How to connect under incomplete information*

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Abstract

We study how players' incomplete information about neighbors affects the structure of a network. In our setup, a player's type is a set of players he would like to be connected with, while a social planer designs mechanisms assigning a undirected network to each profile of types. We suppose that players enter into coalitional contracts either at the ex ante or at the interim stage, and show that the ex ante incentive compatible core and the interim incentive compatible coarse core are both non-empty in the presence of link-specific costs and benefits. Depending on the cost/benefit structure of players' utility functions, two mechanisms turn out to be crucial. According to the first mechanism, a link between any two players is established only if mutual consent is present in the corresponding profile. In the second mechanism, the presence of one player's wish suffices for the link to be built.

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1 Introduction

The way in which agents are connected to each other often shapes their economic success since it provides access to valuable resources such as infor-

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mation or capital. In many social and economic settings, agents' well being crucially depends on who are they connected with and whether these connections are desired or not. In the present paper, we study the impact of these two features on the structure of a network.

More precisely, we consider an environment where players' utility depends solely on who they are directly connected with in a network, and introduce incomplete information as to model players' insufficient knowledge about neighbors. In contrast to recent works of Galeotti et al. (2006) and Kets (2007) where a player's type is his connectivity, we assume in our setup that a player's type is a set of players he would like to be connected with. Naturally then, a mechanism assigns a undirected network to each profile of types. We focus on the existence of incentive compatible mechanisms when coalitions are formed either at the ex ante stage (i.e., before any player receives private information) or at the interim stage (after each player knows his type). Depending on the cost/benefit structure of players' utility functions, two mechanisms turn out to be crucial. According to the first mechanism, a link between any two players is established only if mutual consent is present in the corresponding profile. In the second mechanism, the presence of one player's wish suffices for the link to be built. Specifically, we show that a suitable combination of these mechanisms (taking into account players' link-specific costs and benefits) belongs both to the ex ante incentive compatible core and to the interim incentive compatible coarse core. Thus, these cores are non-empty.

The rest of the paper is organized as follows. The next section contains preliminaries on networks, mechanisms, incentive compatibility and efficiency. In Section 3, we show that any pair-wise measurable and pair-wise efficient mechanism is incentive compatible for a quite general specification of players' utility functions. Our analysis of the situation in which players enter into coalitional contracts at the ex ante stage is presented in Section 4. We start by providing necessary conditions for an ex ante core stable mechanism, and relate constant and ex ante core stable mechanisms. For link-specific costs and benefits, we show that a combination of the two mechanisms mentioned above belongs to the ex ante incentive compatible core. We also elaborate on different examples and illustrate that restricting the blocking coalitions to pairs strictly enlarges the ex ante core, and that there are situations in which the ex ante incentive compatible core consists of more than one mechanism. In Section 5 we show that, again for link-specific costs and benefits, the same combination of the two mechanisms belongs to the interim incentive compatible coarse core. We conclude in Section 6 with some final remarks.

2 Preliminaries

There is a set of players $N = \{1, \ldots, n\}$ that we keep fixed in what follows. Player i's type t_i is a set of players i would like to be connected with, i.e., $t_i \in T_i \subseteq 2^{N \setminus \{i\}}$. There is a *common prior* (probability distribution) q defined over $T = \prod_{i \in N} T_i$. We will use the notation t_{-i} to denote $(t_j)_{j \neq i}$. Similarly $T_{-i} = \prod_{j \in N \setminus \{i\}} T_j$, and for any coalition $S \subseteq N$, $t_S = (t_i)_{i \in S}$. Further, we assume that there are *no redundant types*, i.e., for every $i \in N$ and $t_i \in T_i$, there exists $t_{-i} \in T_{-i}$ such that $q(t_{-i}, t_i) > 0$.

Players are involved in network relationships. More precisely, we denote by g^N the set of all subsets of N of size 2. A (undirected) *network* g is a subset of g^N. The set of all networks is $G = \left\{g \mid g \subseteq g^N\right\}$. For $S \subseteq N$ and

$g \in G$, the *subnetwork* of g on S is defined by $g_{|S} = \{ij \mid ij \in g \text{ and } i, j \in S\}$. For every $i \in N$ and $g \in G$ the set $P_i(g) = \{j \in N \mid ij \in g\}$ consists of all (direct) *neighbors* of i in g.

A *mechanism* is a mapping $\mu : T \to G$ which assigns a (undirected) network to each profile of types. We call a mechanism μ *measurable* wrt (the information available to) a coalition $S \subseteq N$ if for all $t, t' \in T$ we have that $t_S = t'_S$ implies $\mu(t)_{|S} = \mu(t')_{|S}$. Thus, the subnetwork of both $\mu(t)$ and $\mu(t')$ on S remains the same, provided that the types of the players in S do not change. A mechanism is *pair-wise measurable* if it is measurable wrt every coalition S with $|S| = 2$.

