EFFORT, RACE GAPS, AND AFFIRMATIVE ACTION: A GAME-THEORETIC ANALYSIS OF COLLEGE ADMISSIONS

(JOB MARKET PAPER, PART I)

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ABSTRACT. I construct a strategic model of incomplete information where many heterogeneous students compete for seats at colleges and universities of varying prestige. I argue that the model is strategically equivalent to an all-pay auction where agents differ with respect to their cost of bidding. Using tools from auction theory, I characterize equilibrium behavior by deriving a set of equations that approximate equilibrium strategies when the number of players is large. I use the model to analyze the effects of Affirmative Action policies on effort choice and achievement gaps. I also compare the performance of two common implementations of Affirmative Action: quotas and admission preferences. I show that these policies have very different effects on effort, achievement gaps, and allocation of college admissions in equilibrium. The model suggests that admissions preferences (such as those previously used in undergraduate admissions at the University of Michigan) are unambiguously detrimental to effort incentives, and are ineffective as an allocational mechanism. Quotas perform better than admissions preferences, with some positive and some negative effects on effort and achievement gaps. Both policies widen the achievement gap among the best and brightest students.

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1. INTRODUCTION

For several decades, race-conscious admission policies have been used by American colleges and universities with the objective of aiding underrepresented racial minority groups to overcome competitive disadvantages. It has been widely documented that academic gaps between minorities and non-minorities are persistent and pronounced. For example, Hickman [13] shows that the gap between median SAT scores for minorities and non-minorities in 1996 was 150 points (out of 1600 possible); moreover, the median score among minorities (870) was at the 22^{nd} percentile among non-minorities. Among students who attend college, minorities are under-represented at elite schools, and over-represented at lower-tier schools. Using methodology similar to that developed by US News & World Report for its annual publication *America's Best Colleges*, Hickman computes rankings for a set of 1,314 colleges and universities to characterize racial representation in American post-secondary education. Minority students made up 17.65% of all first-time freshmen in 1996. However, they accounted for only 11.04% of enrollment at schools in the top quartile, whereas they made up 29.71% of enrollment in the bottom quartile.[1] These circumstances are viewed by many as residual effects of the institutionalized racism of generations past, and Affirmative Action (AA) in college admissions is meant to address these social ills.

However, despite its intentions, much debate has arisen over the possible effects of AA on the incentives for academic achievement. Supporters claim that it levels the playing field, so to speak. The argument is that AA motivates minority students to achieve at the highest of levels by placing within reach seats in top universities—an outcome previously seen by many as unattainable. In this way, it makes invesstment in costly effort more worthwhile for the beneficiaries of the policy. Opponents claim that by lowering the standards for minority college applicants, AA creates adverse incentives for them to exert less effort in competition for admission to college. By making academic performance less important for one's outcome, they argue, AA creates a tradeoff between promoting equality and maximizing academic performance.

Economic researchers have weighed in on this debate, and there have been some economic theory and experimental findings in support of the former viewpoint: research suggests that an effort-equality tradeoff does not exist, and AA leads to both minority *and* non-minority students exerting more effort.[2] However, existing models rely on unrealistic assumptions about the nature of the competitive disadvantage for minorities, or about the information available to the policy maker. Moreover, there is currently no theoretical framework that allows for an adequate empirical analysis of the implications of AA for academic effort. Fryer and Loury [9] hint at the need

[1] In generating these estimates, the working definition of the term "minority" is the union of the following three race classifications: African-American, Hispanic and American-Indian/Alaskan Native. For a more detailed discussion, see Hickman [13].

[2] See for example Fain [6], Franke [7], Fu [10, 11] and Schotter and Weigelt [23].

for better theory and empirics on AA in their 2005 paper "Affirmative Action and its Mythology" (p. 153):

> "Our view is that confident a priori assertions about how affirmative action affects incentives are unfounded. Indeed, economic theory provides little guidance on what is ultimately a subtle and context-dependent empirical question. First principles, commonsense intuitions and anecdotal evidence are simply inadequate to the task here."

I propose a new model of college admissions to address the deficiencies in the literature. In the model, students are heterogeneous with respect to their academic competitiveness, quantified by a privately-known study cost type. This private type parameterizes a function that determines the utility cost of achieving a given grade. Students observably belong to different demographic groups, and I allow for costs to be asymmetrically distributed across groups.[3] For any student who wishes to go to college, there is a seat open at some institution, but not all are equally desirable. Allocations of college seats are determined by a single entity called "The Board" (short for the college admissions board), that maps academic grades into outcomes according to some rule. The allocation determined by The Board can be thought of as a centralized implementation of the outcome achieved by a matching market in which college candidates submit applications and individual schools reply with acceptance/rejection letters. The Board may incorporate race into its allocation rule, and under the payoffs induced by a given rule, students optimally choose their effort level, based on private costs and competition they face from other students.

I argue that this model of competition for college admissions is strategically equivalent to an all-pay auction. Using analytic tools borrowed from auction theory, I solve for equilibrium behavior in order to assess the implications of race-conscious admission policies for academic incentive structures. A meaningful analysis of college admissions must be applicable to settings where the number of competitors is very large, so I adopt a solution concept which I refer to as an *approximate equilibrium* that delivers analytic and computational tractability.[4]. This is an important technical innovation, as the analysis becomes unwieldy even for moderately large sets of competitors. The approximate equilibrium is a set of equations which characterize academic achievement to arbitrary precision as the number of competitors gets large. These, in turn, characterize equilibrium grade distributions, which allow for a dissection of the various effects of admission policies.

A novel feature of the model is that it allows for comparisons of alternative AA policies. The first variety I study is a quota rule, where fixed quantities of prizes are reserved for allocation to

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[3]My intention is *not* to suggest that asymmetry reflects differences of inherent ability across differing demographic groups. I argue that a more plausible interpretation involves asymmetry arising from socioeconomic factors which happen to be correlated with race. See Section 2 for a full discussion of the asymmetry assumption.

[4]For example, the US National Center for Education Statistics reported that in 2005 over 1.8 million recent high-school graduates enrolled in college.

competitors from each demographic group. The second variety is American-style AA, commonly referred to as an "admission preference" or a "plus-factor" system, where members of one group are effectively awarded extra points based on their group status. I also consider a color-blind admissions rule that ignores race. My objective is to address four research questions. *(i)* What effect do AA policies have on effort incentives? Do they encourage students to study more or less? Do they affect all students' achievement in the same way? *(ii)* What effect do they have on racial academic achievement gaps? Do they widen or narrow the difference in achievement across demographic groups? *(iii)* How effective is AA at achieving proportional enrollment in college? In other words, if there is a behavioral response to a given policy, does it retain a comparable effect one might expect holding behavior constant? And finally, *(iv)* Are there any differences among alternative AA policies in terms of criteria *(i)*-*(iii)*?

Although a complete policy analysis is difficult for a researcher who cannot observe the social choice function, by making some light assumptions on the preferences of the policy maker, one can still guide the policy debate in meaningful ways. Henceforth, I assume that the policy-maker has the following three objectives in selecting an admission policy: *(1)* encouraging high academic performance; *(2)* narrowing the racial achievement gap (or at least to avoid widening it); and *(3)* narrowing the enrollment gap, or in other words, achieving proportional allocations of college seats. However, I make no assumptions on how the policy maker weights these three objectives. Therefore, establishing a preference ranking between two policies will only be possible if one performs better along all 3 criteria. My theoretical framework is convenient for this analysis because the equilibrium grade distributions for each group are sufficient to gauge success along each objective.

The main contribution of this paper is in showing that policy design is a very important issue. The previous literature focuses on the question of AA versus race-neutrality, but this work indicates that the more interesting question may be how alternative AA policies compare to one another. A simple additive admission preference, such as the one formerly used in undergraduate admissions at the University of Michigan, performs very poorly, while a quota does much better. As it turns out, an admission preference erodes academic effort incentives for minorities by artificially increasing their performance level. Rational students take some of the grade boost and simply eat it, rather than using it to bolster their competitive edge. Moreover, by decreasing the return to effort for non-minorities, admissions preferences may also diminish their performance as well. Moreover, in equilibrium the policy merely re-shuffles admissions at the lowest-ranked colleges, leaving allocations at top-tier institutions unchanged from a color-blind outcome. Thus, an additive grade boost accomplishes little, but comes at a potentially high cost. More general admission preference rules can be designed to overcome some of the drawbacks of the Michigan rule.

On the other hand, quotas perform much better by altering the game so that competition for prizes occurs only among members of one's own group. In doing so, they mitigate discouragement effects for disadvantaged minorities that deter investment in effort.[5] Moreover, quotas guarantee representative allocations of seats among colleges of every caliber, by design. However, it also turns out that both forms of AA widen racial academic achievement gaps among the highest performing students, relative to color-blind allocations.

This paper also produces some new contributions to the policy debate. I show that there are meaningful ways in which both the proponents and the opponents of AA are correct. On the one hand, a tradeoff between equality and effort does exist, in the sense that there is always some segment of the population for which achievement diminishes under AA. On the other hand, some variations of AA can indeed overcome discouragement effects for some minority students, potentially producing an increase in average achievement in the minority group, and even in the population as a whole.

Finally, another contribution of this work is in developing a theoretical model of AA amenable empirical analysis. This is an advantage of the auction-like framework of academic competition: it provides access to a powerful set of empirical tools. Since the early 1990s, structural auction econometrics has emerged as one of the top successes in empirical analyses of strategic games of incomplete information. In contrast, Schotter and Weigelt [23] performed an experimental analysis of admissions preferences using a similar model, but within a complete-information paradigm where all heterogeneity is observable. They justify their choice of laboratory experiments by arguing that it is difficult to address the efficiency implications of AA and determine whether observed behavior matches the theory using natural data. However, equilibria in auction games of incomplete information have the appealing characteristic of establishing an intimate, yet parsimonious link between observable behavior and latent unobservable characteristics. This link has long been exploited by auctions researchers to uncover such abstract model primitives as private cost distributions.[6] In the case of AA, a knowledge of the distribution of private costs would provide an invaluable glimpse into the various implications of different college admissions policies. Although an empirical analysis is beyond the scope of the current exercise, this model nevertheless lays the ground work for such an undertaking.[7]

The rest of this paper has the following structure: in Section 2, I briefly discuss the relation between this work and the previous literature on AA. In Section 3, I give an overview of the college competition model, which I specify as a non-cooperative game of incomplete information. In Section 4 I introduce the solution concept of an approximate equilibrium which adds

[5]Discouragement effects are a well-known characteristic of all-pay auctions and asymmetric contests in general. Intuitively, since all competitors forfeit the cost of competing regardless of their payoff, high-cost types recognize that there is little point in exerting much effort since their likelihood of a favorable outcome is low.

[6]An in-depth discussion of the basic tools can be found in Paarsch, Hong and Haley [21].

[7]In ongoing work, Hickman [13] semiparametrically estimates the model using data on US colleges and college entrance test scores.

tractability to the model when the number of players is large. In Section 5, I derive approximate equilibria in different variations of the college admissions game with alternative admission policies. I discuss the qualitative differences in equilibrium actions and outcomes under color-blind allocations, quotas and admissions preferences. In Section 6, I solve the model for a special case where costs are linear in achievement and private types are Pareto distributed. In Section 7, I conclude and discuss avenues for further related research.

2. PREVIOUS LITERATURE

This is the first paper of which I am aware that attempts to address questions *(iii)* and *(iv)* from Section 1. However, there are several papers which have looked at questions *(i)* and *(ii)*. Coate and Loury [5] study a labor-search model of skills acquisition where minority workers strategically interact with employers who have tastes for racial discrimination *á la* Becker. Workers decide whether to forego a fixed skill-acquisition cost and the government mandates minimal minority employment (similar to a quota in my model). The authors find that effort decisions are given by a threshold rule: as long as the benefit exceeds some minimum threshold, workers incur the cost to acquire skills. Moreover, the threshold rule is such that the mandate can be gradually increased over time so that only a desirable effect is produced on minority employment and skills acquisition. The major difference between Coate and Loury and this paper is the choice set. In their model, nature chooses one's skill-acquisition cost, and agents only have a binary choice of whether or not to acquire a fixed skill level at a fixed cost. In contrast, I allow for agents to choose any skill level, meaning that the exact cost incurred is at the agent's discretion. When this is true, under any AA policy, every player's behavior changes, regardless of his private cost type. This creates a tradeoff between equality and effort, and outcome changes are no longer unambiguously desirable.

Two models related to mine are Fain [6] and Fu [10, 11]. Both are two-player strategic models of complete information, where heterogeneity among competitors is observable. The former is a tournament and the latter is an all-pay auction. Both models study an interaction between one advantaged player and one disadvantaged player competing for a single prize, and both find that an admission-preference-like AA rule benefitting the disadvantaged player increases effort exerted by *both* players. They then use these results to argue that colleges will admit a higher-quality body of students if the school gives preference to the minority students by weighting their grades more heavily. I should also mention that Schotter and Weigelt [23] perform an experimental analysis of a two-player model similar to Fain [6], with similar results. However, in generalizing their results to a competition involving many competitors, these authors implicitly assume that *every* beneficiary of the AA policy is at a competitive disadvantage to *every* other student not benefitting from it. However, this assumption is inappropriate in the context of college admissions, where AA is based only on one's observable race, rather than one's unobservable characteristics which determine academic competitiveness.

The current model produces very different results, and this is primarily due to the fact that there are both high-cost types and low-cost types in the minority group, all of whom benefit from AA. There are also high-cost types in the non-minority group who do not receive any benefit from AA. In January of 2008, presidential candidate Barack Obama famously stated to ABC's George Stephanopoulos his view that his daughters should not be treated as disadvantaged in college admissions decisions, and that perhaps white children raised in poverty should benefit from AA. The results of this paper suggest that Mr. Obama's intuition is correct: a common feature in both types of AA considered here is a reduction in effort among both low-cost, advantaged minorities and high-cost, disadvantaged non-minorities. For the former group, AA provides a competitive boost that was not needed; for the latter, AA exacerbates discouragement effects.

