}(\Sigma^*)$. For $(Y, \succ)$, the $Core(Y, \succ) = \Sigma^1$ and $Z = Y - \mathcal{D}(\Sigma^1)$. Now, Σ^2 is the supercore for $(Z, \succ)$ since $Z = \Sigma^2$ and the $Core(Z, \succ) = \Sigma^2$. Thus by applying the above observation we conclude that Σ^2 is the supercore for $(\Delta(S), \succ)$.

[7] Note that given $\sigma^* = (p_i, 1-p_i, 0, 1)$, $\mathcal{D}(\sigma^*) = \{(p_i, 1-p_i\ q, 1-q),\ 0 < q \leq 1\}$.

4 The Binary Relation and the Results

In this section we define a binary relation $\gg$ over $\Delta(S)$ by refining the relation $\succ$, under which the supercore for $(\Delta(S), \gg)$ coincides with Σ^*. Further we also obtain a result concerning the behavior of the non-NE strategy profiles in the abstract system $(\Delta(S), \succ)$.[8]

Definition *A weak dominance system of the mixed extension of a game* Γ^N is a pair $(\Delta(S), \gg)$ where $\gg$ is the binary relation defined on $\Delta(S)$ such that for any two distinct strategy profiles $\sigma', \sigma \in \Delta(S)$:

$$\begin{aligned} \sigma' &\gg \sigma \text{ if there exists } i \in N \text{ such that } \sigma' \in \gamma_i(\sigma) \text{ and } U_i(\sigma') > U_i(\sigma) \text{ or} \\ U_i(\sigma') &= U_i(\sigma) \text{ and } U_j(\sigma'') > U_j(\sigma) \text{ for some } \sigma'' \in \gamma_j(\sigma) \text{ and } j \neq i. \end{aligned}$$

Let us analyze the implications of this definition. Consider $\sigma' \gg \sigma$:

(i) In the case of $U_i(\sigma') > U_i(\sigma)$ we are under relation $\succ$, and hence $\sigma' \gg \sigma$. *(ii)* In the case of $U_i(\sigma') = U_i(\sigma)$, being σ' a NE strategy profile, the rationale for $\sigma' \gg \sigma$ is to consider that player i has a preference for keeping her current payoff in σ, and in σ' she is "protected" from deviations by other player. For instance, consider the strategy profiles in Example 2: $\sigma = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the NE profile $\sigma' = (\frac{1}{2}, \frac{1}{2}, 0, 1)$ where $\sigma' \in \gamma_2(\sigma)$. In this case we have that $U_2(\sigma) = U_2(\sigma') = \frac{1}{2}$. It is easy to find a third profile *e.g.*, $\sigma'' = (1, 0, \frac{1}{2}, \frac{1}{2})$ where $\sigma'' \in \gamma_1(\sigma)$ such that $\frac{1}{2} = U_1(\sigma'') > U_1(\sigma) = \frac{1}{4}$. Since player 1 objects to σ, player 2 objects to σ by choosing σ', and therefore $\sigma' \gg \sigma$. *(iii)* In case of $U_i(\sigma') = U_i(\sigma)$, being σ' and σ two non-NE strategy profiles, we find the following: From any of these profiles there is some player $j \neq i$ who objects to σ and σ'; therefore we have $\sigma' \gg \sigma$ and $\sigma \gg \sigma'$. Thus relation $\gg$ is not asymmetric.

The difference between the conventional relation $\succ$ and the new relation $\gg$ can be also explained in the following terms: Player i has an objection to σ if there exists $\sigma' \in \gamma_i(\sigma)$ in which her payoff is greater than in σ. Hence $\sigma' \succ \sigma$ and $\sigma' \gg \sigma$. Moreover if there is a player $j \neq i$ that has an objection to σ, *e.g.*,

[8] We thank an anonymous referee for motivating us to study this behavior.

by choosing σ'', then player i objects to σ if there exists a $\sigma' \in \gamma_i(\sigma)$ in which her payoff is equal to the one obtained in σ. Hence $\sigma' \gg \sigma$ but not $\sigma' \succ \sigma$.