Next, we introduce the notion of a feasible mechanism. For a coalition $S \subseteq N$, a mechanism ν and a profile of types $t \in T$, let us define the set

$$G_S^{\nu,t} := \{g \in G \mid P_i(g) = P_i(\nu(t)) \text{ for all } i \in N \setminus S\}$$

of all networks that differ from $\nu(t)$ only with respect to links between players in S. We say that a mechanism $\mu_S : T \to G$ is *feasible for S wrt ν* if and only if

(1) μ_S is measurable wrt S, and

(2) $\mu_S(t) \in G_S^{\nu,t}$ for all $t \in T$.

The set of all feasible mechanisms for $S \subset N$ wrt ν is denoted by $\mathcal{F}_S^{\nu}$. Notice that, for $S = N$, $G_S^{\nu,t} = G$ for any mechanism ν. Thus, we write $\mathcal{F}_N$ to denote the set of all feasible mechanisms for N.

Notice that a coalition $S \subset N$ is unable to rearrange links between players in S and players outside S via a feasible mechanism for S. Thus, feasible mechanisms give only 'local' power to coalitions and our notion of measurability of a mechanism correctly supports this idea.

For every $i \in N$, we assume player i's (ex post) *utility* $u_i : G \times T_i \to \mathbb{R}$ to depend on $g \in G$ and on his own type $t_i \in T_i$. Since a network is supposed to be generated by a mechanism, we can finally introduce the notions of incentive compatible and efficient mechanisms. Suppose that a player i is of type $t_i \in T_i$ but reports that he is of type $r_i \in T_i$. Then, player i's *expected utility* is

$$U_i(\mu \mid t_i, r_i) = \sum_{t_{-i} \in T_{-i}} q(t_{-i} \mid t_i) u_i(\mu(t_{-i}, r_i), t_i),$$

where

$$q(t_{-i} \mid t_i) = \frac{q(t_{-i}, t_i)}{\sum_{t'_{-i} \in T_{-i}} q(t'_{-i}, t_i)}.$$

For $r_i = t_i$, i's *interim expected utility* given his type t_i is denoted by

$$U_i(\mu \mid t_i) := U_i(\mu \mid t_i, t_i).$$

Player i's *ex ante expected utility* is then

$$U_i(\mu) = \sum_{t_i \in T_i} q(t_{-i}, t_i) U_i(\mu \mid t_i).$$

A mechanism μ is *incentive compatible* if

$$U_i(\mu \mid t_i) \geq U_i(\mu \mid t_i, r_i) \text{ for all } i \in N \text{ and all } r_i, t_i \in T_i.$$

The set of all feasible mechanisms for $S \subset N$ wrt ν that are also incentive compatible is denoted by $\mathcal{F}_S^{\nu,*}$. Clearly, $\mathcal{F}_S^{\nu,*} \subseteq \mathcal{F}_S^{\nu}$. For $S = N$, we write $\mathcal{F}_N^*$ to denote the set of all incentive compatible feasible mechanisms for N.

A mechanism $\mu \in \mathcal{F}_N$ $(\mu \in \mathcal{F}_N^*)$ is *ex ante (incentive) efficient* if and only if there is no mechanism $\nu \in \mathcal{F}_N$ $(\nu \in \mathcal{F}_N^*)$ such that $U_i(\nu) > U_i(\mu)$ for all $i \in N$.

3 Incentive compatible mechanisms

We start by showing that there are many mechanisms in our setup that are incentive compatible. As it turns out, the two mechanisms mentioned in the

Introduction play a crucial role in determining the set of incentive compatible mechanisms.

According to the first mechanism, $\overline{\overline{\mu}}$, a link between two players is formed only if, in the given profile of types, these two players would like to be connected with each other. In the second mechanism, $\overline{\mu}$, the wish of one of the players suffices for the link to be built.

Thus, for each $t \in T$,

$$\begin{aligned} \overline{\overline{\mu}}(t) &:= \{ij \mid i \in t_j \wedge j \in t_i, i \neq j\}, \\ \overline{\mu}(t) &:= \{ij \mid i \in t_j \vee j \in t_i, i \neq j\}. \end{aligned}$$

Notice that both mechanisms are pair-wise measurable and satisfy the following property.

Pair-wise efficiency A mechanism μ is pair-wise efficient iff for all $i, j \in N$, $i \neq j$ and all $t \in T$,

- $i \in t_j$ and $j \in t_i$ imply $ij \in \mu(t)$, and
- $i \notin t_j$ and $j \notin t_i$ imply $ij \notin \mu(t)$.

As we show next, any pair-wise measurable and pair-wise efficient mechanism is incentive compatible provided the following form of players' utility functions.

For any $i \in N$ and all $(g, t_i) \in G \times T_i$,

$$u_i(g, t_i) := \widetilde{u}_i(P_i(g), t_i)$$

with $\widetilde{u}_i(\cdot, t_i)$ satisfying the following property for each $t_i \in T_i$.

$$\begin{gathered} \text{For all } g, g' \in G: \\ [P_i(g) \cap t_i] \cup [N \setminus (P_i(g) \cap (N \setminus t_i))] \subset [P_i(g') \cap t_i] \cup [N \setminus (P_i(g') \cap (N \setminus t_i))] \\ \Downarrow \\ \widetilde{u}_i(P_i(g), t_i) < \widetilde{u}_i(P_i(g'), t_i). \end{gathered} \tag{A1}$$

The specification just says that a player's utility depends solely on his direct neighbors (and his own type), while (A1) captures the natural idea that a player is strictly better off in a network in which the set of his direct neighbors contains either more friends or less enemies of him.