A final related paper on AA in contests is Franke [7], who analyzes the effect of an admission-preference-like AA policy in a contest with many players. Franke shows that when the policy maker is fully informed on student heterogeneity, he can design a grade-weighting scheme that raises all players' efforts, relative to an equal-treatment (*i.e.*, color-blind) rule. While this is certainly an improvement over a simplistic 2-player model, Franke still relies on the strong assumption of complete information to construct the beneficial policy. In that sense, the paper can be thought of as a characterization of the "first-best" outcome, where no information is hidden from the policy-maker. By contrast, I evaluate the choices available to a policy-maker who cannot observe individual characteristics other than race. The college board can see what grade each student submits, but it cannot observe the cost incurred to achieve that grade. Therefore, in keeping with the Wilson doctrine, I constrain the current theoretical exercise to evaluating policies such as quotas and simple admission preference schemes that can be enacted using only information on race and grades.

3. The Model

I model the competition among high-school students for college admissions as a Bayesian game. Students belong to two demographic groups—minorities and non-minorities—and each is characterized by a privately-known study cost type. Students compete in grades for a set of heterogeneous prizes—seats at colleges/universities of differing quality—and private types determine the costliness of academic achievement. Students have single-unit demands and prizes are allocated by a college admissions board (henceforth, The Board), according to grades. AA enters the model if The Board chooses to base allocations partially on race as well. Students can observe the set of prizes before making decisions, but they must incur a non-recoverable cost associated with academic achievement *before* learning which prize they will receive. A student's payoff at the end of the game is the utility derived from consuming a prize, minus the utility cost of his achieved grade. An equilibrium of the game is characterized by a set of achievement functions that prescribe each student's optimal effort level. A formal description of the components of the game is given below.

3.1. **Costs and Benefits.** The agents are a set $\mathcal{K} = \{1, \ldots, K\}$ of students who observably belong to a minority group $\mathcal{M} = \{1, 2, \ldots, M\}$ or a non-minority group $\mathcal{N} = \{1, 2, \ldots, N\}$, where $M + N = K$. Students are heterogeneous, and each is characterized by a privately-known study cost type $\theta \in [\underline{\theta}, \overline{\theta}]$. Agents view the private types of their opponents as independent random variables, and there is a common prior on private types within each group, $\Theta \sim F_i(\Theta),\ i = \mathcal{M}, \mathcal{N}$. Students have access to a common strategy set $S = \mathbb{R}_+$, comprising grades/test scores. In order to achieve grade level s, an agent must incur a cost $\mathcal{C}(s;\theta)$, which depends on his private type.

The rewards for academic achievement are a set of prizes

$$\mathbf{P}_{\mathcal{K}} = \{p_k\}_{k=1}^{K},$$

where p_k denotes the utility of consuming the k^{th} prize. The prizes are seats at distinct colleges and universities, and students have single-unit demands: they can only attend one school. There enough prizes for every student who competes (*i.e.*, there are enough seats open to serve anyone who wishes to go to college), but no two prizes render the same utility: $p_k \neq p_j,\ k \neq j$. At the end of the game, an agent's payoff is the utility from consuming a prize minus the cost of achievement, or

$$\Pi(s;\theta) = p - \mathcal{C}(s;\theta).$$

The alert reader will notice that I have implicitly assumed agents have identical preferences over differing colleges and universities. However, it is not essential to the model for all students to place the same value on a seat at a given college; the important assumption here is that students rank prize values the same. Without this assumption, a policy discussion concerning admission outcomes is impossible, and the researcher is left with the unsatisfying conclusion that policy has no role to play because fewer minorities attend elite institutions simply because they prefer it that way. An alternative view of the homogeneous ranking assumption is that students have similar preferences over school attributes such as per-pupil spending, graduation rates, student-faculty ratios, *etc.*

3.2. **College Admission Policies.** Grades are mapped into payoffs as the outcome of a matching market with three stages: students send reports of their achievement level to various colleges/ universities, admissions boards make acceptance/rejection decisions, and students choose among the options given to them by the market. I assume that there are no frictions in the matching market, so that its outcome can be implemented by a centralized entity called "The Board," who observes the set of grades $\mathbf{s} = \{s_{\mathcal{M},1}, \ldots, s_{\mathcal{M},M}, s_{\mathcal{N},1}, \ldots, s_{\mathcal{N},N}\}$ achieved by all students and allocates prizes according to some rule.

A simple "color-blind" admission rule is one in which The Board assortatively matches prizes with grades. Thus, the student submitting the highest grade is awarded the most valuable prize, and so on. In what follows, it will be convenient to treat a competition with color-blind admissions as the baseline model. This will facilitate evaluations of different race-conscious policies by comparison to a color-blind rule.

As for AA, consider first a quota system similar to what's known as "Reservation Law" in India. This law mandates that a certain percentage of seats be set apart for allocation only to certain demographic groups. There are many possible quota rules indexed by a number $q \in \{1, 2, \dots, M\}$ of prizes reserved for minorities. However, for simplicity I will consider only the case of a full quota rule, where exactly M prizes are reserved for minorities. Under a full quota rule, students compete only with members of their own group. It is also necessary to specify *how* prizes are selected for reservation. There are many possibilities once again, but for simplicity I will focus on the case where a *representative* set of M prizes is set aside. This can be accomplished by either randomly selecting M prizes from the set $\mathbf{P}_{\mathcal{K}}$, or it can be by first ordering prizes by quality and selecting out every m^{th} prize, where $m = \frac{M+N}{M}$. In what follows, it will be easiest to consider the random selection method, but this is without loss of generality: when the set of prizes is large, the overall effect will be the same.

The form of AA as implemented in the US is different, due to a 1978 Supreme Court ruling that quotas are unconstitutional. Since then, American higher education institutions have since been forced to seek other means by which to implement AA. These alternative implementations are commonly referred to as "admission preferences." I model an admission preference rule as a grade transformation function $\tilde{S} : \mathbb{R}_+ \to \mathbb{R}_+$ that the Board uses to match prizes assortatively with non-minority grades and *transformed* minority grades

$$\{s_{\mathcal{N},1}, \dots, s_{\mathcal{N},N}, \tilde{S}(s_{\mathcal{M},1}), \dots, \tilde{S}(s_{\mathcal{M},M})\}.$$

In other words, under an admission preference The Board views each minority student with a grade of s as if he had submitted a grade of $\tilde{S}(s)$ instead. Regardless of the admission policy, in the event of a tie between two or more scores (some of which may be transformed), they are ordered randomly for the purpose of comparisons.

Before making study decisions, agents observe the set of prizes $\mathbf{P}_{\mathcal{K}}$ and the exogenous admissions rule, $\mathcal{R} \in \{cb \text{ (color-blind)}, q \text{ (quota)}, ap \text{ (admission preference)}\}$. Under the payoff correspondence $\Pi(\mathbf{s}; \theta)$ induced by a particular admission rule, students optimally choose grades based on their own private costs and their opponents' optimal behavior. A *(group-wise) symmetric equilibrium* of the Bayesian game $\Gamma(M, N, \mathbf{P}_{\mathcal{K}}, \mathcal{R})$ is a set of achievement functions $\gamma_i : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$, $i = \mathcal{M}, \mathcal{N}$ which generate optimal grades, given that ones' opponents behave similarly. Equilibrium achievement functions and private cost distributions induce a population grade distribution $G(s)$ and a set of group-specific grade distributions, $G_{\mathcal{M}}$ and $G_{\mathcal{N}}$. These are ultimately the objects of interest from a policy standpoint, as they fully characterize achievement, achievement gaps and allocation gaps in equilibrium.

3.3. **Assumptions.** In order to guarantee existence of a pure-strategy equilibrium, it will be necessary to make the following assumptions on the form of the study cost function:

Assumption 3.1. $\frac{\partial \mathcal{C}}{\partial s} > 0$; $\frac{\partial \mathcal{C}}{\partial \theta} > 0$; $\frac{\partial^2 \mathcal{C}}{\partial s^2} \geq 0$; and $\frac{\partial^2 \mathcal{C}}{\partial s \partial \theta} \geq 0$.

In words, costs are assumed to be convex and increasing in achievement and private type; marginal costs are assumed to be increasing in private costs. This specification of costs lends itself to several interpretations. It could be reflective of an underlying labor-leisure tradeoff where students differ either by preferences for leisure, or by the amount of labor input required to produce a unit of s. Alternatively, it could reflect some psychic cost of exerting mental effort to learn new concepts, where the amount of effort required to produce a given grade differs among students. The above cost structure could also reflect many other external factors affecting students' academic performance such as home conditions, affluence, school quality, and access to things like health-care and tutors.

It is also necessary to make the following assumptions on beliefs:

Assumption 3.2. The private cost distributions $F_{\mathcal{M}}(\theta)$ and $F_{\mathcal{N}}(\theta)$ have continuous and strictly positive densities $f_{\mathcal{M}}(\theta)$ and $f_{\mathcal{N}}(\theta)$, respectively.

One aspect of the model is worth mentioning here. By assuming that private cost distributions are static and exogenous, I am implicitly taking a short-run view of policy implications. One could certainly conceive of a broader model in which The Board designs a policy today so as to affect the evolution of private costs for future generations (*i.e.*, the children of today's college freshmen), but such an undertaking is beyond the scope of the current exercise, and is left for future research. Instead, I shall concentrate on the implications of the policy-maker's choices for actions and outcomes of older school children and today's college candidates, whose private costs can reasonably be viewed as fixed.

For the purpose of the theoretical exercise, it will be helpful to assume the following relation between private cost distributions across groups:

Assumption 3.3. $F_{\mathcal{N}}(\theta) < F_{\mathcal{M}}(\theta)$, $\theta \in (\underline{\theta}, \overline{\theta})$.

In other words, I assume that the game is asymmetric in the sense that private costs in the minority group stochastically dominate non-minority private costs. This asymmetry assumption is not intended to imply that there are fundamental differences in inherent ability across the two groups, as private costs reflect a myriad of environmental factors as well. Rather, it is in keeping with arguments made by proponents of AA regarding systemic competitive disadvantages for minorities, due to various historical factors. For example, White children in the United States, on average, are more affluent and attend primary and secondary schools that are better funded than African-Americans. The idea behind cost asymmetry is that an average minority student must expend more personal effort to overcome the environmental obstacles—poverty, poor health-care, lower quality K-12 education, *etc.*—eroding his competitive edge.[8]

Finally, when considering an admission preference rule, I shall restrict attention to policy functions $\widetilde{S}$ satisfying certain sensibility criteria.

Assumption 3.4. $\widetilde{S}(s)$ is a strictly increasing function lying above the 45°-degree line.

Assumption 3.5. $\widetilde{S}(s)$ is continuously differentiable.

Assumption 3.4 corresponds to the notion that the policy is geared toward assisting minorities, effectively moving each minority student with a grade of s ahead of each non-minority student with a grade of $\widetilde{S}(s) \geq s$. Moreover, it states that a policy-maker will not choose to reverse the ordering of any segment of the minority population, so that some students are awarded prizes of lesser value than other students within their own group whose grades were lower. Assumption 3.5 implies that the policy-maker does not make abrupt jumps in either the assessed grade boost (*i.e.*, $\widetilde{S}$ is continuous), or the marginal grade boost. Aside from characterizing the behavior of a sensible policy-maker, Assumptions 3.4 and 3.5 also guarantee that introducing $\widetilde{S}$ into the model does not interfere with existence of the equilibrium.

3.4. **An Auction-Theoretic View of the Game.** The model defined above is strategically equivalent to a special type of game known in the contests literature as an *all-pay auction*.[9] An all-pay auction is a strategic interaction in which agents compete for a limited resource by incurring some type of unrecoverable cost *before* learning the outcome of the game. Although they are not explicitly implemented in the real world like other auction mechanisms, all-pay auctions have been useful for modeling various winner-take-all games such as political lobbying and R&D races. Likewise, in my model of college competition, high school students cannot recover lost leisure time or disutility incurred by study effort if they discover that they didn't make it into the college they had hoped for.

The Board is analogous to an auctioneer, who auctions off a set of heterogeneous prizes according to a pre-determined mechanism. Students are similar to bidders, and the grades they work for are analogous to bids. The value here in recognizing the connection to auction theory is that I can import a well-developed set of analytic tools for characterizing the equilibrium. For

[8]There is some empirical evidence consistent with this view. Neal and Johnson [19] find that for the Armed Forces Qualification Test, "family background variables that affect the cost or difficulty parents face in investing in their children's skill explain roughly one third of the racial test score differential" (pg. 871). Fryer and Levitt [8] analyze data on racial test-score gaps among elementary school children in an attempt to uncover the causes. They find that by controlling for socioeconomic status and other environmental factors which vary substantially by race, test-score gaps significantly decrease, but not entirely. They test various hypotheses to explain the remainder of the gap, and find that disparities in school quality is the only one not rejected by the data.

[9]Some seminal papers in the contests literature are Tullock [24], Lazear and Rosen [17] and Baye, Kovenock and de Vries [3, 4].

example, as the following proposition shows, I can conclude *a priori* that a monotonic equilibrium exists. As I will shortly demonstrate, existence and monotonicity provide an invaluable step toward analytic and computational tractability of the model when K is large, as it is in college admissions.