Thus, the binary relation we propose to break ties keeps the main feature of the conventional one while at the same time incorporates a preference for keeping or guaranteeing one´s current payoff. This preference for keeping the current payoff may be interpreted as a form of *endowment effect* where the endowment in this context is the payoff in σ.[9] In order to guarantee her endowment, a player objects to σ by using σ' where she keeps her current payoff instead of allowing objections to the prevailing profile σ made by other players that could perhaps (eventually) result in a lower payoff. We believe that this interpretation provides a natural justification to extend the scope of the standard binary relation $\succ$ to the new one $\gg$. Further, various forms of endowment effects or status quo bias have been extensively documented in both the experimental and empirical literature. The endowment effect is also a critical ingredient of Kahneman and Tversky (1979) prospect theory.

Before showing the coincidence between the set of NE strategy profiles and the supercore for $(\Delta(S), \gg)$ we introduce a lemma.

Lemma. $\mathcal{D}(\Sigma^*) \cup \Sigma^*$ in $(\Delta(S), \gg)$ is a closed subset of $\Delta(S)$.

Proof. It is known that Σ^* is a compact subset of $\Delta(S)$;[10] hence, Σ^* is closed. Therefore, it is sufficient to show that the closure of $\mathcal{D}(\Sigma^*)$ is contained in $\mathcal{D}(\Sigma^*) \cup \Sigma^*$.

Let us consider a sequence $\{\sigma_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\Sigma^*)$ such that $\lim_{n\to\infty}\sigma_n = \sigma$. We will see that $\sigma \in \mathcal{D}(\Sigma^*) \cup \Sigma^*$.

Since $\sigma_n \in \mathcal{D}(\Sigma^*)$, there is a NE strategy profile σ_n^* such that for some player $i \in N$, $\sigma_n^* \in \gamma_i(\sigma_n)$ and $U_i(\sigma_n^*) \geq U_i(\sigma_n)$. Taking into account that the set Σ^* is compact and that $\{\sigma_n^*\}_{n\in\mathbb{N}} \subset \Sigma^*$ we can assume without loss of generality the existence of a profile $\sigma^* \in \Sigma^*$ such that $\lim_{n\to\infty}\sigma_n^* = \sigma^*$. (If this is not the case we may replace that sequence by the appropriate subsequence).

[9] This effect is a general phenomenon first uncovered in Knetsch (1989) who demostrated with a simple experiment that preferences are not independent of current entitlements. Several experimental studies have confirmed this important finding (see *e.g.*, Camerer (2003), Kahneman, Knetsch and Thaler (1990)) and many references therein.)

[10] See for example, Fudenberg and Tirole (1991).

Now, let $\mathbb{N}(i) = \{n \in \mathbb{N} : \sigma_n^* \in \gamma_i(\sigma_n)\}$ for each $i \in N$. It is clear that for some $j \in N$ the set $\mathbb{N}(j)$ is countable. Hence, we can choose the subsequences $\{\sigma_n'\}_{n\in\mathbb{N}}$ of $\{\sigma_n\}_{n\in\mathbb{N}}$ and $\{(\sigma^*)_n'\}_{n\in\mathbb{N}}$ of $\{\sigma_n^*\}_{n\in\mathbb{N}}$ such that $(\sigma^*)_n' \in \gamma_j(\sigma_n')$, and $U_j((\sigma^*)_n') \geq U_j(\sigma_n')$ for all $n \in \mathbb{N}$. Therefore, taking the limit on each side in the last expression we have:

$$lim_{n\to\infty} U_j((\sigma^*)_n') \geq lim_{n\to\infty} U_j(\sigma_n').$$

Since $lim_{n\to\infty}(\sigma^*)_n' = \sigma^*$, $lim_{n\to\infty}\sigma_n' = \sigma$, and U_j is a continuous function, we have $U_j(\sigma^*) \geq U_j(\sigma)$. Given that $\sigma^* \in \gamma_j(\sigma)$ if $U_j(\sigma^*) > U_j(\sigma)$ then $\sigma^* \gg \sigma$. On the contrary, suppose that $U_j(\sigma^*) = U_j(\sigma)$. Now if $\sigma \notin \Sigma^*$ then $U_k(\sigma') > U_k(\sigma)$ for some $\sigma' \in \gamma_k(\sigma)$ and $k \in \mathbb{N}$. Clearly $k \neq j$; otherwise we would have $U_j(\sigma') > U_j(\sigma^*)$, which contradicts that $\sigma^* \in \Sigma^*$. Therefore $\sigma^* \gg \sigma$, and either $\sigma \in \mathcal{D}(\Sigma^*)$ or $\sigma \in \Sigma^*$. Thus, the lemma follows. ■

Now, using Example 2 we illustrate that under the binary relation $\succ$, the set $\Sigma^* \cup \mathcal{D}(\Sigma^*)$ is not closed. To see this, consider the sequence $\{\sigma_n\}$ such that $\sigma_n = (\frac{1}{2} + \frac{1}{n}, \frac{1}{2} - \frac{1}{n}, \frac{1}{2}, \frac{1}{2})$ with $n \geq 2$. Note that $\{\sigma_n\} \subset \mathcal{D}(\Sigma^*)$ (see Figure 1). Since $lim_{n\to\infty}\sigma_n = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $\frac{1}{2} = U_2(lim_{n\to\infty}\sigma_n) = U_2(\sigma^*)$ with $\sigma^* = (\frac{1}{2}, \frac{1}{2}, 0, 1)$, then $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \notin \mathcal{D}(\Sigma^*)$ and as $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \notin \Sigma^*$ then $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \notin \Sigma^* \cup \mathcal{D}(\Sigma^*)$.

On the other hand, as shown in the previous lemma under the binary relation $\gg$, the set $\Sigma^* \cup \mathcal{D}(\Sigma^*)$ is closed. More precisely, under $\gg$, we have $\Sigma^* = \{(p,\ 1-p,\ 0,\ 1) : \frac{1}{2} \leq p \leq 1\}$, $\mathcal{D}(\Sigma^*) = \{(p,\ 1-p,\ q,\ 1-q) : \frac{1}{2} < p \leq 1,\ 0 < q \leq 1\} \cup \{(\frac{1}{2},\ \frac{1}{2},\ q,\ 1-q) : 0 < q \leq 1\} \cup \{(p,\ 1-p,\ 0,\ 1) : 0 \leq p < \frac{1}{2}\}$ (this is the shaded area and the thick dotted line in Figure 1 jointly with the thick dotted line in Figure 2) and $\mathcal{P}(\Sigma^*) = \{(p,\ 1-p,\ q,\ 1-q) : 0 \leq p < \frac{1}{2},\ 0 < q \leq 1\}$ (this is the shaded area in Figure 2 minus the shaded area in Figure 1). Since $\mathcal{P}(\Sigma^*) \subset \mathcal{D}(\mathcal{P}(\Sigma^*))$, Σ^* is the supercore for $(\Delta(S), \gg)$.[11]

In what follows we show the coincidence of the supercore and the NE solution under the binary relation $\gg$ for every finite non-cooperative game.

Theorem 1 *Σ^* is the supercore for $(\Delta(S), \gg)$.*

[11] Note that as $\Sigma^* = \mathcal{U}(\Delta(S))$, we have $\Sigma^* \subset \mathcal{U}(\Sigma^*)$ and $\Sigma^* \cap \mathcal{D}(\mathcal{P}(\Sigma^*)) = \emptyset$. Then $\mathcal{U}^2(\Sigma^*) = \Sigma^*$ is equivalent to $\mathcal{P}(\Sigma^*) \subset \mathcal{D}(\mathcal{P}(\Sigma^*))$ (see Section 2).

Proof. Given that any subsolution for $(\Delta(S), \gg)$ contains Σ^*, it is sufficient to prove that Σ^* is a subsolution.[12] That is, $\Sigma^* \subset \mathcal{U}(\Sigma^*)$ and $\Sigma^* = \mathcal{U}^2(\Sigma^*)$.