Proposition 1 *Let* $(\widetilde{u}_1, \ldots, \widetilde{u}_n)$ *be a profile of utility functions satisfying* (A1)*. Then any pair-wise measurable and pair-wise efficient mechanism is incentive compatible.*

Proof. Let μ be as above. By pair-wise efficiency, $\overline{\overline{\mu}}(t) \subseteq \mu(t) \subseteq \overline{\mu}(t)$ holds for all $t \in T$. By pair-wise measurability, for all $t, t' \in T$ and $i, j \in N$, $\mu(t)_{|\{i,j\}} \neq \mu(t')_{|\{i,j\}}$ can happen only if $t_{\{i,j\}} \neq t_{\{i,j\}}$. Then, for all $i \in N$, $r_i, t_i \in T_i$ and $t_{-i} \in T_{-i}$, we have

$$P_i(\mu(t_{-i}, r_i)) \cap t_i \subseteq P_i(\mu(t)) \cap t_i$$

and

$$P_i(\mu(t)) \cap (N \setminus t_i) \subseteq P_i(\mu(t_{-i}, r_i)) \cap (N \setminus t_i).$$

By $\widetilde{u}_i$ satisfying (A1), we have

$$\begin{aligned} U_i(\mu \mid t_i) &= \sum_{t_{-i} \in T_{-i}} q(t_{-i} \mid t_i)\, \widetilde{u}_i(\mu(t), t_i) \\ &\geq \sum_{t_{-i} \in T_{-i}} q(t_{-i} \mid t_i)\, \widetilde{u}_i(\mu(t_{-i}, r_i), t_i) \\ &= U_i(\mu \mid t_i, r_i), \end{aligned}$$

as required for the incentive compatibility of μ. ■

It is easy to see that there are pair-wise measurable mechanisms which are not incentive compatible; as an example, one could take the mechanism μ^- defined as follows: for each $t \in T$, $ij \in \mu^-(t)$ iff $i \notin t_j$ and $j \notin t_i$. On the other hand there are also mechanisms which are pair-wise efficient but not incentive compatible. In order to define such a mechanism, let us take three players and two profiles t' and t'' s.t. $2 \in t'_1 \cap t''_1$, $1 \notin t'_2 \cup t''_2$, $1 \notin t'_3 \cup t''_3$, $3 \notin t'_1$, $3 \in t''_1$. Define now the mechanism ν as follows: For all $i, j \in N$, $i \neq j$,

$$\nu(t)_{|\{i,j\}} = \begin{cases} \overline{\overline{\mu}}(t)_{|\{i,j\}} & \text{if } t \neq t'', \\ \overline{\overline{\mu}}(t)_{|\{i,j\}} & \text{if } t = t'' \text{ and } 3 \in \{i,j\}, \\ \overline{\mu}(t)_{|\{1,2\}} & \text{if } t = t''. \end{cases}$$

Notice that ν is pair-wise efficient, it is not pair-wise measurable, and it provides an incentive for player 1 for cheating at profile t' (he has an incentive to announce that his type is t''_1).

Observe further that neither pair-wise efficiency nor pair-wise measurability is a necessary condition for incentive compatibility. Clearly, on any domain of type profiles one can find a constant mechanism (being incentive compatible and pair-wise measurable) which is not pair-wise efficient. Consider next four players and let $T_1 \times T_2 \times T_3 \times T_4 = \{t, t'\} = \{(2,3,2,\emptyset), (2,3,2,3)\}$. Then the mechanism μ defined by $\mu(t) = \{23\}$, $\mu(t') = \{12, 23\}$ is both pair-wise efficient and incentive compatible but not pair-wise measurable (we have $t_{\{1,2\}} = t_{\{1,2\}}$ but $\mu(t)_{|\{1,2\}} \neq \mu(t)_{|\{1,2\}}$).

4 The ex ante stage

In this section we suppose that players enter into coalitional contracts at the ex ante stage (before any player receives private information) and study the existence of mechanisms that are in the ex ante (incentive compatible) core.

We say that $S \subseteq N$ is an *ex ante (incentive compatible) blocking* for μ if there exists a mechanism $\nu_S \in \mathcal{F}^{\mu}_S$ ($\nu_S \in \mathcal{F}^{\mu,*}_S$) such that $U_i(\nu_S) > U_i(\mu)$ for all $i \in S$. A mechanism μ belongs to the *ex ante (incentive compatible) core* (denoted by EAC ($EAICC$)) if and only if there are no ex ante (incentive compatible) blockings for μ.

There are two remarks with respect to this definition of coalitional blockings we would like to mention. First, notice again that a coalition S can block only via feasible mechanisms and thus, links between members of S and members of $N \setminus S$ do not enter the utility calculations. Second, since a coalitional blocking does not affect the well being of players outside the corresponding coalition, a coalitional blocking can be seen as a necessary step towards an efficient mechanism.

4.1 The ex ante core in the general case

We start by providing some necessary conditions for ex ante core stable mechanisms.