Proposition 3.6. *In the college admissions game $\Gamma(M, N, \mathbf{P}_{\mathcal{K}}, \mathcal{R})$ with $\mathcal{R} \in \{cb, q, ap\}$, there exists a unique symmetric pure-strategy equilibrium $(\gamma_{\mathcal{M}}(\theta), \gamma_{\mathcal{N}}(\theta))$ where achievement is strictly decreasing in private costs; therefore, $G_i(s) = 1 - F_i\left((\gamma_i)^{-1}(s)\right)$.*

Proof: Existence and monotonicity is a straightforward application of Athey [2, Theorem 3] who proves existence and monotonicity of a pure-strategy equilibrium in a general class of auction-related games. The relation between the grade distributions and the achievement functions follows immediately from the fact that achievement is a strictly decreasing function of private cost types. A formal proof of uniqueness is a bit more involved and is under construction. Briefly though, it follows the same logic as Hickman [12, Proposition 3.3, Theorem 3.4]. Given the well-behaved nature of the private cost distributions, it can be shown that any symmetric, monotonic equilibrium must also be differentiable. From differentiability, it follows that the equilibrium achievement functions must satisfy the first-order conditions of an agent's objective function. The first-order conditions define a standard initial value problem, and the fundamental theorem of differential equations can be invoked to show that a unique solution exists. Since any symmetric equilibrium of the college admissions game must be consistent with the unique solution of the first-order conditions, it follows that the symmetric equilibrium is unique. ■

4. Equilibrium Analysis

4.1. **Solution Concept: "Approximate Equilibrium".** In this section I introduce an alternative solution concept that I adopt for tractability. For large $K = M + N$, this equilibrium is analytically and computationally intractable, because a decision-maker's objective function is a complicated sum of functions based on the order statistics of opponents' costs. Agents know that their *ex-post* payoff depends on their rank within the grade distribution, and under monotonicity this is the same as their rank within the realized cohort of opponents. Thus, expected equilibrium payoffs are a weighted average of the prizes, where the weight on the k^{th} best prize is one's probability of being the k^{th} lowest order statistic among K competing private costs.

For simplicity and tractability, I assume that the number of competitors is large enough so that one has a very good idea of one's rank within the realized sample of private costs. I approximate this large, finite model by considering the limiting case as $K \to \infty$, but in order to do so I must first introduce some additional notation. Let μ denote the asymptotic mass of the minority group: as each new agent is created, nature assigns him to group $\mathcal{M}$ with probability μ, and then he draws a private cost from the appropriate group-specific distribution. Given this assumption, with probability 1 the limiting sample of competitors is a dense set on the interval $[\underline{\theta}, \overline{\theta}]$, and each knows with certainty that his sample rank is the same as his rank in the unconditional private cost distribution $\mu F_{\mathcal{M}}(\theta) + (1-\mu)F_{\mathcal{N}}(\theta)$.[10]

For analytic convenience, I also assume that prizes are generated as independent draws from a compact interval $\mathcal{P} = [\underline{p}, \overline{p}] \subset \mathbb{R}_+$ according to a known prize distribution $F_P(p)$ satisfying

Assumption 4.1. F_P has a continuous density $f_P(p)$, which is strictly positive on $\mathcal{P}$; and

Assumption 4.2. (*zero surplus condition*) $\underline{p} = \mathcal{C}(0; \overline{\theta})$.

Even though prize values are *ex-ante* observable, framing them in this way provides an intuitive view of the limiting set of prizes: as $K \to \infty$, $\mathbf{P}_{\mathcal{K}}$ becomes a dense set on $\mathcal{P}$, and the rank of a prize with value p converges to $F_P(p)$.

Assumption 4.2 is necessary because it provides a boundary condition that is used to solve the equilibrium equations. Although the current theoretical exercise focuses solely on the competition among high-school students for college admissions, the zero surplus condition can be thought of as reflecting broader market forces not explicitly included in model. In the broader model, prize values are the additional utility one gains from going to college versus opting out, and $[\underline{\theta}, \overline{\theta}]$ is the set of individuals who demand a college seat, being a subset of a larger group of individuals, some of whom choose the outside option. If schools and firms can freely choose to enter the market and supply either college seats or jobs for unskilled laborers, the marginal college candidate—the highest private cost type opting for college, $\overline{\theta}$—will be just indifferent between attending college and entering the work force as an unskilled laborer. This point highlights a limitation of the current model: it attempts to characterize student behavior *conditional on participation in the post-secondary education market*, and it is not intended to provide insights into the decision of whether to acquire additional education. This aspect of the college admissions problem is left for future research.

With that out of the way, I can treat both agents and prizes as if they belong to a continuum, rather than a finite set. This allows me to avoid framing decisions in terms of the distributions of complicated order statistics, and it reduces each agent's decision problem to a simple objective function expressed in terms of θ, μ, $F_{\mathcal{M}}$, $F_{\mathcal{N}}$ and F_P. Given the well-behaved nature of the model primitives, the maximizers of the finite objective functions converge to the maximizer of the limiting objective function, which allows me to derive what I refer to as an *approximate equilibrium*.

Definition 4.3. Consider a generic game $\Gamma(\mathcal{K}, S, \Pi)$, where $\mathcal{K}$ is the set of players, $S_i \subseteq \mathbb{R}$ is the strategy space for the i^{th} player, and $\Pi(s_1, \ldots, s_K)$ characterizes payoffs on $S = S_1 \times \cdots \times S_K$.

[10]The fact that the limiting sample is a dense set can be seen by applying the following logic: given any two numbers $\theta, \theta' \in [\underline{\theta}, \overline{\theta}]$, where $\theta < \theta'$, the probability mass assigned to the interval (θ, θ') is strictly positive under my assumptions on F_Θ. Therefore, as the number of iid draws from F_Θ gets large, the probability of hitting the interval (θ, θ') at least once approaches one. Thus, a countably infinite random sample of agents will be everywhere dense on $[\underline{\theta}, \overline{\theta}]$.

Given $\delta > 0$, a *δ-approximate equilibrium* is an K-tuple $\mathbf{s}^\delta = (s_1^\delta, \ldots, s_K^\delta)$, such that there exists an equilibrium $\mathbf{s}^* = (s_1^*, \ldots, s_K^*)$ of Γ, where $\|\mathbf{s}^\delta - \mathbf{s}^*\|_{\sup} < \delta$.

The approximate equilibrium concept is more relevant for my purposes than the ε-equilibrium introduced by Radner [22], which is a profile of strategies generating payoffs that are ε close to payoffs in some equilibrium of Γ. The drawback of an ε-equilibrium is that it need not resemble the equilibrium strategies which generate the payoffs being approximated.[11] In my case, the strategies are a principal concern: I wish to concentrate on the effects of admission policies on both payoffs *and* achievement choices. However, there is a connection between the two concepts, as the following remark demonstrates.

Remark 4.4. For a Nash equilibrium $\mathbf{s}^*$ of Γ, if $\mathbf{s}^* \subset U \subset S$, where U is a neighborhood of $\mathbf{s}^*$, and if the payoff function Π is continuous on U, then the set of payoffs generated by ε-equilibria and δ-approximate equilibria form bases of neighborhoods of equilibrium payoffs. That is, given an ε-equilibrium associated with $\mathbf{s}^*$, there exists $\delta > 0$ such that for a δ-approximate equilibrium we have

$$\|\Pi(\mathbf{s}^\delta) - \Pi(\mathbf{s}^*)\|_{\sup} < \varepsilon.$$

Conversely, given a δ-approximate equilibrium associated with $\mathbf{s}^*$, there exists $\varepsilon > 0$ such that for an ε-equilibrium we have

$$\|\Pi(\mathbf{s}^\varepsilon) - \Pi(\mathbf{s}^*)\|_{\sup} < \|\Pi(\mathbf{s}^\delta) - \Pi(\mathbf{s}^*)\|_{\sup}. \square$$

In the context of the college admissions model, I seek to characterize a set of approximate achievement functions $(\gamma_{\mathcal{M}}^\infty(\theta), \gamma_{\mathcal{N}}^\infty(\theta))$ such that, given a fixed tolerance level $\delta > 0$, the functions approximate actual equilibrium achievement to δ-precision for large enough K.

My approximate equilibrium concept is similar to the *oblivious equilibrium* developed by Weintraub, Benkard and Van Roy [25, WBR] to approximate Markov Perfect Equilibria in dynamic oligopoly games. In such models, firms (and researchers) must compute a complex and intractable state transition process in order to exactly determine equilibrium strategies. Instead, WBR assume that firms make nearly optimal decisions based on a long-run average industry statistic which is inexpensive to compute. This issue of computational tractability leads to an alternative interpretation of approximate equilibria. Aside from being a useful approximation of equilibrium behavior, one could also view them as an exact characterization of the behavior of agents with bounded information processing ability. Rather than exactly tracking expected payoffs based on all of the order statistics of a large set of competitors, a cognitively constrained

[11] Radner [22] used an ε-equilibrium to resolve a dilemma in dynamic Cournot oligopoly games. For a fixed set of firms, as long as the number of periods is finite, the unique subgame perfect equilibrium involves static equilibrium strategies being played every period, whereas collusion suddenly becomes possible in the limit. Radner showed that there is a collusive ε-equilibrium of the finite-horizon Cournot game in which cartels are sustainable. Equilibrium payoffs can be replicated to arbitrary precision (*i.e.*, it is nearly optimal to collude if the time horizon is far enough away), even though the ε-equilibrium strategies are excluded from a neighborhood of equilibrium strategies.

agent with private cost θ might find it more attractive to base decisions on his limiting rank $F_i(\theta)$ instead.

5. APPROXIMATE EQUILIBRIA UNDER AFFIRMATIVE ACTION

I shall proceed by deriving the maximizers of an agent's limiting objective function as the natural processes described in Section 4.1 generate increasingly large sets of competitors and prizes. I then prove that the resulting derivations satisfy Definition 4.3 above. From this point on, all discussion and derivations will be in terms of the approximate equilibrium, $(\gamma^{\infty}_{\mathcal{M}}(\theta), \gamma^{\infty}_{\mathcal{N}}(\theta))$, so I shall drop the ∞ superscript for notational ease. Moreover, to avoid tedious verbosity I shall henceforth refer to the approximate equilibrium and the approximate achievement functions simply as "the equilibrium" and "the achievement functions," unless the context requires more specificity. If it becomes necessary to distinguish between the actual equilibrium of a game with K agents and the approximate equilibrium, I shall refer to the former as the "finite equilibrium" and I shall abuse the notation slightly and denote the former by $\gamma(\cdot;K)$, listing K as a parameter. Keeping in mind the processes generating agents and prizes, this notational abuse is not entirely unreasonable: by the law of large numbers, any two randomly generated games $\Gamma(M, N, \mathbf{P}_{\mathcal{K}}, \mathcal{R}),\ M+N=K$, and $\Gamma(M', N', \mathbf{P}'_{K}, \mathcal{R}),\ M'+N'=K$, will be nearly the same (probabilistically) for large K.

Superscripts will henceforth be used to keep track of the admission policy which defines payoffs in the game. Under policy $\mathcal{R} \in \{cb, q, ap\}$, the achievement functions and grade distributions are denoted by $\gamma^{\mathcal{R}}_{\mathcal{M}}(\theta)$, $\gamma^{\mathcal{R}}_{\mathcal{N}}(\theta)$, $G^{\mathcal{R}}$, $G^{\mathcal{R}}_{\mathcal{M}}$, and $G^{\mathcal{R}}_{\mathcal{N}}$. Finally, in what follows it will sometimes be convenient to work with the inverse achievement function, which I denote by $\psi^{\mathcal{R}}_i(s) \equiv \left(\gamma^{\mathcal{R}}_i\right)^{-1}(s)$.

5.1. The Color-Blind Game.

Recall that a color-blind allocation rule means simple positive assortative matching of prizes with grades. I claim (proof to follow later) that in the limit, in equilibrium, this process is equivalent to the Board using the following reward function for a student submitting a grade of s:

$$\begin{aligned}
\pi^{cb}(s) =& F_P^{-1}\left[G^{cb}(s)\right] \\
=& F_P^{-1}\left[\mu G^{cb}_{\mathcal{M}}(s) + (1-\mu)G^{cb}_{\mathcal{N}}(s)\right] \\
=& F_P^{-1}\left[1 - \left(\mu F_{\mathcal{M}}\left[\psi^{cb}(s)\right] + (1-\mu)F_{\mathcal{N}}\left[\psi^{cb}(s)\right]\right)\right].
\end{aligned}$$

Intuitively, on receiving grades from each competitor, the Board observes the population grade distribution $G^{cb}(s)$, and then maps the grade quantiles into the corresponding prize quantiles.

Since individuals' limiting payoffs do not depend on race, it follows that $\gamma^{cb}_{\mathcal{M}}(\theta) = \gamma^{cb}_{\mathcal{N}}(\theta) = \gamma^{cb}(\theta)$; hence, the lack of subscripts on the inverse achievement functions in the third line.[12]

In equilibrium, the limiting net payoff for an agent with cost type θ submitting grade s is

$$\Pi^{cb}(s;\theta) = F_P^{-1}\left[1 - \left(\mu F_{\mathcal{M}}\left[\psi^{cb}(s)\right] + (1-\mu)F_{\mathcal{N}}\left[\psi^{cb}(s)\right]\right)\right] - \mathcal{C}(s;\theta).$$

Differentiating, I get the following FOC:

$$(1) \qquad -\frac{\mu f_{\mathcal{M}}[\psi^{cb}(b)] + (1-\mu)f_{\mathcal{N}}[\psi^{cb}(b)]}{f_P\left(F_P^{-1}\left[1 - \left(\mu F_{\mathcal{M}}\left[\psi^{cb}(s)\right] + (1-\mu)F_{\mathcal{N}}\left[\psi^{cb}(s)\right]\right)\right]\right)}\frac{d\psi^{cb}(s)}{ds} = \mathcal{C}'(s;\theta).$$

Using the fact that $\frac{d\psi^{cb}(s)}{ds} = \frac{1}{(\gamma^{cb})'(\psi^{cb}(s))}$, and the fact that in equilibrium we have $\psi^{cb}(s) = \theta$, I can substitute to get

$$(2) \qquad (\gamma^{cb})'(\theta) = -\frac{\mu f_{\mathcal{M}}(\theta) + (1-\mu)f_{\mathcal{N}}(\theta)}{f_P\left(F_P^{-1}\left[1 - \mu F_{\mathcal{M}}(\theta) - (1-\mu)F_{\mathcal{N}}(\theta)\right]\right)\mathcal{C}'[\gamma^{cb}(\theta);\theta]}.$$

This differential equation partially solves for equilibrium achievement, but a boundary condition is also needed.