Clearly, $\Sigma^* \subset \mathcal{U}(\Sigma^*)$ in $(\Delta(S), \gg)$. If $\Sigma^* = \mathcal{U}(\Sigma^*)$ then Σ^* is a vN&M stable set, and thus Σ^* is a subsolution. So, let us assume that $\mathcal{P}(\Sigma^*) \neq \emptyset$. Since $\Sigma^* = \mathcal{U}(\Delta(S))$, it remains to show that $\Sigma^* = \mathcal{U}^2(\Sigma^*)$ or equivalently that $\mathcal{P}(\Sigma^*) \subset \mathcal{D}(\mathcal{P}(\Sigma^*))$.

Let $\sigma \in \mathcal{P}(\Sigma^*)$. We will show that there exists a $\widetilde{\sigma} \in \mathcal{P}(\Sigma^*)$ such that $\widetilde{\sigma} \gg \sigma$. Since $\sigma \notin \Sigma^*$, σ_i is not the best response to σ_{-i} for some player i. Hence, there is a $\sigma' \in \gamma_i(\sigma)$ such that $U_i(\sigma') > U_i(\sigma)$.

Now, if $\sigma' \in \mathcal{P}(\Sigma^*)$ the proof is complete. If this is not the case then set $\sigma_\lambda = \lambda\sigma + (1-\lambda)\sigma'$ for all $\lambda \in [0, 1)$. By the linearity of U_i we have that $U_i(\sigma_\lambda) > U_i(\sigma)$, and since $\sigma_\lambda \in \gamma_i(\sigma)$, it follows that $\sigma_\lambda \gg \sigma$. Further, by the previous lemma we know that $\mathcal{D}(\Sigma^*) \cup \Sigma^*$ is a closed subset of $\Delta(S)$. Therefore, $\mathcal{P}(\Sigma^*)$ is an open subset of $\Delta(S)$. This implies that there exists an $\varepsilon > 0$ such that the open ball $B(\sigma, \varepsilon) \subset \mathcal{P}(\Sigma^*)$. By choosing a $\lambda \in (0, 1)$ such that $\sigma_\lambda \in B(\sigma, \varepsilon)$ we have that $\sigma_\lambda \in \mathcal{P}(\Sigma^*)$. Since $\sigma_\lambda \gg \sigma$ we conclude that $\sigma \in \mathcal{D}(\mathcal{P}(\Sigma^*))$ and Theorem 1 obtains. ■

Hereafter, whenever we are dealing with the system $(\Delta(S), \gg)$ the dominion of a set A and the sets of elements undominated by any element of A will be denoted respectively as: $\mathcal{D}_{\gg}(A), \mathcal{U}_{\gg}(A)$, and $\mathcal{P}_{\gg}(A)$. The notation $\mathcal{D}(A), \mathcal{U}(A)$ and $\mathcal{P}(A)$ refers to the conventional system $(\Delta(S), \succ)$.

As mentioned in the introduction our analysis allows us to describe the behavior of the non-NE strategy profiles for the system $(\Delta(S), \succ)$.

Let $\sigma' \sim \sigma$ if there exits $i \in N$ such that $\sigma' \in \gamma_i(\sigma)$ and $U_i(\sigma') = U_i(\sigma)$. Then we can establish the following result.

Theorem 2 *Σ^* is not the supercore for $(\Delta(S), \succ)$ if and only if there is a $\sigma \in \mathcal{P}(\Sigma^*)$ such that $\sigma \sim \sigma^*$ for some $\sigma^* \in \Sigma^*$ and σ is not dominated by any of the strategy profiles of $\mathcal{P}(\Sigma^*)$.*

Proof. $(\Longrightarrow)$: If Σ^* is not the supercore for $(\Delta(S), \succ)$ then Σ^* is not a subsolution. Hence $\mathcal{P}(\Sigma^*) \not\subset \mathcal{D}(\mathcal{P}(\Sigma^*))$ and there is a $\sigma \in \mathcal{P}(\Sigma^*)$ such that σ is

[12] Given a subsolution A for $(\Delta(S), \gg)$, since the $Core(\Delta(S), \gg) = \Sigma^*$, Σ^* must be a subset of A.