Proposition 2 *Let* $(\widetilde{u}_1, \ldots, \widetilde{u}_n)$ *be a profile of utility functions satisfying* (A1)*.*

(1) *Let* μ *be a mechanism such that* $\overline{\overline{\mu}}(t') \nsubseteq \mu(t')$ *for some* $t' \in T$ *with* $q(t') > 0$*. Then,* $\mu \notin EAC$*.*

(2) *Let* μ *be a mechanism such that* $ij \in \mu(t')$ *and* $ij \notin \overline{\mu}(t')$ *for some* $i, j \in N$*,* $i \neq j$ *and some* $t' \in T$ *with* $q(t') > 0$*. Then,* $\mu \notin EAC$*.*

Proof. (1) Let μ and t' be as described and suppose on the contrary that $\mu \in EAC$. It follows from $\overline{\overline{\mu}}(t') \nsubseteq \mu(t')$ that there exists a coalition $\{i, j\}$ such that $ij \in \overline{\overline{\mu}}(t')$ and $ij \notin \mu(t')$. Consider the mechanism $\nu_{\{i,j\}} : T \to G$ defined by

$$\nu_{\{i,j\}}(t) = \begin{cases} \mu(t) & \text{if } t \neq t', \\ \mu(t') \cup \{ij\} & \text{otherwise,} \end{cases}$$

and notice that $\nu_{\{i,j\}} \in \mathcal{F}^{\mu}_{\{i,j\}}$. Moreover, for $k, l \in \{i, j\}$, we have $P_i\left(\nu_{\{i,j\}}(t)\right) \cap$

$t_i \supset P_i(\mu(t)) \cap t_i$ and $P_i\left(\nu_{\{i,j\}}(t)\right) \cap (N \setminus t_i) = P_i(\mu(t)) \cap (N \setminus t_i)$. Hence, we have by (A1)

$$U_k\left(\nu_{\{i,j\}}\right) = \sum_{t \in T} q(t)\, \widetilde{u}_k\left(\nu_{\{i,j\}}(t), t_i\right) > \sum_{t \in T} q(t)\, \widetilde{u}_k(\mu(t), t_i) = U_k(\mu)$$

and thus, $\{i,j\}$ is an ex ante blocking (via $\nu_{\{i,j\}}$) for μ. Thus, we have a contradiction to $\mu \in EAC$.

(2) Let μ, i, j and t' be as described and suppose on the contrary that $\mu \in EAC$. Consider the mechanism $\nu_{\{i,j\}} : T \to G$ defined by

$$\nu_{\{i,j\}}(t) = \begin{cases} \mu(t) & \text{if } t \neq t', \\ \mu(t') \setminus \{ij\} & \text{otherwise,} \end{cases}$$

and notice that $\nu_{\{i,j\}} \in \mathcal{F}^{\mu}_{\{i,j\}}$. Moreover, for $k, l \in \{i,j\}$, we have $P_i\left(\nu_{\{i,j\}}(t)\right) \cap t_i = P_i(\mu(t)) \cap t_i$ and $P_i\left(\nu_{\{i,j\}}(t)\right) \cap (N \setminus t_i) \subset P_i(\mu(t)) \cap (N \setminus t_i)$. Hence, we have by (A1)

$$U_k\left(\nu_{\{i,j\}}\right) = \sum_{t \in T} q(t)\, \widetilde{u}_k\left(\nu_{\{i,j\}}(t), t_i\right) > \sum_{t \in T} q(t)\, \widetilde{u}_k(\mu(t), t_i) = U_k(\mu)$$

and thus, $\{i,j\}$ is an ex ante blocking (via $\nu_{\{i,j\}}$) for μ. Thus, we have a contradiction to $\mu \in EAC$. ■

Corollary 1 *Let μ be a constant mechanism. Then, $\mu \in EAC$ implies $\cup_{t' \in T, q(t') > 0} \overline{\overline{\mu}}(t') \subseteq \mu(t) \subseteq \cap_{t' \in T, q(t') > 0} \overline{\mu}(t')$ for all $t \in T$ with $q(t) > 0$.*

Taking into account the statement in Corollary 1, one can easily see that if there are profiles $t, t' \in T$ and two players who like each other at t but hate each other at t', then no constant mechanism can be in the ex ante core.

4.2 Link-specific costs and benefits and non-emptiness of the ex ante incentive compatible core

As we show next, depending on the cost/benefit structure of players' utility functions, the mechanisms $\overline{\overline{\mu}}$ and $\overline{\mu}$ turn out to be crucial for showing the non-emptiness of the ex ante incentive compatible core.

In order to state our existence result, we consider the following specification of players' utility functions. For any $i \in N$ and all $(g, t_i) \in G \times T_i$,

$$u_i(g, t_i) := u_i^{C,D}(P_i(g), t_i)$$

with

$$u_i^{C,D}(P_i(g), t_i) = \sum_{i \in P_i(g) \cap t_i} (c_{ij} + d_{ij}) - \sum_{i \in P_i(g)} d_{ij} \tag{A2}$$

where $c_{ij}, d_{ij} > 0$, $c_{ij} = c_{ji}$, and $d_{ij} = d_{ji}$ for all $i, j \in N$, $i \neq j$. In other words, the above specification of the utility functions takes into account the fact that being linked is always costly ($d_{ij} > 0$), while having a desired link yields an additional utility of $c_{ij} > 0$. Notice that costs and benefits are link-specific and that (A1) is satisfied as well.