By monotonicity, a student with cost type $\overline{\theta}$ is sure to be awarded the lowest quality prize, so the Assumption 4.2 implies the following boundary condition:

$$(3) \qquad \gamma^{cb}(\overline{\theta}) = \mathcal{C}^{-1}(\underline{p};\overline{\theta}).$$

With that, I am ready to prove that the derivations above provide meaningful insights into the equilibrium of a finite college admissions games where the number of competitors is large. The proof is fairly involved, but it is based on simple ideas. I first prove that the finite objective functions converge pointwise in probability to the limiting objective function listed above. By viewing a K-player game as being randomly generated by the natural processes outlined in Section 4.1, one can think of a player's objective function as a random variable; hence the concept of convergence in probability. Using pointwise convergence, I can invoke Egorov's Theorem to deliver uniform convergence of the sequence of finite objective functions. Finally, using uniform convergence, I can invoke the theorem of the maximum to show that the finite maximizers are close to the solution of equation (2) and boundary condition (3) for large K.

Theorem 5.1. *Given $\rho,\ \varepsilon,\ \delta > 0$, there exists $K^* \in \mathbb{N}$, and a set $E \subset [\underline{\theta},\overline{\theta}]$ having (Lebesgue) measure $m(E) < \rho$, such that for any $K \geq K^*$, on any closed subset of $[\underline{\theta},\overline{\theta}] \setminus E$ we have the following:*

[12] The theorist with experience in asymmetric auctions may find this statement puzzling, but one must keep in mind that it merely applies to *limiting* payoffs. In a two-player game, differing behavior arises from the fact that a minority and a non-minority with the same private cost type will view their likely standing in the distribution of realized competition differently, due to the asymmetry in the cost distributions. However, the likely difference between their expected ranks vanishes as the number of players gets large.

(*i*) $\gamma^{cb}(\theta)$ *as defined by equation (2) and boundary condition (3) generates an* ε*-equilibrium of the K-player color-blind game, and*

(*ii*) $\gamma^{cb}(\theta)$ *is a* δ*-approximate equilibrium for the K-player color-blind game, or*

$$\|\gamma^{cb}(\theta) - \gamma_i^{cb}(\theta;K)\|_{\sup} < \delta, \;\; i = \mathcal{M},\mathcal{N}.$$

Proof: For notational ease, I shall drop the "cb" superscripts for the duration of the proof. Also, recall that finite functions are denoted by the presence of a parameter K, whereas limiting functions lack the extra argument. I begin by ordering the sample of K prizes from lowest quality to highest, denoting the k^{th} order statistic by $p_{(k:K)}$. Since $\gamma_i(\theta;K)$ is monotonic for $i = \mathcal{M},\mathcal{N}$, the equilibrium expected payoff function in the K-player game can be written as

$$\Pi_i(s,\theta;K) = \sum_{k=1}^{K} p_{(k:K)} \sum_{\substack{k_i \leq \min\{k,K_i\},\\ k_j = k-k_i}} \left[\binom{K_i-1}{k_i-1} F_i\left(\gamma_i^{-1}[s;K]\right)^{K_i-k_i} \left[1-F_i\left(\gamma_i^{-1}[s;K]\right)\right]^{k_i-1} \right.$$
$$\left. \times \binom{K_j}{k_j} F_j\left(\gamma_j^{-1}[s;K]\right)^{K_j-k_j} \left[1-F_j\left(\gamma_j^{-1}[s;K]\right)\right]^{k_j} \right]$$
$$- \mathcal{C}(s;\theta).$$

In order for a player from group i to win the k^{th} prize, it must be the case that exactly $k-1$ of his opponents have private costs above his own. For each opponent in his own group, this occurs with probability $1-F_i\left(\gamma_i^{-1}[s;K]\right)$, and for each opponent in the other group, this occurs with probability $1-F_j\left(\gamma_j^{-1}[s;K]\right)$. The binomial coefficients and the second summation operator in the expression above are designed to cover all the possible ways in which exactly $k-1$ opponents have higher costs. Thus, the term within the inner summation is the probability of winning the k^{th} prize, and the overall objective function is a weighted sum of all K prizes, giving the expected prize won in equilibrium.

Recall my claim that the limiting equilibrium payoff function is given by

$$\Pi(s,\theta) = F_P^{-1}\left[1-\left(\mu F_{\mathcal{M}}\left[\gamma_{\mathcal{M}}^{-1}(s)\right] + (1-\mu)F_{\mathcal{N}}\left[\gamma_{\mathcal{N}}^{-1}(s)\right]\right)\right] - \mathcal{C}(s;\theta).$$

I wish to show that for large K, it is nearly optimal to act as if one were maximizing $\Pi(s,\theta)$, rather than $\Pi(s,\theta;K)$. Since costs never change with K, I shall drop the cost terms and focus solely on convergence of the gross payoff functions $\pi(s,\theta;K)$ to their limit $\pi(s,\theta)$.

For $l \in [0,1]$, define

$$p(l;K) \equiv \left\{ p_{(t:K)} : t = \operatorname*{argmin}_{k\in\{1,\dots,K\}} \left| l - \frac{k}{K} \right| \right\},$$ [13]

[13] If there are multiple maximizers (there can be at most two) then choose t to be the lesser.

Intuitively, $\{p(l;K)\}_{K=1}^{\infty}$ can be thought of as a random sequence of the t^{th} order statistic in the sample of prizes, where for each K, t is chosen so that $p_{(t:K)}$ approximates the l^{th} sample quantile as closely as possible. Since $\left|l - \frac{i}{K}\right| \leq \frac{1}{2K}$ for all $l \in (0,1)$, in the limit the t^{th} order statistic will be precisely at the l^{th} quantile within the sample of K prizes. Furthermore, since the sample distribution converges to F_P by the law of large numbers, it follows that $\operatorname*{plim}_{K\to\infty} p(l;K) = F_P^{-1}(l)$.

For some $i = \mathcal{M}, \mathcal{N}$, fix $\theta \in [\underline{\theta}, \overline{\theta}]$ and let

$$l = 1 - \mu F_{\mathcal{M}}\left[\gamma_{\mathcal{M}}^{-1}(s)\right] - (1-\mu)F_{\mathcal{N}}\left[\gamma_{\mathcal{N}}^{-1}(s)\right],$$

where $s = \gamma_i(\theta)$. Notice that

$$\operatorname*{plim}_{N\to\infty} p(l;K) = F_P^{-1}\left(1 - \mu F_{\mathcal{M}}\left[\gamma_{\mathcal{M}}^{-1}(s)\right] - (1-\mu)F_{\mathcal{N}}\left[\gamma_{\mathcal{N}}^{-1}(s)\right]\right) = \pi(s,\theta).$$

Moreover, For each K I can rewrite the finite expected gross payoff function as

$$\begin{aligned}\pi_i(s,\theta;K) = p(l;K) \sum_{\substack{k_i \leq \min\{t,K_i\},\\ k_j = t-k_i}} &\left[\binom{K_i-1}{k_i-1} F_i(\theta)^{K_i-k_i}\left[1-F_i(\theta)\right]^{k_i-1}\right.\\ &\left.\times \binom{K_j}{k_j} F_j\left(\gamma_j^{-1}[s;K]\right)^{K_j-k_j}\left[1-F_j\left(\gamma_j^{-1}[s;K]\right)\right]^{k_j}\right]\\ \sum_{\substack{k=1,\dots,N,\\ k\neq t}} p_{(k:K)} \sum_{\substack{k_i \leq \min\{k,K_i\},\\ k_j = k-k_i}} &\left[\binom{K_i-1}{k_i-1} F_i(\theta)^{K_i-k_i}\left[1-F_i(\theta)\right]^{k_i-1}\right.\\ &\left.\times \binom{K_j}{k_j} F_j\left(\gamma_j^{-1}[s;K]\right)^{K_j-k_j}\left[1-F_j\left(\gamma_j^{-1}[s;K]\right)\right]^{k_j}\right].\end{aligned} \tag{4}$$

Let

$$k_i^* \equiv \operatorname*{argmin}_{1\leq k\leq \min\{t,K_i\}} \left|(1-F_i(\theta)) - \frac{k}{K_i}\right|^{14}$$

and note that the following can be extracted from the first term in (4):

$$\begin{aligned}p(l;K)&\left[\binom{K_i-1}{k_i^*-1} F_i(\theta)^{K_i-k_i^*}(1-F_i(\theta))^{k_i^*-1}\right]\\ &\times\left[\binom{K_j}{t-k_i^*} F_j\left(\gamma_j^{-1}[s;K]\right)^{K_j-t-k_i^*}\left[1-F_j\left(\gamma_j^{-1}[s;K]\right)\right]^{t-k_i^*}\right].\end{aligned}$$

The second and third components of the above product represent the probability that exactly $K_i - k_i^*$ group-i players have costs below θ and exactly $K_j - t - k_i^*$ group-j players achieve grades

[14]If there are multiple maximizers (there can be at most two) then choose k_i^* to be the lesser.

below $\gamma_i(\theta;K)$. Letting

$$\mu_i = \begin{cases} (1-\mu) & i = \mathcal{N} \quad \text{and} \\ \mu & i = \mathcal{M}, \end{cases}$$

this can be restated as the probability that fraction

$$\frac{K_i - k_i^*}{K_i} = 1 - \frac{k_i^*}{K_i} \xrightarrow{K} F_i(\theta)$$

of group-i players have costs below θ and fraction

$$\begin{aligned} \frac{K_j - t + k_i^*}{K} &\xrightarrow{K} \mu_j - l + \mu_i(1 - F_i(\theta)) \\ &= \mu_j - 1 + (1-\mu)_i F_i(\theta) + \mu_j F_j(\gamma_j^{-1}[s;K]) + \mu_i(1 - F_i(\theta)) \\ &= \mu_j F_j(\gamma_j^{-1}[s;K]) \end{aligned}$$

of all agents come from group j and achieve equilibrium grades below $\gamma_i(\theta;K)$. In each of the previous two expressions, the convergence over K term follows from the law of large numbers. Since the probability associated with this event is one in the limit, it follows that the pointwise probability limit of (4) is $\pi(s,\theta)$, for $i = \mathcal{M},\mathcal{N}$.

Given that $\{\Pi(s,\theta;K)\}_{K=1}^{\infty}$ is a sequence of measurable functions converging pointwise to $\Pi(s,\theta)$ on a measurable set of finite measure, by Egorov's Theorem it follows that for any $\rho > 0$ there exists a set $E \subset [\underline{\theta},\overline{\theta}]$ having measure $m(E) < \rho$, such that $\{\Pi(s,\theta;K)\}_{K=1}^{\infty} \rightarrow \Pi(s,\theta)$ uniformly on the set $[\underline{\theta},\overline{\theta}] \setminus E$.

This is the same as saying that on the set $[\underline{\theta},\overline{\theta}] \setminus E$, it is nearly optimal to choose one's bid as if one's opponents were adopting a strategy of $\gamma(\theta)$, rather than $\gamma(\theta;K)$. Thus, given $\varepsilon > 0$, there exists K_ε such that for any $K \geq K_\varepsilon$, $\gamma(\theta)$ generates an ε-equilibrium of the K-player finite game. Furthermore, since all of the model primitives are well-behaved–θ is strictly bounded away from zero; $\mathcal{P}$ is compact; $F_{\mathcal{M}}$, $F_{\mathcal{N}}$, and F_P are absolutely continuous; and for each θ the set of undominated bids is compact-valued–I can invoke the Theorem of the Maximum on any compact subset of $[\underline{\theta},\overline{\theta}] \setminus E$ to show that the maximizers of $\Pi(s,\theta;K)$ and $\Pi(s,\theta)$ are close for large K. That is, given $\delta > 0$, there exists K_δ such that for any $K \geq K_\delta$, $\gamma(\theta)$ is a δ-approximate equilibrium of the K-player finite game, or

$$\|\gamma(\theta) - \gamma(\theta;K)\|_{\sup} < \delta.$$

Finally, given $\varepsilon > 0$ and $\delta > 0$, then for any $K \geq K^* \equiv \max\{K_\varepsilon, K_\delta\}$, $\gamma(\theta)$ is a δ-approximate equilibrium which generates an ε-equilibrium of the K-player finite game on any closed subset of $[\underline{\theta},\overline{\theta}] \setminus E$. ■

Before moving on, I should note that Theorem 5.1 can be strengthened slightly, to show that an ϵ-equilibrium and a δ-approximate equilibrium obtains on the entire set $[\underline{\theta},\overline{\theta}]$, rather than on a subset with close to full measure. However, the proof of the stronger version of the theorem invokes results from the econometric theory literature less familiar to economic theorists, concerning uniform convergence in probability of stochastic functions. See the appendix for details.