not dominated by any strategy profile of $\mathcal{P}(\Sigma^*)$. Now, given that σ is not a NE strategy profile, then either $\sigma \in \mathcal{P}_{\gg}(\Sigma^*)$ or $\sigma \in \mathcal{D}_{\gg}(\Sigma^*)$. We next show that $\sigma \in \mathcal{D}_{\gg}(\Sigma^*)$. Assume that $\sigma \in \mathcal{P}_{\gg}(\Sigma^*)$. In the proof of Theorem 1 it is shown that for every $\sigma \in \mathcal{P}_{\gg}(\Sigma^*)$ there exists a $\widetilde{\sigma} \in \mathcal{P}_{\gg}(\Sigma^*)$ such that $\widetilde{\sigma} \in \gamma_i(\sigma)$ and $U_i(\widetilde{\sigma}) > U_i(\sigma)$. Since $\mathcal{P}_{\gg}(\Sigma^*) \subset \mathcal{P}(\Sigma^*)$ then $\widetilde{\sigma} \in \mathcal{P}(\Sigma^*)$ and $\widetilde{\sigma} \succ \sigma$ which contradicts that σ is not dominated by any strategy profile of $\mathcal{P}(\Sigma^*)$. Therefore $\widetilde{\sigma} \in \mathcal{D}_{\gg}(\Sigma^*)$. This implies that there exists $\sigma^* \in \Sigma^*$ such that $\sigma^* \gg \sigma$. But since $\sigma \in \mathcal{P}(\Sigma^*)$ we have that $\sigma^* \not\succ \sigma$ and it must be $\sigma \sim \sigma^*$.

$(\Longleftarrow)$: Since there is $\sigma \in \mathcal{P}(\Sigma^*)$ such that σ is not dominated by any strategy profile of $\mathcal{P}(\Sigma^*)$ then $\mathcal{P}(\Sigma^*) \not\subset \mathcal{D}(\mathcal{P}(\Sigma^*))$. Hence Σ^* is not a subsolution and therefore Σ^* is not the supercore for $(\Delta(S), \succ)$. ■

Summing up, under the binary relation $\succ$ the set $\Delta(S)$ may be partitioned into four sets: Σ^*, $\mathcal{D}(\Sigma^*)$, the set of profiles of $\mathcal{P}(\Sigma^*)$ where each profile is dominated by another profile of $\mathcal{P}(\Sigma^*)$, *and* the set of profiles of $\mathcal{P}(\Sigma^*)$ in which no profile is dominated by another profile of $\mathcal{P}(\Sigma^*)$. In every profile of the latter set, there is a player whose payoff is equal to her payoff in some NE strategy profile.[13] If the latter set is empty then Σ^* coincides with the supercore.[14] Further, if $\mathcal{P}(\Sigma^*) = \emptyset$ then Σ^* coincides with a vN&M stable set for $(\Delta(S), \succ)$.

[13] Since the profiles in this set are non-NE profiles, they are dominated *only* by "some strategy profile dominated in turn by some NE strategy profile."

[14] This is the case for finite games with a finite number of NE strategy profiles (Inarra et al., 2007).

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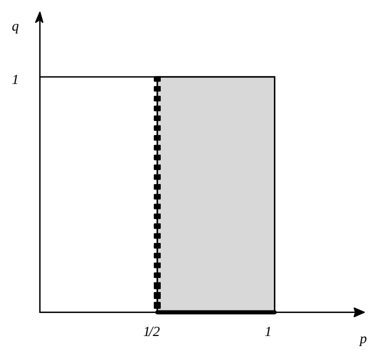

Figure 1

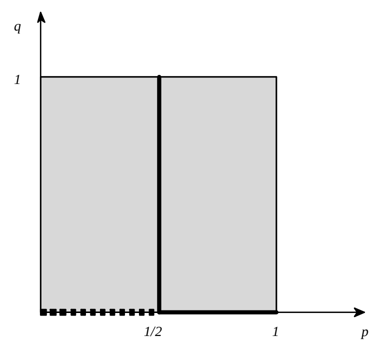

Figure 2