Theorem 1 *Let* $\left(u_1^{C,D}, \ldots, u_n^{C,D}\right)$ *be a profile of utility functions as specified in* (A2)*. Then the ex ante incentive compatible core is non-empty.*

Proof. Consider the mechanism μ^* defined as follows: For all $i, j \in N, i \neq j$ and all $t \in T$,

$$\mu^*(t)_{|\{i,j\}} = \begin{cases} \overline{\mu}(t)_{|\{i,j\}} & \text{if } c_{ij} > d_{ij}, \\ \overline{\overline{\mu}}(t)_{|\{i,j\}} & \text{if } c_{ij} \leq d_{ij}. \end{cases}$$

Notice that μ^* is incentive compatible (it is both pair-wise measurable and pair-wise efficient). We show that $\mu^* \in EAICC$ by proving that there are no ex ante blockings for μ^* at all. Notice that it suffices to show that for each

$S \subseteq N$ and $\nu_S \in \mathcal{F}_S^{\mu^*}$ there exists a player $i \in S$ such that $U_i(\nu_S) \leq U_i(\mu^*)$. For this, we show that $\sum_{i\in S}(U_i(\nu_S) - U_i(\mu^*)) \leq 0$.

For each $i \in S$ and $t \in T$ define the sets

$$
\begin{aligned}
X_t^*(i) &:= \left\{ j \in S \cap t_i \mid ij \in \overline{\overline{\mu}}(t) \wedge ij \notin \nu_S \right\}, \\
X_t^{**}(i) &:= \left\{ j \in S \cap t_i \mid ij \notin \overline{\overline{\mu}}(t) \wedge ij \in \overline{\mu}(t) \wedge ij \notin \nu_S \right\}, \\
Y_t(i) &:= \{ j \in S \cap (N \setminus t_i) \mid ij \in \mu^*(t) \wedge ij \notin \nu_S \}, \\
Z_t^*(i) &:= \{ j \in S \cap (N \setminus t_i) \mid ij \notin \overline{\mu}(t) \wedge ij \in \nu_S(t) \}, \\
Z_t^{**}(i) &:= \{ j \in S \cap (N \setminus t_i) \mid ij \in \overline{\mu}(t) \wedge c_{ij} \leq d_{ij} \wedge ij \in \nu_S(t) \}, \\
W_t(i) &:= \{ j \in S \cap t_i \mid ij \in \overline{\mu}(t) \wedge c_{ij} \leq d_{ij} \}.
\end{aligned}
$$

Then, for each $t \in T$ we have

$$
\begin{aligned}
& u_i(\nu_S(t), t_i) - u_i(\mu^*(t), t_i) \\
= \; & -\sum_{j\in X_t^*(i)} c_{ij} - \sum_{j\in X_t^{**}(i)} c_{ij} + \sum_{j\in Y_t(i)} d_{ij} - \sum_{j\in Z_t^*(i)} d_{ij} - \sum_{j\in Z_t^{**}(i)} d_{ij} + \sum_{j\in W_t(i)} c_{ij}.
\end{aligned}
$$

It follows from the definition of μ^* that for all $i, j \in S$, $i \neq j$, we have $j \in X_t^{**}(i)$ if and only if $i \in Y_t(j)$, and $j \in Z_t^{**}(i)$ if and only if $i \in W_t(j)$. Notice also that that we have $c_{ij} > d_{ij}$ for all $j \in X_t^{**}(i)$, and $c_{ij} \leq d_{ij}$ for all $j \in Z_t^{**}(i)$. Thus,

$$
-\sum_{i\in S} \sum_{j\in X_t^{**}(i)} c_{ij} + \sum_{i\in S} \sum_{j\in Y_t(i)} d_{ij} < 0 \tag{1}
$$

and

$$
-\sum_{i\in S} \sum_{j\in Z_t^{**}(i)} d_{ij} + \sum_{i\in S} \sum_{j\in W_t(i)} c_{ij} \leq 0. \tag{2}
$$

We have then

$$
\begin{aligned}
& \sum_{i\in S}\left(U_i\left(\nu_S\right)-U_i\left(\mu^*\right)\right) \\
= & \sum_{t\in T} q\left(t\right)\left[\sum_{i\in S} u_i\left(\nu_S\left(t\right),t_i\right)-\sum_{i\in S} u_i\left(\mu^*\left(t\right),t_i\right)\right] \\
= & \sum_{t\in T} q\left(t\right)\left[-\sum_{i\in S}\sum_{j\in X_t^*(i)} c_{ij}-\sum_{i\in S}\sum_{j\in Z_t^*(i)} d_{ij}\right] \\
& +\sum_{t\in T} q\left(t\right)\left[-\sum_{i\in S}\sum_{j\in X_t^{**}(i)} c_{ij}+\sum_{i\in S}\sum_{j\in Y_t(i)} d_{ij}\right] \\
& +\sum_{t\in T} q\left(t\right)\left[-\sum_{i\in S}\sum_{j\in Z_t^{**}(i)} d_{ij}+\sum_{i\in S}\sum_{j\in W_t(i)} c_{ij}\right] \\
\leq & \ 0,
\end{aligned}
$$

where the inequality follows from (1) and (2). ■

In view of the proof of Theorem 1 we can conclude that, in particular, there is no feasible mechanism for the grand coalition that blocks μ^*. Thus, we have the following result.