5.2. **Affirmative Action: The Quota Game.** I now depart from the baseline color-blind model, and I derive the approximate equilibrium in the presence of race-conscious admission policies, beginning with quotas. Recall that a quota system in the finite game can be thought of the Board randomly selecting M prizes and setting them aside for allocation to group-$\mathcal{M}$ agents. This effectively splits the single asymmetric competition apart into two separate, symmetric competitions. As K gets large, the sample distributions of prizes reserved for each group both converge in probability to F_P. Thus, the limiting quota rule involves the Board using a set of group-specific reward functions of the form

$$\pi_i^q(b) = F_P^{-1}\left[G_i^q(b)\right],\ i = \mathcal{M}, \mathcal{N}.$$

Intuitively, the Board observes the group-specific grade distributions, and maps the quantiles of these into the corresponding quantiles of the prize distribution.

In equilibrium, the utility for a group-i student with cost θ achieving a grade of s is

$$\Pi_i^q(s,\theta) = F_P^{-1}\left[G_i^q(s)\right] - \mathcal{C}(s;\theta) = F_P^{-1}\left(1 - F_i\left[\psi_i^q(s)\right]\right) - \mathcal{C}(s;\theta).$$

This is identical to payoffs in the color-blind game, except that the unconditional cost distribution has been replaced with F_i for group $i = \mathcal{M},\ \mathcal{N}$. By symmetry then, equilibrium achievement will be determined by

$$(\gamma_i^q)'(\theta) = -\frac{f_i(\theta)}{f_P\left(F_P^{-1}\left(1 - F_i(\theta)\right)\right)\mathcal{C}'(\gamma_i^q(\theta);\theta)} \tag{5}$$

and boundary condition (3).

Theorem 5.2. *Given $\rho,\ \varepsilon,\ \delta > 0$, there exists $K^* \in \mathbb{N}$, and a set $E \subset [\underline{\theta}, \overline{\theta}]$ having (Lebesgue) measure $m(E) < \rho$, such that for any $K \geq K^*$, on any closed subset of $[\underline{\theta}, \overline{\theta}] \setminus E$ we have the following:*

- *(i)* *$\gamma_i^q(\theta),\ i = \mathcal{M}, \mathcal{N}$ as defined by equation (5) and boundary condition (3) generates an ε-equilibrium of the K-player quota game, and*
- *(ii)* *$\gamma_i^q(\theta)$ is a δ-approximate equilibrium for the K-player quota game, or*

$$\|\gamma_i^q(\theta) - \gamma_i^q(\theta;K)\|_{\sup} < \delta,\ \ i = \mathcal{M}, \mathcal{N}.$$

Proof: The logic of the proof is very similar to that of Theorem 5.1, but it is simpler because there is only one distribution to work with. Once again, I drop the "q" superscripts for the remainder of the proof and I begin by ordering the random sample of K prizes from lowest quality to highest, denoting the k^{th} order statistic by $p_{(k:K)}$. Since $\gamma(\theta;K)$ is monotonic, the equilibrium expected payoff function in the K-player game can be written as

$$\Pi(s,\theta;K) = \sum_{k=1}^{K} p_{(k:K)} \left[\binom{K-1}{k-1} F_i(\theta)^{K-k}(1-F_i(\theta))^{k-1} \right] - \mathcal{C}(s;\theta).$$

The first term is a weighted average of the order statistics, where the weights are the probabilities of winning each prize.[15] Recall my claim that the limiting equilibrium payoff function is given by

$$\Pi(s,\theta) = F_P^{-1}\left(1 - F_i(\gamma_i^{-1}(s))\right) - \mathcal{C}(s;\theta).$$

I wish to show that for large K, it is nearly optimal to bid as if one were maximizing $\Pi(s,\theta)$, rather than $\Pi(s,\theta;K)$. Since the cost of submitting a given bid never changes, I drop the second term from each payoff function and focus on convergence of the reward function sequence $\{\pi(\theta;K)\}_{K=1}^{\infty}$ to its limit $\pi(\theta)$.

For $l \in [0,1]$, define

$$p(l;K) \equiv \left\{ p_{(t:K)} : t = \underset{k\in\{1,\dots,K\}}{\operatorname{argmin}} \left| l - \frac{i}{K} \right| \right\},^{16}$$

and once again, $\{p(l;K)\}_{K=1}^{\infty}$ can be thought of as a random sequence of the t^{th} order statistic in the sample of prizes, where for each K, t is chosen so that $p_{(t:K)}$ approximates the l^{th} sample quantile as closely as possible. Note that by the same logic as in the proof of Theorem 5.1, we have $\operatorname*{plim}_{N\to\infty} p(l;K) = F_P^{-1}(l)$. Fix $\theta \in [\underline{\theta},\overline{\theta}]$ and let $l = 1 - F_i(\theta)$. Notice that

$$\operatorname*{plim}_{K\to\infty} p(l;K) = F_P^{-1}(1-F_i(\theta)) = \pi(s,\theta).$$

Moreover, For each K I can rewrite the finite expected gross payoff function as

$$\begin{aligned} \pi(s,\theta;K) =& p(l;K) \left[\binom{K-1}{t-1} F_i(\theta)^{K-t}(1-F_i(\theta))^{t-1} \right] \\ & + \sum_{k=1}^{t-1} p_{(k:K)} \left[\binom{K-1}{k-1} F_i(\theta)^{K-k}(1-F_i(\theta))^{k-1} \right] \\ & + \sum_{k=t+1}^{K} p_{(k:K)} \left[\binom{K-1}{k-1} F_i(\theta)^{K-k}(1-F_i(\theta))^{k-1} \right]. \end{aligned} \tag{6}$$

[15] In order for a player to win the k^{th} prize, there must be exactly $K-k$ competitors with lower costs and $k-1$ with higher costs. The probabilities of these two events are $F_i(\theta)^{K-k}$ and $(1-F_i(\theta))^{k-1}$, respectively. Finally, there are $\binom{K-1}{k-1}$ ways in which the intersection of the two events can occur. Thus, the probability of winning the k^{th} prize is $\binom{K-1}{k-1}F_i(\theta)^{K-k}(1-F_i(\theta))^{k-1}$.

[16] If there are multiple maximizers (there can be at most two) then choose t to be the lesser.

Note that

$$
\begin{aligned}
\left[\binom{K-1}{t-1} F_i(\theta)^{K-t}(1-F_i(\theta))^{t-1}\right] &= \Pr[\text{exactly } K-t \text{ competitors have costs lower than } \theta] \\
&= \Pr[\text{fraction } \frac{K-t}{K} \text{ have lower costs}] \\
&= \Pr[\text{fraction } \left(1-\frac{t}{K}\right) \text{ have lower costs}] \\
&\xrightarrow{K} \Pr[\text{fraction } (1-l) \text{ have lower costs}] \\
&= \Pr[\text{fraction } F_i(\theta) \text{ have lower costs}] \\
&=1,
\end{aligned}
$$

where the convergence over K follows from the law of large numbers. Since probabilities must sum to one, equation (6) reveals that $\pi(\theta;K)$ increasingly resembles $p(l;K)$ as K gets large. Furthermore, since $\operatorname*{plim}_{N\to\infty} p(l;K) = F_P^{-1}(l)$, it follows that the pointwise probability limit of $\pi(\theta;K)$ is $\pi(\theta)$.

With pointwise convergence out of the way, the remainder of the proof is identical to the second half of the proof of Theorem 5.1. ■

As before, by using a more complicated proof technique, this result can be strengthened to demonstrate that γ_i^{ap} generates an ε-equilibrium and a δ-approximate equilibrium on the entire set $[\underline{\theta}, \overline{\theta}]$, rather than on a subset with nearly full measure. See the Appendix for details.

Having solved for the achievement functions, some interesting observations can be made about how achievement changes when moving from a color-blind policy to a quota. It turns out that with quota admissions, and under the assumption that the minority cost distribution stochastically dominates the non-minority cost distribution, the highest performing minority students decrease their academic achievement and the lowest performing students increase it. For non-minorities the change is exactly the opposite: high performers increase their effort and low performers decrease it. This result is formalized in the following proposition.

Proposition 5.3. *If $F_{\mathcal{M}}(\theta) < F_{\mathcal{N}}(\theta)$, $\forall\theta \in (\underline{\theta}, \overline{\theta})$, then when moving from a color-blind admission policy to a quota admission scheme, there exists some $\theta^* \in (\underline{\theta}, \overline{\theta})$ such that*

(i) for minorities, the grade distribution conditional on $\theta \in [\theta^*, \overline{\theta}]$ shifts to the right and the grade distribution conditional on $\theta \in [\underline{\theta}, \theta^*]$ shifts to the left; and

(ii) for non-minorities, the grade distribution conditional on $\theta \in [\theta^*, \overline{\theta}]$ shifts to the left and the grade distribution conditional on $\theta \in [\underline{\theta}, \theta^*]$ shifts to the right.

Sketch Proof: The proof of this result uses the rank-based method for analysis of auctions developed by Hopkins [14], where the equilibrium is derived in terms of the rank of one's private type (cost in this case), rather than one's actual private type. If we define $r = F_i(\theta)$ and let $\hat{\gamma}_i(r)$

denote the achievement function in terms of rank, then as Hopkins showed, there is a simple change of variables relating the two equilibrium functions:

$$\gamma'(\theta) = \frac{d\gamma_i(\theta)}{d\theta} = \frac{d\hat{\gamma}_i(r)}{dr}\frac{dr}{d\theta} = \hat{\gamma}_i'(r) f_i(\theta).$$

First consider the case of minorities. The rank-based equilibrium under color-blind allocations and quotas are, respectively,

$$(\hat{\gamma}^{cb}_{\mathcal{M}})'(r) = -\frac{1}{f_P\left[F_P^{-1}(1-r)\right]\mathcal{C}'\left(\hat{\gamma}^{cb}_{\mathcal{M}}(r); F_\Theta^{-1}(r)\right)}$$

and

$$(\hat{\gamma}^{q}_{\mathcal{M}})'(r) = -\frac{1}{f_P\left[F_P^{-1}(1-r)\right]\mathcal{C}'\left(\hat{\gamma}^{q}_{\mathcal{M}}(r); F_{\mathcal{M}}^{-1}(r)\right)}.$$

From these equations it can be shown that there exists a unique rank $r^* \in (0,1)$ such that $\hat{\gamma}^{q}_{\mathcal{M}}(r^*) = \hat{\gamma}^{cb}_{\mathcal{M}}(r^*)$, and moreover, by stochastic dominance it follows that $(\hat{\gamma}^{q}_{\mathcal{M}})'(r^*) > (\hat{\gamma}^{cb}_{\mathcal{M}})'(r^*)$. Then, by a result proven in Hopkins [14, Lemma 1, Corollary 1, p. 3], result *(i)* follows. Result *(ii)* follows from similar logic, where the inequality is reversed. Finally, given that F_Θ is a mixture of $F_{\mathcal{M}}$ and $F_{\mathcal{N}}$, it follows that a single θ^* holds for both *(i)* and *(ii)*. ■

Theorem 5.3 highlights some interesting facets of a student's decision problem. The intuition behind the result is that, for a high-cost (*i.e.*, low-performing) student, when faced with competing against the overall population there is a discouragement effect coming from the fact that there are relatively many students with lower private costs. Since study costs are non-recoverable, it is not worthwhile for a high cost student to put forth much effort. However, if a high-cost minority student instead competes only with members of his own group, where costs are on average lower, the discouragement effect is mitigated somewhat and effort increases.

For low-cost minorities, the effect is reversed: when competing typically higher-cost competitors, there is less need to aggressively outperform the competition, so effort decreases. For non-minority students in each category, the effects are exactly the opposite, by similar intuition. The discouragement effect for high-cost non-minorities is exacerbated by moving to a quota system, and low-cost non-minorities must compete more aggressively against a set of competitors whose costs are on average lower.

As for the overall effect in terms of average grade change within groups and within the overall population, it is unclear whether general statements can be made for arbitrary prize and cost distributions, so this issue remains an empirical question. The effect on the overall racial achievement gap, as to whether a widening or a narrowing can be expected, is also empirical. However, one qualitative conclusion can be drawn, that a quota system negatively impacts the achievement gap among the set of highest-performing students since low-cost minorities decrease achievement and low-cost non-minorities increase achievement, whereas all like cost types have identical achievement under a color-blind rule. As far as allocations go, one attractive quality of the quota system is that it guarantees an allocation gap of zero, in the sense that the distributions of prizes allocated to each race group will be identical, by construction.

5.3. **Affirmative Action: The Admission Preference Game.** In 1978, the Supreme Court of the United States ruled in the case of Regents of the University of California v. Bakke that quotas explicit quotas are unconstitutional. Subsequently, American higher education admissions boards have been forced to seek other means by which to implement AA. These alternative implementations are sometimes referred to as "admissions preferences." An admission preference rule is modeled as a grade transformation function $\tilde{S} : \mathbb{R}_+ \to \mathbb{R}_+$, where $\tilde{S}(s)$ is increasing and $\tilde{S}(s) \geq s$. In assessing the admission preference, The Board assortatively matches prizes with non-minority grades and transformed minority grades

$$\{s_{w1}, \ldots, s_{wW}, \tilde{S}(s_{m1}), \ldots, \tilde{S}(s_{mM})\}.$$

In what follows, it will be convenient to derive the equilibrium in terms of the inverse grade choice rule. Under an admission preference, the Board repositions group-$\mathcal{M}$ students ahead of their group-$\mathcal{N}$ counterparts with grades of $\tilde{S}(s)$ or less. Thus, the limiting gross payoff function for group $\mathcal{M}$ is

$$\pi^{ap}_{\mathcal{M}}(s) = F_P^{-1}\left[1 - \left((1-\mu)F_{\mathcal{N}}\left[\psi^{ap}_{\mathcal{N}}(\tilde{S}(s))\right] + \mu F_{\mathcal{M}}\left[\psi^{ap}_{\mathcal{M}}(s)\right]\right)\right] \tag{7}$$

and the gross payoff function for group $\mathcal{N}$ is

$$\pi^{ap}_{\mathcal{N}}(s) = F_P^{-1}\left[1 - \left((1-\mu)F_{\mathcal{N}}\left[\psi^{ap}_{\mathcal{N}}(s)\right] + \mu F_{\mathcal{M}}\left[\psi^{ap}_{\mathcal{M}}(\tilde{S}^{-1}(s))\right]\right)\right]. \tag{8}$$

The intuition for the above expressions is as follows: after the preference is assessed, a minority student's standing with respect to his own group doesn't change, but his standing with respect to members of the other group changes in a positive direction according to $\tilde{S}$. For non-minorities, standing with respect to other non-minorities doesn't change, but standing with respect to minorities changes in a negative direction according to $\tilde{S}^{-1}$ (since $\tilde{S}$ lies above the 45°-line, it follows that $\tilde{S}^{-1}$ lies below it).