Proposition 3 *The mechanism μ^* is ex ante (incentive) efficient.*

4.3 Examples

Let us now consider the special case in which costs and benefits are homogeneous across links, i.e., $c_{ij} = c$ and $d_{ij} = d$ hold for all $i, j \in N$, $i \neq j$. By Theorem 1, $\overline{\mu} \in EAICC$ if $c > d$ and $\overline{\overline{\mu}} \in EAICC$ if $c \leq d$.

Notice however, that for $c > d$ there are situations where $\overline{\overline{\mu}}$ is not ex ante (incentive) efficient. Moreover, there are also cases for $c \leq d$ where $\overline{\mu}$ is not ex ante (incentive) efficient either. Thus, as we exemplify next, it may happen that $\overline{\overline{\mu}} \notin EAICC$ when $c > d$ and $\overline{\mu} \notin EAICC$ when $c \leq d$. We also show in the remarks after the corresponding examples that if only pairs are allowed to block a mechanism, then the ex ante core is a strictly larger set; in other words, *the entire network matters.*

Example 1 *Let* $N = \{1,2,3\}$, $T_1 = \{2,3\}$, $T_2 = \{1\}$, *and* $T_3 = \{12,2\}$ *with* $q(2,1,12) = p_1 = \frac{1}{8}$, $q(2,1,2) = p_2 = \frac{3}{8}$, $q(3,1,12) = p_3 = \frac{1}{8}$, $q(3,1,2) = p_4 = \frac{3}{8}$. *For* $c = 4 > 1 = d$ *we have*

$$U_1(\overline{\mu}) = (c-d)(1-p_2) + cp_2 = \tfrac{27}{8} > \tfrac{20}{8} = c(p_1+p_2+p_3) = U_1\left(\overline{\overline{\mu}}\right),$$
$$U_2(\overline{\mu}) = c - d = \tfrac{24}{8} > \tfrac{16}{8} = c(p_1+p_2) = U_2\left(\overline{\overline{\mu}}\right),$$
$$U_3(\overline{\mu}) = 2c(p_1+p_3) + cp_2 + (c-d)p_4 = \tfrac{37}{8} > \tfrac{4}{8} = cp_3 = U_3\left(\overline{\overline{\mu}}\right).$$

Thus, N *is an ex ante incentive compatible blocking (via* $\overline{\mu}$*) for* $\overline{\overline{\mu}}$. ■

Remark 1 *Observe that, in the above situation, no two-player coalition* $\{i,j\} \subset N$ *of players is a blocking (via a corresponding feasible mechanism) for* $\overline{\overline{\mu}}$. *To see this, let us consider the following three possible cases:*
(1) $\{i,j\} = \{1,2\}$. *Notice that* $12 \in \overline{\overline{\mu}}(t)$ *for any profile* $t \in T$ *at which* $2 \in t_1$. *Thus, for any mechanism* $\nu_{\{1,2\}} \in \mathcal{F}^{\overline{\overline{\mu}}}_{\{1,2\}}$ *with* $\nu_{\{1,2\}} \neq \overline{\overline{\mu}}$ *one would have either* $12 \notin \nu_{\{1,2\}}(t)$ *for some* $t \in T$ *at which* $2 \in t_1$, *or* $12 \in \nu_{\{1,2\}}(t')$ *for some* $t' \in T$ *at which* $2 \notin t_1$. *Hence, since* $q(t) > 0$ *for all* $t \in T$, *player 1 will be worse off under* $\nu_{\{1,2\}}$ *in comparison to* $\overline{\overline{\mu}}$.
(2) $\{i,j\} = \{2,3\}$. *Notice that* $23 \notin \overline{\overline{\mu}}(t)$ *for all* $t \in T$ *and that for any* $\nu_{\{2,3\}} \in \mathcal{F}^{\overline{\overline{\mu}}}_{\{2,3\}}$ *with* $\nu_{\{2,3\}} \neq \overline{\overline{\mu}}$ *one would have* $23 \in \nu_{\{2,3\}}(t)$ *for some* $t \in T$. *Clearly then, since* $3 \notin T_2$ *and* $q(t) > 0$ *for all* $t \in T$, *player 3 will be worse off under* $\nu_{\{2,3\}}$ *in comparison to* $\overline{\overline{\mu}}$.
(3) $\{i,j\} = \{1,3\}$. *If* $\nu_{\{1,3\}} \in \mathcal{F}^{\overline{\overline{\mu}}}_{\{1,3\}}$ *is supposed to make player 1 strictly better off in comparison to* $\overline{\overline{\mu}}$, *then one should have* $13 \in \nu_{\{1,3\}}(3,1,2)$ *which implies that, at the profile* $(3,1,2)$, *player 3 is worse off. Thus, in order*

player 3 to be compensated, one should have $13 \in \nu_{\{1,3\}}(2,1,12)$ *as well. To see that this compensation does not suffices, consider the mechanism*

$$\nu^*_{\{1,3\}}(t) = \begin{cases} \overline{\overline{\mu}}(t) \cup \{13\} & \text{if } t = (3,1,2), \\ \overline{\overline{\mu}}(t) \cup \{13\} & \text{if } t = (2,1,12), \\ \overline{\overline{\mu}}(t) & \text{otherwise,} \end{cases}$$

which satisfies the above two conditions. We have then

$$\begin{aligned} U_1\left(\nu^*_{\{1,3\}}\right) &= U_1\left(\overline{\overline{\mu}}\right) - dp_1 + (c-d)\,p_4 = \tfrac{28}{8} > \tfrac{20}{8} = U_1\left(\overline{\overline{\mu}}\right), \\ U_3\left(\nu^*_{\{1,3\}}\right) &= U_3\left(\overline{\overline{\mu}}\right) + (c-d)\,p_1 - dp_4 = \tfrac{4}{8} = U_3\left(\overline{\overline{\mu}}\right). \end{aligned}$$

By noticing that any other feasible mechanism for $\{1,3\}$ *that differs from* $\nu^*_{\{1,3\}}$ *at profiles* $(2,1,2)$ *and/or* $(3,1,12)$ *would provide weakly less ex ante expected utility for the players as* $\nu^*_{\{1,3\}}$*, we conclude that* $\{1,3\}$ *is not a blocking for* $\overline{\overline{\mu}}$*.*

Example 2 *Let* $N = \{1,2,3\}$*,* $T_1 = \{2,3\}$*,* $T_2 = \{1\}$*, and* $T_3 = \{12,2\}$ *with* $q(2,1,12) = p_1 = \frac{1}{8}$*,* $q(2,1,2) = p_2 = \frac{1}{8}$*,* $q(3,1,12) = p_3 = \frac{1}{8}$*,* $q(3,1,2) = p_4 = \frac{5}{8}$*. For* $c = 2 < 4 = d$ *we have*

$$\begin{aligned} U_1\left(\overline{\overline{\mu}}\right) &= c(p_1 + p_2 + p_3) = \tfrac{6}{8} > -\tfrac{12}{8} = (c-d)(1-p_2) + cp_2 = U_1(\overline{\mu}), \\ U_2\left(\overline{\overline{\mu}}\right) &= c(p_1 + p_2) = \tfrac{4}{8} > -2 = c - d = U_2(\overline{\mu}), \\ U_3\left(\overline{\overline{\mu}}\right) &= cp_3 = \tfrac{2}{8} > 0 = 2c(p_1 + p_3) + cp_2 + (c-d)\,p_4 = U_3(\overline{\mu}). \end{aligned}$$

Thus, N *is an ex ante incentive compatible blocking (via* $\overline{\overline{\mu}}$*) for* $\overline{\mu}$*.* ■

Remark 2 *Again, in the above situation, no two-player coalition* $\{i,j\} \subset N$ *of players is a blocking (via a corresponding feasible mechanism) for* $\overline{\mu}$*. Consider the following cases:*

(1) $\{i,j\} = \{1,2\}$*. Recall that any feasible mechanism for* $\{1,2\}$ *rearranges links only between players 1 and 2; any feasible mechanism for* $\{1,2\}$ *that*

differs from $\overline{\mu}$ *should then delete a link between players 1 and 2 at some* $t \in T$*. This however implies, by* $12 \in \overline{\mu}(t)$ *for all* $t \in T$*,* $T_2 = \{1\}$*, and* $q(t) > 0$ *for all* $t \in T$*, that there is no feasible mechanism for* $\{1,2\}$ *that would make player 2 strictly better off in comparison to* $\overline{\mu}$*.*

(2) $\{i,j\} = \{2,3\}$*. We can apply a similar reasoning as above with respect to player 3 to conclude that there is no feasible mechanism for* $\{2,3\}$ *that blocks* $\overline{\mu}$*.*

(3) $\{i,j\} = \{1,3\}$*. If* $\nu_{\{1,3\}} \in \mathcal{F}^{\overline{\mu}}_{\{1,3\}}$ *is supposed to make player 3 strictly better off in comparison to* $\overline{\mu}$*, then one should have* $13 \notin \nu_{\{1,3\}}(3,1,2)$ *which implies that, at the profile* $(3,1,2)$*, player 1 is worse off. Thus, in order player 1 to be compensated, one should have* $13 \notin \nu_{\{1,3\}}(2,1,12)$ *as well. To see that this compensation does not suffices, consider the mechanism*

$$\nu^{**}_{\{1,3\}}(t) = \begin{cases} \overline{\mu}(t) \setminus \{13\} & \text{if } t = (2,1,12), \\ \overline{\mu}(t) \setminus \{13\} & \text{if } t = (3,1,2), \\ \overline{\mu}(t) & \text{otherwise,} \end{cases}$$

which satisfies the above two conditions. We have then

$$U_1\left(\nu^{**}_{\{1,3\}}\right) - U_1(\overline{\mu}) = dp_1 - cp_4 = -\frac{6}{8} < 0.$$

By noticing that no other feasible mechanism for $\{1,3\}$ *that differs from* $\nu^{**}_{\{1,3\}}$ *at profiles* $(2,1,2)$ *and/or* $(3,1,12)$ *would provide a higher ex ante expected utility for player 1 as* $\nu^{**}_{\{1,3\}}$*, we conclude that* $\{1,3\}$ *is not a blocking for* $\overline{\overline{\mu}}$*.*

Finally, there are also situations where both $\overline{\overline{\mu}}$ and $\overline{\mu}$ belong to the ex ante incentive compatible core.