However, the introduction of a preference function $\tilde{S}$ introduces some complications into the analysis. Note that equation (8) only holds for s such that $\tilde{S}^{-1}(s) \geq 0$, because one can only invert grades in the range of the function $\psi^{ap}_{\mathcal{M}}$. If $\tilde{S}$ passes through the origin, this condition is satisfied for every grade in the choice set. On the other hand, there is an interesting class of admission preference rules which do not pass through the origin. An example is an affine rule of the form

$$\tilde{S}(s) = \Delta_1 + \Delta_2 s,$$

where minority students receive a fixed subsidy of Δ_1, regardless of their grade.[17] In that case, non-minorities whose grades are less than $\tilde{S}(0)$ are placed behind *all* minority students, meaning that they compete only with other non-minority students whose grades are less than $\tilde{S}(0)$. This leads to the following proposition:

Proposition 5.4. *In the college admissions game, assume that The Board uses an admission preference rule $\tilde{S}$, where $\tilde{S}(0) = \Delta > 0$. Then it follows that group-$\mathcal{N}$ players with equilibrium grades below $\tilde{S}(0)$ behave as they would under a quota rule.*

Proof: Let θ_Δ denote the non-minority private cost type who's equilibrium grade is Δ and let

$$p_\Delta = F_P^{-1}\left(1 - [(1-\mu)F_{\mathcal{N}}(\theta_\Delta) + \mu F_{\mathcal{M}}(\theta_\Delta)]\right)$$

denote the highest prize awarded to agents whose transformed bids are Δ or less. Also, let $\mathcal{P}_\Delta = [0, p_\Delta]$. On the interval $[\theta_\Delta, \overline{\theta}]$, group $\mathcal{N}$ agents know that they are competing only among themselves for the lowest mass

$$\nu = (1-\mu)F_P(p_\Delta) = (1-\mu)\left[1 - \left((1-\mu)F_{\mathcal{N}}\left[\psi_{\mathcal{N}}^{ap}(\Delta)\right] + \mu F_{\mathcal{M}}\left[\psi_{\mathcal{N}}^{ap}(\Delta)\right]\right)\right]$$

of prizes. Note that ν is also the mass of high-cost, group-$\mathcal{N}$ agents receiving prizes in $\mathcal{P}_\Delta$. From their perspective, it is as if they are playing a quota game where the prize distribution is

$$F_{P_\Delta}(p) = \frac{F_P(p)}{\nu},\ p \in [0, F_P^{-1}(\nu)]$$

and where the distribution over competition is

$$F_{w_\Delta}(\theta) = \frac{F_{\mathcal{N}}(\theta) - (1-\nu)}{\nu},\ \theta \in [\theta_\Delta, \overline{\theta}].$$

Thus, the limiting objective function for high-cost agents from group $\mathcal{N}$ is

$$F_{P_\Delta}^{-1}\left(1 - F_{w_\Delta}\left[\psi_{\mathcal{N}}^{ap}(s)\right]\right) - \mathcal{C}(s;\theta).$$

Since $F_{P_\Delta}^{-1}(r) = F_P^{-1}(\nu r),\ r \in [0,1]$, the objective can be rewritten and rearranged as follows:

$$F_P^{-1}\left[\nu\left(1 - \frac{F_{\mathcal{N}}\left[\psi_{\mathcal{N}}^{ap}(s)\right] - (1-\nu)}{\nu}\right)\right] - \mathcal{C}(s;\theta)$$
$$= F_P^{-1}\left(1 - F_{\mathcal{N}}\left[\psi_{\mathcal{N}}^{ap}(s)\right]\right) - \mathcal{C}(s;\theta),$$

which is exactly the same limiting objective function as under a quota. Since the boundary condition is also the same it follows that on the interval $[\theta_\Delta, \overline{\theta}]$, we have $\gamma_{\mathcal{N}}^{ap}(\theta) = \gamma_{\mathcal{N}}^{q}(\theta)$ and $\theta_\Delta =$

[17] An additive admission preference $\tilde{S}(s) = s + \Delta$ was explicitly used in undergraduate admissions at the University of Michigan, prior to 2004. Admissions decisions were based on an index ranging from 0-120, computed based on applicants' qualifications, with a bonus of 20 points being assessed to all students from underrepresented racial minority groups. This policy was in place until 2003 when the Supreme Court ruled, somewhat vaguely, in a joint opinion on Gratz v. Bollinger and Grutter v. Bollinger, that the bonus was too "narrowly defined" and "mechanical." However, the ruling should not be interpreted as a ban on additive admissions preferences: in the same opinion the court also upheld universities' right to consider race as a "plus factor" in admissions decisions, making it unclear whether the court ruled out all subsidies or that particular level of subsidy.

$\psi^q_{\mathcal{N}}(\Delta)$ from which the result follows. This also provides a boundary condition $\gamma^{ap}_{\mathcal{N}}\left[\psi^q_{\mathcal{N}}(\Delta)\right] = \Delta$ for the solution of $\gamma^{ap}_{\mathcal{N}}$ on the lower interval $[\underline{\theta}, \theta_\Delta]$. ■

Knowing how $\gamma^{ap}_{\mathcal{N}}$ behaves on the upper cost interval (if there is one), I have a boundary condition for non-minorities on the lower interval $[\underline{\theta}, \theta_\Delta]$ for general $\tilde{S}$. Before proceeding, it will be useful to observe that the gross payoff functions satisfy $\pi^{ap}_{\mathcal{M}}(s) = \pi^{ap}_{\mathcal{N}}(\tilde{S}(s))$.

On the lower interval, the limiting objective functions for groups $\mathcal{M}$ and $\mathcal{N}$ are, respectively,

$$F_P^{-1}\left[1 - \left((1-\mu)F_{\mathcal{N}}\left[\psi^{ap}_{\mathcal{N}}(\tilde{S}(s))\right] + \mu F_{\mathcal{M}}\left[\psi^{ap}_{\mathcal{M}}(s)\right]\right)\right] - \mathcal{C}(s;\theta),\ s \geq 0$$

and

$$F_P^{-1}\left[1 - \left((1-\mu)F_{\mathcal{N}}\left[\psi^{ap}_{\mathcal{N}}(s)\right] + \mu F_{\mathcal{M}}\left[\psi^{ap}_{\mathcal{M}}(\tilde{S}^{-1}(s))\right]\right)\right] - \mathcal{C}(s;\theta),\ s \geq \tilde{S}(0)$$

and the FOCs are

$$\mathcal{M}: \quad -\frac{(1-\mu)f_{\mathcal{N}}\left[\psi^{ap}_{\mathcal{N}}(\tilde{S}(s))\right](\psi^{ap}_{\mathcal{N}})'(\tilde{S}(s))\tilde{S}'(s) + \mu f_{\mathcal{M}}\left[\psi^{ap}_{\mathcal{M}}(s)\right](\psi^{ap}_{\mathcal{M}})'(s)}{f_P\left[\Pi^{ap}_{\mathcal{M}}(s)\right]} = \mathcal{C}'(s;\theta)$$

and

$$\mathcal{N}: \quad -\frac{(1-\mu)f_{\mathcal{N}}\left[\psi^{ap}_{\mathcal{N}}(s)\right](\psi^{ap}_{\mathcal{N}})'(s) + \mu f_{\mathcal{M}}\left[\psi^{ap}_{\mathcal{M}}(\tilde{S}^{-1}(s))\right](\psi^{ap}_{\mathcal{M}})'(\tilde{S}^{-1}(s))\frac{1}{\tilde{S}'(\tilde{S}^{-1}(s))}}{f_P\left[\Pi^{ap}_{\mathcal{N}}(s)\right]} = \mathcal{C}'(s;\theta).$$

In equilibrium, it will be true that $\psi^{ap}_i(s) = \theta$ for group i, so by substituting and rearranging I get

$$(\psi^{ap}_{\mathcal{M}})'(s) = -\frac{\mathcal{C}'\left[s;\psi^{ap}_{\mathcal{M}}(s)\right]f_P\left[\Pi^{ap}_{\mathcal{M}}(s)\right]}{\mu f_{\mathcal{M}}\left[\psi^{ap}_{\mathcal{M}}(s)\right]} - \frac{(1-\mu)f_{\mathcal{N}}\left[\psi^{ap}_{\mathcal{N}}(\tilde{S}(s))\right]}{\mu f_{\mathcal{M}}\left[\psi^{ap}_{\mathcal{M}}(s)\right]}(\psi^{ap}_{\mathcal{N}})'(\tilde{S}(s))\tilde{S}'(s) \tag{9}$$

and

$$\begin{aligned}(\psi^{ap}_{\mathcal{N}})'(s)\tilde{S}'(\tilde{S}^{-1}(s)) \\ = -\frac{\mathcal{C}'\left[s;\psi^{ap}_{\mathcal{N}}(s)\right]f_P\left[\Pi^{ap}_{\mathcal{N}}(s)\right]\tilde{S}'(\tilde{S}^{-1}(s))}{(1-\mu)f_{\mathcal{N}}\left[\psi^{ap}_{\mathcal{N}}(s)\right]} - \frac{\mu f_{\mathcal{M}}\left[\psi^{ap}_{\mathcal{M}}(\tilde{S}^{-1}(s))\right]}{(1-\mu)f_{\mathcal{N}}\left[\psi^{ap}_{\mathcal{N}}(s)\right]}(\psi^{ap}_{\mathcal{M}})'(\tilde{S}^{-1}(s)).\end{aligned} \tag{10}$$

Equations (9) and (10), with boundary conditions complete the solution for $(\gamma^{ap}_{\mathcal{M}}, \gamma^{ap}_{\mathcal{N}})$. By evaluating equation (10) at $\tilde{S}(s)$ and substituting it into the FOC for minorities, equation (9) reduces to

$$\mathcal{C}'\left[s;\psi^{ap}_{\mathcal{M}}(s)\right] = \mathcal{C}'\left[\tilde{S}(s);\psi^{ap}_{\mathcal{N}}(\tilde{S}(s))\right]\tilde{S}'(s), \tag{11}$$

which provides a relation between grade selection in the two groups. As it turns out, equation (11) is important for characterizing the effects of $\tilde{S}$ on minority bidding. The solution for equilibrium grades under a general admission preference $\tilde{S}$ is given by Proposition (5.4); equations (10) and (11) and boundary condition $\psi^{ap}_{\mathcal{N}}(\Delta) = \psi^q_{\mathcal{N}}(\Delta) = \theta_\Delta$, where $\Delta = \tilde{S}(0)$. Of course, the following theorem is needed to validate this claim.

Theorem 5.5. *In the college admission game with an admission preference $\tilde{S}$, given $\rho,\ \varepsilon,\ \delta > 0$, there exists $K^* \in \mathbb{N}$, and a set $E \subset [\underline{\theta}, \overline{\theta}]$ having (Lebesgue) measure $m(E) < \rho$, such that for any $K \geq K^*$, on any closed subset of $[\underline{\theta}, \overline{\theta}] \setminus E$ we have the following:*

- *(i) an ε-equilibrium of the K-player admission preference game is generated by $\gamma_i^{ap}(\theta),\ i = \mathcal{M}, \mathcal{N}$ as defined by Proposition (5.4), equation (10), equation (11), boundary condition (3) for non-minorities and boundary condition*

$$\mathcal{C}'\,[0;\theta^*] = \mathcal{C}'\,[\Delta;\theta_\Delta]\,\tilde{S}'(0)$$

for minorities, where $\theta^* = \inf\left\{\theta\ :\ \gamma_{\mathcal{M}}^{ap}(\theta) = 0\right\}$, $\theta_\Delta = \psi_{\mathcal{N}}^{ap}(\Delta)$, and $\Delta = \tilde{S}(0)$; and

- *(ii) $\gamma_i^{ap}(\theta)$ is a δ-approximate equilibrium for the K-player quota game, or*

$$\|\gamma_i^{ap}(\theta) - \gamma_i^{ap}(\theta; K)\|_{\sup} < \delta,\ \ i = \mathcal{M}, \mathcal{N}.$$

Proof: The proof is similar to that for Theorem 5.1. ■

As before, by using a more complicated proof technique, this result can be strengthened to demonstrate that γ_i^{ap} generates an ε-equilibrium and a δ-approximate equilibrium on the entire set $[\underline{\theta}, \overline{\theta}]$, rather than on a subset with nearly full measure. See the Appendix for details.

6. Special Case: Linear Disutility

In order to simplify the analysis, I henceforth assume that the cost function is linear in both θ and s: $\mathcal{C}(s;\theta) = \theta s$. Under this assumption, equation (11) reduces to

$$\psi_{\mathcal{M}}^{ap}(s) = \psi_{\mathcal{N}}^{ap}(\tilde{S}(s))\tilde{S}'(s). \tag{12}$$

As mentioned previously, this equation reveals much about minority grade selection under preference rules. It turns out that the marginal performance subsidy at a grade of $s = 0$ plays an important role, as outlined in the following proposition.