Example 3 *Let* $N = \{1,2,3\}$*,* $T_1 = \{3\}$*,* $T_2 = \{3\}$*, and* $T_3 = \{1,12\}$ *with*

$q(3,3,1) = q(3,3,12) = \frac{1}{2}$. For $c = 3 > 1 = d$ we have

$$U_1\left(\overline{\overline{\mu}}\right) = 3 = U_1\left(\overline{\mu}\right),$$
$$U_2\left(\overline{\overline{\mu}}\right) = 1.5 < 3 = U_2\left(\overline{\mu}\right),$$
$$U_3\left(\overline{\overline{\mu}}\right) = 4.5 > 2.5 = U_3\left(\overline{\mu}\right).$$

It follows from $c > d$ and Theorem 1 that $\overline{\mu} \in EAICC$. We show that $\overline{\overline{\mu}} \in EAICC$ as well. Notice that $\overline{\overline{\mu}}$ gives both player 1 and player 3 their maximal ex ante utilities and thus, it is not worthy for these players to participate in any blocking. The blocking consisting of player 2 is not worthy for him either (his ex ante expected utility is again 1.5). Hence, $\overline{\overline{\mu}} \in EAICC$. ■

Remark 3 *The fact that in the above example both $\overline{\overline{\mu}}$ and $\overline{\mu}$ belong to the ex ante incentive compatible core is not due to the costs and benefits homogeneity. To see this, consider a situation that differs from the one in Example 3 only wrt the costs and benefits, and take for instance $c_{13} = c_{31} = 3 > 1 = d_{13} = d_{31}$ and $c_{23} = c_{32} = 2 < 3 = d_{23} = d_{32}$. One can then easily show that both $\overline{\overline{\mu}}$ (which equals μ^* in this case) and $\overline{\mu}$ belong to the ex ante incentive compatible core.*

5 The interim stage

Consider now the situation where players enter into coalitional contracts at the interim stage, i.e., each player knows his private information (his type) and has some probability assessment over the true information of others. One of the central issues that arises here is about the specification of the information that agents in a coalition are allowed to use in constructing an objection. In what follows, we focus on coarse objections (cf. Wilson (1978)). The coarse core is based then on the assumption that a coalition can focus its objections on an event if and only if this event is commonly known to all members of the coalition.

More precisely, given an event $E \subseteq T$, define

$$E_i = \{t_i \in T_i \mid (t_{-i}, t_i) \in E \text{ for some } t_{-i} \in T_{-i}\}.$$

An event $E \subseteq T$ is *common knowledge for* S if $q\left(t'_{-i} \mid t_i\right) = 0$ holds for all $i \in S$, $t_i \in E_i$ and $\left(t'_{-i}, t_i\right) \notin E$. The set of all common knowledge events for a coalition S is denoted by $\mathcal{E}_S$.

We say that $S \subseteq N$ is an *interim coarse (incentive compatible) blocking* for μ if there is $\nu_S \in \mathcal{F}_S^{\mu}$ ($\nu_S \in \mathcal{F}_S^{\mu,*}$) and $E \in \mathcal{E}_S$ such that $U_i(\nu_S \mid t_i) > U_i(\mu \mid t_i)$ for all $i \in S$ and $t_i \in E_i$. A mechanism μ belongs to the *interim (incentive compatible) coarse core* (denoted by ICC ($IICCC$)) if and only if there are no interim (incentive compatible) coarse blockings for μ.

Theorem 2 *Let* $\left(u_1^{C,D}, \ldots, u_n^{C,D}\right)$ *be a profile of utility functions as specified in* (A2). *Then the interim incentive compatible coarse core is non-empty.*

Proof. Consider the mechanism μ^* as defined in the proof of Theorem 1. Notice that for $\mu^* \in IICCC$ it suffices to show that for each $S \subseteq N$, $\nu_S \in \mathcal{F}_S^{\mu^*}$ and $E \in \mathcal{E}_S$ there exists a player $i \in S$ such that $U_i(\nu_S \mid t_i) \leq U_i(\mu^* \mid t_i)$ for some $t_i \in E_i$. More precisely, it is sufficient to show that $\sum_{i \in S}(U_i(\nu_S \mid t_i) - U_i(\mu^* \mid t_i)) \leq 0$ for some $t \in E$. Hence, we can fix $t \in E$ and precede in the same way as in the proof of Theorem 1. ■

6 Conclusion

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