Proposition 6.1. *In the college admissions game, assume that study costs are of the form $\mathcal{C}(s;\theta) = \theta s$ and The Board uses an admission preference rule $\tilde{S} : \mathbb{R}_+ \to \mathbb{R}_+$, where $\tilde{S}(s) \geq s$ and $\tilde{S}'(s) \geq 0,\ \forall\ s > 0$; moreover, define $\Delta \equiv \tilde{S}(0)$ and $\theta_\Delta \equiv \psi_{\mathcal{N}}^{ap}(\Delta) = \psi_{\mathcal{N}}^{q}(\Delta)$. Then the following results follow:*

- *(i) If $\tilde{S}'(0) \geq (>) \frac{\overline{\theta}}{\psi_{\mathcal{N}}^{ap}(\Delta)}$ the grade achieved by a minority student with the highest possible private cost is non-negative (strictly positive).*
- *(ii) If $\frac{\underline{\theta}}{\psi_{\mathcal{N}}^{ap}(\Delta)} < \tilde{S}'(0) < \frac{\overline{\theta}}{\psi_{\mathcal{N}}^{ap}(\Delta)}$ there is a positive mass $\zeta \in (0, 1)$ of minority students who choose equilibrium grades of zero.*
- *(iii) If $\tilde{S}'(0) < \frac{\underline{\theta}}{\psi_{\mathcal{N}}^{ap}(\Delta)}$ all minority students choose equilibrium grades of zero.*

Proof: From equation (12) it follows that

$$\inf \psi_{\mathcal{M}}(0) = \psi_{\mathcal{N}}(\Delta)\tilde{S}'(0),$$

which solves for the lowest minority cost type who achieves a grade of zero. Statements *(i)*, *(ii)* and *(iii)* then follow from substituting the left-hand side to test whether

$$\overline{\theta} \gtreqless \psi_{\mathcal{N}}(\Delta)\tilde{S}'(0)$$

and

$$\underline{\theta} \gtreqless \psi_{\mathcal{N}}(\Delta)\tilde{S}'(0). \quad \blacksquare$$

At this point I will further simplify the analysis by adopting a simple additive preference policy $\tilde{S}(s) = s + \Delta$. This rule is similar to that previously used at the University of Michigan in undergraduate admissions, so I shall henceforth refer to it as "the Michigan rule." In that case, equation (11) reduces to

$$\psi^{ap}_{\mathcal{M}}(s) = \psi^{ap}_{\mathcal{N}}(s + \Delta). \tag{13}$$

This indicates that a minority student with private cost θ will achieve a grade of exactly Δ less than his non-minority counterpart with the same private cost. In this case, non-minority students whose grades are less than Δ will achieve the same grades as under a quota rule, giving a boundary condition of

$$\psi^{ap}_{\mathcal{N}}(\Delta) = \psi^{q}_{\mathcal{N}}(\Delta) = \theta_{\Delta}, \tag{14}$$

and minority students with costs above θ_{Δ} choose a grade of zero.Equation (13) can in turn be substituted back into the decision problem of $\mathcal{N}$ agents to get

$$F_P^{-1}\left[1 - \left((1-\mu)F_{\mathcal{N}}\left[\psi^{ap}_{\mathcal{N}}(s)\right] + \mu F_{\mathcal{M}}\left[\psi^{ap}_{\mathcal{N}}(s)\right]\right)\right] - \theta s,$$

which gives the familiar FOC:

$$(\gamma^{ap}_{\mathcal{N}})'(\theta) = -\frac{(1-\mu)f_{\mathcal{N}}(\theta) + \mu f_{\mathcal{M}}(\theta)}{f_P\left[F_P^{-1}\left(1 - \left[(1-\mu)F_{\mathcal{N}}(\theta) + \mu F_{\mathcal{M}}(\theta)\right]\right)\right]\theta}. \tag{15}$$

Recall that this is the same as the FOC under the color-blind rule.

Proposition 6.2. *If achievement costs are linear, then under a Michigan admission rule with fixed grade boost Δ, a given competitor from group $\mathcal{M}$ will choose a grade of exactly Δ less than a group-$\mathcal{N}$ competitor with the same private cost, unless the latter's equilibrium grade is less than Δ. In that case, the group-$\mathcal{N}$ student chooses his grade as if a quota system were in place and the group-$\mathcal{M}$ student with a comparable private cost will achieve a grade level of zero.*

Proof: This result follows immediately from equation (13) and from the fact that $\gamma^{ap}_{\mathcal{N}}(\theta) < \Delta$ on the interval $[\theta_{\Delta}, \overline{\theta}]$. $\blacksquare$

This result is significant for several reasons. First, it indicates that with linear costs and an additive subsidy, bidding for group-$\mathcal{N}$ agents with high costs resembles bidding under a quota system. If Δ is not very large, this implies that equilibrium achievement in group $\mathcal{N}$ shifts downward for all students, relative to color-blind allocations. This is because under a quota system, high cost non-minorities decrease their achievement. Behavior for low-cost non-minorities parallels color-blind behavior, but with a lower boundary condition.

Second, it highlights an interesting aspect of the behavioral response among beneficiaries of the admission preference policy. Although the policy-maker may wish for students to use the grade boost to bolster their competitive edge, equation (13) indicates that when costs are linear, a rational student will simply eat the entire grade boost: for $\mathcal{N}$ agents bidding above Δ, $\mathcal{M}$ players with comparable costs achieve the same grades, less Δ and all other group-$\mathcal{M}$ students simply achieve a grade of zero. In other words, under linear costs, an admission preference creates no change in the relative standing between the two groups for low-cost agents.[18] This means that additive grade subsidies also lead to a leftward shift of the minority grade distribution, as well as the overall population grade distribution. The implication is that a Michigan-type admission preference is unambiguously bad for effort choice.

Third, given the above facts about the relation between minority and non-minority achievement under a Michigan rule, it immediately follows that the policy will lead to a widening of the racial achievement gap at every point in the private cost support, relative to a color-blind policy. This is because like types have the same achievement under a color-blind rule, whereas the difference in achievement between a minority and a non-minority with private cost θ under the Michigan rule is $\min\{\Delta, \gamma^{ap}_{\mathcal{M}}(\theta)\}$, which is strictly positive for any $\theta < \overline{\theta}$.

Finally, these facts also imply that an admission preference rule is a somewhat ineffective mechanism for shifting allocations. Indeed, with linear costs, a Michigan rule has *no allocational effect* among students with costs below $\psi^{ap}_{\mathcal{N}}(\Delta)$. The only effect is in re-shuffling allocations in the lower tail of the prize distribution so that minorities get the top mass μ of the set of least desirable prizes. In other words, under the Michigan rule, enrollment at more selective schools will be *identical* to enrollment under a color-blind admission policy. Clearly, additive grade subsidies appear to be an ineffective and costly means of promoting racial equality.

6.1. **Example: Uniform Prizes and Pareto Private Costs.** The results proven so far provide some good intuition for how the model equilibrium is affected by different admission policies, but some results are difficult to derive analytically. For example, it has been shown above that different segments of the population react differently to AA, but it is unclear how the overall population grade distribution changes. Another interesting but analytically difficult question involves a comparison between the alternative AA policies: what must an admission preference grade boost be in order to produce the same average allocative effect as a quota, and how might that grade boost affect performance?

[18] When costs are convex in achievement, the effect is less straightforward (see equation (11)) and complete consumption of the grade boost will not obtain in general. However, it will generally be the case that the beneficiaries of an admission preference will use at least some portion of the grade boost as a direct utility subsidy, rather than using it only to bolster their competitive edge.

To illustrate the model, I consider a simple special case where prizes are distributed uniformly on the interval $[0, 100]$, so that $F_P(p) = \frac{p}{100}$. The fraction of minority college candidates is $\mu = 0.25$ and private costs follow a Pareto distribution, with the upper tail truncated to the interval $[1, 5]$. More precisely, for group i the private cost distribution is

$$F_i(\theta) = \frac{1-\theta^{-\kappa_i}}{1-\overline{\theta}^{-\kappa_i}},\ \kappa_i > 0,\ i = \mathcal{M}, \mathcal{N},$$

where $\kappa_{\mathcal{M}} = 0.1$ and $\kappa_{\mathcal{N}} = 1.5$, so that stochastic dominance holds. All parameters in this section were chosen for purely illustrative purposes, with the exception of Δ, which is discussed below. With the above parameters specified, solving for the equilibrium is a simple mater of integrating differential equations.

A value of Δ was specifically chosen to facilitate comparisons between equilibria under a quota and a Michigan admission preference rule. The key characteristic of a quota is that it ensures that the average prize value allocated to members of each group is the same. Thus, I choose a fixed grade subsidy Δ^* so that the average prize value awarded to each group is the same. This allows me to compare the outcomes under two different AA policies which are implemented so as to achieve a common objective. Computing Δ^* under uniformly-distributed prizes is fairly simple. Recall from the section on admission preferences above that when costs are linear, an additive grade subsidy only alters equilibrium allocations among agents who grade less than Δ. Once again, let θ_Δ denote the player type that submits a grade of Δ for group $\mathcal{N}$ and let $p_\Delta = F_P^{-1}\left[1 - \{(1-\mu)F_{\mathcal{N}}(\theta_\Delta) + \mu F_{\mathcal{M}}(\theta_\Delta)\}\right]$ denote the top prize allocated to players whose transformed grades are Δ or less.

Within the interval $[0, p_\Delta]$, the top μ mass of prizes are awarded to students from group $\mathcal{M}$ and the rest are given to group $\mathcal{N}$. Thus, the average prize given to candidates with costs $c \leq \theta_\Delta$ in group $\mathcal{M}$ and $\mathcal{N}$ are, respectively, $(p_\Delta + \mu p_\Delta)/2$ and $\mu p_\Delta/2$. The average prize awarded to players of either group with private costs above θ_Δ are the same–recall that transformed equilibrium grades are the same for a given θ in this interval–and are given by $(\overline{p} + p_\Delta)/2$. Therefore, the average prize allocated to group $\mathcal{M}$ candidates is

$$[1 - F_{\mathcal{M}}(\theta_\Delta)]\,\frac{p_\Delta + \mu p_\Delta}{2} + F_{\mathcal{M}}(\theta_\Delta)\frac{\overline{p} + p_\Delta}{2} \tag{16}$$

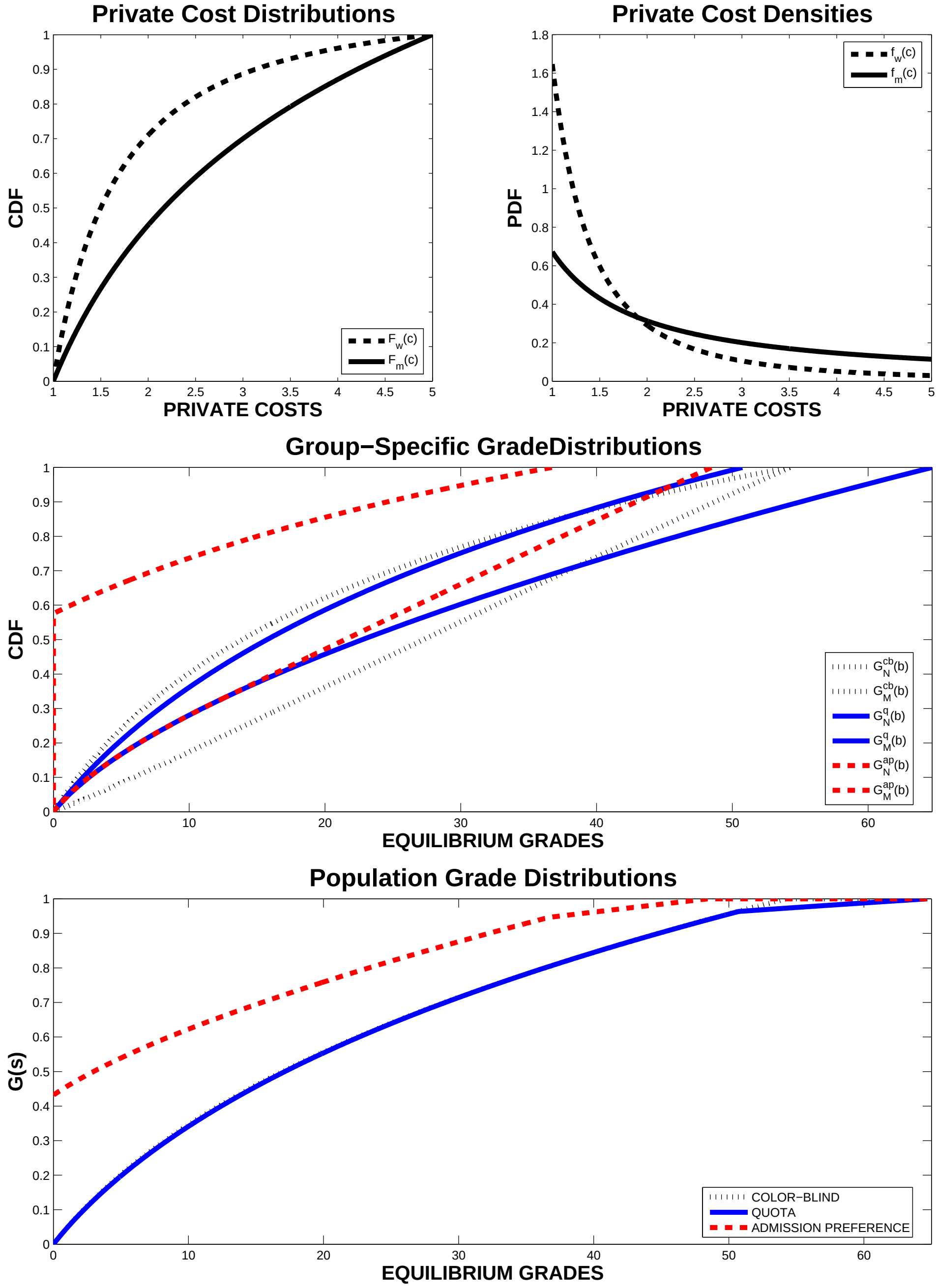
and the average prize for group $\mathcal{N}$ is

$$[1 - F_{\mathcal{N}}(\theta_\Delta)]\,\frac{\mu p_\Delta}{2} + F_{\mathcal{N}}(\theta_\Delta)\frac{\overline{p} + p_\Delta}{2}. \tag{17}$$

Thus, Δ^* is determined by the following equality

$$\begin{aligned}[1 - F_{\mathcal{M}}(\theta_{\Delta^*})]\,&\frac{p_{\Delta^*} + \mu p_{\Delta^*}}{2} + F_{\mathcal{M}}(\theta_{\Delta^*})\frac{\overline{p} + p_{\Delta^*}}{2} \\ &= [1 - F_{\mathcal{N}}(\theta_{\Delta^*})]\,\frac{\mu p_{\Delta^*}}{2} + F_{\mathcal{N}}(\theta_{\Delta^*})\frac{\overline{p} + p_{\Delta^*}}{2}.\end{aligned} \tag{18}$$

FIGURE 1



In the special case of Pareto private costs, Δ^* is 23.15% of the maximal grade achieved by group $\mathcal{M}$. In other words, in order for a Michigan rule to deliver the same average effect as a quota, the subsidy must be quite large.[19]

[19]One might wonder whether this is an artifact of the left-skewed private-cost distribution, but this does not appear to be the case. If the numerical example from this section is re-computed using a normal distribution with an interior mode, or a right-skewed power distribution, results are very similar.

Figure 1 plots several objects of interest. The top two panes are the distributions and densities of private costs. The middle pane displays group-specific grade distributions under each of the three admission policies. A color-blind rule is denoted by a dotted line, a Michigan rule is denoted by a dashed line, and a solid line denotes a quota. For pairs of lines with the same style, the one lying to the left is the grade distribution for minorities. The bottom pane displays population grade distributions, following a similar convention. When comparing two grade distributions, keep in mind that if distribution i lies to the right of distribution j in some region, that indicates an interval of students who are enticed to achieve a higher academic output under policy i.

The middle plot gives an idea of how within-group behavior changes, and also how the achievement gap changes under different policies. As Propositions 6.1 and 6.2 suggest, the picture for an additive admission preference is dismal. A striking feature of the plot is the size of the mass point a policy-maker must settle for in order to equalize average outcomes for each group. Notice also the substantial leftward shift in non-minority grades associated with the quota-comparable Michigan rule. Recall also that allocations in the upper tail of the prize distribution are unaffected, even with this extreme version of the policy. Although producing a more definitive analysis is ultimately an empirical exercise, this example shows that an ill-designed admission preference can lead to a substantial social loss, while producing little positive change.

As outlined in Proposition 5.3, a quota rule produces some interesting benefits, relative to a color-blind system. The middle pane shows that all minorities below the 80^{th} grade percentile and all non-minorities above the 30^{th} grade percentile increase their performance, relative to a color-blind system. Of course, there are also costs involved, as all other students decrease academic output. Although it is difficult to tell from the lower pane, it turns out that a quota produces a very slight stochastic dominance shift in the overall population grade distribution. In contrast, the difference from Michigan-rule grades for both color-blind and quota admissions is quite stark.

Although these observations paint a fairly dismal picture for American-style AA, it is important to note that it is possible to design a more general form of the admission preference rule so as to overcome some of the drawbacks of the Michigan rule. Indeed, the main shortcoming of the Michigan rule is that the additional weight it places on a student's grade does not take into account the student's performance: whether you score 0 or 100, the grade boost is the same.

However, if one were to consider a more general affine admission preference,

$$\tilde{S}(s) = \Delta_0 + \Delta_1 s$$

then as Proposition 6.1 suggests, the slope coefficient could be chosen so as to eliminate the mass point of minority students achieving grades of zero. On the other hand, this may present a problem from the policy-maker's perspective, in that optimal selection of the transformation function parameters would require knowledge of the underlying distributions of private costs.

This highlights another advantage of a quota rule over an admission preference: the former is simple to administer, whereas optimization under the latter may require strong assumptions on the amount of information available for policy decisions.

7. Conclusion

In this paper, I have explored the qualitative implications of different AA policies in college admissions. It turns out that the way in which AA is implemented can have significant effects on effort choice. On the one hand, it appears that the critics of AA are correct in assuming that a tradeoff exists between equality and academic performance incentives, although the exact nature of the tradeoff—whether it results in a socially desirable change—cannot be resolved theoretically. On the other hand, proponents of AA are also correct in assuming that race-conscious admissions can potentially increase academic performance for some minorities by diminishing discouragement effects. However, in the process of leveling the playing field for some minority students, the situation is made worse for disadvantaged (*i.e.*, high-cost) non-minorities. Moreover, advantaged minorities find diminished incentives for academic performance in an environment where the competition they face is less fierce. Once again, the exact nature of the tradeoff cannot be determined without a meaningful empirical analysis.

As for AA and allocations, somewhat more can be said qualitatively about the relation between quotas and admission preferences. Namely, by construction quotas achieve 100% equal allocations in the sense that the racial makeup of student bodies at schools of all quality levels will be reflective of population proportions. On the other hand, the effectiveness of admission preferences in rearranging allocations is hampered by the rational behavioral response to this type of policy. Clearly, a policy-maker should *not* treat behavior as fixed when predicting equilibrium allotments of college seats to different groups under different policies. Indeed, in the case of a Michigan-type additive admission preference, such an assumption may result in a near total nullification of any intended change.

In ongoing work building on the theory presented in this paper, Hickman [13] performs a structural estimation of the model using data on US colleges and college entrance test scores. The object of that exercise is to quantify the social costs and benefits of different AA policies—such as quotas and general admission preferences—in an attempt to better understand the social costs and benefits, and whether a superior alternative can be named.

There are two other interesting directions for further research. First, as alluded to in Footnote 14, an important related question would be how AA might affect educational attainment decisions among minorities. The current model focuses on student behavior, conditional on participation in the higher education market, but there is another interesting group of individuals to consider as well: those whose college/work-force decisions may be affected by a given policy. This question could be addressed by formalizing the "supply-side," being comprised of potential colleges and firms who may enter the market and supply post-secondary education services and unskilled jobs. Such a model might illuminate how the marginal agent (*i.e.*, the individual indifferent between attending college and entering the workforce) is affected by a given college admission policy. This would help to characterize the effect of Affirmative Action on the total mass of minorities enrolling in college.

Finally, the eventual goal for this line of research should be to answer the question of how AA helps or hinders the objective of erasing the residual effects of past institutionalized racism. This will require a more general dynamic model in which the policy-maker is not only concerned with short-term outcomes for students whose private costs are fixed; but also with the long-run evolution of the private cost distributions. Empirical evidence suggests that academic competitiveness is determined by things such as affluence and parents' education. If AA affects performance and outcomes for current minority students in a certain way, the next question is what effect it might that have on their children's competitiveness when the next generation enters high school. If a given policy produces the effect of better minority enrollment and higher achievement in the short-run then one might conjecture that a positive long-run effect will be produced. However, given the mixed picture on the various policies considered in this paper, it seems evident that a long-run model is needed in order to give meaningful direction to forward-looking policy-makers. The theory developed here will hopefully serve as a basis for answering these important questions in the future.

APPENDIX

ALTERNATIVE PROOF OF EQUILIBRIUM APPROXIMATION:

As mentioned in the body of the paper, the various results on equilibrium approximation can be strengthened to demonstrate that the derivations accurately reflect equilibrium actions and outcomes on the entire support of private costs. The cost of the stronger result is application of a more complicated proof which invokes results that may be less familiar to economic theorists. The logic is very similar under all three cases of color-blind admissions, quotas and admission preferences, so here I merely state and prove the alternative claim in the case of a quota rule, where the notation is simplest.

Theorem 7.1. *Given $\varepsilon,\ \delta > 0$, there exists $K^* \in \mathbb{N}$, such that in the college admission game with a quota rule, we have the following:*

(*i*) $\gamma_i^q(\theta),\ i = \mathcal{M}, \mathcal{N}$ *as defined by equation (5) and boundary condition (3) generates an ε-equilibrium of the K-player quota game, and*

(*ii*) $\gamma_i^q(\theta)$ *is a δ-approximate equilibrium for the K-player quota game, or*

$$\|\gamma_i^q(\theta) - \gamma_i^q(\theta; K)\|_{\sup} < \delta, \ \ i = \mathcal{M}, \mathcal{N}.$$

Proof: The first part of the proof involves showing that the finite objective functions converge pointwise in probability to the proposed limiting objective. The argument is identical to the one in the first half of the proof of Theorem 5.2

Using pointwise convergence, I can then invoke Newey's [20, Theorem 2.1] uniform convergence theorem to show that $\tilde{\Pi}(\theta;K)$ converges uniformly in probability to $\Pi(\theta)$ on the entire interval $[\underline{\theta},\overline{\theta}]$. In order to do so, I must first verify a regularity condition and *stochastic equicontinuity* of the sequence $\tilde{\Pi}(\theta;K)$. For this part of the argument, it will be easier to think in terms of l, rather than θ. The regularity condition is that $l(\theta)=1-F_i(\theta)$ must live on a compact interval. Since θ lives on a compact interval and since F_i is continuous and monotonic, $l(\theta)$ attains values of 0 and 1 for finite values of θ.

At this point, all that remains is to verify equicontinuity of the sequence of functions $\tilde{\Pi}(\theta;K)$. For deterministic functions, it is known that pointwise convergence on a compact interval to a continuous limit implies uniform convergence if the sequence is equicontinuous. There is also an analogous condition for sequences of random functions, known as stochastic equicontinuity. In the context of my model, it basically means that for any point θ, $\tilde{\Pi}(\theta;K)$ must be continuous at θ at least with probability close to one for large K.[20] More precisely, $\{\tilde{\Pi}(\theta;K)\}_{K=1}^{\infty}$ is stochastically equicontinuous if for any $\epsilon,\epsilon'>0$ there exists $\tau>0$ such that

$$\begin{aligned}
&\limsup_{K\to\infty}\Pr\left[\sup_{l\in[0,1],\ l'\in B_\tau(l)}\left|\tilde{\Pi}(l;K)-\tilde{\Pi}(l';K)\right|>\epsilon'\right]\\
&=\limsup_{K\to\infty}\Pr\left[\sup_{l\in[0,1],\ l'\in B_\tau(l)}\left|\sum_{i=1}^{K}p_{(k:K)}\left[\binom{K-1}{i-1}(1-l)^{K-i}(l)^{i-1}\right]\right.\right.\\
&\qquad\left.\left.-\sum_{i=1}^{K}p_{(k:K)}\left[\binom{K-1}{i-1}(1-l')^{K-i}(l')^{i-1}\right]\right|>\epsilon'\right]<\epsilon,
\end{aligned}$$

where $B_\tau(l)$ is an open ball centered at l with radius τ.

By similar arguments as above, it is apparent that

$$\sum_{i=1}^{K}p_{(k:K)}\left[\binom{K-1}{i-1}(1-l)^{K-i}(l)^{i-1}\right]\to F_P^{-1}(l)\ \text{ and}$$
$$\sum_{i=1}^{K}p_{(k:K)}\left[\binom{K-1}{i-1}(1-l')^{K-i}(l')^{i-1}\right]\to F_P^{-1}(l').$$

Therefore, I can satisfy stochastic equicontinuity by choosing τ^* so that for all $l\in[0,1]$ and $l'\in B_{\tau^*}(l)$, the following is true:

$$\left|F_P^{-1}(l)-F_P^{-1}(l')\right|<\epsilon'.$$

[20] For a more detailed discussion on stochastic equicontinuity, see Andrews [1, Section 2.1].

Since F_P is continuous and $\mathcal{P}$ is compact, such a τ^* indeed exists. Thus, by Newey's uniform convergence theorem, it follows that for all $\epsilon > 0$, I have

$$\lim_{K\to\infty} \Pr\left[\|\tilde{\Pi}(\theta;K) - \Pi(\theta)\|_{\sup} > \epsilon\right] = 0.$$

In other words, When K is large, the equilibrium grade distribution under a monotonic equilibrium is such that it is nearly optimal to maximize as if one's (equilibrium) objective function were $\Pi(\theta, s) = F_P^{-1}(1 - F_i(\theta)) - \mathcal{C}(s;\theta)$.

This is the same as saying that it is nearly optimal to choose one's grade as if one's opponents were adopting a strategy of $\gamma(\theta)$, rather than $\gamma(\theta;K)$. Thus, given $\varepsilon > 0$, there exists K_ε such that for any $K \geq K_\varepsilon$, $\gamma(\theta)$ generates an ε-equilibrium of the K-player finite game. Furthermore, since all of the model primitives are well-behaved–θ is strictly bounded away from zero and lives in a compact set, $\mathcal{P}$ is compact, F_i and F_P are absolutely continuous and for each θ the set of undominated grades is compact-valued–the Theorem of the Maximum implies that the maximizers of $\tilde{\Pi}(s,\theta;K)$ and $\Pi(s,\theta)$ are close for large K. That is, given $\delta > 0$, there exists K_δ such that for any $K \geq K_\delta$, $\gamma(\theta)$ is a δ-approximate equilibrium of the K-player finite game, or

$$\|\gamma(\theta) - \gamma(\theta;K)\|_{\sup} < \delta.$$

Finally, given $\varepsilon > 0$ and $\delta > 0$, then for any $K \geq K^* \equiv \max\{K_\varepsilon, K_\delta\}$, $\gamma(\theta)$ is a δ-approximate equilibrium which generates an ε-equilibrium of the K-player finite game. ■

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